Evolving to the Impatience Trap:

The Example of the Farmer-Sheriff Game[1]

BY DAVID K. LEVINE, SALVATORE MODICA, FEDERICO WEINSCHELBAUM AND FELIPE ZURITA

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Abstract

The literature on the evolution of impatience, focusing on one-person decision problems, finds that evolutionary forces favor the more patient individuals. This paper shows that in the context of a game, this is not necessarily the case. In particular, it offers a two-population example where evolutionary forces favor impatience in one group while favoring patience in the other. Moreover, not only evolution but also efficiency may prefer impatient individuals. In our example, it is efficient for one population to evolve impatience and for the other to develop patience. Yet, evolutionary forces move the wrong populations.

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1. Introduction

Why are we often more impulsive than we might like to be? To take one of many examples: although the "cost" of getting a copy of a new book or the last model of a computer decreases substantially with time, few people choose to wait. Moreover, in some cases there are people that spend the night in line to be the first buyers. From the perspective of evolution this poses a puzzle: evolution favors the very long run. Given the great variation in patience and self-control in the population, will not evolutionary forces favor those more willing to wait? Should we not evolve towards ever-greater patience and absence of impulsivity? Indeed, Blume and Easley [1992] and more recently Bottazzi and Dindo [2011][2] show in the context of a wealth accumulation problem that evolution favors the patient so strongly that it favors the patient over the smart.

One explanation is the natural explanation, for example by Chowdhry [forthcoming] that we are impatient because we may not live to see tomorrow. However this does not in itself explain why we should evolve impatience: even a very patient individual will behave impatiently in the face of uncertain life.

Here we explore an alternative explanation of the evolution of impatience. In an investment problem short-sightedness is dysfunctional. The same is not true in a game. Preferences can act as a form of commitment device. For example, a reputation for laziness is very desirable in order to avoid requests for referee reports or letters of recommendation. In a repeated game an impatient player can not be threatened with future punishment, and so is harder to exploit.

The idea of impatience as commitment is a subtle one. Successful commitment - as lovers of Dr. Strangelove will know – requires two elements: credibility and publicity. Evolutionary forces by building impatience into preferences makes impatient behavior credible. But how does this help against an opponent that cannot directly observe preferences? Certainly it is reasonable that preferences might be inferred from past behavior - but then there is an incentive even for a patient player to build a reputation for impatience, and it is not so clear why evolution would favor the inflexibility of commitment over the flexibility of pretense. Moreover, it is interesting to note that

[2] See especially the discussion in their Section 4.

pretense requires patience to work. Building a reputation is something that an impatient player would not choose to do. The patient can mimic an impatient, but the impatient will not mimic anyone.

To attack this issue, we make the simplifying assumption that a player's play is observed only at the end of his life. This eliminates any incentive for a patient player to mimic an impatient player. Why then would evolution favor impatience over patience? The answer is that while the player does not gain fitness from being impatient, his children do. The player herself does not care about this, only about her own utility given her patience. Evolutionary selection on the other hand is highly dependent on the consequences of parental action for the children. If other players are able to infer a player’s patience *ex post* from his play, if they can observe who her children are, and if they understand that patience is hereditary – then children potentially benefit from the parent’s impatience.

We explore these issues in the context of a simple game designed to illustrate both how impatience can emerge as an evolutionary outcome and also to understand how different social roles may result in different degrees of patience. Indeed despite anecdotal evidence - the behavior of Charles Sheen comes to mind - that the rich may be as impulsive as the poor there is statistical evidence, for example in Cunha and Heckman [2009] that there is a strong connection between economically unsuccessful families and impatience and lack of self-control.

This paper is designed to further advance the literature on the evolution of preferences. The evolution of altruism has been much studied, for example, in Bowles [2001]. They have been studied in the context of cultural evolution by Bisin and Topa [2004] and the broader issue of cultural versus other forms of transmission have been studied by Bisin [2001]. Other deep issues about kinship and selection have been examined by Alger and Weibull [2010]. Authors such as Ely [2001] and Dekel, Ely and Yilankaya [2007] have examined the theoretical underpinings of evolutionary equilibrium when preferences evolve, relating evolutionary outcomes to equilibria of the fitness game. However the evolution of impatience (as opposed to patience) has not been much studied.

There are a variety of subtle issues about impulsive behavior and self-control that have been explored in the behavioral economics literature - see for example Fudenberg and Levine [2006]. However, we do not examine these issues of commitment, present bias and time consistency here - rather we focus on the simpler question of why intertemporal preferences with a low geometric discount factor might emerge in an evolutionary setting.

We also look at the inefficiency of equilibrium, which has a natural interpretation when the model is viewed as a buyer-seller model. On this we elaborate in the conclusions.

In all the above-mentioned cases, the gains from impatience are private. However, there are also cases in which there are social gains from impatience. An example of this is provided in the literature on conflict.[3] In this literature people can satisfy their desires either by producing or by appropriating others' production (that is, through conflict). In general, resources spent in conflict are a social waste. Thus, it is best for society that people do not engage in appropriation by conflict; as a second best, it is best that those who do it be more impatient, so that they do not invest much in technologies that are detrimental to social welfare. This is an extreme case that can be explained in our model. An alternative, less extreme case, is for example, is the case of speculators. They could have a social function, namely helping the alignment of prices, yet they do appropriate part of the gains from investments.

The rest of the paper is organized as follows. In Section 2 we develop the model. In Section 3 we analyze the equilibrium of the evolutionary process. In Section 4 we discuss efficiency issues. Finally, we conclude in Section 5.

2. The Model

There is a continuum of players divided into two populations, *Farmers* who constitute a fraction ϕ of the population and *Sheriffs* who are the other $1-\phi$ of the population. Each round Farmers and Sheriffs are randomly matched where the probability of a meeting between a Farmer and a Sheriff is $2\phi(1-\phi)$. The remaining Farmers and Sheriffs are unmatched. All players have an initial endowment of one bushel of wheat, and fitness is linear in wheat.

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[3] See Hirshleifer [1991]. Rent seeking is a particularly interesting special case of conflict that has received much attention at least since Tullock [1967] and Krueger [1974].

A *round* consists of either a one-person or two-person game that has three periods.

Unmatched Farmer [Investment Game]:

- Period 1: invest $k_I \in [0,1]$, consume $1 - k_I$
- Period 2: receive and consume output $y_I = Ak_I{}^{\alpha}$, where $\alpha A \leq 1$ and $0 < \alpha < 1, A > 0$.
- Period 3: nothing

Unmatched Sheriff:

- Period 1: consume endowment of 1
- Period 2: nothing
- Period 3: nothing

Farmer-Sheriff Game:

- Period 1a: Sheriff invests $k_S \in [0,1]$, consumes $1 - k_S$ and states a demand $d_S \geq 0$.
- Period 1b: Farmer invests $k_F \in [0,1]$, consumes $1 - k_F$ and agrees to pay the Sheriff $d_F \geq 0$.
- Period 2: Farmer produces output $y_F = Ak_F{}^{\alpha} + G$, consumes $y_F - d_F$ and the Sheriff consumes d_F where $G \geq 0$ is the "gain to trade" from the match.
- Period 3: if $d_F \geq d_S$ nothing; if $d_F < d_S$ the Sheriff issues a punishment that costs the Farmer $ABk_S{}^{\alpha}$ where $B > 1$. This latter assumption implies that it is easier to destroy output than to produce it.

Note that we allow the punishment to result in negative fitness.

