# Factor models and the credit risk of a loan portfolio

Palombini, Edgardo

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**Abstract**

Factor models for portfolio credit risk assume that defaults are independent conditional on a small number of systematic factors. This paper shows that the conditional independence assumption may be violated in one-factor models with constant default thresholds, as conditional defaults become independent only including a set of observable (time-lagged) risk factors. This result is confirmed both when we consider semi-annual default rates and if we focus on small firms. Maximum likelihood estimates for the sensitivity of default rates to systematic risk factors are obtained, showing how they may substantially vary across industry sectors. Finally, individual risk contributions are derived through Monte Carlo simulation.

**Keywords:** Asset correlation, factor models, loss distribution, portfolio credit risk, risk contributions.

**JEL Classification:** G21, C13, C15

Address: Fondo Interbancario di Tutela dei Depositi, Via del Plebiscito 102, 00186 Rome, Italy, email:epalombini@fitd.it[1]

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# 1 Introduction

Estimating the credit risk of a portfolio of financial instruments requires two types of inputs: 1) exposure-specific variables like probability of default, loss given default and exposure at default; 2) the multivariate distribution of future credit events or, in other words, some assumptions on both the marginal distribution and the dependence structure of variables that may trigger defaults. The second type of inputs includes the estimation of correlations between these variables, which are often interpreted as the obligors' asset values, as in the credit risk model introduced by Merton (1974).

Correlations are particularly important because they limit the possibility to fully diversify credit risk at a portfolio level. For large and diversified portfolios they represent the main source of uncertainty about future losses, but their estimation is quite challenging, due to the fact that asset returns are not observable. The existing literature offers three alternatives to solve this problem. First, a factor structure can be assumed for asset returns, and asset correlations are derived from the covariance matrix of the chosen factors (Kealhofer and Bohn, 2001). Second, the rapid development of the credit derivatives market allows to derive the correlations implicit in the prices of such instruments (Tarashev and Zhu, 2006). Third, asset correlations can be obtained from historical data on defaults and credit transitions.

When default data are used, asset correlations can be derived from default correlations using non parametric approaches as in Lucas (1995) or in Gordy (2000). Alternatively, asset correlations are obtained directly from default data, through maximum likelihood estimation; this approach is introduced by Gordy and Heitfield (2002) and then used in various papers, among which Düllmann and Scheule (2003) and Rösch and Scheule (2004). An important advantage of parametric methodologies is that structural restrictions arising from theoretical assumptions can be easily incorporated in the model. Furthermore, deterministic explanatory variables like macroeconomic factors can be considered together with unobservable random factors.

Many empirical works and applications, including the Basel II regulatory framework, are based on the model proposed by Vasicek (1997). Asset returns are assumed to be jointly Gaussian distributed and defaults occur when they fall below a constant threshold. A crucial assumption of one-factor models is that defaults are independent conditional on the (single) common risk factor. As suggested by Schönbucher (2000) this assumption should be carefully checked, as a violation could lead to an underestimation of tail probabilities.

Our empirical results show that in the one-factor framework the hypothesis of conditional independence between defaults is violated, and that the introduction of a set of deterministic explanatory variables is necessary. This evidence is confirmed also when we use semi-annual data on default rates, thus obtaining estimates based on a considerably longer time series of defaults. Similar results are also found estimating a specific model for small enterprises, even if sensitivities to economic conditions tend to decrease.

This paper is organized as follows. Section 2 describes the two-state factor model used in the following analysis. Section 3 presents our empirical results, how they vary according to the industry and the size of the obligor, and also according to the length of the horizon considered. In Section 4 individual contributions to portfolio credit risk are obtained through Monte Carlo simulation.

# 2 A credit risk model with factor structure

Consider a portfolio of exposures where the normalized return on the assets of obligor $i$ at time $t$, with $i \in \{1, ..., n_t\}$ and $t \in \{1, ..., T\}$, is driven by both observable (time-lagged) and unobservable risk factors:

$$R_{i,t} = \boldsymbol{\alpha}'\mathbf{Z}_{t-1} + U_{i,t} \tag{1}$$

where the vector $\mathbf{Z}_{t-1}$ represents a set of information available at time $t-1$ to estimate the default probability of obligor $i$. It may include obligor-specific variables as well as factors that affect the credit quality of the whole portfolio. The unobservable variable $U_{i,t}$ is influenced by a one-dimensional systematic factor $Y_t$ and by a firm-specific component $E_{i,t}$:

$$U_{i,t} = \sqrt{\delta}Y_t + \sqrt{1-\delta}E_{i,t} \tag{2}$$

where $Y_t$ and $E_{i,t}$ follow standard normal distributions. All variables in equations (1) and (2) are assumed to be independent and each $E_{i,t}$ is also independent across obligors. Therefore, conditional on the realizations of common risk factors, defaults are independent. From equation (2) it follows that the sensitivity of asset returns to the systematic unobservable risk factor depends on the time-invariant parameter $\delta$, which represents the so-called asset correlation.

The default of a firm occurs when the return on its assets falls below a threshold $k$, and let the random variable $D_{i,t}$ be a default indicator, i.e., a variable that is equal to one if firm $i$ is in default at time $t$ and zero otherwise:

$$D_{i,t} = 1 \Longleftrightarrow R_{i,t} < k \tag{3}$$

Conditional on the realization of common risk factors, the default probability can be expressed as:

$$\begin{aligned} p_{i,t}\left(\mathbf{z}_{t-1}, y_t\right) &= P\left(R_{i,t} < k \mid \mathbf{z}_{t-1}, y_t\right) \\ &= P\left(\boldsymbol{\alpha}'\mathbf{z}_{t-1} + U_{i,t} < k\right) \\ &= P\left(\boldsymbol{\alpha}'\mathbf{z}_{t-1} + \sqrt{\delta}y_t + \sqrt{1-\delta}E_{i,t} < k\right) \\ &= \Phi\left(\frac{k - \boldsymbol{\alpha}'\mathbf{z}_{t-1} - \sqrt{\delta}y_t}{\sqrt{1-\delta}}\right) \end{aligned} \tag{4}$$

where $\Phi$ denotes the standard normal cumulative density function.

