This is a postprint of:

A unified approach to approximate solutions in games and multiobjective programming

F. Patrone & S.H. Tijs

Journal of Optimization Theory and Applications, 1987, vol. 52, nr. 2, pp. 273-278

A Unified Approach to Approximate Solutions in Games and Multiobjective Programming[1]

F. PATRONE[2] AND S.H. TIJS[3]

Abstract. Some consequences are derived from a theorem of Tijs, which in turn give results about approximate solutions in Nash equilibrium theory and in multiobjective programming. Weak conditions are described under which it is possible to replace an infinite strategy set, an infinite alternative set or an infinite set of criteria by a finite subset without loosing all approximate solutions of the problem under consideration.

Key Words. Game Theory, Nash equilibrium, multiobjective programming, ε - Pareto solutions.

---

[1]This work was done during the period when S.H. Tijs was Visiting Professor of the Italian National Research Council at the Mathematical Department of the University of Pavia.

[2]Department of Mathematics, University of Pavia, Pavia, Italy.

[3]Department of Mathematics, Catholic University, Nijmegen, The Netherlands.

1. Introduction

Aim of this work is to show how it is possible to derive from theorem 1.1 below by Tijs (Ref. 1) some consequences, which in turn give results about approximate solutions, both in Nash equilibria problems (Refs. 1 and 2) and in multiobjective programming (Refs. 3 and 4).

For $x,y \in \mathbb{R}^m$, $x \leq y$ will mean that the inequality holds coordinatewise: $x_i \leq y_i$ for all $i \in \{1,2,\ldots,m\}$.
We say that $V \subset \mathbb{R}^m$ is upper bounded if there exists an $a \in \mathbb{R}^m$ such that $x \leq a$ for all $x \in V$. We denote by 1_m the vector $(1,1,\ldots,1) \in \mathbb{R}^m$.

A starting point for the following is the next

<u>Theorem 1.1</u>. (Tijs, Ref. 1). Let $V \subset \mathbb{R}^m$ be upper bounded. Then for each $\varepsilon > 0$ there exists a finite subset $W \subset V$ such that W ε-dominates V, that is $\forall_{v \in V} \exists_{w \in W} [v \leq w + \varepsilon 1_m]$

One can reformulate this theorem by introducing on $\mathbb{R}^m$ the semimetric $\sigma = \mathbb{R}^m \times \mathbb{R}^m \to [0,\infty)$ (See Ref. 5, p. 61) defined by

$$\sigma(x,y) = \max\{0, x_1-y_1, x_2-y_2, \ldots, x_m-y_m\}$$

for all $x,y \in \mathbb{R}^m$. We say that $V \subset \mathbb{R}^m$ is σ-bounded if there is an $a \in \mathbb{R}^m$ and an $\varepsilon \geq 0$ such that V is a subset of the (σ,ε)-ball

$$B_\varepsilon(a) = \{x \in \mathbb{R}^m : d(x,a) \leq \varepsilon\}$$

with center a. Then $V \subset \mathbb{R}^m$ is σ-bounded iff V is upper bounded. Now theorem 1.1 can be read as follows:
If V is a σ-bounded set of $\mathbb{R}^m$, then V is σ-precompact i.e. for each $\varepsilon > 0$ there exists a finite number of (σ,ε)-balls in $\mathbb{R}^m$ with centers in V, which cover V or

$$V \subset \bigcup_{w \in W} B_\varepsilon(w) \quad \text{for some finite } W \subset V.$$

2. Consequences and applications

From theorem 1.1 it is easy to derive the following

Proposition 2.1. Let X be a set and Y a finite set.

Let $\Phi : X \times Y \to \mathbb{R}$ be an upper bounded function. Then for each $\varepsilon > 0$, there exists a finite subset $Z \subset X$ such that

$$\forall_{x \in X} \exists_{z \in Z} \forall_{y \in Y} [\Phi(x,y) \leq \Phi(z,y) + \varepsilon]$$

Proof. Let $Y = \{y_1, y_2, \ldots, y_m\}$. Define

$$V = \{v \in \mathbb{R}^m : \exists_{x \in X} [(v_1, v_2, \ldots, v_m) = (\Phi(x,y_1), \Phi(x,y_2), \ldots, \Phi(x,y_m))]\}$$

and apply theorem 1.1. ∎

Corollary 2.1. (Ref. 2, lemma 4.3). Let Y be a finite set and let F be an upper bounded family of real functions on Y. Then, for each $\varepsilon > 0$, there exists a finite subfamily $G \subset F$ such that G (ε, Y)-dominates F, that is

$$\forall_{f \in F} \exists_{g \in G} \forall_{y \in Y} [f(y) \leq g(y) + \varepsilon]$$

Proof. In the previous proposition, take $X = F$ and let $\Phi : X \times Y \to \mathbb{R}$ be defined by $\Phi(f,y) = f(y)$. ∎

Application 2.1. For applications of the above result, to show the existence of approximate Nash equilibria, see Tijs (Ref. 2), in particular theorem 4.1.