A player's preferences depend on fitness and are characterized by a discount factor δ_F, δ_S. Discounting takes place between periods. In the Investment game the objective function of the Farmer is $1 - k_I + \delta_F y_I$. In the Unmatched Sheriff game the objective function of the Sheriff is 1. In the Farmer-Sheriff game the objective function of the Farmer is

$$1 - k_F + \delta_F(y_F - d_F) - \delta_F{}^2 \mathbf{1}_{d_F < d_S} ABk_S{}^{\alpha}$$

where $\mathbf{1}_{d_F<d_S}$ is the indicator function that evaluates to 1 when $d_F < d_S$ and 0 otherwise, and that of the Sheriff

$$1 - k_S + \delta_S d_F .$$

Entering each match the Farmer and the Sheriff know their own discount factor and have independent common knowledge beliefs about the discount factor of the other player given by probability measures $\mu_F(\delta_S), \mu_S(\delta_F)$. Except in the Farmer-Sheriff game, these beliefs are irrelevant. We assume that at the end of each round strategies during the round are commonly observed.

Notice that this assumption means that it is observed how a matched farmer "would have played" if she had been unmatched and how an unmatched sheriff "would have played" if she had been matched. What we have in mind is that players in actuality play more than once and are sometimes matched and sometimes not so that in fact their play is observed in both contingencies, however the notation to make this formal is quite cumbersome and results in the same model.

To see what is captured by this game, consider first the case $G = 0$. In this case the Sheriffs do not contribute to social welfare beyond their own endowment: only Farmers are socially productive in the sense that they can make investments resulting in an increase in wheat. However Sheriffs can appropriate some of the output of Farmers. In this sense the model has a predator-prey flavor. Notice, however, that the model is formulated so that there is no intrinsic distortion in the predation: the amount that the Sheriffs can appropriate is independent of how much is produced by Farmers. The predation takes place through threat of punishment: Farmers must choose whether or not to comply with the Sheriffs' demands. If Farmers fail to comply with the demand of the Sheriff then they are punished. The level of punishment depends on the investment made by the Sheriff. Notice that there is no commitment issue for the Sheriff: the more patient they are the more they will invest in punishment – and as we will see Sheriffs will evolve towards a high degree of patience.

This game is unlike the Peasant-Dictator[4] game where the Dictator faces a commitment problem – but one that is not sensitive to patience. Here it is Farmers who

[4] See, for example, Van Huyck, Battalio and Walter [1995].

face a commitment problem: punishment takes place with a delay. Because of the delay a less patient Farmer is less willing to give in to demands by the Sheriff, and if the Sheriff knows this, she will demand less. Hence there is a commitment problem on the part of the Farmer.

So far we have discussed the case $G = 0$. Here Sheriffs have no social function and are merely predators or parasites. If we think of the Sheriffs as landlords and the Farmers as peasants, generally landlords provide some services, ranging from protection to improvements to the capital stock. This we capture – somewhat crudely – through $G > 0$. This means that there is a positive surplus accruing to a match with a Sheriff. Notice that the output from the match accrues to the Farmer, not the Sheriff. Here the model becomes one of potentially beneficial trade – but the only mechanism the Sheriff has for appropriating some of the gains to trade is by threatening the Farmer. Unfortunately this mechanism is not related to the gain to trade: the amount the Sheriff can appropriate does not depend on how good the match is. This captures a situation that sometimes occurs in practice: if one party owns the enforcement mechanism, why not appropriate the most that can be appropriated rather than some sort of amount determined by efficiency considerations? Why should a large politically connected monopolist merely appropriate what the market is willing to pay, when they can have a nice piece of tax revenue to go with it?

One interpretation when $G > 0$ is that the Sheriffs are buyers and the Farmers sellers, the amount of wheat provided to the Sheriff/buyer represents the quality of a product and G the gains to trade. Here the Farmer/sellers have an incentive to cheat the Sheriff/buyers – and the only recourse that the Sheriff/buyers have is to retaliate against a Farmer/seller who provides low quality. Hence the quality provided will be in proportion to the ability of the Sheriff/buyer to punish the Farmer/seller. In a sense this provides the opposite from the case where $G = 0$: in that case the Sheriffs are parasites. In the buyer/seller context they are buyers who may receive too little share of the surplus to provide them with adequate incentives.

3. Equilibrium

3.1 Equilibrium of a Match

We turn now to studying subgame perfect equilibria of the different matches. First, and this is a critical point, information about a player's strategy becomes public only after the match ends, at which point the player dies and does not play again, so the only consideration a player has is utility received during the match given preferences.

In the investment game the objective function for the Farmer is $1 - k_I + \delta_F A {k_I}^{\alpha}$, the first order condition is $\alpha \delta_F A {k_I}^{\alpha - 1} - 1 = 0$, from which the optimum is $k_I = (\alpha A)^{1/(1-\alpha)} {\delta_F}^{1/(1-\alpha)}$.

In the Farmer-Sheriff game the objective function of the Farmer is $1 - k_F + \delta_F \left(A {k_F}^{\alpha} - d_F + G \right)$ if $d_F \geq d_S$ or

$$1 - k_F + \delta_F \left(A {k_F}^{\alpha} - d_F + G \right) - {\delta_F}^2 AB {k_S}^{\alpha}$$

if $d_F < d_S$. Notice that this is rigged so that the optimal investment choice of the Farmer is independent of d_F, whether or not there is punishment, the Farmer's beliefs and is the same as when the Farmer is unmatched: $k_F = k_I = (\alpha A)^{1/(1-\alpha)} {\delta_F}^{1/(1-\alpha)}$. Notice that more impatient Farmers produce less so are potentially less fit than more patient Farmers. Hence it is by no means a foregone conclusion that evolutionary forces will favor the less patient Farmer.

In choosing how much to pay, clearly the Farmer should choose either $d_F = 0$ and get $1 - k_F + \delta_F \left(A {k^{\alpha}}_F + G \right) - {\delta_F}^2 AB {k_S}^{\alpha}$ or $d_F = d_S$ and get $1 - k_F + \delta_F \left(A {k_F}^{\alpha} - d_S + G \right)$, whichever is larger – again regardless of beliefs.

The optimal play of the Sheriff depends on his beliefs. As this will be the case we make use of, we solve only for the case in which these beliefs are a point mass $\overline{\delta}_F$. Then the Sheriff should choose the largest demand consistent with payment: $d_S = \overline{\delta}_F AB {k_S}^{\alpha}$. The (believed) utility of the Sheriff is then $1 - k_S + \delta_S \overline{\delta}_F AB {k_S}^{\alpha}$. Finally, k_S is chosen by the Sheriff to maximize his utility, so that $k_S = (\alpha AB)^{1/(1-\alpha)} (\overline{\delta}_F \delta_S)^{1/(1-\alpha)}$. The corresponding demand is

$$\begin{aligned} d_S &= \overline{\delta}_F AB((\alpha AB)^{1/(1-\alpha)} (\overline{\delta}_F \delta_S)^{1/(1-\alpha)})^{\alpha} \\ &= \alpha^{\alpha/(1-\alpha)} (AB)^{1/(1-\alpha)} {\overline{\delta}_F}^{1/(1-\alpha)} {\delta_S}^{\alpha/(1-\alpha)} \end{aligned}$$

The amount demanded by the Sheriff is an increasing function of both the discount factor of the Sheriff – since a patient Sheriff will invest more – and the (believed) discount factor of the Farmer – since a patient Farmer is more susceptible to a threat.

