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Report No. 9555 (December 1995)

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Abstract. To an economy with production where inactive agents are allowed, we introduce a strategic game, the send and claim game. We show that strong Nash equilibria (Pareto optimal Nash equilibria) correspond to the core elements (imputations) of the classical NTU-game connected with the economy. Also reduced economies are introduced. The NTU-games associated with reduced economies are proved to be related to the Pelegs' reduction for NTU-games.

Key words: Reduced Economy, Reduced Game, Core, Strong Nash equilibrium

AMS-classification: 90D30

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1. Introduction.

In the model of a production economy we will discuss in this paper, one has a finite set of agents, each agent has an initial endowment of resources and each *coalition of agents* has access to some production technologies. A production technology describes the way resources can be converted into products. Furthermore , each agent has a utility function on the set of product bundles. It describes the agent's appreciation for a certain mix of products.

Traditionally, an NTU-game is associated with a production economy (Shapley and Shubik (1969), Scarf (1967)). It describes, for every coalition of agents, the utility levels that can be obtained by actions, feasible for that coalition. In this paper we also associate a strategic game with each production economy. In the strategic game each agent (player) mentions a collection of coalitions he wishes to form (production units), sends a part of his resource bundle to each of these coalitions and claims a part of the products. A coalition, understood as a production unit, is said to *form* if and only if all the members of the coalition agree with its formation and the total product bundle claimed from the coalition can be produced from the resources sent to the coalition. If a coalition forms, the agents obtain their claimed product bundle but if a coalition is not formed, the resources sent to this production unit are no longer useful. In this way each agent obtains product bundles from each coalition that forms and of which he is a member. His appreciation for the sum of the bundles he obtains, is his payoff. We call these strategic games *send and claim games*.

If a production economy, as described before, is given and moreover a proper coalition $N\backslash S$ of agents with *fixed claimed utility levels* is given, one can consider a *reduced production economy (with respect to the claimed utility levels)*. In this economy the agents in the coalition $N\backslash S$ have withdrawn from the decision making process but they can be called back for the purpose of production. If some of them are called back, they must be given a product bundle of at least their claimed utility level.

With such a reduced production economy we also associate an NTU-game and a strategic (send and claim) game. In this paper we investigate the relations between the NTU-games generated by a production economy and a reduced economy. The NTU-game of a reduced production economy turns out to be the reduced game of the NTU-game, as defined by Peleg (1985), of the original production economy *under the condition that the inactive agents do not exaggerate their claims.* [1] Furthermore, we will prove the existence of a canonical relation between core elements and imputations of the NTU-game associated with a (reduced) production economy on one side and strong Nash equilibria and Pareto optimal Nash equilibria in the associated send and claim games on the other side. In the last section we show how exchange economies in the sense of Debreu (1959), linear production models in the sense of Owen (1976) and market game models of Shapley and Shubik (1969) fit quite naturally into our model.

This paper is in the same spirit as Borm and Tijs (1993). The main differences are that players are not claiming 'utility levels' but product bundles. An other difference is the potential presence of 'inactive' agents.

We finish this introduction with a simple but instructive example.

Example. We consider a linear production model in the sense of Owen (1976). There are three agents, two resources and three products. The technology matrix A, the initial endowments $\{b_i\}_{i=1,2,3}$ and the price vector $c \in \mathbf{R}^3_+$ are

$$A := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad b_1 = b_2 := \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad b_3 := \begin{bmatrix} 0 \\ 2 \end{bmatrix} \; c := [5,\ 2,\ 16]\,.$$

All coalitions have the same production possibilities.

The associated NTU-game (in fact a TU-game) has the values

S	(1)	(2)	(3)	(12)	(13)	(23)	(123)
$v(S)$	10	10	4	20	32	32	42

Let us give some possible strategies in the send and claim game:

(a) Player 1 sends $(1,0)$ to coalition $\{1\}$ and claims 5 and he sends $(1,0)$ to $\{1,3\}$ and claims 6. Player 2 sends $(2,0)$ to coalition $\{2,3\}$ and claims 12. Player 3 sends

[1] to be made precise later on.

(0, 1) to the coalitions $\{1,3\}$ and $\{2,3\}$ and claims 10 from both coalitions. Note that the players claim money and not a vector of products. This means that in our model there is only one product 'money' and there is linear utility for money. The products in the Owen model are used to 'make money' using the fixed price vector c.

The claim of player 1 in coalition $\{1\}$ can be met and the payoff is 5. To coalition $\{1,3\}$ the resource bundle $(1,1)$ is sent and a total payoff of 16 is claimed. This is also feasible. In coalition $\{23\}$ the total claim is 22 and the total resource bundle is $(2,1)$. The best product bundle that can be produced is $(1,0,1)$ and this yields only 21. So the players obtain $(5+6,\ 0,\ 10) = (11,0,10)$. The strategy profile is not a Nash equilibrium, as player 2 can do better by sending his resources to coalition $\{2\}$ and claim 10. The payoff $(11,0,10)$ is also not in the core of the associated TU-game.

