ALGORITHMS FOR THE LINEAR COMPLEMENTARITY PROBLEM WHICH ALLOW AN ARBITRARY STARTING POINT*

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1. Introduction

The linear complementarity problem with data $q \in R^n$ and $M \in R^{n\times n}$ consists in finding two vectory s and z in R^n such that

(1.1) $s = Mz + q$,

(1.2) $s, z \geq 0$,

(1.3) $s_i z_i = 0$, $i = 1, 2, \ldots, n$.

We denote this problem LCP or LCP(q,M). We only consider vectors (s,z) satisfying (1.1). Two vectors s and z are said to be feasible if they are nonnegative (1.2) and are said to be complementary if they satisfy (1.3).

The LCP is an important problem in mathematical programming [see, e.g., Garcia and Gould (1980) for references]. Lemke (1965) first presented a solution for this problem. His ideas were later

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exploited by Scarf (1967) in his work on fixed point algorithms. The relationship between the LCP and the fixed point problem is well described by Eaves and Scarf (1976) and by Eaves and Lemke (1981).

Recently, Van der Laan and Talman (1979, 1981) proposed a class of variable dimension restart algorithms for approximating fixed points. These methods allow a start at an arbitrary point in the domain of the fixed point problem. One among several directions is followed to leave the starting point. These directions define a collection of cones of variable dimensions in which the search for an approximate fixed point takes place. Properties of the function govern the movement of the procedure between the conical regions. In each region movement occurs through simplicial pivoting, but continuous path-following could be applied too [see Allgower and Georg (1980)].

The intimate relation between the fixed point problem and the LCP raises the question of the significance of Van der Laan and Talman's work for the LCP. We show that the ideas behind their variable dimension fixed point algorithms yield an interesting class of LCP algorithms. An important feature of these algorithms is that they can be initialized at any nonnegative point z^0. When $z^0 = 0$, the algorithms reduce to Lemke's original algorithm (Lemke, 1965). Similar ideas can be used to modify other LCP algorithms, like the variable dimension algorithm of Van der Heyden (1980) [see also Yamamoto (1981)], to accept an arbitrary starting point. Flexibility in the choice of the starting point is desirable, e.g., in using prior information on the solution, in sensitivity analysis, and when solving nonlinear complementarity problems via a succession of approximating LCP's [Josephy (1979)].

Several authors have presented LCP algorithms which allow an arbitrary starting point. Eaves (1978) and Garcia and Gould (1980) present procedures based on homotopies. Reiser (1978), in an appendix to his dissertation, states two ways to transform an LCP with arbitrary starting point into one to which Lemke's algorithm can be applied. Our approach unifies the two Reiser algorithms in that the first Reiser algorithm becomes a special case in our framework, while another instance in our class of algorithms is very close to Reiser's second algorithm. This relationship with Reiser's work mirrors the similarity that exists between the Reiser and the Van der Laan and Talman fixed point algorithms [Reiser (1981)].

The paper is organized as follows. In section 2 we motivate our algorithm by interpreting the artificial variable in Lemke's algorithm as a measure of infeasibility. We then define the positions of our algorithm and the line segments which are followed to reach successive positions and which form a piecewise linear path leading to a solution. The procedure itself is explained in section 3, where we deal with convergence issues. In section 4, we discuss implementation and show that our algorithm can be seen as applying Lemke's algorithm to a transformed problem.

2. Movements and positions

We only consider pairs (s,z) satisfying (1.1) with z feasible. Let us take a starting point (s^0,z^0) . We define t_0 as

(2.1) $$t_0 = \max\ (t_j : j \in I(n+k)) ,$$

where, for any positive integer h , $I(h)$ denotes the index set $\{1,2,\ldots,h\}$, and where

$$t_i = -s_i \quad \text{for} \quad i \in I(n) ,$$
$$= \sum_{j \in P_h} s_j \quad \text{for} \quad i = n+h, \quad h \in I(k) ,$$

$\{P_h : h \in I(k)\}$ being an arbitrary partition of the set $I^+(n) = \{ i \in I(n) : z_i^0 > 0 \}$. The quantity t_0 measures the the infeasibility of the starting point (s^0,z^0) by checking for the nonnegativity of s^0 and for its complementarity with z^0 . z^0 is a solution for the LCP if and only if $t_0 \leq 0$. t_0 is negative at

(s^0,z^0) only if $(s^0,z^0) = (q,0)$ is a solution and $q > 0$. The largest infeasibility at z^0 defines the initial value of t_0.