Notice that the true beliefs of the Farmer are irrelevant to this equilibrium. At the end of the match, the strategy of the Farmer is revealed, and in particular $k_I = (\alpha A)^{1/(1-\alpha)} \delta_F{}^{1/(1-\alpha)}$, so that the discount factor can be inferred by inverting this function: $\delta_F = k_I{}^{1-\alpha} / (\alpha A)$.

3.2. The Evolutionary Process: Two Types

We now wish to consider the co-evolution of preferences as measured by the discount factors and the number of Farmers and Sheriffs. In the analysis overall fitness of a particular population does not depend on preferences, but on the total, undiscounted expected utility over the life of the individual.[5]

For simplicity we consider first the case where there are two possible preferences: either patient preferences with discount factor one – corresponding to maximizing the same total fitness objective function as evolutionary fitness – or impatient preferences with some $0 < \delta < 1$, that is $\delta_F, \delta_S \in \{\delta, 1\}$.

In this simple model there are four types of individuals: patient Farmers, patient Sheriffs, impatient Farmers and impatient Sheriffs. At the end of each round each group gives birth to offspring who are identical in preferences and type: offspring are commonly observed. Since beliefs going into a round are fixed no player has any incentive to do other than maximize with respect to his true preferences; as we observed above this means at the end of a round players' preferences can be inferred from behavior, so the preferences of offspring are known with certainty – and equal to their true value. In this context: why should not evolution simply favor patient players as they maximize fitness. The reason for this is that individuals simply maximize with respect to their own preferences and do not take account of how this will effect subsequent

[5] A few words may be useful about fitness. Fitness is meant to be what evolution favors, and it is not utility. Take a simple example: there are two people. One is miserable in a solid brick house and the other is happy in the woods. The morning after a freezing night the first guy is, as always, unhappily complaining over his coffee, but the second is dead. The former are preferences, the latter is fitness. Fitness is an objective measure independent of preferences and is in general an elusive concept. In our case, however, preferences only enter as discount factors, hence removing them yields the desired measure of fitness.

generations. In particular, for fixed Sheriff beliefs it is costly in fitness for an individual Farmer to maximize with respect to a discount factor less than one. However, by doing so, she (involuntarily) establishes that her offspring are impatient – and this means that subsequent Sheriffs will demand less from her offspring. While the impatient Farmer loses through her impatience, her offspring benefit, and this creates a potential evolutionary force towards impatience.

Recall that ϕ is the fraction of the population who are Farmers; let ψ denote the fraction of Farmers who are impatient;[6] and let ψ_S denote the fraction of the Sheriffs who are impatient. Let $V_F(\delta_F), V_S(\delta_S)$ denote the evolutionary fitness of Farmers and Sheriffs as a function of their preferences. To compute this, we compute fitness in the different matches. The fitness of an unmatched Farmer is

$$V_F^U(\delta_F) = 1 + \alpha^{\alpha/(1-\alpha)} A^{1/(1-\alpha)} {\delta_F}^{\alpha/(1-\alpha)} - (\alpha A)^{1/(1-\alpha)} {\delta_F}^{1/(1-\alpha)},$$

while in the Farmer-Sheriff game it is

$$\begin{aligned} V_F^{FS}(\delta_F, \delta_S) &= V_F^U(\delta_F) - d_S + G \\ &= V_F^U(\delta_F) - \alpha^{\alpha/(1-\alpha)}(AB)^{1/(1-\alpha)} {\delta_F}^{1/(1-\alpha)} {\delta_S}^{\alpha/(1-\alpha)} + G \end{aligned}$$

The fitness of an unmatched sheriff is one, while in the Farmer-Sheriff game it is

$$\begin{aligned} & V_S^{FS}(\delta_F, \delta_S) = 1 - k_S + d_S \\ & = 1 + \alpha^{\alpha/(1-\alpha)}(AB)^{1/(1-\alpha)} {\delta_F}^{1/(1-\alpha)} ({\delta_S}^{\alpha/(1-\alpha)} - \alpha {\delta_S}^{1/(1-\alpha)}) \end{aligned}$$

Our model of evolution is the standard replicator dynamics based on evolutionary fitness. If ϕ_j is the population fraction of group j, V_j is the fitness of the group and $\bar{V}$ is the average fitness of the population, then

$$\dot{\phi}_j = \phi_j(V_j - \bar{V}).$$

Our analysis is greatly aided by the observation that Sheriffs evolve strictly towards greater patience:

Proposition 1: $\dot{\psi}_S < 0$

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[6] Anticipating, we omit the subscript F for the Farmers on ψ.

Proof: It suffices to show that $V_S^{FS}(\delta_F, \delta_S)$ is increasing in δ_S. We compute

$$D_{\delta_S} V_S^{FS}(\delta_F, \delta_S) =$$
$$\alpha^{\alpha/(1-\alpha)}(AB)^{1/(1-\alpha)}{\delta_F}^{1/(1-\alpha)} \frac{\alpha}{1-\alpha} \delta_S^{-1}({\delta_S}^{\alpha/(1-\alpha)} - {\delta_S}^{1/(1-\alpha)}) > 0$$

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The interesting case in the long-run, therefore, has only three types: patient Sheriffs, and both patient and impatient Farmers. In this case, on which we now focus, we can compute the overall fitnesses of a (patient) Sheriff to be

$$V_S = 1 + \alpha^{\alpha/(1-\alpha)}(AB)^{1/(1-\alpha)}(1-\alpha)\{(1-\psi)\phi + \psi\phi\delta^{1/(1-\alpha)}\}$$

while that of Farmers is given by

$$\begin{aligned} V_F(\delta_F) = 1 + \alpha^{\alpha/(1-\alpha)} A^{1/(1-\alpha)} {\delta_F}^{\alpha/(1-\alpha)}(1-\alpha\delta_F) \\ -(1-\phi)\alpha^{\alpha/(1-\alpha)}(AB)^{1/(1-\alpha)}{\delta_F}^{1/(1-\alpha)} + (1-\phi)G \end{aligned}.$$

Notice that this depends on how many farmers there are, but not, of course, what type they are. The replicator dynamics can now be summarized by two equations:

$$\dot{\psi} = \psi(1-\psi)[V_F(\delta) - V_F(1)]$$

$$\dot{\phi} = \phi(1-\phi)\{[V_F(\delta) - V_S] - (1-\psi)[V_F(\delta) - V_F(1)]\}$$

Theorem 2*: Suppose* $B^{1/(1-\alpha)}\alpha < (1-\alpha)(B^{1/(1-\alpha)} - 1)$*. Then for any* $0 < \delta < 1$ *there exists an open set of* G*'s such that there is a unique interior steady state and it is dynamically stable. At the steady state*

$$\phi = \phi^* \equiv 1 - \frac{1-\alpha-\delta^{\alpha/(1-\alpha)}(1-\alpha\delta)}{B^{1/(1-\alpha)}(1-\delta^{1/(1-\alpha)})}$$

Proof: In Appendix A.

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Notice that ϕ^* does not depend on G. Notice also that the hypothesis $B^{1/(1-\alpha)}\alpha < (1-\alpha)(B^{1/(1-\alpha)} - 1)$ is not vacuous since for any $B > 1$ it is satisfied for sufficiently small α. We can also compute

$$D_\delta \phi^* \equiv \frac{\alpha}{1-\alpha}\delta^{\alpha/(1-\alpha)}\frac{\delta^{-1}-1}{B^{1/(1-\alpha)}(1-\delta^{1/(1-\alpha)})}$$
$$+\frac{\phi\,*}{(1-\alpha)(1-\delta^{1/(1-\alpha)})}\delta^{\alpha/(1-\alpha)}) > 0$$

so that if the impatient Farmers are less impatient there will be more of them at the steady state.