As the number of borrowers grows, if the share of the largest exposure vanishes to zero, the default rate $DR_t$ converges to the conditional default probability by the strong law of large numbers:

$$DR_t \to \mathrm{E}\left[DR_t \mid \mathbf{z}_{t-1}, y_t\right] = p_{i,t}\left(\mathbf{z}_{t-1}, y_t\right) \tag{5}$$

$$\Phi^{-1}\left(DR_t\right) \to \frac{k - \boldsymbol{\alpha}'\mathbf{z}_{t-1}}{\sqrt{1-\delta}} - \frac{\sqrt{\delta}}{\sqrt{1-\delta}}y_t \tag{6}$$

For a given $q \in (0,1)$, let $\psi_q(Y)$ denote the $q^{th}$ percentile of the distribution of the random variable $Y$. If the expected default rate is monotonically decreasing in the systematic random factor, the $q^{th}$ percentile of its distribution can be expressed as (Gordy 2003):

$$\psi_q(\mathrm{E}\left[DR_t \mid \mathbf{z}_{t-1}, Y_t\right]) = \mathrm{E}\left[DR_t \mid \mathbf{z}_{t-1}, \psi_q(Y_t)\right] \tag{7}$$

Given a database of default data, assuming that the random factor is independently and identically distributed over time, the parameters in equation (4) can be estimated maximizing the log-likelihood function:

$$\ln L = \sum_{t=1}^{T} \ln \int_{-\infty}^{+\infty} \left\{ \prod_{i \in N_t} p_{i,t}\left(\mathbf{z}_{t-1}, y_t\right)^{d_{i,t}} \left(1 - p_{i,t}\left(\mathbf{z}_{t-1}, y_t\right)\right)^{1-d_{i,t}} \right\} d\Phi\left(y_t\right) \tag{8}$$

The integrals in equation (8) can be solved approximately using the adaptive Gaussian quadrature described in Pinheiro and Bates (1995). It follows from the general theory of maximum likelihood estimation that the estimates exist asymptotically, and are consistent as well as asymptotically normally distributed (Davidson and MacKinnon, 1993).

# 3 Empirical analysis

## 3.1 Data

This paper uses a time series of default data coming from the "Base Informativa Pubblica" (BIP), a database provided by the Bank of Italy containing quarterly time series for the number of defaults and the number of loans that were not in default at the beginning of the period, starting in 1990. The number of defaults in a given quarter is given by the number of borrowers that become "adjusted bad debtors" during that quarter.[2]

In the BIP database borrowers can be distinguished according to several variables like industry sector and geographical location. In this paper we focus on non-financial companies, as divided between the fifteen industries listed in Table 1, which also shows the average number of firms and the average default rate.

## 3.2 Parameter estimates

The credit risk model described in Section 2 relies on the crucial assumption that, conditional on common risk factors, defaults are independent. If this were not the case, i.e. if systematic factors did not capture all the correlation between defaults, maximizing the log-likelihood function in equation (8) would lead to an underestimation of tail probabilities.

As a test of our model we can use the fact that, under the conditional independence assumption, as the number of borrowers grows the default rate $DR_t$ converges to the conditional default probability. Since the number of borrowers in our dataset is large enough to represent a reasonable approximation of an infinitely granular portfolio, we can use (6)

[2]According to the definition used by the Bank of Italy a borrower becomes an "adjusted bad debtor" if reported to the Italian Central Credit Register: a) as a bad debt by the only bank that disbursed credit; b) as a bad debt by one bank and as having an overshoot by the only other bank exposed; c) as a bad debt by one bank and the amount of the bad debt is at least 70% of its exposure towards the banking system, or as having overshoots equal to or more than 10% of its total loans outstanding; d) as a bad debt by at least two banks for amounts equal to or more than 10% of its total loans outstanding.

Further information about the dataset and the Italian Central Credit Register is available at the Bank of Italy's website.

Table 1: Industry sectors
Average number of borrowers and
average default rate (1990-2008)

| | | $N$ | $\%DR$ |
|---|---|---|---|
| 1 | Agricultural, forestry and fishery products | 48,651 | 1.85 |
| 2 | Metal products, except transport equip. | 31,742 | 1.68 |
| 3 | Food and tobacco products | 17,113 | 2.45 |
| 4 | Textiles, clothing and footwear | 30,602 | 2.93 |
| 5 | Paper and paper products | 10,793 | 2.18 |
| 6 | Building and construction | 95,055 | 2.77 |
| 7 | Trade services, recovery and repair services | 172,580 | 2.38 |
| 8 | Lodging and catering services | 35,893 | 2.21 |
| 9 | Other market services | 117,178 | 1.84 |
| 10 | Fuel, power products and chemical products | 6,850 | 1.67 |
| 11 | Ores and metals and non-metallic minerals and products | 15,450 | 2.09 |
| 12 | Agricultural and industrial machinery and transport equip. | 22,631 | 1.91 |
| 13 | Electrical goods, office and data processing machines | 16,351 | 2.17 |
| 14 | Other manufactured products | 34,594 | 2.07 |
| 15 | Transport and communication services | 28,480 | 2.37 |

and run a regression of $\Phi^{-1}(DR_t)$ on a constant and a set of potential explanatory variables. Then, we can investigate the properties of the residuals, in particular checking if the assumptions of normality and serial independence are violated.

We start considering the one-factor version of the Vasicek model, thus imposing $\boldsymbol{\alpha}' = 0$, as this model is used in many empirical works and applications, including the Basel II regulatory framework. In fact, when correlation estimates are obtained through non-parametric approaches, as in Gordy (2000) or de Servigny and Renault (2002), asset returns are assumed to be jointly Gaussian distributed and defaults occur when they fall below a constant threshold.

If we run a regression of $\Phi^{-1}(DR_t)$ on a constant, the correlogram of residuals provides strong evidence of serial correlation. Table 2 shows autocorrelations and partial autocorrelations up to 12 lags, together with the Ljung-Box Q-statistics for the null hypothesis of no autocorrelation, which is strongly rejected. This can be explained by the fact that default rates are likely to depend on their past values due to the persistence of economic shocks and because of possible contagion effects.

