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THE OPTIMAL EXPLOITATION OF A NATURAL RESOURCE WHEN THERE IS FULL COMPLEMENTARITY

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We analyse optimal growth for an economy in the possession of an exhaustible resource when the economy's non-resource output is produced by means of capital and the utilization of the resource. The optimal trajectories are sketched for the case where these factors of production are complements.

1. Introduction

As Stiglitz (1974) has pointed out 'there are at least three economic forces offsetting the limitations imposed by natural resources: technical change, returns to scale and the substitution of man-made factors of production (capital) for natural resources'. Consequently the impact of each of these factors is analysed. It can however be doubted whether all these forces exist in reality. Returns to scale, especially in the case of energy and capital, are not likely to occur. The evidence on substitutability is mixed. In the Griffin and Gregory (1976) study capital and energy are found to be substitutes, whereas Magnus (1979) concludes that these factors of production are complements rather than substitutes. [See Griffin (1981) and Berndt and Wood (1981, 1979) for a discussion on the subject.]

It is extremely difficult to make definite statements about the long run. Nevertheless it seems interesting to try to characterize optimal time-paths of the economy when substitution possibilities are ruled out, and this is the aim of the present article. In order to make the results comparable with Stiglitz's conclusions, we shall use the same utility function. For simplicity labour will be neglected but all results can easily be generalized.

The plan of this article is as follows. In section 2, Stiglitz's model is briefly reviewed. Section 3 presents the model with complementarity and the necessary conditions for an optimum are given. It also provides some preliminary results. In section 4, properties of the optimum are derived. Finally, section 5 contains the conclusions.

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2. The Stiglitz model

The model Stiglitz uses can be described as follows. The economy's non-resource output (Y) is produced by means of three factors of production: capital (K), the withdrawal from the resource (E) and labour (L). Production takes place according to a Cobb–Douglas technology, possibly exhibiting technical progress at an exogenously given rate λ. The time index t is omitted where there is no danger of confusion. Hence,

$$Y=e^{\lambda t}K^{\alpha_1}L^{\alpha_2}E^{\alpha_3}, \qquad \alpha_1+\alpha_2+\alpha_3=1. \tag{1}$$

Non-resource output is allocated to consumption (C) and investments ($\dot{K}\equiv dK/dt$). Depreciation is neglected.

$$Y=C+\dot{K}. \tag{2}$$

The initial stock of capital is given: $K(0)=K_0$. The initial size of the pool or the stock of the natural resource is denoted by S_0 and it is given as well. Exploitation is assumed to be costless. The resource constraint reads

$$\int_0^\infty E(t)\,dt\leqq S_0. \tag{3}$$

The supply of labour grows at a rate n,

$$L=e^{nt}L_0. \tag{4}$$

The economy's objective is to maximize

$$I=\int_0^\infty e^{-\delta t}(C/L)^\nu/\nu\,dt, \tag{5}$$

where δ is the constant rate of time preference and ν is a constant with $\nu<1$, $\nu\neq 0$. Subsequently the optimal trajectories are characterized. The calculations are carried out for the special case of the logarithmic utility function ($\nu=0$). It is straightforward (but rather tedious) to generalize the results. They are summarized in Proposition 1:

Proposition 1. (1) *A necessary and sufficient condition for the existence of a solution of the problem posed above is*

$$\delta+n\nu>(\lambda+\alpha_2 n)\nu/(1-\alpha_1).$$

(2) *As t tends to infinity,* Y/K, $\dot{E}/E$ *and* $\dot{C}/C$ *approach, respectively,*

$$\frac{(\delta+n\nu)(1-\alpha_2)+(\lambda+\alpha_2 n)(1-\nu)}{\alpha_1(1-\alpha_1-\alpha_3\nu)},$$

$$-\frac{(\delta+n\nu)(1-\alpha_1)+(\lambda+\alpha_2 n)\nu}{\alpha_1(1-\alpha_1-\alpha_3\nu)},$$

$$-\frac{(\delta+n\nu)\alpha_3+(\lambda+\alpha_2 n)}{1-\alpha_1-\alpha_3\nu}.$$

Proof. The proof will not be given here. It is a straightforward generalization of the proof in Stiglitz (1974).