Each component of the vector $t = (t_j : j \in I(n+k))$ is associated with a direction that can be followed to leave z^0. The directions associated with the first n components of t are the unit directions:

$$d^i = u^i \quad \text{for} \quad i \in I(n) ,$$

u^i denoting the ith unit vector in R^n. These also are the directions that can be followed to leave the starting point in Lemke's algorithm $(z^0 = 0)$. Leaving z^0 along d^i amounts to increasing z_i. With t_{n+h}, $h \in I(k)$, we associate direction d^{n+h} where

$$d_i^{n+h} = -z_i^0 \quad \text{for} \quad i \in P_h ,$$

$$= 0 \quad \text{for} \quad i \in I(n)-P_h .$$

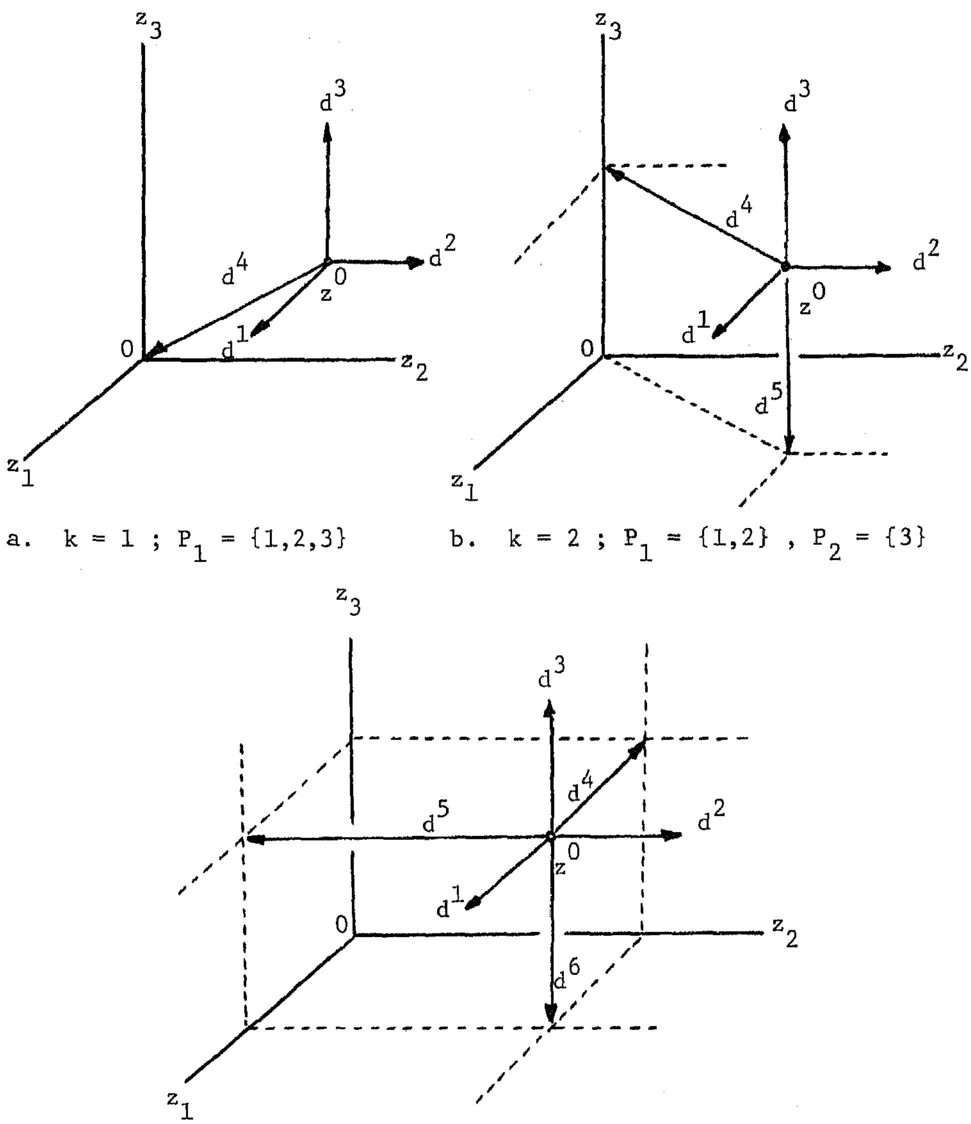
A movement along d^{n+h} amounts to decreasing all coordinates of z with indices in P_h. The directions are illustrated in figure 1.

