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Determinateness of Two-Person Games.

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Sunto. – *Viene discussa la determinatezza per giochi a due persone, con funzioni guadagno non necessariamente limitate. È ben noto che le estensioni miste dei giochi bimatrici $m \times \infty$, qualora la mat.ice guadagno per il secondo giocatore sia superiormente limitata, sono determinati. In questo lavoro vengono provati vari teoremi che garantiscono l'esistenza di equilibri approssimati per le estensioni c-miste di giochi bimatrici $m \times \infty$, con matrice guadagno per il secondo giocatore inferiormente limitata. Viene anche mostrato che tutte le estensioni miste di giochi bimatrice semi-infiniti sono determinate, purché le matrici guadagno siano superiormente o inferiormente limitate.*

1. – Introduction.

Game theory started with the fundamental paper of John von Neumann [4], in which he proved that mixed extensions of finite matrix games are completely determined i.e. they have a value and both players possess optimal mixed strategies. Semi-infinite matrix games were considered for the first time by A. Wald [11]. He proved that mixed extensions of bounded semi-infinite matrix games are (weakly) determined i.e. they possess a value, but an optimal mixed strategy for the player with an infinite number of pure strategies not necessarily exists. He also gave an example of an $\infty \times \infty$ matrix game which is not weakly determined. Tijs [9] proved that also c-mixed extensions of unbounded semi-infinite matrix games possess a value. In his papers [2], [3], J. Nash introduced finite bimatrix games and proved that they have an equilibrium point in mixed strategies. This extended the von Neumann result for finite zero-sum games to general non-cooperative finite two-person games. Here a natural question arises, which is until now not completely solved. Are semi-infinite bimatrix games (weakly) determined? For semi-infinite $(m \times \infty)$-bimatrix games,

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where the second player has an upper bounded payoff matrix an affirmative answer is given in Tijs [7] (Cf. [6]). The main aim of this work is to consider this question in more detail. In section 3 we first tackle the problem: what is precisely the meaning of weakly determinateness for non zero-sum games if we allow unbounded payoff functions? The reason of considering unbounded payoff functions is illustrated together with the definition 3.4 below, which gives an answer to the previous question. In section 4 many sufficient conditions are given for the weak determinateness of c-mixed extensions and in section 5 it is proved that all full mixed extensions of one sided $(m\times\infty)$-bimatrix games are determined. In section 2 we recall some definitions and well-known facts.

2. – Preliminaries.

For surveys of minimax theorems, concerning determinateness of zero-sum games, we refer the reader to [12] and to the chapter 5 of the book [5]. A survey of ε-equilibrium point theorems for n-person games is given in [10] and a systematic study of all kind of mixed extensions for semi-infinite and infinite two-person games can be found in [7]. Semi-infinite zero-sum games are also considered in [1], where sufficient conditions can be found for the existence of conservative strategies. Finally motivations to introduce approximate equilibria are discussed in [6].

A two-person game (in normal form) is an ordered quadruplet (X, Y, K_1, K_2), where X and Y are the non-empty strategy sets of the players 1 and 2 respectively, and where $K_i\colon X\times Y\to\overline{\boldsymbol{R}}$ is the payoff function of the player i, which assigns to a strategy pair (x, y) an extended real number $K_i(x, y)$. Such a game is played as follows: player 1 chooses a strategy $x\in X$ and player 2 a strategy $y\in Y$. Then the payoff to player i is $K_i(x, y)$. In the case $K_2=-K_1$ we have a zero-sum game. A pair $(\bar{x}, \bar{y})\in X\times Y$ is a (Nash-Cournot) equilibrium for the game if $K_1(\bar{x}, \bar{y}) = \max_x K_1(x, \bar{y})$, $K_2(\bar{x}, \bar{y}) = \max_y K_2(\bar{x}, y)$. We shall indicate by $E(X, Y, K_1, K_2)$ the (possibly empty) set of the equilibria of the game. For a zero-sum game the existence of an equilibrium is equivalent to the following facts:

1) the values exists: namely

$$v(X, Y, K_1, K_2) := \sup_x \inf_y K_1(x, y) = \inf_y \sup_x K_1(x, y)$$

2) the sets of optimal strategies

$$O_1(X, Y, K_1, K_2) := \{\bar{x} \in X : \inf_y K_1(\bar{x}, y) = v(X, Y, K_1, K_2)\}$$

$$O_2(X, Y, K_1, K_2) := \{\bar{y} \in Y : \sup_x K_1(x, \bar{y}) = v(X, Y, K_1, K_2)\}$$