3. Complementarity

Here we modify the previous model in some respects. Firstly, labour will not be mentioned explicitly. This is done for notational simplicity. Nothing essential is lost in doing so. More importantly, it is now assumed that capital and the resource good are complements in production. This is expressed by introducing a constant elasticity (σ) of demand for the resource good with respect to capital use. Heating of buildings, petrol for transportation purposes are obvious examples of this interrelationship, although the constancy of the elasticity is clearly a far-going assumption made for expository purposes. The economy's technology is now described by

$$Y=e^{\lambda t}K^{\alpha}, \tag{6}$$

$$E=\phi K^{\sigma}, \tag{7}$$

where $\lambda \geqq 0$, $0<\alpha<1$, $\phi>0$, $\sigma \geqq 1$. Furthermore, eqs. (2) and (3) should hold. Finally the objective functional (5) prevails. We wish to apply Pontryagin's maximum principle [see e.g. Takayama (1974)]. In defining the Lagrangean there is no problem except possibly with respect to the resource constraint. But this constraint can be written as

$$\int_0^\infty (b\,e^{-bt}S_0-E)\,dt \geqq 0,$$

for some $b>0$. Hence the Lagrangean can now be written as follows:

$$\mathscr{L}=e^{-\delta t}C^{\nu}/\nu+p_1(Y-C)+p_2(b e^{-bt}S_0-E)$$
$$+p_3(e^{\lambda t}K^{\alpha}-Y)+p_4(E-\phi K^{\sigma}). \tag{8}$$

Obviously in an optimum all variables will be positive. Hence non-negativity constraints are not taken into account in the Lagrangean. p_1 corresponds to eq. (2) and is continuous, p_2 corresponds to the resource constraint and is constant. The meaning of p_3 and p_4 is clear. The necessary conditions evolving from the application of the maximum principle are

$$\partial\mathscr{L}/\partial C=0\Rightarrow e^{-\delta t}C^{\nu-1}=p_1, \tag{9}$$

$$\partial\mathscr{L}/\partial K=-\dot{p}_1\Rightarrow e^{\lambda t}p_3\alpha K^{\alpha-1}-p_4\phi\sigma K^{\sigma-1}=-\dot{p}_1, \tag{10}$$

$$\partial\mathscr{L}/\partial E=0\Rightarrow -p_2+p_4=0, \tag{11}$$

$$\partial\mathscr{L}/\partial Y=0\Rightarrow p_1-p_3=0. \tag{12}$$

After substitution the system is reduced to

$$p_1=e^{-\delta t}c^{\nu-1}, \tag{13}$$

$$-\dot{p}_1=p_1\alpha e^{\lambda t}K^{\alpha-1}-p_2\phi\sigma K^{\sigma-1}. \tag{14}$$

We first show that Stiglitz's results on convergence do not hold in the present model.

Suppose Y/K is constant. Then $e^{\lambda t}K^{\alpha-1}$ is constant and $\dot{K}/K=\lambda/(1-\alpha)$. It follows from (7) that $\dot{E}/E=\sigma(\dot{K}/K)$. Upon substitution it follows that $\dot{E}/E=\sigma\lambda/(1-\alpha)>0$. This is incompatible with the resource constraint.

Suppose $\dot{E}/E$ is constant and equals β (<0). Then, from (7), $\dot{K}/K=\beta/\sigma$ and $\dot{Y}/Y=\lambda+(\alpha\beta/\sigma)$ from (6). Substitution into (2) yields

$$C=e^{(\lambda+(\alpha\beta/\sigma))t}K_0^{\alpha}-e^{(\beta/\sigma)t}K_0.$$

Hence if the economy would choose β close enough to zero a growing rate of consumption would be realized in the case of the presence of technical progress. It is however straightforward that the conditions (13) and (14) are not satisfied. To see this, remark that in the case at hand C is eventually behaving as

$$e^{(\lambda+(\alpha\beta/\sigma))t},$$

since $\beta<0$ and $\alpha<1$. It follows that $\dot{p}_1/p_1$ approaches a constant. But

$$e^{\lambda t}K^{\alpha-1}$$

goes to infinity if t goes to infinity and this occurs at a rate different from the rate at which $p_2\phi\sigma K^{\sigma-1}/p_1$ will eventually behave. Hence we have obtained a contradiction.

Suppose finally that $\dot{C}/C$ is constant. Then $\dot{p}_1/p_1$ is constant from (13) and the argument used above can be repeated.

A few final remarks are in order. Mirrlees (1967) has studied this type of model for the case of no natural resource or, if one wishes, of an abundant natural resource. He has found that $\delta \geqq \lambda v$ is a necessary and sufficient condition for the existence of an optimum. It will be shown below that if a trajectory fulfils the necessary conditions and if $\delta > \lambda v$ this trajectory is optimal. Non-existence of an optimal solution in the model Mirrlees studied is caused by possible profitability of postponing consumption to infinity. This profitability arises when the rate of time preference is too small relative to technical progress. It is easily seen that the existence of a natural resource does not remove this difficulty. Hence we conclude that the condition $\delta > \lambda v$ is a necessary condition in our case too. Therefore it will be assumed to hold throughout.