(b) Every player i sends his resource bundle to the one-person coalition $\{i\}$ and claims 10, 10 and 4 respectively. The payoff is now $(10,10,4)$ and the strategy profile is a Nash equilibrium. It is, however, not a Pareto optimal Nash equilibrium (see the next strategy profile). Furthermore , the payoff vector $(10,10,4)$ is not efficient and therefore certainly not an imputation (or core element) of the TU-game.

(c) Player 1 sends $(2,0)$ to coalition $\{1\}$ and claims 10 and player 2 and 3 send their resources to coalition $\{2,3\}$. They claim 12 and 20, respectively. Both coalitions can yield the claimed money and the payoff $(10,12,20)$ is an imputation that weakly dominates the payoff under (b). It is not a core allocation and the strategy profile is also not a strong Nash equilibrium as coalition $\{1,3\}$ can improve by sending their resources to $\{1,3\}$ and claiming $(11,-,21)$.

(d) The players send their resources to the same coalitions as under (c) but claim 10, 10 and 22. Now the payoff is $(10,10,22)$ is a core element of the TU-game and the strategy profile is a strong Nash equilibrium in the send and claim game.

2. Basic definitions and main results.

First we introduce the model of a production economy as we will use it subsequently. There are r resources numbered 1 up to r and p products numbered 1 up to p. A *resource bundle* is a vector in $\mathbf{R}^r_+$ and a *product bundle* is a vector in $\mathbf{R}^p_+$.

A *technology* is a correspondence $P: \mathbf{R}^r_+ \twoheadrightarrow \mathbf{R}^p_+$. The set $P(a)$ consists of the product bundles that can be produced from a resource bundle a. We assume:

$P(0) = \{0\}$ (no creation), $0 \in P(a)$ for all resource bundles a (no production is possible), and $P(a) + P(a') \subseteq P(a + a')$ for every pair of resource bundles $a, a' \in \mathbf{R}^r_+$ (superadditivity of production). Furthermore, we assume that $P(a)$ is a compact subset of $\mathbf{R}^p_+$ for every resource bundle $a \in \mathbf{R}^r_+$.

From superadditivity and $0 \in P(a)$ for all $a \in \mathbf{R}^r_+$ we derive immediately the monotonicity of P i.e. $P(a) \subseteq P(a')$ for all resource bundles $a \leq a'$. Namely, if $x \in P(a)$ and $a \leq a'$, we have $0 \in P(a' - a)$ and $x \in P(a')$ by superadditivity.

In a production economy $\mathcal{E}$ we have a (finite) set of agents N and each agent $i \in N$ has an initial endowment $\omega_i \in \mathbf{R}^r_+$ and a *continuous nonnegative utility function* u_i on $\mathbf{R}^p_+$ with $u_i(0) = 0$. Each non-empty coalition $T \subseteq N$ (production unit) has a technology P_T. We assume $P_T \subseteq P_S$ whenever $T \subseteq S$ (monotonicity of production possibilities). A production economy is, therefore, given by $\mathcal{E} = \langle N, \{\omega_i, u_i\}_{i \in N}, \{P_T\}_{T \subseteq N} \rangle$.

For a coalition $T \subseteq N$, a *T-feasible plan* is a vector of product bundles $\{b_i\}_{i \in T}$ satisfying the condition $\sum_{i \in T} b_i \in P_T(\sum_{i \in T} \omega_i)$.

So, T-feasibility means that the product bundle $\sum_{i \in T} b_i$ can be produced from the total initial endowment of coalition T.

The *NTU-game associated with the production economy* $\mathcal{E}$ has player set N and coalitional values

$$V(T) := \{z \in \mathbf{R}^T \mid \text{there is a } T\text{-feasible plan } \{b_i\}_{i \in T} \text{ with } u_i(b_i) \geq z_i \text{ for all } i \in T\}.$$

Notice that $V(T)$ is closed (by the compactness of the set of T-feasible plans and the continuity of the utility functions), comprehensive (by definition) and that the set $\{z \in \mathbf{R}^T_+ \mid z \in V(T)\}$ is compact.

With a production economy $\mathcal{E}$ we also associate a *strategic game* Γ. The player set is N and a typical strategy σ_i of player $i \in N$ consists of three components:

(a) a non-empty collection C_i of coalitions containing i,

(b) a set of resource bundles $\{a_{i,T}\}_{T \in C_i}$ with $\sum_{T \in C_i} a_{i,T} \leq \omega_i$,

(c) a set of product bundles $\{b_{i,T}\}_{T \in C_i}$.

The coalitions in C_i are the coalitions player i wishes to form (as production units). To each of these coalitions (production units) T player i sends a part of his resource bundle namely $a_{i,T}$ and claims a part of the production, namely $b_{i,T}$. If a strategy n-tuple $\sigma = (\sigma_1, \ldots, \sigma_n)$ is given, we define the collection $D(\sigma)$ of *σ-feasible coalitions* by

$$D(\sigma) := \{T \subseteq N \mid T \in \bigcap_{i \in T} C_i, \qquad \sum_{i \in T} b_{i,T} \in P_T(\sum_{i \in T} a_{i,T})\}.$$

So, a coalition T is σ-feasible if all players in T wishes to form the coalition T and the claimed product bundles can be produced from the resources sent to T. Denote by $D_i(\sigma)$ the collection of coalitions in $D(\sigma)$ containing the agent i. The payoff to player i (under the strategy n-tuple σ) is $\quad K_i(\sigma) := u_i(\sum_{T \in D_i(\sigma)} b_{i,T})$.