are both non-empty. We call a zero-sum game (weakly) determined if its value exists. In the next section se introduce determinateness for non zero-sum games. A game is called semiinfinite if one of the strategy spaces is a finite set and the other one a countable infinite set. If X consists of m elements and the payoff functions are real valued, then the game can be described by two real $(m \times \infty)$-matrices $A = [a_{ij}]_{i=1,j=1}^{m\ \ \infty}$, $B = [b_{ij}]_{i=1,j=1}^{m\ \ \infty}$, by numbering the trategies. If $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots\}$, then $a_{ij} = K_1(x_i, y_j)$, $b_{ij} = K_2(x_i, y_j)$, for $i \in \{1, \ldots, m\}$, $j \in \mathbf{N}$. Given such a bimatrix game, the c-mixed extension of (A, B) is the game (S^m, S^c, E_A, E_B), so defined:

$$S^m := \left\{p \in \mathbf{R}^m : p \geqslant 0, \sum_{i=1}^{m} p_i = 1\right\}$$

$$S^c := \left\{q \in \mathbf{R}^\infty : q \geqslant 0, \sum_{j=1}^{\infty} q_j = 1, q_j = 0 \text{ if } j \text{ is large}\right\}$$

$$E_A(p, q) := \sum_{i=1}^{m} \sum_{j=1}^{\infty} p_i a_{ij} q_j = pAq$$

and

$$E_B(p, q) := pBq \qquad \text{for all } (p, q) \in S^m \times S^c .$$

An $(m \times \infty)$-matrix A is called upper bounded (lower bounded) if there exists a $k \in \mathbf{R}$ such that $a_{ij} \leqslant k$ $(a_{ij} \geqslant k)$ for all (i, j). A is called one-sided bounded if it is upper or lower bounded. If A and B are both one-side bounded then the full mixed extensions $(S^m, S^\infty, E_A, E_B)$ can be defined. Here

$$S^\infty := \left\{q \in \mathbf{R}^\infty : q \geqslant 0, \sum_{j=1}^{\infty} q_j = 1\right\}$$

and $E_A(p, q) = pAq$, $E_B(p, q) = pBq$ for all $p \in S^m$, $q \in S^\infty$.

3. – Determinateness of general two-person games.

In sections 3 and 4 we shall assume that the payoff functions of the games are real valued.

Aim of this section is mainly to arrive to a suitable definition of determinateness for general (non necessarily zero-sum) games: to do this we seek inspiration at zero-sum games. Namely theorem 3.3 below shows the equivalence between determinateness of zero-sum games (i.e. the existence of the value) and the non emptiness of the sets of «approximate equilibria». This seems to be a reasonable way to define determinateness for general games, about which we cannot speak of value. In general for a two-person game a point $(\bar{x}, \bar{y})$ is called an $(\varepsilon_1, \varepsilon_2)$ equilibrium point, $\varepsilon_i \geqslant 0$, if

$$K_1(x; \bar{y}) \leqslant K_1(\bar{x}, \bar{y}) + \varepsilon_1 \quad \text{for all } x \in X ,$$

$$K_2(\bar{x}, y) \leqslant K_2(\bar{x}, \bar{y}) + \varepsilon_2 \quad \text{for all } y \in Y .$$

A Nash-Cournot equilibrium point represents a stable pair of strategies in the sense that no one player is motivated to change his choice, if the other one plays his fixed strategy. So an ε equilibrium point can be viewed as a pair that every player has only a small (for small ε) incentive to unilaterally deviate. And, as usual, the introduction of approximate solution concepts is motivated, also in this setting, by the difficulties to practically compute the actual solutions. (See also [6]).

In the sequel we shall denote by $E^{\varepsilon_1,\varepsilon_2}(X, Y, K_1, K_2)$ the set of $(\varepsilon_1, \varepsilon_2)$ equilibria of the game (X, Y, K_1, K_2). For zero-sum games we have the following proposition, for which the proof can be found in [8].

PROPOSITION 3.1. – *Let (X, Y, K_1, K_2) be a zero-sum game. Then the following assertions are equivalent*:

(i) *The game has a finite value.*

(ii) *For each $\varepsilon_1, \varepsilon_2 > 0$ the game possesses an $(\varepsilon_1, \varepsilon_2)$ equilibrium point.*

For zero-sum games with a finite value and for $\varepsilon \geqslant 0$ the sets of ε-optional strategies of the players are given by

$$O_1^\varepsilon(X, Y, K) := \{\bar{x} \in X : \inf_y K(\bar{x}, y) \geqslant v - \varepsilon\} ,$$

$$O_2^\varepsilon(X, Y, K) := \{\bar{y} \in Y : \sup_x K(x, \bar{y}) \leqslant v + \varepsilon\} .$$

(Here $K = K_1 = -K_2$).