In order to eliminate any serial correlation we include the (percentage) lagged default rate in the model. The coefficient of the new variable is highly significant, the model's fit improves

Table 2: Linear regression for default rate (1)

| Parameter estimates | | | Correlogram of residuals | | | | |
|---|---|---|---|---|---|---|---|
| Variable | Estimate | $Pr > \|t\|$ | Lag | AC | PAC. | Q-Stat | p-value |
| Intercept | -2.015 | 0.000 | 1 | 0.897 | 0.897 | 17.842 | 0.000 |
| | | | 2 | 0.721 | -0.427 | 30.058 | 0.000 |
| Tests for zero-mean residuals | | | 3 | 0.507 | -0.207 | 36.479 | 0.000 |
| Test | Statistic | p-value | 4 | 0.305 | 0.016 | 38.946 | 0.000 |
| Sign | -0.5 | 1.000 | 5 | 0.102 | -0.208 | 39.246 | 0.000 |
| Wilkoxon | 0 | 1.000 | 6 | -0.072 | -0.034 | 39.406 | 0.000 |
| | | | 7 | -0.210 | -0.008 | 40.872 | 0.000 |
| Tests for normality of residuals | | | 8 | -0.304 | -0.060 | 44.235 | 0.000 |
| Test | Statistic | p-value | 9 | -0.375 | -0.135 | 49.852 | 0.000 |
| Shapiro-Wilk | 0.876 | 0.018 | 10 | -0.406 | 0.042 | 57.153 | 0.000 |
| D'Agostino-Stephens | 3.606 | 0.165 | 11 | -0.406 | -0.050 | 65.373 | 0.000 |
| | | | 12 | -0.378 | -0.055 | 73.535 | 0.000 |

and the correlogram shows that we have almost eliminated correlation between residuals (Table 3). To verify if residuals are normally distributed we perform two statistical tests, Shapiro-Wilk and D'Agostino-Pearson (see D'Agostino and Stephens, 1986 for a discussion). In both cases the normality assumption is rejected at very high confidence levels, suggesting that we should extend our set of explanatory variables.

Table 3: Linear regression for default rate (2)

| Parameter estimates | | | Correlogram of residuals | | | | |
|---|---|---|---|---|---|---|---|
| Variable | Estimate | $Pr > \|t\|$ | Lag | AC | PAC. | Q-Stat | p-value |
| Intercept | -2.396 | 0.000 | 1 | 0.376 | 0.376 | 2.987 | 0.084 |
| $\%DR_{t-1}$ | 0.165 | 0.000 | 2 | 0.193 | 0.061 | 3.829 | 0.147 |
| | | | 3 | -0.102 | -0.225 | 4.077 | 0.253 |
| Tests for zero-mean residuals | | | 4 | 0.056 | 0.185 | 4.157 | 0.385 |
| Test | Statistic | p-value | 5 | -0.159 | -0.233 | 4.858 | 0.434 |
| Sign | -3 | 0.238 | 6 | -0.140 | -0.087 | 5.446 | 0.488 |
| Wilkoxon | -23.5 | 0.325 | 7 | -0.229 | -0.057 | 7.159 | 0.412 |
| | | | 8 | -0.102 | -0.055 | 7.534 | 0.480 |
| Tests for normality of residuals | | | 9 | -0.077 | 0.023 | 7.769 | 0.558 |
| Test | Statistic | p-value | 10 | -0.037 | -0.074 | 7.831 | 0.645 |
| Shapiro-Wilk | 0.848 | 0.008 | 11 | 0.059 | 0.127 | 8.012 | 0.712 |
| D'Agostino-Stephens | 13.185 | 0.001 | 12 | -0.046 | -0.201 | 8.138 | 0.774 |

In order to choose an additional independent variable we consider the log-changes of a broad set of key economic indicators for Italy - including GDP, industrial production, new orders to manufacturing, gross fixed capital formation, household confidence, a composite leading indicator and real interest rates - as measured at various quarterly lags and over different time horizons ranging from two to four quarters. It turns out that the semi-annual log-change of household confidence calculated at the end of the previous year is the variable that, together with a constant and the lagged default rate, maximizes our model's fit. An analysis of the possible reasons why household confidence shows the best predictive power among the selected macroeconomic variables goes beyond the scope of this paper, but this result is probably due to its anticipatory nature, as confirmed by the fact that the leading economic indicator has a similar performance.

Table 4 shows that all coefficients are significant with the expected sign (default rates increase when economic conditions deteriorate) and, finally, that the normality hypothesis for the residuals cannot be rejected even at quite low confidence levels. We also check the zero-mean assumption for our residuals performing the sign test and the Wilkoxon signed rank test; in both cases the null hypothesis cannot be rejected at very low confidence levels.

Table 4: Linear regression for default rate (3)

| Parameter estimates | | | Correlogram of residuals | | | | |
|---|---|---|---|---|---|---|---|
| Variable | Estimate | $Pr > \|t\|$ | Lag | AC | PAC | Q-Stat | p-value |
| Intercept | -2.441 | 0.000 | 1 | 0.218 | 0.218 | 1.003 | 0.317 |
| $\%DR_{t-1}$ | 0.187 | 0.000 | 2 | 0.167 | 0.126 | 1.633 | 0.442 |
| $Houseconf_{t-1}$ | -1.806 | 0.000 | 3 | 0.003 | -0.060 | 1.633 | 0.652 |
| Tests for zero-mean residuals | | | 4 | -0.178 | -0.201 | 2.445 | 0.655 |
| Test | Statistic | p-value | 5 | -0.019 | 0.068 | 2.455 | 0.783 |
| Sign | 0 | 1.000 | 6 | -0.296 | -0.271 | 5.090 | 0.532 |
| Wilkoxon | -3.5 | 0.899 | 7 | -0.110 | -0.014 | 5.486 | 0.601 |
| | | | 8 | -0.313 | -0.291 | 9.016 | 0.341 |
| Tests for normality of residuals | | | 9 | -0.109 | 0.039 | 9.490 | 0.393 |
| Test | Statistic | p-value | 10 | 0.027 | -0.005 | 9.522 | 0.483 |
| Shapiro-Wilk | 0.969 | 0.775 | 11 | -0.054 | -0.062 | 9.675 | 0.560 |
| D'Agostino-Stephens | 1.004 | 0.605 | 12 | 0.176 | 0.026 | 11.542 | 0.483 |

We finally have a model that meets all the assumptions discussed in section 2, thus we can derive risk estimates as quantile measures of our loss distribution using equation (7). In order to do so, we need to estimate the correlation parameter that measures the sensitivity of asset returns to the random risk factor, the so-called asset correlation. This can be done either considering the variance of the residuals of our regression or maximizing the log-likelihood function in equation (8).