We call this strategic game Γ the *send and claim game* associated with the economy $\mathcal{E}$.

In a *reduced economy* $\mathcal{E}^{S,z}$ there is a subgroup of agents who are no longer actively involved in the economical decision making process. They only claim a product bundle of a certain utility level for the use of their resources and production technologies.

More precisely; $\qquad \mathcal{E}^{S,z} := \langle S, N \backslash S, \{\omega_i, u_i\}_{i \in N}, \{P_T\}_{T \subseteq N}, z \in \mathbf{R}_+^{N \backslash S} \rangle$.

A reduced economy is an economy as we defined before with an additional structure. There is a partition of the player set N into the set S of active players and the set $N \backslash S$ of inactive players. Each of the inactive players has a utility level he claims for the use of his resources and the technologies partially under his control. We say that $\mathcal{E}^{S,z}$ is the *reduction of the economy $\mathcal{E}$ on S at z*. Note that the collection of classical economies can be seen as aproper subset of the class of reduced economies.

For a coalition $T \subseteq S$, a *T-feasible plan* consists of a coalition $Q \subseteq N \backslash S$, a vector of product bundles $\{b_i\}_{i \in T}$ and a vector of product bundles $\{g_j\}_{j \in Q}$ with $u_j(g_j) \geq z_j$ for all $j \in Q$ and $\qquad \sum_{i \in T} b_i + \sum_{j \in Q} g_j \in P_{T \cup Q}(\sum_{i \in T} \omega_i + \sum_{j \in Q} \omega_j)$.

With a reduced economy $\mathcal{E}^{S,z}$ we associate an NTU-game $(S, V^{S,z})$ defined by:

$$V^{S,z}(T) := \{w \in \mathbf{R}^T \mid \text{there is a } T\text{-feasible plan } \{b_i\}_{i \in T}, \{g_j\}_{j \in Q}$$
$$\text{with } u_i(b_i) \geq w_i \text{ for all } i \in T\}.$$

Also for the reduced economies we define a *send and claim game* $\Gamma^{S,z}$, with player set S and typical strategy for player i given by

(a) a non empty collection C_i of coalitions in N, containing i. We write these coalitions as $T \cup Q$ with $i \in T \subseteq S$ and $Q \subseteq N \backslash S$.

(b) a vector of resource bundles $\{a_{i,T\cup Q}\}_{T\cup Q\in C_i}$ with $\sum_{T\cup Q\in C_i} a_{i,T\cup Q} \leq \omega_i$,

(c) a vector of product bundles $\{b_{i,T\cup Q}\}_{T\cup Q\in C_i}$.

If σ is a strategy S-tuple, the collection $\bar{D}(\sigma)$ of temporarily σ-feasible coalitions consists, by definition, of the coalitions $T \cup Q$ with the properties that

(a) $T\cup Q \in \bigcap_{i\in T} C_i$ (only the players in T have to give their consent to the formation of $T \cup Q$),

(b) *there are product bundles* $\{g_j\}_{j\in Q}$ with $u_j(g_j) \geq z_j$ for all players $j \in Q$ and

$$\sum_{i\in T} b_{i,T\cup Q} + \sum_{j\in Q} g_j \in P_{T\cup Q}\left(\sum_{i\in T} a_{i,T\cup Q} + \sum_{j\in Q} \omega_j\right).$$

Applying $\bar{D}(\sigma)$ directly has a drawback: it may happen that two different coalitions in $\bar{D}(\sigma)$ have a player $j \in N\backslash S$ in common. In this case the initial endowment ω_j will be claimed by two coalitions and this is of course not possible. Therefore, we make the collection $\bar{D}(\sigma)$ *consistent* by deleting coalitions containing players of $N\backslash S$ claimed by more than one coalition of $\bar{D}(\sigma)$. We are left with a maybe smaller collection $D(\sigma)$ of feasible production units. Then the payoff functions $\{K_i\}_{i\in S}$ are defined, as before, as the utility levels of the total product bundle each of the players in S obtains.

3. A relation between reduction of economies and reduction of NTU-games.

It may be clear that, if (some of) the inactive players have unreasonably high claims, the active players will refuse to take care for their needs. Therefore we introduce the following reasonability constraint on the claims z of the inactive agents:

The vector $z \in \mathbf{R}_+^{N\backslash S}$ is called a *reasonable claim* if, for every coalition $T \supset S$ and every product bundle $b \in P_T\left(\sum_{i\in T} \omega_i\right)$, a vector of product bundles $\{g_j\}_{j\in N\backslash T}$ exists with $u_j(g_j) \geq z_j$, $j \in N\backslash T$ and $\sum_{j\in N\backslash T} g_j + b \in P_N(\sum_{i\in N} \omega_i)$.

Reasonability means that, if a coalition T containing S (the set of active agents) is able to produce a bundle b, the resources of the remaining players (in N) can be used to increase the production from b to a product bundle b' and that the difference bundle $b' - b$ is enough to satisfy the claims of the players outside T.