The key observation here is that at a stable interior steady state in the long-run there is a positive fraction of farmers who are impatient: evolution leads to impatience. Furthermore, Appendix A shows that if the fraction of the population who are Farmers falls below $\phi\,*$ the fraction of Farmers who are impatient grow, and the fraction of the population of Farmers rises above $\phi\,*$ the fraction of Farmers who are patient grow. That is: many Sheriffs favor the impatient since impatience reduces the demands of the Sheriffs, while few Sheriffs favor the patient since patience leads to more productive investment. The problematic aspect of this analysis is that with only two possible discount factors the level of impatience δ is specified exogenously. A more satisfactory analysis would allow many different possible levels of impatience and ask which level emerges endogenously. We turn to this next.

3.3 The Evolutionary Process: Many Types

It is not very natural to suppose that the only possible preferences are given by two discount factors $\delta, 1$. Suppose instead that there are individuals with every discount factor in the interval $\delta \in [0,1]$. The general case is intractable, but a simple approximation gives us insight into the dynamics and enables us to determine a steady state value of δ.

First observe that as with the case with two types, Sheriffs with $\delta = 1$ always have higher fitness than those with lower discount factors, so in the long run the Sheriffs will evolve towards patience. As before, the interesting case is where there is a single group of patient Sheriffs, and we will focus on this case.

Next suppose that there is a density function over discount factors ψ_δ and that we are near an interior steady state, the case of interest. Then as the steady state is approached the density function must approach a spike as every type of Farmer evolves towards the optimal discount factor. The replicator dynamic is given by

$$\dot{\psi}_\delta = \psi_\delta(V_F(\delta) - \overline{V}_F),$$

where $\overline{V}_F$ is the mean fitness of farmers. Since the distribution of types is very concentrated near the mean value δ_F we may introduce an approximation. First, we may approximate the mean fitness $\overline{V}_F$ by the fitness V_F evaluated at the mean discount factor δ_F.

$$\begin{aligned}\dot{\psi}_\delta &= \psi_\delta(V_F(\delta) - \overline{V}_F) \\ &\approx \psi_\delta(V_F + DV_F[\delta - \delta_F] - V_F) \\ &= \psi_\delta DV_F[\delta - \delta_F]\end{aligned}$$

After a short interval of time τ the system will evolve according to

$$\begin{aligned}\psi_\delta(t+\tau) &\approx \psi_\delta(t) + \dot{\psi}_\delta(t)\tau \\ &\approx \psi_\delta(t) + \psi_\delta(t)DV_F[\delta - \delta_F]\tau\end{aligned}$$

We can then compute the mean discount factor by integrating:

$$\begin{aligned}\delta_F(t+\tau) &= \int \delta\psi_\delta(t+\tau)d\delta \\ &\approx \int \delta[\,\psi_\delta(t) + \psi_\delta(t)DV_F[\delta - \delta_F]\tau\,]\,d\delta \\ &= \int \delta\psi_\delta(t)d\delta + \int \delta\psi_\delta(t)DV_F[\delta - \delta_F]\tau d\delta \\ &= \delta_F(t) + DV_F\tau\int \delta\psi_\delta(t)[\delta - \delta_F]d\delta \\ &= \delta_F(t) + \sigma^2(t)DV_F\tau\end{aligned}$$

This then gives the approximate dynamic equation for the mean discount factor of the Farmers as

$$\dot{\delta}_F \approx \sigma^2(t)DV_F.$$

The fact that the variance σ^2 is time varying does not matter for our stability analysis, so we hold it fixed, and study the dynamic equation

$$\dot{\delta}_F = \sigma^2 DV_F$$

which is simply the continuous time best response dynamic – that is the mean moves in the direction of increasing fitness. The dynamics of ϕ are the replicator dynamic, now based on the mean discount factor, so

$$\dot{\phi} = \phi(1-\phi)(V_F - V_S).$$

Theorem 3: *Assume* $G > (\alpha AB)^{1/(1-\alpha)}$*. Then there is a unique interior steady state and it is dynamically stable.*

Proof: In Appendix B.

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Notice that like Theorem 2, for stability Theorem 3 requires that G not be too small. However, unlike Theorem 2 it does not place an upper bound on G. From the proof of Theorem 2 in Appendix A it transpires that the reason for the upper bound on G does not involve stability, but rather is needed to insure the existence of an interior steady state. To understand what is going on, recall that by Proposition 1 ϕ^* does not depend on G. As we increase G holding fixed the other parameters this increases the utility of the Farmers, while not changing the utility of the Sheriffs. Hence once G is big enough at ϕ^* – regardless of the value of ψ Farmers of both types will do better than Sheriffs, and so the number of Farmers will be increasing. This implies that there is no interior steady state: to the right of ϕ^* patient Farmers are favored over impatient ones. However, this is an artifact of the fact that there are only two types. If the impatient Farmers were less impatient – that is to say, if δ were larger, we saw that this would shift ϕ^* to the right, and so for this higher value of δ there could be a steady state. Once we endogenize δ_F Theorem 3 shows that this is the right intuition: regardless of how large G is there is always a steady state.

We now establish some results concerning the steady state.

Theorem 4: *(1) The steady state value of* ϕ *is larger than* $1/2$*, and larger the larger is* G.

The comparative statics with respect to G *and* B *are the following:*

(2) $D_G\delta_F > 0, D_G\phi > 0$, $D_B\delta_F < 0$, *and for sufficiently large* G, $D_B\phi < 0$.

Proof: In Appendix B.

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4. Efficiency and the Impatience Trap

We now turn to the issue of welfare. Our measure of welfare is the average fitness for the whole population. Our goal is to show how an inefficient impatience trap arises in which the wrong population becomes impatient.

To compute the average fitness of the entire population, observe that: there is a fraction ϕ^2 of unmatched farmers with fitness ${V_F}^U(\delta_F)$; a fraction $(1-\phi)^2$ of unmatched sheriffs with fitness 1; and a fraction $2\phi(1-\phi)$ of matched farmers and sheriffs who share a total fitness of $V_F^{FS}(\delta_F,\delta_S)+V_S^{FS}(\delta_F,\delta_S)$. Therefore expected average fitness is

$$
\begin{aligned}
V &= \phi^2 \int_0^1 V^U{}_F(\delta_F) f_F(\delta_F)\, d\delta_F + (1-\phi)^2 1 \\
&+\phi(1-\phi) \int_0^1 \int_0^1 \left(V_F^{FS}(\delta_F,\delta_S) + V_S^{FS}(\delta_F,\delta_S) \right) f_F(\delta_F) f_S(\delta_S)\, d\delta_F d\delta_S
\end{aligned} \quad (1)
$$

We think of the social planner as choosing a distribution over discount factors for Farmers and Sheriffs, $f_F(\delta_F), f_S(\delta_S)$ respectively (which may and in fact will be Dirac delta functions), and what fraction ϕ of the population is assigned the role of a Farmer, in order to maximize fitness. In turn, each individual chooses his optimal level of investment. Since the planner is constrained to choose discount factors, we refer to this as the *second best*.