Table 5 shows our maximum likelihood estimates for the entire portfolio and for each industry sector separately[3]. The highest (unconditional) default probabilities, indicated by the largest default thresholds, are found for sectors 4 and 15. Comparatively low default probabilities are found for sectors 9 and 10. Asset correlation, as expressed by the sensitivity of asset returns to the unobservable common risk factor is never significant at the 99% level, except for the whole portfolio, and it is not significant at the 90% level in three cases; its highest values are 0.42% for sector 12 and 0.38% for sector 2. Serial dependence between default rates turns out to be quite relevant in industries 1, 6 and 9, whilst it is not significant at the 95% level for sector 4. The log-change in household confidence performs fairly well, as it is significant at the 99% level in 11 sectors and it is below the 90% level only for Building and construction and (marginally) for Fuel, power products and chemical products.

Table 5: Maximum likelihood estimates

| Industry | Constant | | Asset correlation | | $\%DR_{t-1}$ | | $Houseconf_{t-1}$ | |
|---|---|---|---|---|---|---|---|---|
| | Est. | $Pr > \|t\|$ | %Est. | $Pr > \|t\|$ | Est. | $Pr > \|t\|$ | Est. | $Pr > \|t\|$ |
| 1 | -2.472 | 0.000 | 0.279 | 0.034 | 0.203 | 0.000 | -0.944 | 0.002 |
| 2 | -2.451 | 0.000 | 0.379 | 0.017 | 0.128 | 0.000 | -1.150 | 0.001 |
| 3 | -2.214 | 0.000 | 0.023 | 0.452 | 0.104 | 0.000 | -0.493 | 0.002 |
| 4 | -2.029 | 0.000 | 0.367 | 0.013 | 0.045 | 0.059 | -0.929 | 0.004 |
| 5 | -2.314 | 0.000 | 0.301 | 0.043 | 0.121 | 0.000 | -0.925 | 0.004 |
| 6 | -2.474 | 0.000 | 0.228 | 0.011 | 0.223 | 0.000 | -0.273 | 0.221 |
| 7 | -2.380 | 0.000 | 0.134 | 0.012 | 0.156 | 0.000 | -0.723 | 0.000 |
| 8 | -2.473 | 0.000 | 0.156 | 0.036 | 0.194 | 0.000 | -0.483 | 0.031 |
| 9 | -2.618 | 0.000 | 0.114 | 0.016 | 0.210 | 0.000 | -0.781 | 0.000 |
| 10 | -2.518 | 0.000 | 0.171 | 0.168 | 0.144 | 0.000 | -0.450 | 0.104 |
| 11 | -2.400 | 0.000 | 0.051 | 0.231 | 0.137 | 0.000 | -0.792 | 0.000 |
| 12 | -2.362 | 0.000 | 0.424 | 0.017 | 0.113 | 0.000 | -1.154 | 0.001 |
| 13 | -2.300 | 0.000 | 0.154 | 0.056 | 0.116 | 0.000 | -0.699 | 0.004 |
| 14 | -2.320 | 0.000 | 0.277 | 0.018 | 0.114 | 0.000 | -1.006 | 0.001 |
| 15 | -2.105 | 0.000 | 0.297 | 0.021 | 0.053 | 0.022 | -0.578 | 0.042 |
| All | -2.413 | 0.000 | 0.092 | 0.009 | 0.163 | 0.000 | -0.726 | 0.000 |

In general, the results in Table 5 suggest that sensitivities to risk factors vary across borrowers, according to the industry in which they operate. If this were the case, the portfolio composition could significantly affect our risk estimates and risk contributions would differ across industries. This issue can be formally investigated through statistical tests, as the ratio of any estimate in Table 5 with its standard error produces a t-value, with approximate

[3]Serial independence and normality of residuals were also checked at industry level.

Table 6: Statistical tests for ML estimates
$H_0$ :Industry estimates are equal to portfolio estimates

| Industry | Intercept | Asset corr. | $\%DR_{t-1}$ | $Houseconf_{t-1}$ |
|---|---|---|---|---|
| 1 | 4.504 | 6.527 | 7.405 | 3.239 |
| 2 | 2.688 | 8.544 | 5.836 | 5.803 |
| 3 | 23.969 | 7.075 | 16.731 | 5.286 |
| 4 | 27.965 | 8.765 | 20.521 | 2.879* |
| 5 | 7.176 | 6.440 | 6.431 | 2.791* |
| 6 | 5.384 | 6.908 | 12.911 | 7.775 |
| 7 | 3.504 | 3.165 | 1.655* | 0.055* |
| 8 | 5.655 | 3.705 | 6.996 | 4.317 |
| 9 | 22.485 | 1.790* | 12.567 | 1.170* |
| 10 | 8.085 | 2.794* | 3.481 | 4.073 |
| 11 | 1.442* | 3.423 | 6.709 | 1.388* |
| 12 | 3.440 | 8.908 | 8.181 | 5.642 |
| 13 | 10.408 | 3.324 | 10.214 | 0.461* |
| 14 | 7.444 | 7.288 | 9.263 | 4.325 |
| 15 | 23.830 | 7.372 | 20.236 | 2.172* |

* indicates that the null hypothesis cannot be rejected at the 99% level

degrees of freedom computed as the number of observations minus one. Table 6 shows the values of the relevant t-statistics under the null hypothesis that sensitivities estimated at industry level are equal to those estimated for the entire portfolio, which is rejected at the 99% confidence level in most cases.

Considering the important role played by common risk factors, it may be useful to perform some sensitivity analysis of portfolio credit risk with respect to their coefficients. In particular, we consider the values of such coefficients at the 90th, 95th and 99th percentiles of their distribution and calculate 99.9% VaR using equation (7)[4]. Estimated sensitivities to systematic risk factors may have a significant impact on portfolio credit risk: when we consider 90% level values our risk measure may increase by more than 7%, but if we consider 99% level values its increases range between 24% and 26% (Table 7).

## 3.3 Portfolio credit risk and time horizon

In this section we repeat the empirical analysis of section 3.2 using semi-annual default data. Estimating portfolio credit risk over short horizons (less than one year) may be useful for

[4]For observable risk factors the results of sensitivity analysis are time-varying, as they depend on the value of the corresponding variable.