If the claims are reasonable, the NTU-game connected with a reduced economy is the reduced game in the sense of Peleg of the NTU-game associated with the original economy (cf. Peleg (1985)).

Theorem 1. *If $\mathcal{E}$ is a production economy, $S \subseteq N$ is a coalition and $z \in \mathbf{R}_+^{N\setminus S}$ is a* reasonable *claim, then the game $(S, V^{S,z})$ is the reduced game (in the sense of Peleg) of the game (N, V) associated with the economy $\mathcal{E}$.*

Proof: Peleg's reduced game of an NTU-game (N, V) with respect to a vector $z \in \mathbf{R}^{N\setminus S}$ is defined by

$$V_{\mathrm{P}}^{S,z}(T) := \begin{cases} \{w_T \in \mathbf{R}^T \mid (w_T, z_Q) \in V(T \cup Q) \text{ for a coalition } Q \subseteq N\setminus S\}, & T \subsetneq S, \\ \{w_S \in \mathbf{R}^S \mid (w_S, z_{N\setminus S}) \in V(N),\} & \text{if } T = S. \end{cases}$$

Let us start with the easy part by showing that $V^{S,z}(T) = V_{\mathrm{P}}^{S,z}(T)$ for coalitions $T \subsetneq S$ and $V^{S,z}(S) \supseteq V_{\mathrm{P}}^{S,z}(S)$.

If $w_T \in V_{\mathrm{P}}^{S,z}(T)$ for any coalition $T \subseteq S$. then there exists a coalition $Q \subseteq N\setminus S$ (and $Q = N\setminus S$ if $T = S$) with $(w_T, z_Q) \in V(T \cup Q)$. From the definition of $V(T \cup Q)$ we find that there is $T \cup Q$-feasible plan $\{b_i\}_{i\in T} \cup \{g_j\}_{j\in Q}$ with $u_i(b_i) \geq w_i$ for $i \in T$ and $u_j(g_j) \geq z_i$ for $j \in Q$. This is a T-feasible plan in the reduced economy and therefore, $w_T \in V^{S,z}(T)$. For $T \subsetneq S$ we can follow the argument in the reverse direction and find the equality of $V^{S,z}(T)$ and $V_P^{S,z}(T)$.

We are left to show that $V^{S,z}(S) \subseteq V_{\mathrm{P}}^{S,z}(S)$. Suppose $w \in V^{S,z}(S)$. Then there is a coalition $Q \subseteq N\setminus S$ and vectors of product bundles $\{b_i\}_{i\in S}$ and $\{g_j\}_{j\in Q}$ with $u_j(g_j) \geq z_j = u_j$ for all $j \in Q$, $u_i(b_i) \geq w_i$ for all $i \in S$ and

$$\textstyle\sum_{i\in S} b_i + \sum_{j\in Q} g_j \in P_{S\cup Q}\left(\sum_{i\in S} \omega_i + \sum_{j\in Q} \omega_j\right).$$

If $\bar{Q}$ is the complement of $S \cup Q$ and $b = \sum_{i\in S} b_i + \sum_{j\in Q} g_j$, the reasonability condition on z gives a vector of product bundles $\{\bar{g}_k\}_{k\in\bar{Q}}$ with $u_k(\bar{g}_k) \geq z_k$ for all $k \in \bar{Q}$ satisfying the feasibility condition $\sum_{i\in S} b_i + \sum_{j\in Q} g_j + \sum_{k\in\bar{Q}} \bar{g}_k \in P_N\left(\sum_{i\in N} \omega_i\right)$. Then $(w_S, z_{N\setminus S}) \in V(N)$ and $w \in V_{\mathrm{P}}^{S,z}(S)$. ◁

The following diagram pictures the result of Theorem 1. The horizontal arrows denote the transition from a (reduced) economy to an NTU-game. The vertical arrows exhibit the transition to reduced economies and reduced games.

$$\begin{array}{ccccc} \mathcal{E} & \longrightarrow & & & (N, V) \\ \downarrow & & & & \downarrow \\ \mathcal{E}^{S,z} & \longrightarrow & (N, V^{S,z}) & = & (N, V_P^{S,z}) \end{array}$$

4. Relations between equilibria in send and claim games and allocations in the associated NTU-game

In this section we will prove that strong Nash equilibria of send and claim games are canonically related to core allocations in the NTU-game associated with a production economy and that the same is true for Pareto optimal Nash equilibria and imputations. *For the rest of the paper we assume that $z_{N\backslash S}$ is a* reasonable *claim.*

The following theorem gives a relation between the core elements of $(S, V^{S,z})$ and the *strong Nash equilibria* of the associated send and claim game $\Gamma^{S,z}$.

As usual, the *core of an* NTU-game (N, V) is defined as the set of points $z \in V(N)$ whereupon no coalition $T \subseteq N$ can weakly improve.

A strategy profile of a strategic game Γ is a *strong Nash equilibrium* if no coalition T can deviate and increase the payoff of at least one player in T, while the payoffs of the other players in T do not decrease.

Accordingly, we state the following theorem.