The following proposition describes relations between ε-optimal strategies and $(\varepsilon_1, \varepsilon_2)$ equilibria for zero-sum games with finite value. The proof is straightforward and it is left to the reader.

PROPOSITION 3.2. – *Let $\varepsilon_1 \geqslant 0$, $\varepsilon_2 \geqslant 0$ and (X, Y, K) be a zero-sum game with a finite value. Then*

(i) $O_1^{\varepsilon_1}(X, Y, K) \times O_2^{\varepsilon_2}(X, Y, K) \subset E^{\varepsilon_1+\varepsilon_2, \varepsilon_1+\varepsilon_2}(X, Y, K, -K)$;

(ii) $E^{\varepsilon_1, \varepsilon_2}(X, Y, K, -K) \subset O_1^{\varepsilon_1+\varepsilon_2}(X, Y, K) \times O_2^{\varepsilon_1+\varepsilon_2}(X, Y, K)$;

(iii) $O_i(X, Y, K) = \bigcap_{\varepsilon > 0} O_i^{\varepsilon}(X, Y, K)$ *for* $i = 1, 2$;

(iv) $E(X, Y, K, -K) = \bigcap_{\varepsilon_1 > 0, \varepsilon_2 > 0} E^{\varepsilon_1, \varepsilon_2}(X, Y, K, -K)$.

Let now look at a zero-sum game with value $v = \sup_x \inf_y K(x, y) = +\infty$. Then for each $k \in \boldsymbol{R}$ and $\varepsilon > 0$ there is a point $(\bar{x}, \bar{y}) \in X \times Y$ such that $k \leqslant K(\bar{x}, \bar{y}) \leqslant K(\bar{x}, y) + \varepsilon$ for all $y \in Y$.
If player 1 wants at least a payoff k, and ε is small, then such a point $(\bar{x}, \bar{y})$ weakly guarantees that, because the incentive for player 2 to unilaterally deviate from $\bar{y}$ is small as he can only increase his payoff by at most ε. For a general two-person game we define now the set of (k, ε) equilibria by

$$E^{k,\varepsilon}(X, Y, K_1, K_2) := \{(\bar{x}, \bar{y}) \in X \times Y \colon K_1(\bar{x}, \bar{y}) \geqslant k, \quad K_2(\bar{x}, \bar{y}) \geqslant \sup_y K_2(\bar{x}, y) - \varepsilon\}$$

(where we think to k as a number large enough).

If $E^{k,\varepsilon}(X, Y, K_1, K_2) \neq \emptyset$ for all $k \in \boldsymbol{R}$ and for all $\varepsilon > 0$, then the players can reach a weakly stable agreement.

Similarly, we define

$$E^{\varepsilon,k}(X, Y, K_1, K_2) := \{(\bar{x}, \bar{y}) \in X \times Y \colon K_2(\bar{x}, \bar{y}) \geqslant k, \quad K_1(\bar{x}, \bar{y}) \geqslant \sup_x K_1(x, \bar{y}) - \varepsilon\}.$$

as the set of the (k, ε) equilibria.

In the next theorem we give a characterization of determinateness of zero-sum games.

THEOREM 3.3. – *Let (X, Y, K) the game. Then the following assertions are equivalent:*

(i) *The game is determined*

(ii) *One of the following properties holds*:

(D.1) $E^{\varepsilon_1,\varepsilon_2}(X, Y, K) \neq \emptyset$ *for all* $\varepsilon_1, \varepsilon_2 > 0$.

(D.2) $E^{k,\varepsilon}(X, Y, K) \neq \emptyset$ *for all* $k \in \boldsymbol{R}$, $\varepsilon > 0$.

(D.3) $E^{\varepsilon,k}(X, Y, K) \neq \emptyset$ *for all* $\varepsilon > 0$, $k \in \boldsymbol{R}$.

PROOF. – At first we note that, by proposition 3.1, property (D.1) is equivalent to the fact that the games has a real value. Now suppose that (D.3) holds. Then for every $n \in \boldsymbol{N}$ and $\varepsilon = 1$ we can take $(x^n, y^n) \in E^{\varepsilon,n}(X, Y, K)$. From

$$-K(x^n, y^n) \geqslant n\,, \qquad K(x^n, y^n) \geqslant \sup_x K(x, y^n) - 1$$

it follows that, for each $n \in \boldsymbol{N}$,

$$\inf_y \sup_x K(x, y) \leqslant \sup_x K(x, y^n) \leqslant -n + 1\,,$$

which implies that the value $\inf_y \sup_x K(x, y)$ equals $-\infty$. If the value is $-\infty$, then (D.3) holds and in a similar way $v = +\infty$ is equivalent to (D.2).