4. The optimal path

In this section some properties of the optimal path are derived. Unfortunately, the necessary conditions look rather complicated. We can however make some positive statements. In aid of the propositions to be made the following lemma turns out to be useful.

Lemma 1. For any two paths (K, C) and $(\bar{K}, \bar{C})$, satisfying

$$\dot{K} = e^{\lambda t} K^{\alpha} - C, \qquad \dot{\bar{K}} = e^{\lambda t} \bar{K}^{\alpha} - \bar{C},$$

and for any interval $[a, b]$, where $K(a) = \bar{K}(a)$,

$$\int_a^b e^{-\delta t}(u(C) - u(\bar{C}))\, dt \geqq \int_a^b e^{-\delta t} u'(C)(K - \bar{K})$$

$$\times (\alpha e^{\lambda t} K^{\alpha-1} - \delta - (1-v)\dot{C}/C)\, dt$$

$$-(K(b) - \bar{K}(b))\, e^{-\delta t} u'(C(b)),$$

where

$$u(C) = C^{v}/v \quad \textit{and} \quad u'(C) = du/dC.$$

Proof. The proof is straightforward, using strict concavity of the functions involved and integrating by parts.

Proposition 2. Define

$$\tilde{K}(t)=\{(\alpha/(\delta+(1-\nu)\lambda))\,e^{\lambda t}\}^{1/(1-\alpha)}.$$

If $(\hat{K},\hat{C})$ *is optimal, then* $\dot{\hat{K}}<0$ *for all t such that* $\hat{K}(t)>\tilde{K}(t)$.

Proof. Suppose the proposition is false. However, there is a moment at which $\hat{K}(t)=\tilde{K}(t)$ in view of the limited availability of the resource. Now construct an alternative path (K,C) in the following way (illustrated in fig. 1).

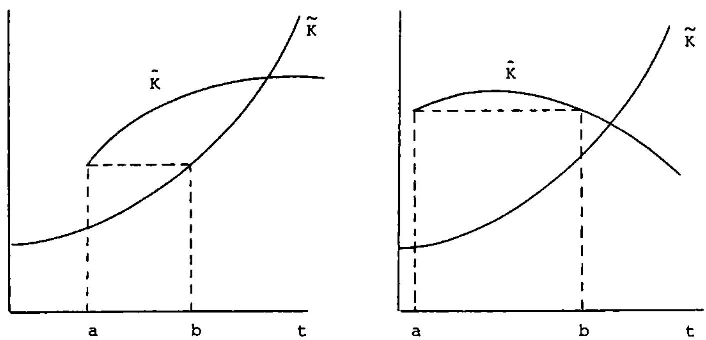

Fig. 1

From $t=a$ on the alternative capital stock is kept constant at the $\hat{K}(a)$ level. We consider two cases:

(1) $\hat{K}(a)=\tilde{K}(t)$ for some $t=b$ and $\hat{K}(b)>\hat{K}(a)$. In this case follow the constant path $\hat{K}(a)$ until time b and from b on the alternative path is set equal to $\tilde{K}$ until $\tilde{K}(t)=\hat{K}(t)$. As long as K is constant, $\alpha e^{\lambda t}K^{\alpha-1}-\delta-(1-\nu)\dot{C}/C$ is negative because $K(t)>\tilde{K}(t)$. If $K(t)=\tilde{K}(t)$, then this expression is also negative as can be verified by straightforward calculations. Now it is easy to apply Lemma 1, substituting $\hat{K}(t)$ for $\bar{K}(t)$ and realizing that $\hat{K}(t)>K(t)$. Finally, the alternative path uses less of the resource.

(2) $\hat{K}(a)=\hat{K}(b)$ for some b and $\hat{K}(b)>\tilde{K}(b)$. In this case, keep K constant up to time b and follow $\hat{K}$ from b on. The same argument as used in case 1 now applies.

Hence in both cases the optimal path can be overtaken by an alternative path that uses less of the resource. This contradicts optimality of $\hat{K}$. □

The necessary conditions (2), (6), (13) and (14) can be written as

$$\dot{K} = e^{\lambda t} K^{\alpha} - C, \tag{15}$$

$$(1-v)\dot{C}/C + \delta = \alpha e^{\lambda t} K^{\alpha-1} - p_2 \phi \sigma e^{\delta t} C^{1-v} K^{\sigma-1}. \tag{16}$$

At first sight nothing in these necessary conditions seems to exclude the possibility of having 'bulges' in the optimal time-path of the stock of capital.