Theorem 5: *The second best distribution is given by*

$$
\phi = \min\left\{ 1, \frac{1}{2} + \frac{1}{2} \frac{1 - A^{\frac{1}{1-\alpha}} \left(\alpha^{\frac{1}{1-\alpha}} - \alpha^{\frac{\alpha}{1-\alpha}} \right)}{G} \right\},
$$

whereas f_F *and* f_S *assign point mass at* $\delta_F = 1$ *and* $\delta_S = 0$*, respectively.*

Proof: The social planner chooses the investment levels k_F and k_S indirectly, by choosing the discount factors. The implemented investment satisfies:

$$k_F = k_I = (\alpha A)^{1/(1-\alpha)} {\delta_F}^{1/(1-\alpha)}$$

$$k_S = (\alpha AB)^{1/(1-\alpha)} (\overline{\delta}_F \delta_S)^{1/(1-\alpha)}.$$

In terms of investments, fitness is given by:

$$V_F^U(\delta_F) = 1 + Ak_F^\alpha - k_F$$
$$V_F^{FS}(\delta_F, \delta_S) + V_S^{FS}(\delta_F, \delta_S) = 2 + Ak_F^\alpha + G - k_F - k_S$$

Given that fitness is strictly decreasing in k_S, the optimal distribution assigns point mass to the value of δ_S which implements $k_S = 0$, namely $\delta_S = 0$. Similarly, fitness is maximized when Farmers choose to maximize net output, which they do if $\delta_F = 1$. Both conclusions hold irrespective of ϕ. Hence, we may find the optimal value of this latter parameter by maximizing Equation (1) when f_F and f_S are evaluated at their optimal values, that is, they assign point mass at $\delta_F = 1$ and $\delta_S = 0$, respectively. Thus, the objective becomes

$$V = \phi^2(1 + Ak_F^\alpha - k_F) + (1 - \phi)^2 + \phi(1 - \phi)(2 + Ak_F^\alpha - k_F + G - k_S)$$

which is maximized as asserted.

☑

The intuition for the optimal discount factors is simple: Sheriffs' investments are a social waste, which they would not do if they become extremely impatient. On the other hand, Farmers are productive, and they would choose the optimal investment if they were extremely patient. In fact, in the language of Hirshleifer Sheriffs obtain their wealth through conflict; in the language of Tullock and Krueguer, Sheriffs are rent-seekers. In contrast, Farmers obtain their wealth through production.

As for the optimal fraction of Farmers, it is less than 1 because there is a social gain of G whenever a Farmer and a Sheriff meet. The fraction of the matched population is maximized at

$$\phi = \frac{1}{2}.$$

In the spirit of the rent seeking literature this is saying that societies, optimally, would have rent seekers only if when matched to productive agents they were to increase "social output" (that is, $G > 0$). Otherwise, if $G = 0$, it would be optimal not to have rent seekers.

A related question has to do with the optimal mix of Farmers and Sheriffs when the social planner does not choose their discount factors, but instead when they are at their equilibrium values. The first order condition for this constrained maximization problem gives

$$\phi = \min\left\{1, \frac{1}{2} + \frac{1}{2}\frac{y_F - G - k_F}{G - k_S}\right\}$$

which is sufficient provided $G - k_S > 0$.[7] It is less than 1 for G large enough, and tends to $1/2$ as G grows.

The fact that steady state $\phi > 1/2$ (see Theorem 4) implies that if G is large enough, in the steady state there are inefficiently many Farmers, and too few Sheriffs. The intuition is that this arises because the Sheriff's have to pay to collect a share of G.

This is what we call the impatience trap. We see it as a trap when interpreting the model as one where Sheriffs are Buyers and Farmers are Sellers, viewing k_S as the short run cost of enforcing reliability and G as the long run gain of partnership and trust. Note that inefficiency worsens the larger is G.

In this interpretation, the final result of Theorem 4 says that if the gains to trade G are large enough increasing the effectiveness of punishment will raise the steady state number of Sheriff/Buyers, thus reducing inefficiency. We will come back on this point in the conclusions.

5. Extensions

In all our analysis we have focused in a particular sequence of the game. Now we discuss the results under alternative sequences of the game.

In the equilibrium analysis of the model there is a "non-standard" result, namely, that in a bargaining situation being impatient might be better. Usually we get the opposite result (for example, Rubinstein's model). The reason for this has to do with the structure of the game. Here, punishments are applied in the future and as such, a more patient Farmer is more influentiable by threats, weakening his bargaining position to the advantage of Sheriffs that get paid more. In contrast, in Rubinstein's model being more patient means that the cost associated to the delay to reach an agreement is smaller, strengthening the bargaining position.

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[7] When the second order condition does not hold (that is $G - k_S < 0$) the optimal solution is $\phi = 1$.

Under the current game structure (namely, k_F and k_S are chosen after a meeting is produced, with the knowledge of the opponent's type) we can distinguish two effects: (E1, or direct effect) Meeting with a more patient Farmer renders any given investment k_S by the Sheriff more productive (privately), since a higher demand d will be accepted by the Farmer, and (E2, or indirect effect) The Sheriff may take further advantage of this by conditioning his investment level on δ_F. In our case, k_S increases in δ_F. E1 makes d depend on δ_F, and E2 makes k_S depend on δ_F.

If the Sheriff makes his investment decision k_S before knowing his opponent's δ_F, E1 remains and E2 goes away. The main result would still obtain, although Sheriffs would have a lower expected utility implying a higher equilibrium ϕ. This would also be the case if the Sheriff were to make his investment decision before knowing if he would be matched or not, although admittedly resulting in an even lower fitness.

If the Sheriff makes both, his investment decision and his demand before knowing his opponent's δ_F, both effects go away. Besides not obtaining the effect we want, this case is also cumbersome to analyze because in equilibrium there would be demands that are not accepted by the more impatient farmers.

Both effects would also disappear if the punishment were to take place in period 2 rather than in period 3. Indeed, the discount factor affects the relationship between promised punishment and willingness to accept demands exclusively because punishments are promised to happen at a future date.

Regarding the welfare, the results are independent on the sequence of the game. The efficient distribution of ϕ, δ_S and δ_F is independent of the sequence. Moreover, given the efficient distribution all the results are independent of the sequence.

There are alternative characterizations, and real world situations, where in games less patient people do better than patient people. For example, Blaydes [2004] uses a version of Fearon's [1998] model to explain the division of cartel profits within the OPEC. In the model there is a first step in which there is a bargaining that determines the payoffs of a static game that is infinitely repeated. To enforce the "efficient" outcome in the infinitely repeated game, more impatient players need a higher "static" payment. Thus, impatience is also in this case a source of bargaining strength.

6. Conclusion

We have shown that impatience survives evolutionary forces when it keeps down punishment by the opponents. This is in contrast to the single-person investment context where (Blume and Easley, 1992) the patient beats the informed.

When interpreting the model as one of buyer and seller, where the Farmer is the Seller and the Sheriff is the Buyer, we see G as the long run gains of partnership, not fully exploited in equilibrium owing to the presence of too many impatient sellers.

To put this discussion in context, the underlying issue here is: What makes a good business environment? The most common, reasonable short answer is “competence and reliability.” The model of this paper has something to say about reliability, which is another face of patience. A reliable business does not “take the money and run” meaning a reliable seller must be patient. In our model suppose that potential gains from trade G are large. Never the less the share that may be claimed by buyers in the form of d_S may be limited. In an underdeveloped economy the cost of investing in punishing recalcitrant sellers in the face of resource constraints may be large. The result can be an evolutionary stable impatience trap, in which the equilibrium is inefficient and sellers have little money to run with because there are too few buyers to spoil.