Observe that, from the proof of theorem 3.3, it follows that at most one of the properties (D.i) holds. This is no longer the case for general two-person games. If we look at the $(2 \times \infty)$-bimatrix game (A, B) with

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 2 & \cdots \end{bmatrix}$$

then $(e^1, e^1) = ((1, 0), (1, 0, \ldots)) \in S^2 \times S^c$ is an element of $E^{\varepsilon_1,\varepsilon_2}(S^2, S^c, E_A, E_B)$ for all $\varepsilon_1, \varepsilon_2 \geqslant 0$ and $(e^2, e^k) \in E^{\varepsilon,k}(S^2, S^c, E_A, E_B)$ for all $\varepsilon > 0$ and $k \in \boldsymbol{R}$. Hence for this game (D.1) and (D.3) both hold. Some general two-person games have another property of attractive outcomes for both players, which lacks all zero-sum games. If, for instance, the players have the same unbounded payoff function, then for the game the set $E^{k_1,k_2}(X, Y, K_1, K_2)$ of (k_1, k_2) equilibria is non-empty for each $k_1, k_2 \in \boldsymbol{R}$, where we set

$$E^{k_1,k_2}(X, Y, K_1, K_2) := \{(\bar{x}, \bar{y}) \in X \times Y\colon\ K_1(\bar{x}, \bar{y}) \geqslant k_1,\ K_2(\bar{x}, \bar{y}) \geqslant k_2\}\,.$$

We are now ready to introduce the following

DEFINITION 3.4. – Let (X, Y, K_1, K_2) be a two-person game. Then the game is called determined if at least one of the following four properties holds:

(D.1) $E^{\varepsilon_1,\varepsilon_2}(X, Y, K_1, K_2) \neq \emptyset$ for all $\varepsilon_1, \varepsilon_2 > 0$

(D.2) $E^{k,\varepsilon}(X, Y, K_1, K_2) \neq \emptyset$ for all $k \in \boldsymbol{R}$, $\varepsilon > 0$

(D.3) $E^{\varepsilon,k}(X, Y, K_1, K_2) \neq \emptyset$ for all $\varepsilon > 0$, $k \in \boldsymbol{R}$

(D.4) $E^{k_1,k_2}(X, Y, K_1, K_2) \neq \emptyset$ for all $k_1, k_2 \in \boldsymbol{R}$.

There is an important motivation for the definition 3.4. Starting from a game (X, Y, K_1, K_2) we can construct equivalent games $(X, Y, f_1 K_1, f_2 K_2)$ if $f_i\colon \boldsymbol{R} \to \boldsymbol{R}$ are strictly increasing functions. The games are equivalent because the underlying preference structures, for the outcomes, of the two players are the same: and obbiously the sets of the Nash-Cournot equilibria coincide. This fact at first further motivates the introduction and the study of the case of unbounded payoff functions; consequently the possibility for the value (of a zero-sum game) to be $+\infty$ and finally the definition 3.4. It can be in fact easily proved that if a game is determined in the sense of definition 3.4, then an equivalent game with bounded payoff functions has $(\varepsilon_1, \varepsilon_2)$ equilibria for every $\varepsilon_1, \varepsilon_2 > 0$. Conversely, if $(X, Y, f_1 K_1, f_2 K_2)$ has $(\varepsilon_1, \varepsilon_2)$ equilibria for every $\varepsilon_1, \varepsilon_2 > 0$, and the K_i are lower bounded, then (X, Y, K_1, K_2) is determined in the sense of definition 3.4.

We conclude this section with the following remark. Suppose we have two games (X, Y, K_1, K_2) and $(X, Y, \bar{K}_1, \bar{K}_2)$ such that $\bar{K}_1(x, y) = K_1(x, y) + c_1$, $\bar{K}_2(x, y) = K_2(x, y) + c_2$ for all x, y and for some real constants c_1 and c_2. Then one of the two games is determined if and only if the other one is determined. This fact will be often used in the following sections.

4. – Determinateness of c-mixed extensions of semi-infinite games.

In this section we concentrate our attention to $(m \times \infty)$-bimatrix games. We shall give nine theorems containing sufficient conditions for the determinateness of the c-mixed extensions (S^m, S^c, E_A, E_B) of an $(m \times \infty)$-bimatrix game (A, B). For such a game we denote by $E^{\varepsilon_1,\varepsilon_2}(A, B)$, $E^{\varepsilon,k}(A, B)$, $E^{k,\varepsilon}(A, B)$, $E^{k_1,k_2}(A, B)$ the corresponding approximate equilibrium sets of the game (S^m, S^c, E_A, E_B) and we say that (A, B) is determined if (S^m, S^c, E_A, E_B) is determined.