A solution $K(t)$ of the system of differential equations (13) and (14) shows a bulge if there is an interval $[a, b]$, where $K(a) = K(b)$ and $K(t) \geqq K(a)$ for all $a \leqq t \leqq b$, with $K(t) > K(a)$ for at least one t in the interval. For the case of no technical progress it is easy to show that only one bulge can occur and that, if it occurs, this only happens in the very beginning of the planning period.

Proposition 3. If $\lambda = 0$, a bulge can only occur in the beginning of the planning period.

Proof. In fig. 2a a bulge is sketched.

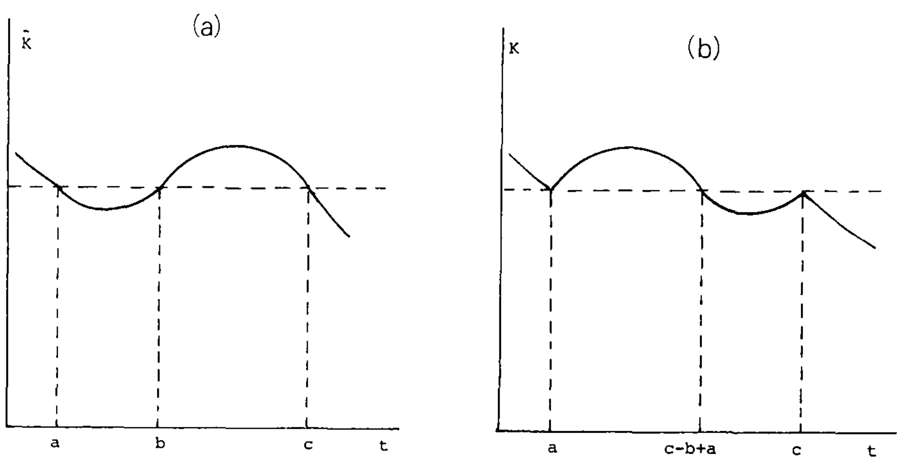

Fig. 2

Suppose the optimal path can be depicted as in fig. 2a. Then an alternative path (K, C) can be constructed that is better. This is shown in fig. 2b. In fig. 2b the optimal path is copied but on different segments. Both paths are identical up to time a and from time c on.

After reaching a we copy the optimal path from $[b, c]$ on the interval $[a, c-b+a]$ and from $c-b+a$ to c we take the optimal path of interval $[a, b]$. Total utility on both paths is compared. By $\hat{u}$ we denote the utility of the optimal path obtained between a and c and by u the utility of the alternative path. In the subsequent proof the resource good does not play a part because both paths require the same amount of it.

$$\hat{u}=\int_a^b e^{-\delta t}u(\hat{C})\,dt+\int_b^c e^{-\delta t}u(\hat{C})\,dt,$$

$$u=\int_a^{c-b+a} e^{-\delta t}u(C)\,dt+\int_{c-b+a}^{c} e^{-\delta t}u(C)\,dt.$$

For $a\leqq t\leqq c-b+a$, $K(t)=\hat{K}(t+b-a)$, $\dot{K}(t)=\dot{\hat{K}}(t+b-a)$, and for $c-b+a\leqq t\leqq c$, $K(t)=\hat{K}(t-c+b)$, $\dot{K}(t)=\dot{\hat{K}}(t-c+b)$.

For $a\leqq t\leqq c-b+a$ put $t'=t+b-a$. Then $t=a$ implies $t'=b$, and $t=c-b+a$ implies $t'=c$. For $c-b+a\leqq t\leqq c$ put $t'=t+b-c$. Then $t=c-b+a$ implies $t'=a$, and $t=c$ implies $t'=b$. It follows that

$$u=\int_a^b e^{-\delta(t+c-b)}u(\hat{C}(t+c-b))\,dt+\int_b^c e^{-\delta(t-b+a)}u(\hat{C}(t-b+a))\,dt,$$

and

$$u-\hat{u}=\int_a^b e^{-\delta t}(1/\nu)\{(e^{(\delta/\nu)(c-b)}\hat{K}^{\alpha}-e^{(-\delta/\nu)(c-b)}\dot{\hat{K}})^{\nu}-(\hat{K}^{\alpha}-\dot{\hat{K}})^{\nu}\}\,dt$$

$$+\int_b^c e^{-\delta t}(1/\nu)\{(e^{(-\delta/\nu)(a-b)}\hat{K}^{\alpha}-e^{(-\delta/\nu)(a-b)}\dot{\hat{K}})^{\nu}-(\hat{K}^{\alpha}-\dot{\hat{K}})^{\nu}\}\,dt.$$