Figure 1 shows that the directions $D = (d^i : i \in I(n+k))$, when drawn through z^0, partition R^n into relatively open conical regions $C(P) = \{z : z = z^0 + Dy ,\ y \in R^{n+k} ,\ y_j > 0 \text{ for } j \in P\}$. To maintain the feasibility of z we require that

$$(2.2) \qquad y_j \leq 1 \quad \text{when} \quad j > n .$$

A vector y is said to be <u>feasible</u> if it is nonnegative, satisfies (2.2), but does not meet $y_j > 0$ for all $j \in P_h \cup \{n+h\}$, $h \in I(k)$. The latter condition ensures that the correspondence between y and z is one-to-one. In what follows, we equivalently refer to z or to its unique representation in terms of a feasible y.

The algorithm maintains a generalized form of complementarity between leading infeasibilities in maximand (2.1) and directions. Except for boundary issues, <u>t_0-complementarity</u> between the vectors



a. k = 1 ; P_1 = {1,2,3}

b. k = 2 ; P_1 = {1,2} , P_2 = {3}

c. k = 3 ; P_1 = {1}, P_2 = {2}, P_3 = {3}

FIGURE 1. The directions d^j , $1 \leq j \leq n+k$, in three special cases of our algorithm (n = 3 ; k = 1, 2, and 3).

t and y requires

$$y_j = 0 \quad \text{or} \quad t_j = t_0 \ , \quad j \in I(n+k) \ .$$

The algorithm thus moves in cones defined by directions associated with leading infeasibilities. We now motivate the definition of t_0-complementarity on the boundary of the nonnegative orthant in z-space.

Assume that $t_0 = t_{n+h}$ is the unique leading infeasibility at z^0. The only movement allowed by t_0-complementarity is to move along direction d^{n+h} by increasing y_{n+h}. As soon as in this movement another infeasibility, say t_j, ties t_{n+h} as the leading infeasibility $(t_0 = t_{n+h} = t_j)$, further movement along d^{n+h} is infeasible for it would lead to points (t,y) verifying $t_j = t_0 > t_{n+h}$ with $y_{n+h} > 0$. These points would not be t_0-complementary. The only movement consistent with t_0-complementarity is to then move in t_0-complementary fashion into cone $C(\{n+h,j\})$ by increasing y_j while maintaining $t_0 = t_{n+h} = t_j$. The latter restriction removes the degree of freedom introduced by moving into a higher dimensional region. Another possibility arising when leaving z^0 along d^{n+h} is that t_{n+h} remains the unique leading infeasibility so that, for all $0 \leq y_{n+h} \leq 1$ and for all $j \in I(n+k)-\{n+h\}$, $t_0 = t_{n+h} > t_j$. Once the boundary $y_{n+h} = 1$ is reached, further movement along d^{n+h} generates infeasible y's. The algorithm then keeps $y_{n+h} = 1$ and allows t_{n+h} to differ from t_0 by removing t_{n+h} from maximand (2.1). The definition of t_0 on the boundary of the feasible y-region is then completed as follows:

(2.3) $$t_0 = \max\ (t_j : j \in I(n+k)\ ,\ y_j < 1 \ \text{ when } \ j > n)\ .$$

In order to maintain t_0-complementarity during the movement of the algorithm, we need to generalize the notion of t_0-complementarity by also calling the pair (t_{n+h}, y_{n+h}) t_0-complementary when $y_{n+h} = 1$.

(2.4) <u>Definitions</u>. A component t_j is said to be <u>nonbasic</u> if

$t_j = t_0$. t_0 is said to be nonbasic when $t_0 = 0$. y_j is nonbasic when $y_j = 0$ or when $y_j = 1$ and $j > n$. The vectors t and y are said to be <u>t_0-complementary</u> when for each $j \in I(n+k)$ either y_j or t_j is nonbasic.

t_0-complementarity is one of two properties which will be shown to define a piecewise linear path to a solution. The second property constrains the components of t which are not involved in the computation of t_0 , namely those associated with components of y assuming their upper bound. We motivate the second property by returning to a situation discussed earlier. Let us imagine that the algorithm leaves the initial point z^0 along direction d^{n+h} and that this movement is pursued until $y_{n+h} = 1$. During this movement t_0-complementarity requires that t_{n+h} remains the largest component of t . Upon reaching the boundary, t_{n+h} disappears from maximand (2.3) and t_0 decreases discontinuously if the second largest component of t is strictly smaller than t_{n+h} . Assuming this to be the case, t then verifies $t_{n+h} > t_0 = t_j$, $j \in I(n+k) - \{n+h\}$. If we like the algorithm to terminate with a solution when $t_0 = 0$, we must require that the inequality $t_{n+h} > t_0$ be maintained while $y_{n+h} = 1$. If at a later stage t_{n+h} again becomes equal to t_0 , then the algorithm continues in t_0-complementary fashion by decreasing y_{n+h} from 1 while maintaining $t_{n+h} = t_0$. We now formally introduce the lines followed by the algorithm.