At first we recall two well known theorems: in the proofs the theorems of von Neumann and of Nash, respectively, for finite games play a role.

THEOREM 4.1 (cf. [9], theorem 4.5). – *Let* $(A, -A)$ *be an* $(m \times \infty)$-*(bi)matrix game. Then the game is c-determined.*

THEOREM 4.2 (Cf. [8], corollary 3.2). – *Let* (A, B) *be an* $(m \times \infty)$-*bimatrix game with* B *upper bounded. Then for each* $\varepsilon > 0$ *the set* $E^{0,\varepsilon}(A, B)$ *is non-empty.*

Now we start with a string of new almost equilibrium theorems. In most of those theorems the payoff matrix B of player 2 is lower bounded, contrary to the condition in theorem 4.2, and one or more conditions are also laid on A, the payoff matrix of player 1.

THEOREM 4.3. – *Let* A *and* B *be both lower bounded* $(m \times \infty)$-*matrices. Then* (A, B) *is c-determined.*

PROOF. – Without loss of generality we suppose that $B \geqslant 0$ and $a_{ij} \geqslant 1$ for all $i \in \{1, 2, \dots, m\}$, $j \in \boldsymbol{N}$.

We distinguish three cases.

Case 1. – B is also upper bounded

Case 2. – B and A are not upper bounded

Case 3. – B is not upper bounded, A is upper bounded.

In case 1 property (D.1) holds in view of theorem 4.2. In case 2 we prove that (D.4) holds. Take $k_1, k_2 \in \boldsymbol{R}$. Let row $e^{i'}A$ be unbounded for matrix A and let row $e^{i''}B$ of B be unbounded. Let player 1 play $\hat{p} = \frac{1}{2}e^{i'} + \frac{1}{2}e^{i''} \in S^m$, which corresponds to playing rows i' and i'' with probability $\frac{1}{2}$ if the rows are different, and which corresponds to the pure strategy i' if $i' = i''$. Take $j', j'' \in \boldsymbol{N}$ such that $a_{i',j'} \geqslant 4k_1$, $b_{i'',j''} \geqslant 4k_2$ and let player 2 play strategy $\hat{q} = \frac{1}{2}e^{j'} + \frac{1}{2}e^{j''}$.

Due to the non-negativity of A and B we have $E_A(\hat{p}, \hat{q}) \geqslant k_1$, $E_B(\hat{p}, \hat{q}) \geqslant k_2$.

Hence, $(\hat{p}, \hat{q}) \in E^{k_1,k_2}(A, B)$. Thus, (D.4) holds in case 2. In the third case we prove that (D.3) holds. Let $0 < \varepsilon < 1$, $k \in \boldsymbol{R}$. Take (i', j') such that $a_{i',j'} \geqslant \sup\limits_{i,j} a_{ij} - \frac{1}{2}\varepsilon$.

Then we can find $\delta \in (0, 1)$ such that $(1 - \delta)^2 a_{i',j'} \geqslant \sup\limits_{i,j} a_{i,j} - \varepsilon$. Let $e^{i''}B$ be unbounded and take $j'' \in \boldsymbol{N}$ such that $b_{i'',j''} \geqslant k \cdot \delta^{-2}$. Define $x^* := \delta e^{i''} + (1 - \delta)e^{i'}$ and $y^* := \delta e^{j''} + (1 - \delta)e^{j'}$. Then, obviously, $(x^*, y^*) \in E^{\varepsilon,k}(A, B)$.

In the following theorem B is lower bounded and there is a weak condition on A.

THEOREM 4.4. – *Let B be a lower bounded $(m \times \infty)$-matrix. Let A be not an upper bounded $(m \times \infty)$-matrix. Then (A, B) is c-determined.*

PROOF. – W.l.o.g. we suppose $b_{ij} \geq 1$ for all (i, j). We consider two cases.

Case 1. – There is an $i \in \{1, \ldots, m\}$ such that

$$\sup_{j \in N} a_{ij} = \sup_{j \in N} b_{ij} = +\infty$$

Case 2. – For each non-upper bounded row of A the corresponding row of B is bounded.

For case 1 we prove that (D.4) holds. Take $k_1, k_2 \in \boldsymbol{R}$.

Let i^* be such that $\sup_{j \in N} a_{i^*, j} = \sup_{j \in N} b_{i^*, j} = +\infty$.

Choose $j^1 \in \boldsymbol{N}$ such that $b_{i^*, j^1} \geq 2k_2$ and let $j^2 \in \boldsymbol{N}$ be such that $a_{i^*, j^2} \geq 2(k_1 + |a_{i^*, j^1}|)$.

Then $(e^i, \frac{1}{2} e^{j^1} + \frac{1}{2} e^{j^2})$ is a (k_1, k_2)-equilibrium point.