Hence

$$u-\hat{u}=-(1-e^{-\delta(a-b)})\int_b^c e^{-\delta t}u(\hat{C})\,dt-(1-e^{-\delta(c-b)})\int_a^b e^{-\delta t}u(\hat{C})\,dt.$$

From the optimality of $\hat{C}$ we know that

$$\int_b^c e^{-\delta t}u(\hat{C})\,dt>\int_b^c e^{-\delta t}u(\bar{C})\,dt=-(1/\delta)u(\bar{C})(e^{-\delta c}-e^{-\delta b}),$$

where $\bar{C}$ corresponds with a $\bar{K}$ that is constant from b to c. Secondly we have

$$\int_a^b e^{-\delta t}u(\hat{C})\,dt<\int_a^b e^{-\delta t}u(\bar{C})\,dt=-(1/\delta)u(\bar{C})(e^{-\delta b}-e^{-\delta a}).$$

It follows that $\hat{u}-u<0$ by straightforward calculations. This contradicts the optimality of $\hat{C}$. □

Proposition 3 is easily interpreted. In the absence of technical progress a bulge such as the one depicted in fig. 2a in the interval $[b, c]$ will benefit consumption. But in view of the positive rate of time preference more utility is obtained when such a bulge occurs earlier. In the presence of technical progress however this is not true in its generality since in the interval $[b, c]$ the economy receives also the benefits of larger technical progress compared with technical progress at earlier dates. But we can prove the following proposition:

Proposition 4. There exists $T>0$ such that $\dot{K}<0$ for all $t>T$.

Proof. Suppose the proposition is false. Consider fig. 3.

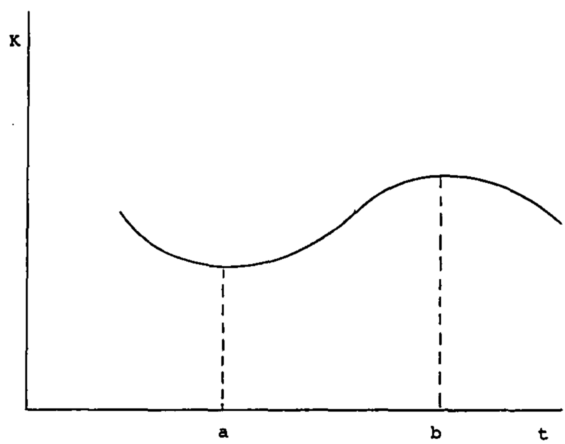

Fig. 3

In a and b, $\dot{K}=0$. Hence $C=e^{\lambda t}K^{\alpha}$ for $t=a$ and $t=b$. It follows from (15) that

$$\dot{C}=e^{\lambda t}K^{\alpha}(\lambda+\alpha\dot{K}/K)-\ddot{K}.$$

$\ddot{K}>0$ in a, and hence $\dot{C}/C<\lambda$. For b we have $\dot{C}/C>\lambda$. Consider the function

$$f(t)=\alpha\, e^{\lambda t}K^{\alpha-1}-e^{(\delta+\lambda(1-\nu))t}K^{\sigma-1+\alpha(1-\nu)}p_2\phi\sigma,$$

where $K(t)$ fulfils the necessary conditions (15) and (16). For $t=a$, $\dot{K}=0$, $C=e^{\lambda t}K^{\alpha}$ and $f(a)=(1-\nu)\dot{C}/C+\delta$. This also holds for $t=b$. It follows that $f(a)<(1-\nu)\lambda+\delta$ and $f(b)>(1-\nu)\lambda+\delta$. Hence for at least one t in the interval $[a, b]$ where the stock capital is increasing, $f(t)$ must have a positive derivative:

$$f'(t)=(\alpha(\alpha-1)\,e^{\lambda t}K^{\alpha-1}$$
$$-e^{(\delta-\lambda v+\lambda)t}(\alpha-\alpha v+\sigma-1)K^{\alpha(1-v)+\sigma-1}p_2\phi\sigma)\dot{K}/K$$
$$+\alpha\lambda\,e^{\lambda t}K^{\alpha-1}-e^{(\delta-\lambda v+\lambda)t}(\delta-\lambda v+\lambda)K^{\alpha(1-v)+\sigma-1}p_2\phi\sigma.$$