Our Theorem 4 points to an instrument that can potentially be used to reduce inefficiency, namely raising the effectiveness of punishment in the hand of the buyers, the parameter B. This is not simple – it would indeed be not credible if it were. For B is often nothing but social pressure on the unreliable producers. Said otherwise, the problem is to raise awareness of the long run nature of the benefits of business, and this links unreliability to the other component of a good business environment - competence or education. In the way of prescriptions for development we are not uncovering something new. On the other hand the model seems to be the first to uncover the source of the problem's persistence: the inefficient equilibrium we have is not simply undone by evolutionary forces.

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Appendix A: Proof of Theorem 2

From the text, the dynamical system is given by

$$\dot{\psi} = \psi(1-\psi)[V_F(\delta) - V_F(1)]$$

$$\dot{\phi} = \phi(1-\phi)\{[V_F(\delta) - V_S] - (1-\psi)[V_F(\delta) - V_F(1)]\}.$$

From the fitnesses in the text, we can compute the fitness differences

$$\begin{aligned} V_F(\delta) - V_S &= (1-\phi)G + \alpha^{\alpha/(1-\alpha)} A^{1/(1-\alpha)} \times \\ &\quad \{\delta^{\alpha/(1-\alpha)}(1-\alpha\delta) - B^{1/(1-\alpha)}\delta^{1/(1-\alpha)} \\ &\quad + \phi B^{1/(1-\alpha)}\{\delta^{1/(1-\alpha)} - (1-\alpha)(1-\psi+\psi\delta^{1/(1-\alpha)})\} \end{aligned}$$

$$\begin{aligned} V_F(\delta) - V_F(1) = \alpha^{\alpha/(1-\alpha)} A^{1/(1-\alpha)} &\times \{(\delta^{\alpha/(1-\alpha)}(1-\alpha\delta) - 1 + \alpha) \\ &+ (1-\phi)B^{1/(1-\alpha)}(1-\delta^{1/(1-\alpha)})\} \end{aligned}.$$

Lemma A1: *For* $1 > \psi > 0$ *we have* $\dot{\psi} >=< 0$ *as* $\phi <=> \phi^*$ *where*

$$\phi^* = 1 - \frac{1-\alpha-\delta^{\alpha/(1-\alpha)}(1-\alpha\delta)}{B^{1/(1-\alpha)}(1-\delta^{1/(1-\alpha)})}$$

lies between 0 and 1.

Proof: The computation of ϕ^* comes from solving $V_F(\delta) - V_F(1) = 0$, and we may also compute

$$D_\phi[V_F(\delta) - V_F(1)] \propto -\phi B^{1/(1-\alpha)}(1-\delta^{1/(1-\alpha)}) < 0$$

from which the signs follow.

Rewriting

$$1 - \phi^* = \frac{1-\delta^{1/(1-\alpha)} - \left[\delta^{\alpha/(1-\alpha)}(1-\delta) + \alpha(1-\delta^{1/(1-\alpha)})\right]}{B^{1/(1-\alpha)}(1-\delta^{1/(1-\alpha)})}$$

we can see that since $B \geq 1$ the numerator of the RHS is smaller than the denominator implying $1-\phi^* < 1$, so that ϕ^* cannot be negative. We may also write the numerator of $1-\phi^*$ as

$$f(\delta) \equiv 1-\alpha-(\delta^{\alpha/(1-\alpha)}-\alpha\delta^{1/(1-\alpha)}).$$

We then compute

$$f(0) \equiv 1-\alpha$$
$$f(1) \equiv 0$$
$$f'(\delta) \equiv -\frac{\alpha}{1-\alpha}\delta^{-1}\delta^{\alpha/(1-\alpha)}(1-\delta) < 0$$

from which it follows that $f(\delta) \geq 0$, and so $\phi^* \leq 1$.

☑

Lemma A2: $\dot{\phi} \propto a + b\phi + c\psi + d\phi\psi$ *where the factor of proportionality is* $A^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}$ *and*

$$a = \tilde{G} - \alpha - (B^{1/(1-\alpha)} - 1)$$
$$b = (B^{1/(1-\alpha)}\alpha - \tilde{G})$$
$$c = \delta^{\alpha/(1-\alpha)}(1-\alpha\delta) - 1 + \alpha + B^{1/(1-\alpha)}(1-\delta^{1/(1-\alpha)})$$
$$d = -B^{1/(1-\alpha)}\alpha(1-\delta^{1/(1-\alpha)})$$

with $\tilde{G} = G / A^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}$.

Proof: Direct computation using the fitness differences. ☑

Corollary A3: $d < 0, c + d \geq 0$

Proof: $d < 0$ is immediate. For $c + d$ we compute

$$c + d = f(\delta) \equiv$$
$$\delta^{\alpha/(1-\alpha)}(1-\alpha\delta) + (1-\alpha)[B^{1/(1-\alpha)}(1-\delta^{1/(1-\alpha)}) - 1]$$
$$f(0) = (1-\alpha)[B^{1/(1-\alpha)} - 1] > 0$$
$$f(1) = 0$$

and the derivative

$$f'(\delta) = \delta^{\alpha/(1-\alpha)} \times$$
$$\left\{\frac{\alpha}{1-\alpha}(\delta^{-1}-1) - (1-\alpha)B^{1/(1-\alpha)}\delta\right\}$$

The part in brackets is decreasing, and this implies that $f(\delta)$ is single peaked. Hence it follows from the boundary conditions that $f(\delta) \geq 0$.

☑

Lemma A4: *An interior steady state exists if and only if*

$$X \equiv (1-\phi^*)\tilde{G} + (1-\alpha) - B^{1/(1-\alpha)}[1-\alpha\phi^*] < 0$$

$$Y \equiv (1-\phi^*)\tilde{G} + (1-\alpha) - B^{1/(1-\alpha)}[1-\phi^* + (1-\alpha)\phi^* \delta^{1/(1-\alpha)}] > 0$$

and if it exists it is unique.

Proof: If there is an interior steady state by Lemma A1 it must occur for $\phi = \phi^*$. This implies that the fitness of both types of farmers is equal, so that the sign of $\dot{\phi}$ is determined by

$$V_F(1) - V_S \propto f(\psi) \equiv$$
$$(1-\phi)\tilde{G} + (1-\alpha) - (1-\phi)B^{1/(1-\alpha)}$$
$$-B^{1/(1-\alpha)}(1-\alpha)\{\phi^* + \psi\phi^*(\delta^{1/(1-\alpha)}-1)\}$$

This is linear and increasing in ψ. Hence there is an interior steady state if and only if $f(0) < 0$, $f(1) > 0$, and in that case because $f(\psi)$ is linear, it is unique. The conditions in the Lemma follow from the expression for $f(\psi)$.

☑

Lemma A5: *A sufficient condition for an interior steady state* ϕ^*, ψ^* *to be stable is* $b < 0$.

Proof: It is sufficient that in the system linearized at the steady state the trace be negative and the determinant positive. Disregarding irrelevant factors, the matrix of the linearized system is

$$M = \begin{bmatrix} 0 & e \\ c + d\phi^* & b + d\psi^* \end{bmatrix}$$

where

$$e = -B^{1/(1-\alpha)}(1 - \delta^{1/(1-\alpha)}) < 0\,.$$

Hence the sufficient condition is $c + d\phi^* > 0$ and $b + d\psi^* < 0$. By Corollary A3 $d < 0, c + d \geq 0$ and $\phi^* < 1$ implies $c + d\phi^* > 0$, so the remaining condition is sufficient. Since $d < 0$ it is in turn sufficient that $b < 0$.