For case 2 we prove that (D.2) holds. Let $k \in \boldsymbol{R}$, $\varepsilon \in (0, 1)$.

Take $i \in \{1, 2, \ldots, m\}$ with $\sup_{j \in N} a_{ij} = +\infty$. Let $j^1 \in \boldsymbol{N}$ be such that $b_{ij^1} \geq \sup_{i,j} b_{ij} - \frac{1}{2}\varepsilon$. Let $\delta \in (0, 1)$ be such that $(1 - \delta) \cdot (\sup_{ij} b_{ij} - \frac{1}{2}\varepsilon) \geq \sup_{i,j} b_{ij} - \varepsilon$. Let $j^2 \in \boldsymbol{N}$ be such that $a_{ij^2} \geq \delta^{-1}(k + |a_{ij^1}|(1 - \delta))$. Then $(e^i, (1 - \delta) e^{j^1} + \delta e^{j^2})$ is a (k, ε)-equilibrium point.

In the next theorem we put only conditions on the payoff matrix of player 2.

THEOREM 4.5. – *Let (A, B) be an $(m \times \infty)$-bimatrix game with B lower bounded. Then we have*

(i) *If all rows of B are unbounded, then (A, B) is c-determined.*

(ii) *If one row of B is bounded, then (A, B) is c-determined.*

PROOF. – Without loss of generality we suppose that $b_{ij} \geq 1$ for all (i, j).

(i) Suppose that all rows of B are unbounded. Let $k \in \boldsymbol{R}$. We prove that $E^{0,k}(A, B) = \emptyset$. For each $i \in \{1, 2, \ldots, m\}$ we take $j(i) \in \boldsymbol{N}$ such that $b_{i, j(i)} \geq mk$. Let $\hat{q} = m^{-1} \cdot \sum_{i=1}^{m} e_{j(i)}$. Then $\hat{q} \in S^c$ and $pB\hat{q} \geq k$ for all $p \in S^m$. Let $\hat{p} \in S^m$ be a best response to $\hat{q}$ i.e. $\hat{p}A\hat{q} = \sup_{p \in S^m} pA\hat{q}$. Then $(\hat{p}, \hat{q}) \in E^{0,k}(A, B)$.

(ii) Suppose that exactly one row of B is bounded, say the m-th row. Take $0<\varepsilon<1$ and $k\in\boldsymbol{R}$. We will show that at least one of the sets $E^{0,\varepsilon}(A,B)$ and $E^{0,k}(A,B)$ is non empty. Let δ be such that $0<\delta\leqslant 1-(\beta-\frac{1}{2}\varepsilon)^{-1}(\beta-\varepsilon)$, where $\beta=\sup_{j\in\boldsymbol{N}} b_{mj}\geqslant 1$.

For each $i\in\{1,2,\ldots,m-1\}$ we take $j(i)\in\boldsymbol{N}$ such that $b_{i,j(i)}\geqslant(m-1)k\delta^{-1}$. We take $j(m)\in\boldsymbol{N}$ such that $b_{m,j(m)}\geqslant\beta-\frac{1}{2}\varepsilon$.

Let $\hat{q}=(1-\delta)e^{j(m)}+(m-1)^{-1}\sum_{=1}^{m-1}\delta e^{j(i)}$. Then

$$e^{i}B\hat{q}\geqslant b_{i,j(i)}\hat{q}_{j(i)}\geqslant k \qquad \text{if } i\in\{1,2,\ldots,m-1\}$$

$$e^{m}B\hat{q}\geqslant b_{m,j(m)}\hat{q}_{j(m)}\geqslant(\beta-\tfrac{1}{2}\varepsilon)(1-\delta)\geqslant\beta-\varepsilon\geqslant e^{m}Bq-\varepsilon$$

for all $q\in S^{c}$.

If $e^{m}A\hat{q}=\sup_{p\in S^{m}} pA\hat{q}$, then $(e^{m},\hat{q})\in E^{0,\varepsilon}(A,B)$.

If $e^{i}A\hat{q}=\sup_{p\in S^{m}} pA\hat{q}$ and $i\neq m$, then $(e^{i},\hat{q})\in E^{0,k}(A,B)$.

Now we consider the pairs (ε,k) with $\varepsilon=t^{-1}$, $k=t$ for $t\in\boldsymbol{N}$. We have proved that, for each $t\in\boldsymbol{N}$ one of the sets $E^{0,t^{-1}}(A,B)$ and $E^{0,t}(A,B)$ is non-empty. If there is a subsequence of non-empty sets of the sequence

$$E^{0,1}(A,B)\,,\qquad E^{0,2^{-1}}(A,B)\,,\,\ldots$$

then (D.1) holds. Otherwise, (D.3) holds.