The first term is negative (since $v<1, \sigma\geqq 1$). Hence, for at least one t the sum of the final two terms should be positive. This condition amounts to having at least one t for which

$$K<(\alpha\lambda/p_2\phi\sigma(\delta-\lambda v+\lambda))^{1/(\sigma-\alpha v)}e^{-(\delta-\lambda v)t/(\sigma-\alpha v)}. \tag{17}$$

Since in the interval $[a,b]$ the stock of capital is increasing we have $C<e^{\lambda t}K^{\alpha}$. Using this inequality and (17), (15) and (16), it is found that

$$(1-v)\dot{C}/C+\delta>\alpha\pi^{\alpha-1}\{1-\lambda/(\delta-\lambda v+\lambda)\}\,e^{(\delta(1-\alpha)+\lambda(\sigma-v))t/(\sigma-\alpha v)}, \tag{18}$$

where

$$\pi=\{\lambda\alpha/p_2\phi\sigma(\delta+\lambda(1-v))\}.$$

The trajectories of K and $\dot{C}/C$ are depicted in fig. 4.
The downward sloping curve in fig. 4a is the representation of the right-hand side of (17). The upward sloping curve in fig. 4b is the right-hand side of (18). This expression is positive and unbounded since $\pi>0$, $\delta>\lambda v$, $\sigma\geqq 1>v$. It has

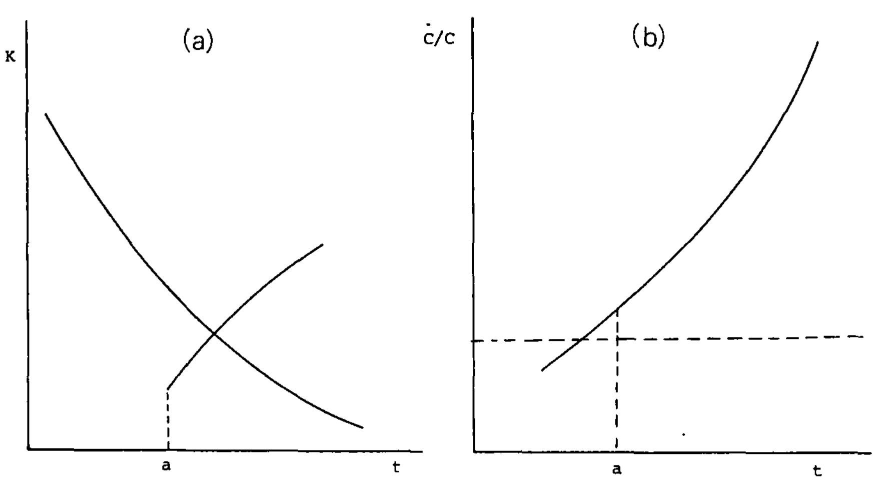

Fig. 4

been assumed that $\dot{K}>0$ for an infinite series of times. Hence there are infinitely many points such as a. At such points (17) holds and (18) holds as well. But we must also have $\dot{C}/C<\lambda$ at points like a. Eventually this will contradict (18). □

Knowing that the stock of capital is eventually decreasing, we can easily prove the following propositions. It has already been noted that the necessary conditions look rather complicated so that one cannot expect to obtain explicit solutions. However, it is possible to calculate upper and lower bounds of some of the variables. In the sequel $(-\delta+\lambda v)/(\sigma-\alpha v)$ and $(\lambda\sigma-\alpha\delta)/(\sigma-\alpha v)$ will be denoted by μ and ψ, respectively.

Proposition 5. For all $\bar{\mu}>\mu$ there exists T such that, for all $t>T$, $K(t)<\mathrm{e}^{\bar{\mu}t}$.

Proof. Denote the right-hand side of (16) by $f(t)$ and differentiate with respect to time,

$$f'(t)=\alpha\,\mathrm{e}^{\lambda t}K^{\alpha-1}(\lambda+(\alpha-1)\dot{K}/K)$$
$$-p_2\phi\sigma C^{1-v}K^{\sigma-1}\mathrm{e}^{\delta t}(\delta+(1-v)\dot{C}/C+(\sigma-1)\dot{K}/K).$$

$\dot{K}/K$ is negative eventually. Hence $f'(t)>0$ if $\dot{C}/C\leqq-\delta/(1-v)$. Therefore $\dot{C}/C>-\delta/(1-v)$ eventually. Using the fact that eventually $C>\mathrm{e}^{\gamma t}K^{\alpha}$ and using (16) again the proposition is established. □

Proposition 6. For all $\underline{\psi}<\psi$ there exists T such that, for all $t>T$, $C>\mathrm{e}^{\underline{\psi}t}$.