(2.5) <u>Definition</u>. A <u>line of our algorithm</u> consists of a set of t_0-complementary points such that

a. exactly one variable in each pair (t_j, y_j) is nonbasic;

b. $t_j > t_0$ when $y_j = 1$, $j > n$;

c. $t_0 > 0$.

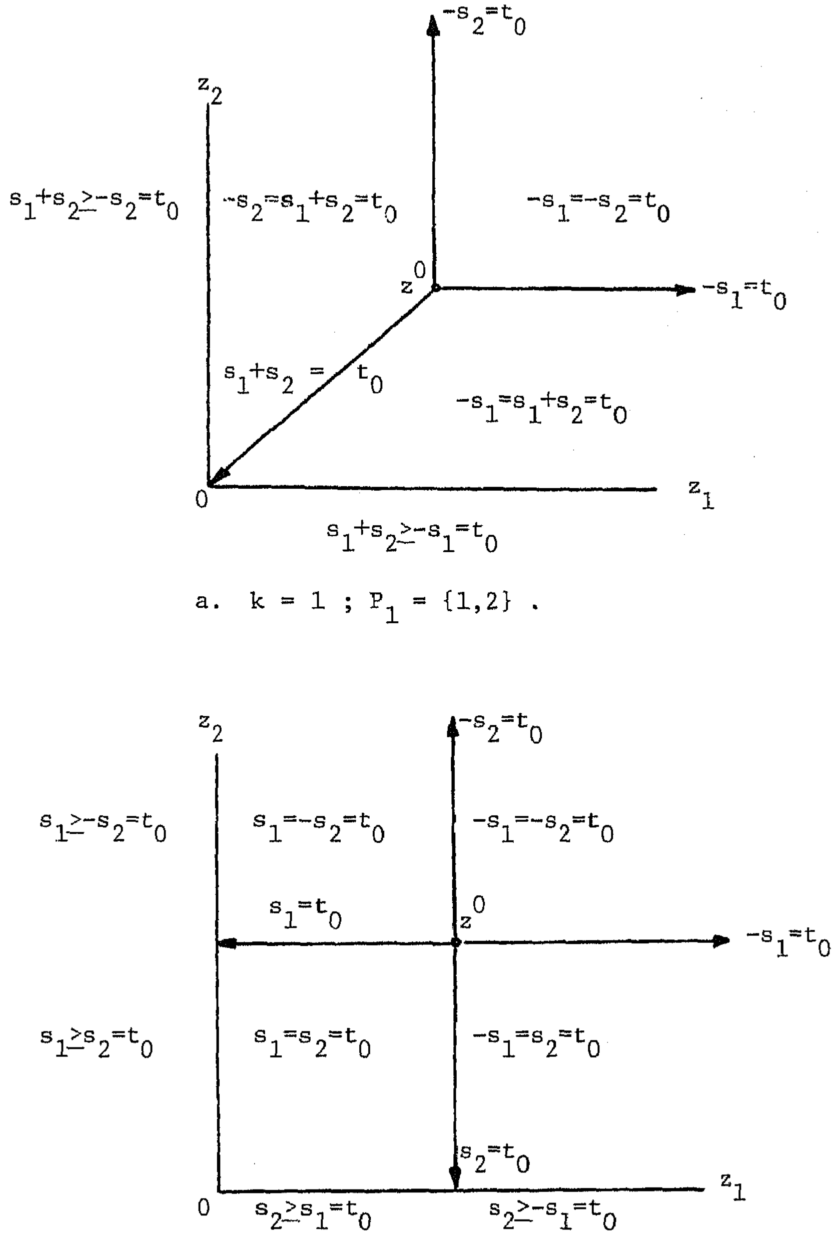
Note that by definition of t_0 , $t_j \leq t_0$ when $y_j = 0$, while the algorithm requires that $t_j \geq t_0$ when $y_j = 1$, $j > n$. In all other cases ($0 < y_j$ and $y_j < 1$ when $j > n$) , $t_j = t_0$. The algorithm thus imposes various types of constraints on the t-variables in different regions of z-space. Figure 2 illustrates these constraints for the case $n = 2$ and $z^0 > 0$.

In order for the set of points satisfying (2.5) to form a line, we need to impose the following nondegeneracy assumption.