Table 7: Sensitivity of $VaR_{99.9\%}$ to ML estimates
(% default rates for 2009)

| | Percentile | | | |
|---|---|---|---|---|
| | 50th | 90th | 95th | 99th |
| All | 2.246 | | | |
| Random factor | | 2.378 | 2.676 | 2.786 |
| $\%DR_{t-1}$ | | 2.419 | 2.688 | 2.827 |
| $Houseconf_{t-1}$ | | 2.327 | 2.668 | 2.779 |

two reasons: first, it allows to generate "short-term" portfolio loss distributions, which can be used to price financial instruments with quarterly or semi-annual cash flows (e.g. portfolio credit derivatives); second, it gives the opportunity to obtain one-year loss distributions using parameters estimated from a much larger number of observations.

Portfolio credit risk increases with horizon because of 1) an increase in default probabilities with constant asset correlations or 2) an increase in both (Lucas, 1995). In other words, defaults tend to cluster over longer horizons due solely to an increase in the marginal probabilities of default events, or also because they are more firm-specific over short horizons, but more triggered by systematic factors over longer time periods. For instance, Zhou (1997) shows that, as the horizon lengthens, an increase in default correlations may be solely due to an increase in default probabilities, with constant asset correlations.

Our results for semi-annual default rates confirm the empirical findings of section 3.2. Data exhibit serial correlation that can be eliminated including lagged default rates in the model, but residuals remain highly non-normal (Tables 8 and 9).

We repeat the selection process aimed at finding an additional explanatory variable and, as in section 3.2, the lagged semi-annual log-change of household confidence turns out to be the one that maximizes our regression's fit. The coefficient of the new variable is highly significant, with the expected sign, and we can no longer reject the hypothesis that residuals are normally distributed (Table 10).

Table 11 shows our maximum likelihood estimates for the entire portfolio and for each industry sector separately[5]. Asset correlation, as expressed by the sensitivity of asset returns to the unobservable common risk factor is now significant at the 99% level for eight industries (and for the whole portfolio); its highest value is 0.37% for sectors 4 and 6. Serial dependence between default rates is even more relevant, as it is always significant at the 99% level. On the contrary, the log-change in household confidence shows a lower predictive power, as it is

[5]Serial independence and normality of residuals were also checked at industry level.

Table 8: Linear regression for semi-annual default rate (1)

| Parameter estimates | | | Correlogram of residuals | | | | |
|---|---|---|---|---|---|---|---|
| Variable | Estimate | $Pr > \|t\|$ | Lag | AC | PAC. | Q-Stat | p-value |
| Intercept | -2.292 | 0.000 | 1 | 0.944 | 0.944 | 36.604 | 0.000 |
| | | | 2 | 0.872 | -0.172 | 68.732 | 0.000 |
| Tests for zero-mean residuals | | | 3 | 0.796 | -0.067 | 96.225 | 0.000 |
| Test | Statistic | p-value | 4 | 0.715 | -0.073 | 119.07 | 0.000 |
| Sign | -3 | 0.418 | 5 | 0.604 | -0.326 | 135.86 | 0.000 |
| Wilkoxon | -11.5 | 0.870 | 6 | 0.498 | 0.049 | 147.63 | 0.000 |
| | | | 7 | 0.400 | 0.004 | 155.49 | 0.000 |
| Tests for normality of residuals | | | 8 | 0.301 | -0.115 | 160.07 | 0.000 |
| Test | Statistic | p-value | 9 | 0.207 | 0.073 | 162.32 | 0.000 |
| Shapiro-Wilk | 0.893 | 0.002 | 10 | 0.104 | -0.250 | 162.91 | 0.000 |
| D'Agostino-Stephens | 8.101 | 0.017 | 11 | 0.010 | -0.017 | 162.91 | 0.000 |
| | | | 12 | -0.068 | 0.122 | 163.18 | 0.000 |

Table 9: Linear regression for semi-annual default rate (2)

| Parameter estimates | | | Correlogram of residuals | | | | |
|---|---|---|---|---|---|---|---|
| Variable | Estimate | $Pr > \|t\|$ | Lag | AC | PAC. | Q-Stat | p-value |
| Intercept | -2.639 | 0.000 | 1 | 0.114 | 0.114 | 0.5197 | 0.471 |
| $\%DR_{t-1}$ | 0.304 | 0.000 | 2 | 0.088 | 0.076 | 0.8395 | 0.657 |
| | | | 3 | 0.024 | 0.006 | 0.8640 | 0.834 |
| Tests for zero-mean residuals | | | 4 | 0.273 | 0.268 | 4.1295 | 0.389 |
| Test | Statistic | p-value | 5 | -0.018 | -0.084 | 4.1441 | 0.529 |
| Sign | -5.5 | 0.099 | 6 | -0.114 | -0.155 | 4.7446 | 0.577 |
| Wilkoxon | -56.5 | 0.401 | 7 | -0.040 | -0.004 | 4.8213 | 0.682 |
| | | | 8 | -0.008 | -0.063 | 4.8246 | 0.776 |
| Tests for normality of residuals | | | 9 | 0.193 | 0.260 | 6.7449 | 0.664 |
| Test | Statistic | p-value | 10 | -0.042 | -0.016 | 6.8378 | 0.741 |
| Shapiro-Wilk | 0.898 | 0.003 | 11 | -0.140 | -0.209 | 7.9298 | 0.720 |
| D'Agostino-Stephens | 14.695 | 0.001 | 12 | -0.048 | -0.008 | 8.0651 | 0.780 |

significant at the 99% level in only six sectors and it is below the 90% level in four cases. In general, however, the results in Table 11 confirm that sensitivities to risk factors substantially vary across industries.

Table 10: Linear regression for semi-annual default rate (3)

| Parameter estimates | | |
|---|---|---|
| Variable | Estimate | $Pr > \|t\|$ |
| Intercept | -2.653 | 0.000 |
| $\%DR_{t-1}$ | 0.315 | 0.000 |
| $Houseconf_{t-1}$ | -0.379 | 0.000 |

| Tests for zero-mean residuals | | |
|---|---|---|
| Test | Statistic | p-value |
| Sign | -4.5 | 0.188 |
| Wilkoxon | -25.5 | 0.706 |

| Tests for normality of residuals | | |
|---|---|---|
| Test | Statistic | p-value |
| Shapiro-Wilk | 0.966 | 0.300 |
| D'Agostino-Stephens | 1.100 | 0.577 |

| Correlogram of residuals | | | | |
|---|---|---|---|---|
| Lag | AC | PAC | Q-Stat | p-value |
| 1 | -0.072 | -0.072 | 0.2082 | 0.648 |
| 2 | 0.109 | 0.105 | 0.7017 | 0.704 |
| 3 | 0.018 | 0.033 | 0.7152 | 0.870 |
| 4 | 0.236 | 0.232 | 3.1454 | 0.534 |
| 5 | -0.058 | -0.032 | 3.2950 | 0.655 |
| 6 | 0.020 | -0.035 | 3.3134 | 0.769 |
| 7 | -0.040 | -0.053 | 3.3903 | 0.847 |
| 8 | 0.007 | -0.052 | 3.3926 | 0.907 |
| 9 | 0.053 | 0.086 | 3.5382 | 0.939 |
| 10 | 0.117 | 0.149 | 4.2678 | 0.934 |
| 11 | -0.214 | -0.206 | 6.7984 | 0.815 |
| 12 | -0.007 | -0.070 | 6.8014 | 0.870 |