Corollary 2.2. Let F be a finite family of upper bounded real functions, defined on a set X. Then, for each $\varepsilon > 0$ there exists a finite subset

$Z \subset X$ such that Z (ε, F)-dominates X, that is

$$\forall_{x \in X} \exists_{z \in Z} \forall_{f \in F} [f(x) \leq f(z) + \varepsilon]$$

Proof. Still from proposition 2.1: take $Y = F$ and let $\Phi : X \times Y \to \mathbb{R}$ be defined as $\Phi(x,f) = f(x)$. ∎

Application 2.2. The above result can be applied to multiobjective programming, when we look at non dominated solutions (called also Pareto solutions). Given a set $F = \{f_1, f_2, \ldots, f_m\}$ of m upper bounded criteria, then we can reduce the set of possible alternatives X to a finite set Z, if we are interested only in approximate maximization (up to ε). Moreover, if we remove from Z F-dominated points (if any), then we are left with a non-empty set of ε-optimal solutions for Pareto problems as considered e.g. in Loridan (Ref. 4).

It is well known that often a finiteness hypothesis can be replaced be a compactness-like hypothesis. So, along this line, we can prove the following

Proposition2.2. Let (Y,d) be a pre-compact metric space, and let X be a set. Let $\Phi : X \times Y \to \mathbb{R}$ be upper bounded and equicontinuous in y with respect to x, that is

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{y',y'' \in Y} \forall_{x \in X} [d(y',y'') < \delta \Rightarrow |\Phi(x,y') - \Phi(x,y'') < \varepsilon].$$

Then, for each $\varepsilon > 0$ then exists a finite set $Z \subset X$ such that

$$\forall_{x \in X} \exists_{z \in Z} \forall_{y \in Y} [\Phi(x,y) \leq \Phi(z,y) + \varepsilon] \tag{1}$$

Proof. From equicontinuity, given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $y', y'' \in Y$ with $d(y',y'') < \delta$ we have

$$|\Phi(x,y') - \Phi(x,y'')| < \frac{1}{3}\varepsilon \quad \text{for all } x \in X \tag{2}$$

The family of the δ-balls $U_\delta(y') = \{y \in Y : d(y,y') < \delta\}$, as y' varies, is a covering of Y, from which we can extract, by pre-compactness a finite one. Let $Y^\varepsilon = \{y_1, y_2, \ldots, y_m\} \subset Y$ be such that $\bigcup_{i=1}^{m} U_\delta(y_i) \supset Y$ and let Φ^ε be the restriction of Φ to $X \times Y^\varepsilon$.
Apply to Φ^ε proposition 2.1 with $\frac{1}{3}\varepsilon$ in the role of ε. Then with the aid of (2) we obtain the result. ∎

Corollary 3.2. (Ref. 2, proposition 4.3).

Let (Y,d) be a pre-compact metric space. Let F be an equicontinuous and upper bounded family of real functions on Y. Then, for each $\varepsilon > 0$, there exists a finite subfamily $G \subset F$ such that G (ε,Y)-dominates F.

Proof. Take $X = F$, let $\Phi(f,y) = f(y)$ and apply proposition 2.2. ∎

Corollary 2.4. Let X be a set. Let (F,d) be an upper bounded family of real functions defined on X, pre-compact with respect to the metric d, and assume that d has the following property:

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{f',f''\in F}\ \forall_{x\in X}\ [d(f',f'') < \delta \Rightarrow |f'(x)-f''(x)| < \varepsilon]$$

Then, for each $\varepsilon > 0$ there exists a finite subset $Z \subset X$ such that Z (ε,F)-dominates X.

Proof. Direct from proposition 2.2 with $Y = F$ and $\Phi(x,f) = f(x)$. ∎

Remark 2.1. The assumption on d in corollary 2.4 clearly means that it is a metric of uniform convergence on X.

Application 2.3. For what concerns applications of corollary 2.3 for games we still refer to Tijs (Ref. 2). From corollary 2.4 we can learn that a finite subset of points in X is sufficient for ε-dominance even in the case of an infinite set of upper bounded criteria F if for each $\varepsilon > 0$ this set of criteria can be divided into a finite number of groups of criteria, where in each group the criteria are ε-near to each other.

Remark 2.2. Without the hypothesis of pre-compactness as in proposition 2.2, in general we do not obtain ε-dominance.

As an example, let X and Y be the canonical (Hilbertian) basis in ℓ^2, let Φ be the inner product and $\varepsilon < 1$. Then there is no finite set $Z \subset X$ such that (1) holds.

References

[1] TIJS, S.H., ε-Equilibrium Point Theorems for Two-Person Games, Methods of Operations Research, Vol. 26, pp. 755-766, 1977.

[2] TIJS, S.H., Nash Equilibria for Noncooperative n-Person Games in Normal Form, SIAM REVIEW, Vol. 23, pp. 225-237, 1981.

[3] WHITE, D.J., Optimality and Efficiency, John Wiley, Chichester, 1982.

[4] LORIDAN, P., ε-Solutions in Vector Minimization Problems, Journal of Optimization Theory and Applications, Vol. 43, pp. 265-276, 1984.

[5] NACHBIN, L., Topology and Order, D. van Nostrand Comp., Princeton, New Jersey, 1965.