(2.6) Assumption. At most $n+k+1$ among the $2(n+k)+1$ variables (t_0, t, y) are nonbasic at any given point (t,y) .

We indicate in section 4 that this assumption is similar to a nondegeneracy assumption in linear programming and thus can be satisfied with the usual perturbation techniques. One component of t is nonbasic by definition of t_0 . t_0-complementarity imposes $n+k-1$ additional restrictions on the vector (t,y) so that one degree of freedom remains. The set of points (if any) satisfying definition (2.5) with a fixed set of nonbasic variables do form a line segment.

Let us examine the endpoints of the lines of our algorithm. An endpoint is reached when a basic variable becomes nonbasic. If there is no discontinuity in the value of t_0 and if t_0 is still basic, there is by nondegeneracy exactly one pair of variables which are both nonbasic. This gives rise to two types of position for the algorithm. At a position of type a we have that, for some $j \in I(n+k)$, $y_j = 0$ and $t_j = t_0 > 0$. At a position of type b we have that, for some $j > n$, $y_j = 1$ and $t_j = t_0 > 0$. If an endpoint is reached where t_0 is nonbasic, then it will be shown to be a solution. The latter is also true if t_0 becomes nonpositive during a discontinuous decrease at the endpoint. If after a discontinuous decrease t_0 is still positive, there is one nonbasic pair (y_h, t_h) with $t_h = t_0 > 0$ and $y_h = 0$ for some $h \in I(n+k)$. The endpoint is a position of type a . This completes our



a. $k = 1$; $P_1 = \{1,2\}$.

b. $k = 2$; $P_1 = \{1\}$, $P_2 = \{2\}$.

FIGURE 2. The constraints imposed on the variables in different regions of z-space for a 2-dimensional example (n=2). We have omitted the inequalities that are implicit in the definition of t_0 .

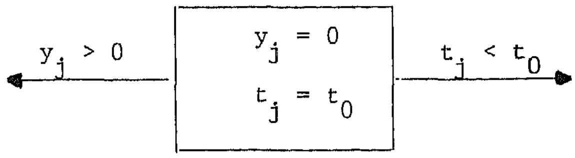
classification of endpoints into positions of type a or b and t_0-complementary points with $t_0 \leq 0$.

We now prove the important fact that if t_0 becomes nonpositive at an endpoint, then a solution has been found. Since $t_0 \geq \max(t_i = -s_i : i \in I(n))$, it is clear that $s \geq 0$ at such an endpoint. We still need to argue that $s_i = 0$ whenever $z_i > 0$. We distinguish two cases. If $i \in P_h$ and $y_{n+h} < 1$, then $s_i = 0$ follows from the fact that $0 \geq t_0 \geq t_{n+h} = \sum_{i \in P_h} s_i \geq 0$. If $y_{n+h} = 1$, then the positivity of z_i requires the positivity of y_i along the line leading to the endpoint. Hence, t_i is nonbasic along the line: $-s_i = t_i \geq 0$ (since $t_0 > 0$ along the line). This inequality is still valid at the endpoint and implies $s_i = 0$.

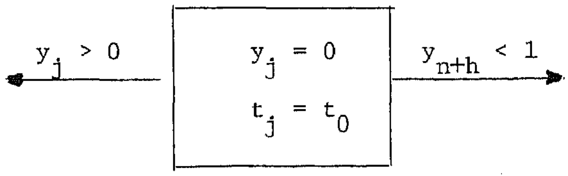
We illustrate the incidence between positions and lines of our algorithm in Figure 3. The algorithm leaves the initial position along the unique line incident to it. Every other position, which is not a solution, has two lines incident to it. If the position is reached along one line, then the algorithm leaves it along the other line. Solutions can be shown to be incident to only one line of our algorithm.

3. Convergence issues

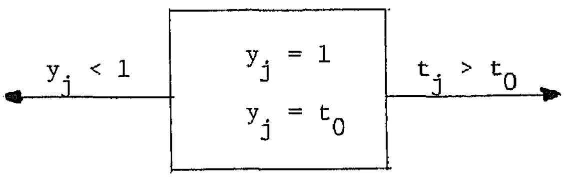
The previous section set the stage for an application of the well-known Lemke-Howson argument. The initial position is incident to one line of the algorithm. Every other position which is not a solution is incident to two lines of our algorithm. The Lemke-Howson argument proves that if lines are followed without turning back no position will ever be visited twice. The number of lines is finite, hence, so is the number of positions. The algorithm thus either stops at a solution for the LCP or follows an unbounded line. Following Lemke (1965), we present a class of matrices--characterized by Garcia (1973)--for which the algorithm finds a solution for any right-hand side vector q . We then show that for copositive plus matrices [Lemke (1965)] the existence of an un-



i. Position of type a: at least one y-variable is basic. No discontinuity in value of t_0 .



ii. Position of type a: all y-variables are nonbasic. Let $n+h = \operatorname{argmin}(t_{n+j} : y_{n+j} = 1)$. Discontinuous increase in value of t_0 when leaving position along line drawn at the right of the position.



iii. Position of type b: $j > n$. No discontinuity in value of t_0 .