We are now ready to prove that all $(2\times\infty)$-bimatrix games with one-sided bounded B are c-determined.

THEOREM 4.6. – *Let (A,B) be a $(2\times\infty)$-bimatrix game and let B one-sided bounded. Then (A,B) is c-determined.*

PROOF. – If B is upper bounded, then (D.1) is satisfied by theorem 4.2. Hence, suppose $B\geqslant 0$. If both rows of B are unbounded, then (A,B) is c-determined by theorem 4.5 (i) and in the case that one row is unbounded the c-determinateness follows from theorem 4.5 (ii).

In theorem 4.5 (ii) we found a proof of the c-determinateness of a bimatrix game, where one of the rows of the lower bounded matrix B was bounded. One cannot adapt the proof of that theorem to settle determinateness of a bimatrix game where B has two or more bounded rows: every bounded row requires that we look for a column to play with probability near to 1 (in the proof $1-\delta$). In the next two theorems more rows of B may be bounded but then also a condition is required for the corresponding rows of A.

THEOREM 4.7. – *Let (A, B) be an $(m\times\infty)$-bimatrix game with lower bounded B and for all $i\in\{1, 2, ..., m\}$ we have:*

(i) *if row e_iB is bounded, then row e_iA upper bounded but not bounded.*

(ii) *if row e_iB is unbounded, then row e_iA is lower bounded. Then (A, B) is c-determined.*

PROOF. – We may suppose that there are bounded and unbounded rows in B. Otherwise, theorem 4.2 guarantees the c-determinateness. Furthermore we may suppose that $B\geqslant 0$ and that it are precisely the first m_1 rows of B which are bounded. Also we may suppose that $a_{ij}\leqslant 0$ for each $i\leqslant m_1$ and $j\in\boldsymbol{N}$. Take $k\in\boldsymbol{N}$. We shall prove that there is a $(0, k)$-equilibrium point for (A, B). Let

$$\alpha=\inf\{a_{ij}\colon m_1\leqslant i\leqslant m,\ j\in\boldsymbol{N}\}\ .$$

For $i\in\{1, 2, ..., m_1\}$ we take $j(i)\in\boldsymbol{N}$ such that $a_{ij(i)}\leqslant m(\alpha-1)$. For $i\in\{m_1+1, ..., m\}$ we take $j(i)\in\boldsymbol{N}$ such that $b_{i,j(i)}\geqslant mk$. Let $\hat{q}:=m^{-1}\sum_{k=1}^{m}e^{j(k)}\in S^c$.

Then, for all $i\in\{1, ..., m_1\}$ we have

$$e^iA\hat{q}=m^{-1}\sum_{k=1}^{m}a_{i,j(k)}\leqslant m^{-1}a_{i,j(i)}\leqslant m^{-1}m(\alpha-1)=\alpha-1\ .$$

For all $i\in\{m_1+1, ..., m\}$: $e^iA\hat{q}\geqslant\alpha$.

Let i^* be such that $e^{i^*}A\hat{q}=\max_i e^iA\hat{q}$. Then $i^*>m_1$ and then

$$e^{i^*}B\hat{q}\geqslant m^{-1}b_{i^*,j(i^*)}\geqslant k\ .$$

Hence, $(e^{j^*}, \hat{q})$ is a $(0, k)$-equilibrium point.

A strong result is obtained in the following theorem. In the rather complicated proof we make an essential use of theorem 4.2 and condition (ii).

THEOREM 4.8. – *Let (A, B) be an $(m\times\infty)$-bimatrix game, where we have:*

(i) *B is lower bounded*

(ii) *there is at least one $i\in\{1, 2, ..., m\}$ with e_iA and e_iB bounded*

(iii) *if a row of B is bounded, then the corresponding row of A is upper bounded.*

Then (A, B) is c-determined.

PROOF. – W.l.o.g. we suppose that there are $m_1, m_2 \in \boldsymbol{N}$ with $1 \leqslant m_1 \leqslant m_2 \leqslant m$ such that

(a) $e^i A$ and $e^i B$ are bounded rows for $1 \leqslant i \leqslant m_1$

(b) $e^i B$ is bounded, $e^i A \leqslant 0$, $e^i A$ is unbounded for $m+1 < < i \leqslant m_2$

(c) $e^i B$ is non-negative and unbounded for $m_2 + 1 \leqslant i \leqslant m$.

If $m_2 = m$, then the c-determinateness follows from theorem 4.2.

So we suppose in the following that $1 \leqslant m_1 \leqslant m_2 < m$ (where we do not exclude the possibility that $m_1 = m_2$).