Theorem 2. *For each core element $\hat{w}$ of $(S, V^{S,z})$ there is a strong Nash equilibrium $\hat{\sigma}$ of $\Gamma^{S,z}$ with $K_i(\hat{\sigma}) = \hat{w}_i$ for all $i \in S$ and conversely, if $\hat{\sigma}$ is a strong Nash equilibrium of $\Gamma^{S,z}$, then $\{K_i(\hat{\sigma})\}_{i \in S}$ is a core element of $(S, V^{S,z})$.*

Proof: If $\hat{w}$ is a core element of $(S, V^{S,z})$, we have in particular that $\hat{w} \in V^{S,z}(S)$. Because $z_{N\backslash S}$ is a reasonable claim and therefore Theorem 1 holds, there exists a vector $\{\hat{b}_i\}_{i \in S}$ and a vector $\{\hat{g}_j\}_{j \in N\backslash S}$ of product bundles such that $u_i(\hat{b}_i) \geq \hat{w}_i$ for all $i \in S$ and $u_j(\hat{g}_j) \geq z_j$ for all agents $j \in N\backslash S$. Moreover, $\sum_{i \in S} \hat{b}_i + \sum_{j \in N\backslash S} \hat{g}_j \in P_N(\sum_{i \in N} \omega_i)$. As $\hat{w}$ is a core element, we have $u_i(\hat{b}_i) = \hat{w}_i$ for all agents $i \in S$. We define the following strategy $\hat{\sigma}_i$ for $i \in S$:

(a) $\hat{C}_i := \{N\}$,

(b) $\hat{a}_{i,N} := \omega_i$ for all players $i \in S$,

(c) $\hat{b}_{i,N} := \hat{b}_i$ for all players $i \in S$.

In strategy $\hat{\sigma}_i$ each of the players $i \in S$ wishes only to form the grand coalition N, sends his resources to the grand coalition and claims $\hat{b}_i$.

It is immediately clear that $D(\hat{\sigma}) = \{N\}$ and $K_i(\hat{\sigma}) = u_i(\hat{b}_i) = \hat{w}_i$ for all $i \in S$. Therefore we are left to prove that $\hat{\sigma}$ is a strong Nash equilibrium i.e. that no coalition of players can win by deviating. Suppose that a coalition $T \subseteq S$ deviates from $\hat{\sigma}$. The new strategies σ_i, $i \in T$ are supposed to have the following components:

(a') C_i (with the usual property),

(b') $\{a_{i,U\cup Q}\}_{U\cup Q\in C_i}$ (in total at most ω_i),

(c') $\{b_{i,U\cup Q}\}_{U\cup Q\in C_i}$.

Let $D := D\left(\sigma_T, \hat{\sigma}_{S\setminus T}\right)$ be the consistent collection of feasible coalitions. We distinguish two cases, namely $N \in D$ and $N \notin D$.

If $N \in D$, no other coalition in D has agents with $N\setminus S$ in common and such a coalition must even be a subcoalition of T (players j in $S\setminus T$ play the strategy $\hat{\sigma}_j$ and allow only the coalition N). We write $D' := D\setminus\{N\}$ and D'_i for the family of coalitions in D' containing player i. If the resource bundles $\sum_{U\in D'_i} a_{i,U}$ are sent to T, it is possible to produce in T the bundle $\sum_{i\in T}\sum_{U\in D'_i} b_{i,U}$ for :

$$\begin{aligned} b' := \sum_{i\in T}\sum_{U\in D'_i} b_{i,U} = & \sum_{U\in D'}\sum_{i\in U} b_{i,U} \\ & \in \sum_{U\in D'} P_U\left(\sum_{i\in U} a_{i,U}\right) && \text{(as } U \in D\text{)} \\ & \subseteq \sum_{U\in D'} P_T\left(\sum_{i\in U} a_{i,U}\right) && \text{(by monotonicity of technologies)} \\ & \subseteq P_T\left(\sum_{U\in D'}\sum_{i\in U} a_{i,U}\right) && \text{(by superadditivity)} \\ & = P_T\left(\sum_{i\in T}\sum_{U\in D'_i} a_{i,U}\right) =: P_T(a'). \end{aligned}$$

Since $N \in D$ we have

$b'' := \sum_{i\in T} b_{i,N} + \sum_{i\in S\setminus T} \hat{b}_{i,N} + \sum_{j\in N\setminus S} g_j \in P_N\left(\sum_{i\in T} a_{i,N} + \sum_{j\in N\setminus T}\omega_j\right) =$
$: P_N(a'')$

for some vector of product bundles $\{g_j\}_{j\in N\setminus S}$ with $u_j(g_j) \geq z_j$ for all $j \in N\setminus S$. The product bundle $b' + b''$ can be produced in N from the resource bundle $a' + a''$ (we use the monotonicity $P_T \subseteq P_N$ and the superadditivity of P_N).

In this way we have constructed an N-feasible plan with utility levels $K_i(\sigma_T, \hat{\sigma}_{N\setminus T})$ for players $i \in T$ and utility levels $K_j(\hat{\sigma}) = \hat{w}_j$ for players $j \in S\setminus T$. Therefore, the deviation $(\sigma_T, \hat{\sigma}_{S\setminus T})$ cannot be a weak improvement for T, as $\hat{w}$ is a core element.