Table 11: Maximum likelihood estimates - semi-annual data

| Industry | Constant | | Asset correlation | | $\%DR_{t-1}$ | | $Houseconf_{t-1}$ | |
|---|---|---|---|---|---|---|---|---|
| | Est. | $Pr > \|t\|$ | %Est. | $Pr > \|t\|$ | Est. | $Pr > \|t\|$ | Est. | $Pr > \|t\|$ |
| 1 | -2.715 | 0.000 | 0.174 | 0.019 | 0.398 | 0.000 | -0.655 | 0.000 |
| 2 | -2.715 | 0.000 | 0.361 | 0.001 | 0.286 | 0.000 | -0.616 | 0.005 |
| 3 | -2.465 | 0.000 | 0.063 | 0.123 | 0.196 | 0.000 | -0.338 | 0.014 |
| 4 | -2.331 | 0.000 | 0.372 | 0.001 | 0.131 | 0.000 | -0.567 | 0.007 |
| 5 | -2.595 | 0.000 | 0.145 | 0.069 | 0.271 | 0.000 | -0.560 | 0.003 |
| 6 | -2.669 | 0.000 | 0.374 | 0.000 | 0.381 | 0.000 | -0.169 | 0.382 |
| 7 | -2.628 | 0.000 | 0.090 | 0.001 | 0.307 | 0.000 | -0.448 | 0.000 |
| 8 | -2.697 | 0.000 | 0.097 | 0.027 | 0.356 | 0.000 | -0.087 | 0.519 |
| 9 | -2.830 | 0.000 | 0.138 | 0.001 | 0.397 | 0.000 | -0.323 | 0.012 |
| 10 | -2.748 | 0.000 | 0.049 | 0.570 | 0.275 | 0.000 | -0.197 | 0.284 |
| 11 | -2.647 | 0.000 | 0.060 | 0.186 | 0.275 | 0.000 | -0.429 | 0.003 |
| 12 | -2.640 | 0.000 | 0.364 | 0.002 | 0.263 | 0.000 | -0.501 | 0.019 |
| 13 | -2.565 | 0.000 | 0.136 | 0.025 | 0.242 | 0.000 | -0.323 | 0.042 |
| 14 | -2.594 | 0.000 | 0.245 | 0.002 | 0.255 | 0.000 | -0.447 | 0.013 |
| 15 | -2.386 | 0.000 | 0.275 | 0.003 | 0.121 | 0.000 | -0.139 | 0.463 |
| All | -2.652 | 0.000 | 0.084 | 0.000 | 0.315 | 0.000 | -0.378 | 0.000 |

## 3.4 Portfolio credit risk and firm size

The results of Section 3.2 are not necessarily valid for small companies, as they have, on average, a less diversified portfolio of assets than large companies, thus they may show a lower sensitivity to systematic risk factors. In the Basel II IRB approach, for instance, asset correlation is an increasing function of the size of the obligor, as measured by its yearly turnover[6]. The results in Düllmann and Scheule (2003) and Lopez (2003) show that asset correlations increase with firm size, whilst Dietsch and Petey (2004) find a U-shaped relationship between the two variables. These conclusions, however, can be influenced by the fact that small enterprises operate mainly in industries that are less cyclical, whereas large firms prevail in sectors more dependent on macroeconomic conditions, leading small companies to show lower systematic risk.

The database used in this paper allows to distinguish only loans to sole proprietorships with up to 5 employees as a proxy for the small business sector. Even if such a category corresponds more to a definition of micro-enterprise, it can be used to repeat the analysis of section 3.2 and check if the results vary with firm size.

As for larger firms, if we run a regression of $\Phi^{-1}(DR_t)$ on a constant, the correlogram of residuals provides strong evidence of serial correlation, which can be eliminated including the (percentage) lagged default rate in the model (Tables 12 and 13).

Table 12: Linear regression for default rate - small firms (1)

| Parameter estimates | | | Correlogram of residuals | | | | |
|---|---|---|---|---|---|---|---|
| Variable | Estimate | $Pr > \|t\|$ | Lag | AC | PAC. | Q-Stat | p-value |
| Intercept | -2.008 | 0.000 | 1 | 0.804 | 0.804 | 14.321 | 0.000 |
| | | | 2 | 0.626 | -0.057 | 23.510 | 0.000 |
| Tests for zero-mean residuals | | | 3 | 0.438 | -0.135 | 28.291 | 0.000 |
| Test | Statistic | p-value | 4 | 0.275 | -0.061 | 30.301 | 0.000 |
| Sign | -0.5 | 1.000 | 5 | 0.146 | -0.024 | 30.909 | 0.000 |
| Wilkoxon | 0 | 1.000 | 6 | -0.026 | -0.238 | 30.929 | 0.000 |
| | | | 7 | -0.185 | -0.150 | 32.065 | 0.000 |
| Tests for normality of residuals | | | 8 | -0.336 | -0.151 | 36.155 | 0.000 |
| Test | Statistic | p-value | 9 | -0.444 | -0.096 | 44.010 | 0.000 |
| Shapiro-Wilk | 0.917 | 0.098 | 10 | -0.394 | 0.276 | 50.896 | 0.000 |
| D'Agostino-Stephens | 5.576 | 0.062 | 11 | -0.357 | -0.060 | 57.268 | 0.000 |
| | | | 12 | -0.369 | -0.279 | 65.050 | 0.000 |

It is interesting to notice that, with respect to larger firms, departures from normality are less evident, suggesting that for some sub-portfolios it may not be necessary to include additional variables to meet the conditional independence assumption. The coefficient of the

[6] Asset correlation is a strictly increasing function of the obligor's turnover only if this is not below 5 million euro, and not above 50 million euro.