FIGURE 3. The incidence between positions (with $t_0 > 0$) and lines of our algorithm. Notice that in case ii, the line drawn at the right of the position is defined only if $\{n+i : y_{n+i} = 1\}$ is non-empty. If the set is empty, we are at the initial position. This position is the only position (with $t_0 > 0$) incident to one line of our algorithm.

bounded line implies that the LCP is not feasible. The point behind both results is that they hold for any starting point z^0 in R^n_+. [Garcia and Gould (1980) discuss the possibility of convergence for a particular set of starting points.]

(3.1) Theorem. Let M satisfy the property that LCP(q,M) admits the unique solution $z = 0$ both when $q = 0$ and when $q = e$, where $e = (1,1,...1)^t$. Then no line of our algorithm is unbounded.

Proof. An unbounded line of our algorithm implies the existence of a (2n+1)-directional vector $(\bar{t}_0, \bar{s}, \bar{z})$ verifying the following conditions:

(3.2)
a. $\bar{s} = M\bar{z}$ with $\bar{z} \geq 0$;
b. if $\bar{z}_i > 0$ then $-\bar{s}_i = \bar{t}_0$;
c. if $\bar{z}_i = 0$ then $-\bar{s}_i \leq \bar{t}_0$;
d. $\bar{t}_0 \geq 0$.

[Notice that the directional vector $\bar{y}$ associated with $\bar{z}$ always has $\bar{y}_j = 0$ for $j > n$, for we can't leave the nonnegative orthant in z-space. Hence, $\bar{y}_i = \bar{z}_i$ for $i \in I(n)$.] It is clear that $\bar{z}$ is nonzero. If $\bar{t}_0 = 0$ then $\bar{s}$ is nonnegative and complementary with $\bar{z}$, which itself is nonnegative. $\bar{z}$ represents a nontrivial solution for LCP(0,M), which is impossible. If $\bar{t}_0 > 0$, we rescale $\bar{s}$ and $\bar{z}$ so that $\bar{t}_0 = 1$. $\bar{z}$ satisfies the inequalities $M\bar{z} + e \geq 0$, where the i^{th} inequality is an equality if $\bar{z}_i > 0$. LCP(e,M) thus admits a nonzero solution, which again contradicts our assumption.

(3.3) Theorem. Let M be copositive plus: $u^tMu \geq 0$ when $u \geq 0$, with $u^tMu = 0$ implying $(M+M^t)u = 0$. If the algorithm generates an unbounded line then the LCP is infeasible.

Proof. The LCP is infeasible if $s = Mz + q$, s and $z \geq 0$, is an infeasible linear system. By Farkas's lemma this infeasibility is equivalent with the existence of a nonnegative vector u such that $u^tM \leq 0$ and $u^tq < 0$.

The arguments of Theorem (3.1) show that an unbounded line implies the existence of a vector $(\bar{t}_0, \bar{s}, \bar{z})$ verifying (3.1). If $\bar{t}_0 > 0$, then $\bar{z}^tM\bar{z} = \bar{z}^t\bar{s} = -(\bar{z}^te)\bar{t}_0 < 0$ since $\bar{z}$ is nonzero. This contradicts the copositive plus character of M . Hence $\bar{t}_0 = 0$.

A zero value for $\bar{t}_0$ implies that $\bar{z}^tM\bar{z} = 0$ and, hence, $M^t\bar{z} = -M\bar{z} \leq 0$, since $-M\bar{z} = -\bar{s} \leq \bar{t}_0e = 0$. $\bar{z}$ is our candidate Farkas direction. To conclude our proof, we only need to show that $\bar{z}^tq < 0$.