If $N \notin D$ and $U\cup Q \in D$, a product bundle $\sum_{i\in U} b_{i,U\cup Q} + \sum_{j\in Q} g_{j,Q}$ wherein $\{g_{j,Q}\}_{j\in Q}$ is a vector of product bundles with $u_j(g_{j,Q}) \geq z_j$ for all $j \in Q$ can be produced in

$U \cup Q$ from the resource bundle $\sum_{i\in U} a_{i,U\cup Q} + \sum_{j\in Q} \omega_j$. The same product bundle can be produced from the same resource bundle in $\bar{U} \cup \bar{Q}$, the union of the coalitions in D (monotonicity of production technologies). Then the total product bundle $\sum_{U\cup Q\in D} (\sum_{i\in U} b_{i,U\cup Q} + \sum_{j\in Q} g_{j,Q})$ can be produced in $\bar{U}\cup\bar{Q}$ from the resource bundle $\sum_{U\cup Q\in D} (\sum_{i\in U} a_{i,U\cup Q} + \sum_{j\in Q} \omega_j)$. As there are no inconsistencies in D, we find, in this way, a $\bar{U}$-feasible plan. The payoff $K_i(\sigma_T, \hat{\sigma}_{S\setminus T})$ is equal to $u_i(\sum_{U\cup Q\in D_i} b_{i,U\cup Q})$ for players in $\bar{U}$ and this cannot be a weak improvement for $\bar{U}$ upon $\hat{w}$. Furthermore, $\bar{U}$ must be a subset of T, as players outside T only agree with coalitions N.

For the players $j \in T\setminus\bar{U}$ the payoff $K_j(\sigma_T, \hat{\sigma}_{S\setminus T}) = 0$ and certainly not an improvement. Hence, we find that the strategy profile $\hat{\sigma}$ does not allow a profitable deviation by any coalition T: $\hat{\sigma}$ is a strong Nash equilibrium.

Conversely, if $\hat{\sigma}$ is a strong Nash equilibrium of $\Gamma^{S,z}$, a T-feasible plan $\{b_i\}_{i\in T}$ and $\{g_j\}_{j\in Q}$ (i.e. $u_j(g_j) \geq z_j$ for $j \in Q$ and $\sum_{i\in T} b_i + \sum_{j\in Q} g_j \in P_{T\cup Q}(\sum_{i\in T\cup Q} \omega_i)$) also defines a deviation $\sigma_i := \langle C_i := \{T \cup Q\}, a_{i,T\cup Q} := \omega_i, b_{i,T\cup Q} := b_i\rangle$. If the T-feasible plan gives an weak improvement upon $\hat{w} := K(\hat{\sigma})$ then the deviation of T from $\hat{\sigma}$ to $(\sigma_T, \hat{\sigma}_{S\setminus T})$ gives the same weak improvement in the payoff. As $\hat{\sigma}$ is a strong Nash equilibrium, such a deviation does not exist. Therefore $\hat{w}$ is a core element of $(S, V^{S,z})$.

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Comments.

(1) The core of an NTU-game satisfies the reduced game property (cf. Peleg (1985)). Therefore z_S is a core element of $(S, V^{S,z})$, if z is a core allocation of (N, V). The strong Nash equilibrium $\hat{\sigma}$ associated with z reduces to a strong Nash equilibrium $\hat{\sigma}_S$ in the reduced game $\Gamma^{S,\hat{\sigma}}$ according to Peleg and Tijs (1995) but $\Gamma^{S,z}$ is *not* the same as the reduced game in the sense of Peleg and Tijs. In the game $\Gamma^{S,z}$ the active players can change the strategies of the inactive players, in the reduced game $\Gamma^{S,\hat{\sigma}}$ this is not allowed.

(2) If $z_{N\setminus S}$ are reasonable claims and *the utility functions u_i are strictly monotonic* (up to now we even did not assume that they are monotonic !), $\hat{w}$ is a core element of $(S, V^{S,z})$ if and only if $(\hat{w}, z_{N\setminus S})$ is a core element of (N, V).

Proof: If $\hat{w}$ is a core element of $(S, V^{S,z})$ and $(\hat{w}, z_{N\setminus S})$ is not a core element of (N, V), there is a coalition T and a T-feasible plan $\{b_i\}_{i\in U}$, $\{g_j\}_{j\in Q}$ with $T = U \cup Q$,

$u_i(b_i) \geq \hat{w}_i$ for all $i \in U$, $u_j(g_j) \geq z_j$ and $\sum_{i\in U} b_i + \sum_{j\in Q} g_j \in P_{U\cup Q}\left(\sum_{i\in U\cup Q} \omega_i\right)$. There is, moreover, at least one strict inequality. If $u_i(b_i) > \hat{w}_i$ for some $i \in U$, we have a U-feasible plan in $\mathcal{E}^{S,z}$ that is a weak improvement upon $\hat{w}$. Hence, $u_i(b_i) = \hat{w}_i$ for all agents $i \in U$ and $u_j(b_j) > z_j$ for at least one agent $j \in Q$. By the continuity and strict monotonicity of the functions u_i there is a product bundle $\delta \geq 0$, $\delta \neq 0$, such that $g_j - \delta \geq 0$ and still, $u_j(g_j - \delta) > z_j$ and $u_i(b_i + |U|^{-1}\delta) > u_i(b_i) = \hat{w}_i$ for all agents $i \in U$. This gives also an improvement for U upon $\hat{w}$. This is impossible, as $\hat{w}$ is a core element.