☑

Theorem 2: *Suppose* $B^{1/(1-\alpha)}\alpha < (1-\alpha)(B^{1/(1-\alpha)} - 1)$. *Then for any* $0 < \delta < 1$ *there exists an open set of* G*'s such that there is a unique interior steady state and it is dynamically stable. At the steady state*

$$\phi = \phi^* \equiv 1 - \frac{1 - \alpha - \delta^{\alpha/(1-\alpha)}(1 - \alpha\delta)}{B^{1/(1-\alpha)}(1 - \delta^{1/(1-\alpha)})}$$

Proof: The characterization of ϕ^* is in Lemma A1. For sufficiently small $\varepsilon > 0$ we can choose

$$\tilde{G} = \frac{B^{1/(1-\alpha)}[1 - \alpha\phi^*] - (1-\alpha) - \varepsilon}{(1 - \phi^*)} > 0\,.$$

The first condition from Lemma A4 for an interior steady state is

$$X \equiv -\varepsilon < 0$$

Moreover

$$\begin{aligned} Y &= X + B^{1/(1-\alpha)}(1-\alpha)\phi^*(1 - \delta^{1/(1-\alpha)}) \\ &= -\varepsilon + B^{1/(1-\alpha)}(1-\alpha)\phi^*(1 - \delta^{1/(1-\alpha)}) \end{aligned}$$

which is positive for ε sufficiently small. Hence for such choices of $\tilde{G}$ an interior steady state exists.

Turning to stability, by Lemma A5, we require $b < 0$, by Lemma A2 this condition is

$$B^{1/(1-\alpha)}\alpha < \tilde{G}\,.$$

Notice that

By the assumption that $B^{1/(1-\alpha)}\alpha > (1-\alpha)(B^{1/(1-\alpha)}-1) - \varepsilon$ this implies that $\tilde{G} > B^{1/(1-\alpha)}\alpha - \varepsilon$, so that $b < 0$ for ε sufficiently small.

☑

Appendix B: Proof of Theorems 3 and 4

As in the model with two types we can compute the fitnesses

$$V_S = 1 + (AB)^{1/(1-\alpha)}(1-\alpha)\alpha^{\alpha/(1-\alpha)}\phi{\delta_F}^{1/(1-\alpha)}$$

$$V_F = 1 + (1-\phi)G + \alpha^{\alpha/(1-\alpha)}A^{1/(1-\alpha)}{\delta_F}^{\alpha/(1-\alpha)}\{1 - \alpha\delta_F - (1-\phi)B^{1/(1-\alpha)}\delta_F\}\,.$$

Define $\tilde{\alpha} = \alpha/(1-\alpha)$, $\tilde{B} = B^{\tilde{\alpha}+1}$ and as in Appendix A $\tilde{G} = \tilde{\alpha}G/(\alpha \mathrm{A})^{\tilde{\alpha}+1}$. Note since $B > 0, \alpha > 0$ that $\tilde{B} > 1$. Normalizing $\sigma^2 = 1$[8] this enables us to write the dynamical system as

$$\dot{\delta}_F = \frac{(\alpha \mathrm{A})^{\tilde{\alpha}+1}}{1-\alpha}\delta_F^{\tilde{\alpha}-1}\left[(1-\delta_F) - \alpha^{-1}(1-\phi)\tilde{\mathrm{B}}\delta_F\right]$$

$$\dot{\phi} = \phi(1-\phi)h(\phi,\delta_F)$$
$$h(\phi,\delta_F) \equiv (\alpha \mathrm{A})^{\tilde{\alpha}+1}\delta_F^{\tilde{\alpha}}\left[\alpha^{-1} - \delta_F - \alpha^{-1}\tilde{B}\delta_F + \phi\tilde{B}\delta_F\right] + (1-\phi)G$$

Lemma B1: *There is a unique interior steady state.*

Proof: Combining $\dot{\phi}/(\phi(1-\phi)) = 0$ and $\dot{\delta}_F = 0$ yields

$$f(\delta_F) \equiv \tilde{B}(1+\alpha^{-1}\tilde{B})\delta_F^{\tilde{\alpha}+2} - (1+\alpha^{-1})\tilde{B}\delta_F^{\tilde{\alpha}+1} + \tilde{G}\delta_F - \tilde{G} = 0$$

and letting $\xi = 1 - \phi$

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[8] This is relevant only to the stability analysis, and since that is based on a sign argument, the magnitude does not matter.

$$g(\xi) \equiv \xi\left[(1+\alpha^{-1})\tilde{B} + \tilde{G}(1+\alpha^{-1}\tilde{B}\xi)^{\tilde{\alpha}+1}\right] - (\tilde{B}-1) = 0 .$$

We show that each has a unique zero in $(0,1)$.

Examining g first, we have $g(0) = -(\tilde{B}-1) < 0$ and $g(1) = \alpha^{-1}\tilde{B} + \tilde{G}(1+\alpha^{-1}\tilde{B})^{\tilde{\alpha}+1} + 1 > 0$. Moreover g is the sum of a constant and two increasing functions, so it is increasing, and hence has a unique zero in $(0,1)$.

Turning to f, we see that $f(0) = -\tilde{G} < 0$ and $f(1) = \alpha^{-1}\tilde{B}(\tilde{B}-1) > 0$, so that there is at least one solution by continuity. To prove uniqueness, observe that

$$f'(\delta_F) = (\tilde{\alpha}+2)\tilde{B}(1+\alpha^{-1}\tilde{B})\delta_F^{\tilde{\alpha}+1} - (\tilde{\alpha}+1)(1+\alpha^{-1})\tilde{B}\delta_F^{\tilde{\alpha}} + \tilde{G}$$

Hence $f'(0) = \tilde{G} > 0$, and

$$\begin{aligned} f'(1) &= (\tilde{\alpha}+2)\tilde{B}(1+\alpha^{-1}\tilde{B}) - (\tilde{\alpha}+1)(1+\alpha^{-1})\tilde{B} + \tilde{G} \\ &= (1+\alpha^{-1})\tilde{B} + \alpha^{-1}(\tilde{\alpha}+2)\tilde{B}(\tilde{B}-1) + \tilde{G} > \tilde{G} \end{aligned}$$

The second derivative is

$$f''(\delta_F) = \tilde{B}\delta_F^{\tilde{\alpha}-1}\left[(\tilde{\alpha}+2)(\tilde{\alpha}+1)(1+\alpha^{-1}\tilde{B})\delta_F - (\tilde{\alpha}+1)\tilde{\alpha}(1+\alpha^{-1})\right].$$

This is negative below $\delta^0 \equiv \tilde{\alpha}(1+\alpha^{-1})/(\tilde{\alpha}+2)(1+\alpha^{-1}\tilde{B}) < 1$ and positive above. So f' decreases to its minimum at δ^0 then increases. There are two possibilities: $f'(\delta^0) \geq 0$ or $f'(\delta^0) < 0$. In the first case f increases from $f(0) < 0$ to $f(1) > 0$ so has a unique zero. In the second case it increases to a local maximum at $\delta^1 \in (0,\delta^0)$, then decreases, then, since $f'(1) > 0$ increases again to $f(1) > 0$. A unique zero follows provided that $f(\delta^1) < 0$. Since from 0 to δ^0, and in particular from 0 to δ^1, f is concave, it follows that $f(\delta^1) < f(0) + f'(0)\delta^1 = -\tilde{G} + \tilde{G}\delta^1 = -\tilde{G}(1-\delta^1) < 0$.