Consider the unique endpoint of the unbounded line, say (t_0^*, s^*, z^*) , where

$$s^* = Mz^* + q \ , \qquad z^* \geq 0 \quad \text{and} \quad t_0^* > 0 \ .$$

Premultiplication with $\bar{z}^t$ yields

$$\begin{aligned} \bar{z}^ts^* &= \bar{z}^tMz^* + \bar{z}^tq \\ &= -\bar{s}^tz^* + \bar{z}^tq \ . \end{aligned}$$

Because of t_0-complementarity at (s^*, z^*) and along the unbounded line we have $-s_i^* = t_0^*$ whenever $\bar{z}_i > 0$, for even if nonbasic at z^* , y_i is basic along the line. Hence, $\bar{z}^ts^* = -(\bar{z}^te)t_0^* < 0$ implying that

$$-\bar{s}^tz^* + \bar{z}^tq < 0 \ .$$

If we can argue that $\bar{s}^tz^* = 0$, then our result is obtained.

If $\bar{s}_i > 0 = -\bar{t}_0$, then y_i is nonbasic along the unbounded line $(\bar{y}_i = \bar{z}_i = 0)$. At the same time,

$$\sum_{j\varepsilon P_h} \bar{s}_j \geq \bar{s}_i > 0 = \bar{t}_0 \ ,$$

where $i \in P_h$. The first inequality follows from the nonnegativity of $\bar{s}$. Inequality $\sum_{j \in P_h} \bar{s}_h > \bar{t}_0$ implies that $y_{n+h} = 1$ along the unbounded line, and thus at its endpoint z^* . Since $y_{n+h} = 1$ and $y_i = 0$ along the line, we have $z_i = 0$ along the line, and hence $z_i^* = 0$ at the endpoint. This concludes the argument establishing that $\bar{s}^t z^* = 0$.

4. Implementation

We introduce the matrix $E = (E_{hj})$ to identify the partition $\{P_h : h \in I(k)\}$:

$$E_{hj} = 1 \quad \text{if} \quad j \in P_h ,$$
$$= 0 \quad \text{otherwise} .$$

t can then be written in matrix form as

$$t = \begin{bmatrix} -s \\ \\ Es \end{bmatrix} = \begin{bmatrix} -M \\ \\ EM \end{bmatrix} z + \begin{bmatrix} -q \\ \\ Eq \end{bmatrix} . \tag{4.1}$$

We introduce nonnegative vectors to represent the deviations of t from t_0:

$$t = \begin{bmatrix} e^1 \\ \\ e^2 \end{bmatrix} t_0 - \begin{bmatrix} t^{1-} \\ \\ t^{2-} \end{bmatrix} + \begin{bmatrix} 0 \\ \\ t^{2+} \end{bmatrix},$$

e^1 and $t^{1-} \in R^n_+$, e^2 , t^{2-} and $t^{2+} \in R^k_+$, e^i (i=1,2) representing a vector of ones. We partition $y = (y^1, y^2)^t$, $y^1 \in R^n_+$, $y^2 \in R^k_+$, and introduce the corresponding partition for $D = (D^1, D^2)$. We write the feasibility constraint on y^2 as

$$v^2 + y^2 = e^2 \quad , \quad v^2 \text{ and } y^2 \geq 0 ,$$

and append it to (4.1) . The latter system can be written

$$(4.2)\qquad \begin{bmatrix} t^{1-} \\ t^{2-} \\ v^2 \end{bmatrix} = \begin{bmatrix} MD^1 & MD^2 & 0 \\ -EMD^1 & -EMD^2 & I \\ 0 & -I & 0 \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \\ t^{2+} \end{bmatrix} + \begin{bmatrix} e^1 \\ e^2 \\ 0 \end{bmatrix} t_0 + \begin{bmatrix} q^0 \\ -Eq^0 \\ e^2 \end{bmatrix},$$

where $q^0 = Mz^0+q$. t_0-complementarity between t and y is equivalent with the ordinary complementarity between (t^{1-}, t^{2-}, v^2) and (y^1, y^2, t^{2+}) . The starting point is $(y^1, y^2, t^{2+}) = 0$. Our algorithm can thus be seen as a projection of Lemke's algorithm applied to an enlarged problem. Notice also that assumption (2.6) is satisfied when linear system (4.2) is nondegenerate. Classical perturbation techniques applied to (4.2) ensure nondegeneracy. Finally, the discontinuity of t_0 , as described in figure 3 (case ii), reduces to a trivial pivot step in the enlarged system. In the pivot step that corresponds in figure 3 (case ii) to a movement along the line appearing at the right-hand side of the position, t_0 increases by an amount equal to the samllest positive component of t^{2+} . All basic components of t^{2+} are decreased by that amount whereas all basic components of (t^{1-}, t^{2-}) are increased by a similar amount. The components of (y^1, y^2) are not affected by this pivot step.