The converse follows from the reduced game property for core elements of NTU-games under Pelegs' reduction and Theorem 1. $\triangleleft$

(3) Theorem 2 does not say that $(S, V^{S,z})$ has a non-empty core or that $\Gamma^{S,z}$ has a strong Nash equilibrium. It only says that both occur together.

The next theorem gives a similar relation between the set of imputations of $(S, V^{S,z})$ and the Pareto optimal Nash equilibria of $\Gamma^{S,z}$.

In an NTU-game (N, V) a point $x \in V(N)$ is an *imputation* if x_i cannot be improved by one person coalitions and x cannot be improved by the grand coalition N. We call these properties *individual rationality* and *efficiency,* respectively. We avoid the term Pareto optimality because it may cause confusion with the next definition.

A Nash equilibrium $\hat{\sigma}$ in a strategic game is called *Pareto optimal* if there is no strategy profile σ such that $K(\sigma)$ dominates $K(\hat{\sigma})$ weakly.

Theorem 3. *If $z_{N\setminus S}$ are reasonable claims and $\hat{w}$ is an imputation of $(S, V^{S,z})$, then there exists a Pareto optimal Nash equilibrium $\hat{\sigma}$ of $\Gamma^{S,z}$ with $K_i(\hat{\sigma}) = \hat{w}_i$ for all players $i \in S$. Conversely, the payoff vector $\{K_i(\hat{\sigma})\}_{i\in S}$ of a Pareto optimal Nash equilibrium $\hat{\sigma}$ is an imputation of $(S, V^{S,z})$.*

Proof: The proof follows the same lines as the proof of Theorem 2. If $\hat{w}$ is an imputation, there is, by Theorem 1, an S-feasible plan $\{\hat{b}_i\}_{i\in S}$, and $\{\hat{g}_j\}_{j\in N\setminus S}$ with $u_i(\hat{b}_i) \geq \hat{w}_i$ for all agents $i \in S$.

Efficiency of $\hat{w}$ shows that $u_i(\hat{b}_i) = \hat{w}_i$ for all agents $i \in S$ (otherwise the grand coalition S has a weak improvement). We define the strategy $\hat{\sigma}$ as before. Then $D(\hat{\sigma}) = \{N\}$ and $K_i(\hat{\sigma}) = \hat{w}_i$ for all agents $i \in S$. Suppose that player i deviates from $\hat{\sigma}_i$ to a strategy σ_i given by

(a) C_i, (b) $\{a_{i,U\cup Q}\}_{U\cup Q\in C_i}$ with $\sum_{U\cup Q\in C_i} a_{i,U\cup Q} \le \omega_i$,

(c) $\{b_{i,U\cup Q}\}_{U\cup Q\in C_i}$.

We define as before $D := D(\sigma_i, \hat{\sigma}_{-i})$.

If D contains N, only the coalition $\{i\}$ can further be a coalition in D if $K_i(\sigma_i, \hat{\sigma}_{-i}) > 0$. If we send the resource bundle $a_{\{i\}}$ also to N and claim the product bundle $b_{i,N} + b_{i,\{i\}}$, we have an S-feasible plan that gives the player i the payoff $K_i(\sigma_i, \hat{\sigma}_{-i})$ and to the other players j the payoff $K_j(\hat{\sigma}) = \hat{w}_j$. This cannot be an improvement for player i, as $\hat{w}$ is efficient.

If $N \notin D$, the collection D consists of some coalitions of the form $\{i\} \cup Q$. In this case we send all resource bundles to $\{i\} \cup \bar{Q}$ where $\bar{Q}$ is the union of all coalition Q with $\{i\} \cup Q \in D$. From these resource bundles the product bundle $\sum_{\{i\}\cup Q\in D} b_{i,\{i\}\cup Q}$ can be produced together with product bundles $\{g_j\}_{j\in\bar{Q}}$ with $u_j(g_j) \ge z_j$, $j \in \bar{Q}$. This gives an $\{i\}$-feasible plan assigning to player i a utility level $K_i(\sigma_i, \hat{\sigma}_{-i})$. This cannot be more than $\hat{w}_i$, as $\hat{w}$ is individually rational. We find that $\hat{\sigma}$ is a Nash equilibrium.