☑

Lemma B2: *If* $G > (\alpha AB)^{1/(1-\alpha)}$ *then the interior steady state is stable.*

Proof: As in the proof of Lemma A5 it is sufficient that in the system linearized at the steady state the trace be negative and the determinant positive. Disregarding irrelevant factors, the matrix of the linearized system is

$$M = \begin{pmatrix} \partial\dot{\delta}_F / \partial\delta_F & \partial\dot{\delta}_F / \partial\phi \\ \partial h / \partial\delta_F & \partial h / \partial\phi \end{pmatrix}$$

Consequently it is sufficient that $\partial\dot{\delta}_F / \partial\delta_F, \partial h / \partial\phi < 0$ and $\partial\dot{\delta}_F / \partial\phi > 0, \partial h / \partial\delta_F < 0$.

We compute

$$\frac{\partial\dot{\delta}_F}{\partial\delta_F} = \frac{(\alpha\mathrm{A})^{\tilde{\alpha}+1}}{1-\alpha}\left[(\tilde{\alpha}-1)\delta_F^{\tilde{\alpha}-2}(1-\delta_F) - \delta_F^{\tilde{\alpha}-1} - \tilde{\alpha}\alpha^{-1}(1-\phi)\tilde{\mathrm{B}}\delta_F^{\tilde{\alpha}-1}\right]$$

Using the fact that when $\dot{\delta} = 0$ we have $\alpha^{-1}(1-\phi)\tilde{\mathrm{B}} = (1-\delta)/\delta$, from which one obtains

$$\frac{\partial\dot{\delta}_F}{\partial\delta_F} = -\frac{(\alpha\mathrm{A})^{\tilde{\alpha}+1}}{1-\alpha}\delta_F^{\tilde{\alpha}-2} < 0.$$

Next

$$\frac{\partial\dot{\delta}_F}{\partial\phi} = \frac{(\alpha\mathrm{A})^{\tilde{\alpha}+1}}{1-\alpha}\alpha^{-1}\tilde{\mathrm{B}}\delta_F^{\tilde{\alpha}} > 0.$$

Using the definition of h we have

$$\frac{\partial h}{\partial\delta_F} = \alpha^{\tilde{\alpha}}\mathrm{A}^{\tilde{\alpha}+1}\tilde{\alpha}\delta^{\tilde{\alpha}-1}\left\{1-(1+(\alpha^{-1}-\phi)\tilde{\mathrm{B}})\delta_F\right\}$$

Using the steady state condition

$$\delta_F = \frac{1}{1+\alpha^{-1}\tilde{B}(1-\phi)}$$

the expression in brackets becomes $-\delta\phi\tilde{\mathrm{B}}\tilde{\alpha}^{-1}$, so that

$$\frac{\partial h}{\partial\delta_F} = -\alpha^{\tilde{\alpha}}A^{\tilde{\alpha}+1}\tilde{B}\delta^{\tilde{\alpha}} < 0.$$

Finally, compute

$$\frac{\partial h}{\partial\phi} = (\alpha\mathrm{A})^{\tilde{\alpha}+1}\tilde{B}\delta_F^{\tilde{\alpha}+1} - G$$

Since $\delta < 1$, $(\alpha \mathrm{A})^{\tilde{\alpha}+1}\tilde{B}\delta_F^{\tilde{\alpha}+1} - G < (\alpha \mathrm{A})^{\tilde{\alpha}+1}\tilde{B} - G$, which is negative for $G > (\alpha \mathrm{A})^{\tilde{\alpha}+1}\tilde{B}$, that is to say for the condition of the Lemma $G > (\alpha AB)^{1/(1-\alpha)}$.

☑

Theorem 3 follows directly from Lemmas B1 and B2.

Lemma B3: *The steady state* $\phi > 1/2$.

Proof: Using $(1+\alpha^{-1}\tilde{B}\xi)^{\tilde{\alpha}+1} > 1+\alpha^{-1}\tilde{B}\xi$ it is easily checked that $g(1/2) > 0$ and g is increasing in ξ which implies that if $\xi \geq 1/2$ then $g(\xi) > 0$. It follows that the steady state value of ξ is less than ½, so that the steady state value of $\phi = 1-\xi$ is greater than ½. The last assertion follows from the fact that g is larger for all ξ the larger is G.

☑

Lemma B4: $D_G\delta_F > 0, D_G\phi > 0$

Proof: It suffices to show this for $\tilde{G}$ as given the other parameters $\tilde{G}$ is an increasing linear function of G. From the definitions of f, g the former is decreasing and the latter increasing in $\tilde{G}$. In the proof of Lemma B1 we showed that both f, g cross the horizontal axis from below. The implicit function theorem then gives the desired result.

☑

Lemma B5: $D_B\delta_F < 0$, *for sufficiently large* G $D_B\phi < 0$.

Proof: It suffices to show the result with respect to $\tilde{B}$ as this is an increasing function of B. By inspection $D_{\delta_F} f > 0$ and $\partial_\xi g > 0$, so $D_\phi g < 0$. We compute

$$D_{\tilde{B}} f = \delta_F^{\tilde{\alpha}+1}\left(2\alpha^{-1}\delta_F\tilde{B} + \delta_F - (1+\alpha^{-1})\right) > 2\alpha^{-1} + \delta_F - (1+\alpha^{-1}) > \delta_F > 0.$$

It is also the case that $\delta_F\tilde{B} > 1$ in the steady state. This follows from the fact that

$$f(1/\tilde{B}) = \tilde{B}^{-(\tilde{\alpha}+1)}(1-\tilde{B})(1+\tilde{G}\tilde{B}^{\tilde{\alpha}}) < 0.$$

Hence $D_{\tilde{B}}\delta_F < 0$.

Finally

$$D_{\tilde{B}} g = \xi\left[(1+\alpha^{-1}) + \tilde{G}\xi\alpha^{-1}(\tilde{\alpha}+1)(1+\alpha^{-1}\tilde{B}\xi)^{\tilde{\alpha}}\right] - 1.$$

We can write $g(\xi) = 0$ as

$$\xi\left[\frac{(1+\alpha^{-1})\tilde{B}}{\tilde{G}} + (1+\alpha^{-1}\tilde{B}\xi)^{\tilde{\alpha}+1}\right] = \frac{\tilde{B}-1}{\tilde{G}}.$$

The expression in brackets is bounded below by 1, so that as $\tilde{G} \to \infty$ it must be that one $\xi \to 0$. Rewriting the expression as

$$\tilde{G}\xi = \frac{\tilde{B}-1}{\frac{(1+\alpha^{-1})\tilde{B}}{\tilde{G}} + (1+\alpha^{-1}\tilde{B}\xi)^{\tilde{\alpha}+1}}$$

we see that as $\tilde{G} \to \infty$, $\xi \to 0$ the RHS approaches $\tilde{B}-1$, and so $\Gamma\xi \to \tilde{B}-1$. Hence as $\tilde{G} \to \infty$ we have $D_{\tilde{B}}g \to -1$. The implicit function theorem then gives the second result.

☑

Theorem 4 now follows directly from Lemmas B3, B4 and B5.