It is clear that the last k equations in (4.2) can be handled implicity as they represent upper bounds on y^2 . We now indicate that a similar implicit treatment can be given to the middle k equations. Adding appropriate sums of the first n equations to these middle k equations, they can be written

$$(4.3)\qquad Et^{1-} + t^{2-} - t^{2+} - (Ee^1 + e^2)\ \ t_0 = 0 .$$

These equations are of the GVUB type [Schrage (1978)] since every variable with a positive coefficient appears only once in (4.3) . At a position $t_0 > 0$ and $t^{2+} \geq 0$, so that at least one among the variables t_h^{2-} and $(t_j^{1-} : j \in P_h)$ is basic. This implies that the basis matrix, after suitable permutation of its columns, contains an identity submatrix of order k . This property allows an implicit treatment of these equations so that every pivot step in system (4.2) involves the updating of a basic submatrix of order n , rather than $n+2k$ in an explicit treatment of (4.2) .

There may exist instances of the LCP where the freedom to arbitrarily choose a partition of $I^+(n)$ could be exploited. One such instance occurs when the matrix M presents the special structure

$$M = \begin{bmatrix} A, & 0 & \cdots & 0 \\ 0 & A_2 & & 0 \\ \vdots & & \ddots & \\ B_1 & B_2 & \cdots & A_k \end{bmatrix} .$$

Every submatrix A_h could then be associated with a partition set P_h . However, in the absence of special structure, it is reasonable to expect the algorithm to treat all coordinates symmetrically. This points us to the two extreme cases, $k = 1$ and $k = |I^+(n)|$.

When $k = |I^+(n)|$, every set P_h is a singleton. If $z^0 > 0$ (4.2) can be written

$$(4.4) \quad \begin{bmatrix} t^{1-} \\ t^{2-} \\ v^2 \end{bmatrix} = \begin{bmatrix} M & -M & 0 \\ -M & M & I \\ 0 & -I & 0 \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \\ t^{2+} \end{bmatrix} + \begin{bmatrix} e \\ e \\ 0 \end{bmatrix} t_0 + \begin{bmatrix} q^0 \\ -q^0 \\ z^0 \end{bmatrix} .$$

If $z_i^0 = 0$, feasibility in row $2n+i$ sets $v_i^2 = y_i^2 = 0$, so that equation $n+i$ can be eliminated, as indicated in (4.2) . This case is analogous to one of Reiser's algorithms [Reiser (1978)].

When $k = 1$, (4.2) becomes

$$(4.5) \quad \begin{bmatrix} t^{1-} \\ t^{2-} \\ u^2 \end{bmatrix} = \begin{bmatrix} M & -Mz^0 & 0 \\ -e^+M & e^+Mz^0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \\ t^{2+} \end{bmatrix} + \begin{bmatrix} e \\ 1 \\ 0 \end{bmatrix} t_0 + \begin{bmatrix} q^0 \\ -e^+q^0 \\ 1 \end{bmatrix},$$

where t^{2-}, t^{2+}, u^2 , and y^2 are scalars, and where $e^+ = (e_i^+)$ with $e_i^+ = 1$ if $z_i^0 > 0$, and $e_i^+ = 0$ otherwise. The second Reiser algorithm considers only the first n equations of (4.5) . That algorithm corresponds to movements along t_0-complementary lines where $t_0 = \max(-s_1, -s_2,\ldots,-s_n, 0)$ as compared with $t_0 = \max(-s_1, -s_2,\ldots,-s_n, \sum_{i\in I^+(n)} s_i)$ for our algorithm. The complementarity conditions along a line in Reiser's algorithm are

$$(4.6) \qquad (t^{1-})^t\, y^1 = 0 \quad \text{and} \quad t_0 y^2 = 0 \ .$$

In this setting $t_0 = 0$ no longer identifies a solution. The algorithm termimates either when y^2 reaches its upper bound of 1 or when t_0 and t^{1-} are all nonbasic. In the first case, $t_0 = 0$ by complementarity along a line so that the first n equations of (4.5) can be written $t^{1-} = My^1 + q$. Since (t^{1-}, y^1) is also complementary it is a solution for the LCP. In the second case, $(t_0, t^{1-}) = 0$ and it is easily seen that $(s,z) = (0, y^1 + (1-y^2)z^0)$ is a solution for the LCP.

We conclude with examining the special case where $z^0 = 0$. Equation (4.2) then becomes

$$t^{1-} = My^1 + e^1\, t_0 + q \ .$$

Our algorithm requires t^{1-} and y^1 to remain complementary and this special case thus reduces to Lemke's original algorithm.

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