In a similar way one can prove that $\hat{\sigma}$ is Pareto optimal Nash. Suppose that σ is a strategy profile with $K_i(\sigma) \ge K_i(\hat{\sigma})$ for all $i \in N$. Let D be $D(\sigma)$. If $T \cup Q \in D$, a product bundle $\sum_{i\in T} b_{i,T\cup Q} + \sum_{j\in Q} g_{j,T\cup Q}$ can be produced in $T \cup Q$ from the resource bundle $\sum_{i\in T} a_{i,T\cup Q} + \sum_{j\in Q} \omega_j$ If $\bar{T} \cup \bar{Q}$ is the union of the coalitions in D, the product bundle $\sum_{T\cup Q\in D}\left(\sum_{i\in T} b_{i,T\cup Q} + \sum_{j\in Q} g_{j,T\cup Q}\right)$ can be produced in $\bar{T} \cup \bar{Q}$ from the resource bundle $\sum_{T\cup Q\in D}\left(\sum_{i\in T} a_{i,T\cup Q} + \sum_{j\in Q} \omega_j\right)$. For $j \in \bar{Q}$ there is exactly one coalition $T \cup Q \in D$ with $j \in Q$ (consistency of D). Call $\bar{g}_j := g_{j,T\cup Q}$ if $j \in Q$ and $T \cup Q \in D$. Then $\sum_{T\cup Q\in D}\left(\sum_{i\in T} b_{i,T\cup Q}\right) + \sum_{j\in\bar{Q}} \bar{g}_j$ can be produced from $\sum_{T\cup Q\in D}\left(\sum_{i\in T} a_{i,T\cup Q}\right) + \sum_{j\in\bar{Q}} \omega_j$. As the claims $z_{N\setminus S}$ are reasonable, we can produce in N this product bundle together with a product bundle $\sum_{k\in N\setminus(S\cup\bar{Q})} \bar{g}_k$ with $u_k(\bar{g}_k) \ge z_k$ for all $k \notin S \cup \bar{Q}$. Note that all players in S obtain the same product bundles as under σ. Then we have an N-feasible action with the same utility levels as the payoff $K(\sigma) \ge \hat{w}_i$. No coordinate can be strictly larger, as $\hat{w}$ is efficient.

Conversely, if $\hat{\sigma}$ is Pareto optimal Nash and $K(\hat{\sigma})$ is not individually rational, there is a one-person coalition (i) with a feasible action $\langle (i) \cup Q, b_i, \{g_j\}_{j\in Q}\rangle$ that gives player i a higher utility level than $K_i(\hat{\sigma})$. The deviation by player i to the strategy $\langle C_i = (i) \cup Q, a_{i,(i)\cup Q} = \omega_i, b_i\rangle$ gives player i also this higher payoff (notice that N is no longer a feasible coalition !) and this is impossible, as $\hat{\sigma}$ is a Nash equilibrium. In the same way, if $K(\hat{\sigma})$ is not efficient, we find that $\hat{\sigma}$ is not *Pareto optimal* Nash. ◁

5. Applications of the theory.

In this section we show how we can apply the theory of section 2 to models one can find already in the literature.

(I) **Exchange Economies.** [Debreu (1956)] In an exchange economy $\mathcal{E} := \langle N, \{u_i, \omega_i\}\rangle$ the resources are also the products. The technologies P_T are the same for every coalition $T \subseteq N$: $P_T(a) = P(a) := \{x \in \mathbf{R}^p \mid 0 \le x \le a\}$. It is clear that $P(0) = \{0\}$, that $P(a) + P(a') \subseteq P(a + a')$ and $P(a)$ is closed and bounded for every $a \in \mathbf{R}^p_+$.

(II) **Linear Production Games** [Owen (1975)] In a linear production situation with technology matrix $A \ge 0$, price vector c and resource bundles $\{b_i\}_{i\in N}$ there is only one product (the TU-utility money). For every coalition $T \subseteq N$ $P_T(a) := \{\langle c, x\rangle \mid Ax \le a,\, x \ge 0\}$ and $\omega_i := b_i$ for every $i \in N$. The utility functions u_i are the same for all players: $U(x) = x$ for every amount of money $x \ge 0$.

(III) **Market Games** [Shapley and Shubik (1969)] In a market game according to Shapley and Shubik, we have a set of agents N, each agent has an initial endowment ω_i and a continuous, monotonic and concave utility function U_i on commodity bundles. To fit this model in our model we assume that the initial endowments commodity are the resource bundles and that there is one product, 'money'. For a resource bundle $a \in \mathbf{R}^p_+$ we define

$$P_T(a) := \{x \in \mathbf{R}_+ \mid x \le \textstyle\sum_{i\in T} U_i(b_i) \quad \text{with} \quad \sum_{i\in T} b_i \le a\}.$$

We have the utility function $u_i(x) := x$ for all agents $i \in N$. Notice that the utility functions in our model ($\{u_i\}_{i\in N}$) are not the same as in the Shapley/Shubik model ($\{U_i\}_{i\in N}$). The superadditivity of P_T follows from the concavity of the functions U_i, $i \in N$: $x \in P_T(a)$ and $x' \in P_T(a')$ gives $x \le \sum_{i\in T} U_i(b_i)$ and $x' \le \sum_{i\in T} U_i(b'_i)$ for some bundles $\{b_i\}_{i\in T}$ and $\{b'_i\}_{i\in T}$ with $\sum_{i\in T} b_i \le a$ and $\sum_{i\in T} b'_i \le a'$. Then, $x + x' \le \sum_{i\in T}(U_i(b_i) + U_i(b'_i)) \le \sum_{i\in T} U_i(b_i + b'_i)$ and $\sum_{i\in T}(b_i + b'_i) \le a + a'$. Then $x + x' \in P_T(a + a')$.

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