Proof. Suppose there exist $\underline{\psi}<\psi$ and t_0 such that, for all $t>t_0, C\leqq\mathrm{e}^{\underline{\psi}t}$. K is decreasing eventually. Hence, for large enough t,

$$\mathrm{e}^{\lambda t}K^{\alpha}<C\leqq\mathrm{e}^{\underline{\psi}t}.$$

Substitution of $K<\mathrm{e}^{(\underline{\psi}-\lambda)t/\alpha}$ and $C\leqq\mathrm{e}^{\underline{\psi}t}$ into (16) gives that $\dot{C}/C$ is unbounded, contradicting $C\leqq\mathrm{e}^{\underline{\psi}t}$.

Therefore there exists for all $\underline{\psi}<\psi$ and for all t_0 a t such that $C>\mathrm{e}^{\underline{\psi}t}$. If this inequality holds eventually the proposition is proved. Suppose that there exists $\underline{\psi}<\psi$ such that for all t_0 there is a t such that $C<\mathrm{e}^{\underline{\psi}t}$. But as we have seen above, as long as $C\leqq\mathrm{e}^{\underline{\psi}t}$ the growth rate of C is speeding up and hence, once $C>\mathrm{e}^{\underline{\psi}t}$, C will never get below $\mathrm{e}^{\underline{\psi}t}$. □

Proposition 7. For all $\bar{\psi}>\psi$ there exists T such that, for all $t>T$, $C<\mathrm{e}^{\bar{\psi}t}$.

Proof. Suppose that the proposition does not hold and that there exists

$\bar{\psi} > \psi$ such that for all $t > T$, for some T, $C > e^{\bar{\psi}t}$. Then the stock of capital will be eaten up in finite time, which cannot be optimal. This can be seen as follows: $\bar{\psi} = \psi + \varepsilon$ for some $\varepsilon > 0$. $K < e^{\bar{\mu}t}$ eventually for all $\bar{\mu} > \mu$. Hence $\bar{\mu}$ can be chosen such that $\varepsilon > \bar{\mu} - \mu$. Then $\bar{\psi} > \alpha\bar{\mu} + \lambda$. Since $e^{\lambda t}K^{\alpha} < e^{\lambda t}e^{\alpha\bar{\mu}t}$ and $C > e^{\bar{\psi}t}$ by assumption, $\dot{K}$ is unbounded from below and K will become negative, a contradiction.

Therefore, if the proposition were not true, there is an infinite series of times such that $C < e^{\bar{\psi}t}$. Consider fig. 5.

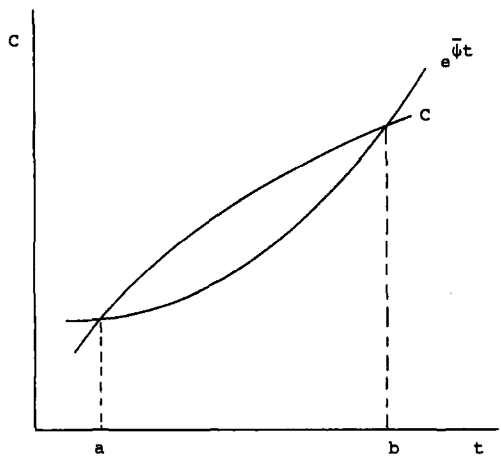

Fig. 5

Somewhere in the interval $[a, b]$ we must have $\dot{C}/C = \bar{\psi}$. By taking the interval far enough on the time axis one can ensure that at the point where $\dot{C}/C = \bar{\psi}$, $\delta + (1-\nu)\dot{C}/C + (\sigma - 1)\dot{K}/K < 0$. This implies that at such an instant of time $(\mathrm{d}\dot{C}/C)/\mathrm{d}t > 0$ (see the proof of Proposition 5). Hence the inequality $C > e^{\bar{\psi}t}$ will persist, a contradiction. □

Proposition 8. For all $\underline{\mu} < \mu$ there exists T such that, for all $t > T$, $K(t) > e^{\underline{\mu}t}$.