Table 13: Linear regression for default rate - small firms (2)

| Parameter estimates | | | Correlogram of residuals | | | | |
|---|---|---|---|---|---|---|---|
| Variable | Estimate | $Pr > \|t\|$ | Lag | AC | PAC. | Q-Stat | p-value |
| Intercept | -2.380 | 0.000 | 1 | 0.238 | 0.238 | 1.1990 | 0.274 |
| $\%DR_{t-1}$ | 0.164 | 0.000 | 2 | 0.116 | 0.063 | 1.5027 | 0.472 |
| | | | 3 | 0.093 | 0.055 | 1.7081 | 0.635 |
| Tests for zero-mean residuals | | | 4 | -0.078 | -0.125 | 1.8631 | 0.761 |
| Test | Statistic | p-value | 5 | 0.170 | 0.220 | 2.6673 | 0.751 |
| Sign | -3 | 0.238 | 6 | 0.078 | 0.000 | 2.8520 | 0.827 |
| Wilkoxon | -15.5 | 0.523 | 7 | -0.077 | -0.123 | 3.0484 | 0.880 |
| | | | 8 | -0.015 | -0.020 | 3.0568 | 0.931 |
| Tests for normality of residuals | | | 9 | -0.247 | -0.209 | 5.5045 | 0.788 |
| Test | Statistic | p-value | 10 | -0.144 | -0.050 | 6.4348 | 0.778 |
| Shapiro-Wilk | 0.897 | 0.050 | 11 | -0.001 | 0.045 | 6.4348 | 0.843 |
| D'Agostino-Stephens | 3.417 | 0.181 | 12 | -0.120 | -0.066 | 7.2983 | 0.837 |

semi-annual log-change in household confidence is not significant at the 99% level, confirming that small firms are less influenced by systematic risk factors, but the normality hypothesis for the residuals cannot be rejected even at quite low confidence levels (Table 14).

Table 14: Linear regression for default rate - small firms (3)

| Parameter estimates | | | Correlogram of residuals | | | | |
|---|---|---|---|---|---|---|---|
| Variable | Estimate | $Pr > \|t\|$ | Lag | AC | PAC | Q-Stat | p-value |
| Intercept | -2.388 | 0.000 | 1 | 0.079 | 0.079 | 0.131 | 0.718 |
| $\%DR_{t-1}$ | 0.162 | 0.000 | 2 | -0.108 | -0.114 | 0.391 | 0.822 |
| $Houseconf_{t-1}$ | -0.314 | 0.092 | 3 | 0.250 | 0.273 | 1.887 | 0.596 |
| | | | 4 | -0.056 | -0.134 | 1.967 | 0.742 |
| Tests for zero-mean residuals | | | 5 | -0.015 | 0.083 | 1.973 | 0.853 |
| Test | Statistic | p-value | 6 | 0.087 | -0.020 | 2.202 | 0.900 |
| Sign | -2 | 0.481 | 7 | 0.109 | 0.178 | 2.591 | 0.920 |
| Wilkoxon | -9.5 | 0.702 | 8 | 0.015 | -0.049 | 2.599 | 0.957 |
| | | | 9 | -0.190 | -0.184 | 4.041 | 0.909 |
| Tests for normality of residuals | | | 10 | -0.212 | -0.265 | 6.054 | 0.811 |
| Test | Statistic | p-value | 11 | -0.034 | -0.004 | 6.114 | 0.866 |
| Shapiro-Wilk | 0.938 | 0.272 | 12 | -0.151 | -0.147 | 7.480 | 0.824 |
| D'Agostino-Stephens | 0.428 | 0.808 | | | | | |

# 4 Risk contributions

Risk managers are typically interested in estimating not only the overall risk of a portfolio, but also the contribution of individual exposures to such risk. In fact, individual risk contributions are a a key input for setting pricing limits, calculating risk-adjusted performances and for capital allocation decisions.

Marginal risk contributions can be represented as conditional expected losses (Tasche, 1999). Let $L_i$ be the loss from obligor $i$ and $L$ the total loss in the portfolio; assuming $P(L = VaR_\alpha) > 0$, the marginal $VaR_\alpha$ contribution from obligor $i$ can be expressed as:

$$E(L_i \mid L = VaR_\alpha) \tag{9}$$

In the model described in section 2 individual expected losses are equal for all exposures, as both default thresholds and sensitivities to common risk factors are assumed to be constant across borrowers. The empirical results in section 3.2 show that these assumptions are acceptable when calculating an overall portfolio risk measure. On the contrary, when we estimate individual risk contributions, the same assumptions should be relaxed, as incremental risk for a borrower with a higher PD (or asset correlation) is, *ceteris paribus*, greater than incremental risk for a borrower with a lower PD (or asset correlation).

In this section we consider a fictitious portfolio of 7,500 loans, where all industries are equally represented, and we estimate risk contributions for each sector. In order to do that, we use maximum likelihood estimates of correlations and default probabilities to approximate the loss distribution by Monte Carlo simulation.

We draw an independent standard normal variable for each exposure and a vector from the multivariate normal distribution of common risk factors $\mathbf{Y} \sim N(0, \mathbf{\Omega})$, where $\mathbf{\Omega}$ is the covariance matrix of the empirical estimates of systematic factors in different industries. We calculate the asset return for each exposure, we compare it to the default threshold, and determine the default indicator (one for default, zero otherwise). The portfolio loss for this draw is given by the sum of default indicators of all exposures, thus assuming that loss given default and exposure at default are always equal to one. To estimate the portfolio loss distribution, to be interpreted as the distribution of the number of defaults in the portfolio, we repeat this process many times.

Standard Monte Carlo methods are affected by serious problems in estimating marginal risk contributions, as these can be represented as conditional expected losses on subportfolios, conditioned on low-probability events. To address the problems deriving from the rarity of large losses, we use an adaptation of the importance sampling procedure developed by Glasserman and Li (2005) to estimate the tail of our loss distribution. The basic idea is moving from the real probability measure $P$ to a new probability measure $\tilde{P}$, under which large losses are more frequent. In particular, the probability for the total loss to be above a certain value $l$ can be expressed as:

$$P(L > l) = E\left(1_{\{L>l\}}\right) = \tilde{E}\left(1_{\{L>l\}}\frac{dP}{d\tilde{P}}\right) \tag{10}$$

where $\tilde{\mathrm{E}}$ denotes the expectation under $\tilde{P}$ and $\frac{dP}{d\tilde{P}}$ is the likelihood ratio between the two probability measures.