Proof. Suppose that the proposition is false. Then there exists $\underline{\mu} < \mu$ such that for all T, $K(t) \leqq e^{\underline{\mu}t}$ for some $t > T$.

$$\dot{K} = e^{\lambda t}K^{\alpha} - C < e^{(\lambda + \alpha\underline{\mu})t} - e^{\underline{\psi}t},$$

for an infinite series of times. Hence for each $M > 0$ there exists t such that $\dot{K}(t) < -M$, since $\underline{\psi}$ can be chosen such that $\underline{\psi} > \lambda + \alpha\underline{\mu}$. At such instants of time we surely have $\dot{K}/K < \underline{\mu}$ and the inequality $K \leqq e^{\underline{\mu}t}$ will persist. Therefore K becomes negative, a contradiction. □

As for the economic interpretation of the previous propositions, we restrict ourselves to the asymptotic growth rate of consumption (ψ) since the 'growth' rate of the stock of capital just guarantees that non-resource output suffices to sustain consumption. It is clear that a higher rate of technical progress will ensure that consumption grows faster, and that the rate of time preference has the opposite effect. This is in accordance with what classical economic growth theory tells us. This observation also holds for the elasticity of marginal utility. Then we are left with α/σ. The asymptotic growth rate of consumption is a decreasing function of this ratio since $\delta > \lambda v$. α is the elasticity of non-resource output with respect to capital, and σ is the elasticity of resource demand with respect to capital use. Hence α/σ is approximately the percentage increase of production when resource use is increased by one percent. The larger this rate is the more this economy is benefited in all respects since the resource is a limiting factor. It is therefore obvious that the economy will in view of the relatively high rate of time preference prefer to have more consumption now and chose a smaller growth rate in the future.

Finally we prove that the necessary conditions are also sufficient conditions for an optimum provided that $\delta > \lambda v$.

Proposition 9. Let $\{\hat{K}, \hat{C}, \hat{S}, \hat{E}\}$, *where* $S(t) = S_0 - \int_0^t \hat{E}(t)\mathrm{d}t$, *filfil conditions* (2), (3), (6), (7), (13) *and* (14). *Let* $\delta > \lambda v$. *Then* $\{\hat{K}, \hat{C}, \hat{S}, \hat{E}\}$ *constitutes an optimum.*

Proof. Let $\{K, C, S, E\}$ constitute an alternative feasible program,

$$
\begin{aligned}
J_T &= \int_0^T (\mathrm{e}^{-\delta t}(\hat{C}^v - C^v)/v)\,\mathrm{d}t \\
&> \int_0^T \mathrm{e}^{-\delta t}\hat{C}^{v-1}(\hat{K} - K)(\alpha\,\mathrm{e}^{\lambda t}\hat{K}^{\alpha-1} - \delta - (1-v)\dot{\hat{C}}/C\,\mathrm{d}t \\
&\quad -(\hat{K}(T) - K(T))\,\mathrm{e}^{-\delta T}\hat{C}(T)^{v-1} \\
&= \int_0^T \mathrm{e}^{-\delta t}\hat{C}^{v-1}(\hat{K} - K)(\mathrm{e}^{\delta t}\hat{C}^{1-v}\hat{K}^{\sigma-1}p_2\phi\sigma)\,\mathrm{d}t \\
&\quad -(\hat{K}(T) - K(T))\,\mathrm{e}^{-\delta T}\hat{C}(T)^{v-1} \\
&\geqq \int_0^T p_2(\phi\hat{K}^\sigma - \phi K^\sigma)\,\mathrm{d}t - \hat{K}(T)\,\mathrm{e}^{-\delta T}\hat{C}(T)^{v-1} \\
&\geqq \int_0^T p_2(\hat{E} - E)\,\mathrm{d}t - \hat{K}(T)\,\mathrm{e}^{-\delta T}\hat{C}(T)^{v-1} \\
&= p_2(S(T) - \hat{S}(T)) - \hat{K}(T)\,\mathrm{e}^{-\delta T}\hat{C}(T)^{v-1}.
\end{aligned}
$$

Since exploitation is costless, on an optimal trajectory the entire resource stock will be exhausted. Hence

$$\lim_{T\to\infty} p_2(S(T)-\hat{S}(T))=p_2S(T)\geqq 0.$$

It follows from Propositions 5 and 6 that

$$\lim_{T\to\infty} \hat{K}(T)\,e^{-\delta T}\hat{C}(T)^{\nu-1}=0. \quad \square$$

5. Conclusions

In this article the impact on the theory of exhaustible resources has been considered of the introduction of complementarity between capital and a resource good as factors of production. The main aim was to characterize the optimal trajectories of exploitation, the rate of consumption and the stock of capital. Apart from this we have gained the insight that the shadow price of the capital stock (p_1) is eventually decreasing, even if there is no technical progress. The future outlook is pessimistic in so far as all capital will be eaten up. However, in the presence of technical progress a growing rate of consumption is realizable and will indeed be optimal if technical progress is large enough.

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