The procedure suggested by Glasserman and Li (2005) for credit risk models with factor structure consists of two steps: 1) shifting the distribution of systematic factors; 2) increasing conditional default probabilities.

In our model the estimated coefficients of the unobservable common risk factors are very low, thus we focus only on the second step of the importance sampling procedure. In particular, we consider a new (increased) default probability for industry $j$, with $j \in \{1, ..., 15\}$:

$$p\left(\theta, Y_j\right) = \frac{p\left(Y_j\right)e^{\theta}}{1 + p\left(Y_j\right)\left(e^{\theta} - 1\right)} \tag{11}$$

where $\theta$ is the solution to the equation:

$$\frac{\partial}{\partial\theta}\psi\left(\theta, \mathbf{Y}\right) = VaR_{\alpha} \tag{12}$$

with

$$\psi\left(\theta, \mathbf{Y}\right) = \sum_{j=1}^{15} \log\left(1 + p\left(Y_j\right)\left(e^{\theta} - 1\right)\right) \tag{13}$$

The likelihood ratio that allows to move from the probability measure $\tilde{P}$ back to the original measure can be expressed as:

$$\lambda = \exp\left(-\theta\left(\mathbf{Y}\right)L + \psi\left(\theta\left(\mathbf{Y}\right), \mathbf{Y}\right)\right) \tag{14}$$

The $VaR_{\alpha}$ contribution for obligors in sector $j$ becomes:

$$VaR_{\alpha,j} = \frac{\tilde{E}\left(L_j\lambda 1_{\{L=VaR_{\alpha}\}}\right)}{\tilde{E}\left(\lambda 1_{\{L=VaR_{\alpha}\}}\right)} \tag{15}$$

Table 15 reports the $VaR_{99.9\%}$ contributions for each industry in our portfolio. The results confirm the key role played by sensitivities to common risk factors (both observable and random) in determining individual risk contributions to portfolio credit risk. For instance, sector 6 shows the third lowest unconditional PD (as indicated by the constant in Table 5) but also a quite high autoregressive component, which reflects in its risk contribution. On the contrary, sector 4 has the highest unconditional PD, but its risk contribution is reduced by low serial dependence. As for asset correlation, we can notice that sector 3 shows a high default probability, but a much lower risk contribution (in relative terms) due to its low sensitivity to the random risk factor. (Tables 5 and 15).

Table 15: $VaR_{99.9\%}$ contributions
Importance sampling estimates for 2009
(as % default rate)
99.9% confidence intervals are in parentheses

| Industry | Annual | | Semi-annual | |
|---|---|---|---|---|
| 1 | 0.1908 | (0.000473) | 0.0848 | (0.000379) |
| 2 | 0.1665 | (0.000445) | 0.0788 | (0.000373) |
| 3 | 0.1680 | (0.000413) | 0.0904 | (0.000381) |
| 4 | 0.1696 | (0.000449) | 0.1538 | (0.000536) |
| 5 | 0.2045 | (0.000488) | 0.0866 | 0.000380) |
| 6 | 0.1761 | (0.000455) | 0.0988 | (0.000431) |
| 7 | 0.1704 | (0.000425) | 0.0783 | (0.000355) |
| 8 | 0.1561 | (0.000409) | 0.0683 | (0.000333) |
| 9 | 0.1170 | (0.000351) | 0.0567 | (0.000305) |
| 10 | 0.1147 | (0.000356) | 0.0474 | (0.000277) |
| 11 | 0.1369 | (0.000376) | 0.0664 | (0.000328) |
| 12 | 0.1976 | (0.000497) | 0.0903 | (0.000403) |
| 13 | 0.1786 | (0.000439) | 0.0839 | (0.000372) |
| 14 | 0.1960 | (0.000473) | 0.0906 | (0.000395) |
| 15 | 0.2404 | (0.000533) | 0.1156 | (0.000452) |

When we consider semi-annual data, observable risk factors are less important in determining risk contributions, as lagged default rates are lower with respect to yearly data, but also because changes in macroeconomic conditions do not fully display their effects over short horizons. At the same time, sensitivities to random factors have a relatively greater impact. For instance, the risk contribution of sector 6 is very high compared to its PD, because of a significant asset correlation, whereas industries 7 and 11 display low risk contributions for the opposite reason (Tables 11 and 15).

## 5 Conclusions

This paper shows that measuring portfolio credit risk using the one-factor version of the Vasicek model with constant default thresholds may lead to an underestimation of tail probabilities, as a single systematic risk factor cannot capture all the correlation between asset returns, leading to a violation of the conditional independence assumption. Instead, we prove through statistical tests that factor models require the inclusion of deterministic explanatory variables like macroeconomic factors and autoregressive components, together with unobservable random factors. In fact, default rates are influenced by macroeconomic conditions with a certain time-lag, but they also depend on their past values due to the persistence of economic shocks and because of possible contagion effects.

Sensitivities to risk factors vary across borrowers, according to the industry in which they operate, therefore a change in the composition of a portfolio may significantly affect our risk estimates. Serial dependence between default rates is quite relevant in all industries and it is not significant at the 99% level in only two cases. Macroeconomic conditions, as measured by the log-change in household confidence, also play a major role in most cases, but they are not significant at the 90% level for two sectors. Finally, asset correlation, as expressed by the sensitivity of asset returns to the unobservable common risk factor, is never significant at the 99% level, except for the whole portfolio, and it is not significant at the 90% level in three cases.

The main results obtained with annual default rates are confirmed by semi-annual data, thus using a considerably longer series of observations. When we increase the frequency of our data the autoregressive component becomes more important, but it is still not sufficient to explain all the correlation between defaults.

Portfolio credit risk and sensitivities to systematic risk factors vary according to the size of the borrower. If we compare the results obtained for small companies with the coefficients derived for larger firms, we observe that serial dependence between default rates remains significant, but macroeconomic conditions play a less important role.

Default thresholds and sensitivities to common risk factors can be assumed to be constant across borrowers when calculating a single measure for portfolio credit risk. These assumptions must be relaxed when we estimate individual risk contributions, which represent a fundamental input for pricing, calculating risk-adjusted performances and for capital allocation decisions. Maximum likelihood estimates are used to derive the risk contribution for each industry sector through Monte Carlo simulation and our results confirm the key role played by sensitivities to common risk factors (both observable and random).

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