Let $k \in \boldsymbol{N}$, $\varepsilon > 0$. Let δ be a real number such that

$$0 < \delta < \min \{\tfrac{1}{4}\alpha^{-1}\varepsilon, \tfrac{1}{4}\beta^{-1}\varepsilon, 1\} \tag{4.1}$$

where

$$\alpha = 1 + \sup \{|a_{ij}| : 1 \leqslant i \leqslant m_1, j \in \boldsymbol{N}\} \tag{4.2}$$

$$\beta = 1 + \sup \{|b_{ij}| : 1 \leqslant i \leqslant m_1, j \in \boldsymbol{N}\} . \tag{4.3}$$

For each $i \in \{m_2 + 1, m_2 + 2, \dots, m\}$ we take $j(i) \in \boldsymbol{N}$ such that

$$b_{i,j(i)} \geqslant \delta^{-1}(m - m_1) k \tag{4.4}$$

For each $i \in \{m_1 + 1, m_1 + 2, \dots, m_2\}$ we take $j(i) \in \boldsymbol{N}$ such that

$$a_{i,j} \leqslant - \delta^{-1}(m - m_1)\alpha . \tag{4.5}$$

Define $q^* := (m - m_1)^{-1} \sum_{i=m_1+1}^{m} e^{j(i)}$. Let

$$S_*^m := \{p \in S^m : p_i = 0 \text{ if } i > m_1\}$$

and let $(\hat{p}, \hat{q}) \in S_*^m \times S^c$ be such that

$$\hat{p} A \hat{q} \geqslant p A \hat{q} \qquad \text{for all } p \in S_*^m \tag{4.6}$$

$$\hat{p} B \hat{q} \geqslant \hat{p} B q - \tfrac{1}{2}\varepsilon \qquad \text{for all } q \in S^c . \tag{4.7}$$

Such a pair $(\hat{p}, \hat{q})$ exists in view of theorem 4.2 applied to the $(m_1 \times \infty)$-bimatrix game $([a_{ij}]_{i=1,j=1}^{m_1,\ \infty}, [b_{ij}]_{i=1,j=1}^{m_1,\ \infty})$.

Let $q^{**} := (1 - \delta)\hat{q} + \delta q^* \in S^c$ and let $i^*\{1, 2, \dots, m\}$ be such that

$$e^{i^*} A q^{**} = \max_i e^i A q^{**} . \tag{4.8}$$

By (4.2), for all $i \in \{1, 2, ..., m_1\}$: $e^i A q^{**} \geq 1 - \alpha$ and by (*b*) and (4.5) for all $i \in \{m_1 + 1, m_1 + 2, ..., m_2\}$: $e^i A q^{**} \leq \delta(m - m_1)^{-1} a_{i,j(i)} \leq -\alpha$. This implies together with (4.8) that $i^* \notin \{m_1 + 1, ..., m_2\}$.

Hence, we have to consider two cases: $i^* > m_2$ and $i^* \leq m_1$.

Case 1. – Suppose $i^* > m_2$. Then (e^{i^*}, q^{**}) is a $(0, k)$-equilibrium point because in view of (*c*) and (4.4):

$$e^{i^*} B q^{**} \geq \delta(m - m_1)^{-1} b_{i^*, j(i^*)} \geq k$$

and because (4.8) holds.

Case 2. – Suppose $i^* \leq m_1$. We prove that in this case $(\hat{p}, q^{**})$ is an $(\varepsilon, \varepsilon)$-equilibrium point.

Firstly, note that, in view of (4.2) and (4.1), for $x \in S^m_*$:

$$(4.9) \qquad |x A \hat{q} - x A q^{**}| = \delta |x A (\hat{q} - q^*)| \leq \leq \delta(|x A \hat{q}| + |x A q^*|) \leq 2\delta\alpha \leq \tfrac{1}{2}\varepsilon$$

and that by (4.3) and (4.1):

$$(4.10) \qquad |\hat{p} B \hat{q} - \hat{p} B q^{**}| \leq 2\beta\delta \leq \tfrac{1}{2}\varepsilon .$$

Using (4.8), (4.9) with $x = e^{i^*}$, (4.6) and (4.9) with $x = \hat{p}$, we obtain for all $p \in S^m$:

$$(4.11) \qquad p A q^{**} \leq e^{i^*} A q^{**} \leq e^{i^*} A \hat{q} + \tfrac{1}{2}\varepsilon \leq \hat{p} A \hat{q} + \tfrac{1}{2}\varepsilon \leq \hat{p} A q^{**} + \varepsilon .$$

In view of (4.7) and (4.10) we obtain for all $q \in S^c$:

$$(4.12) \qquad \hat{p} B q \leq \hat{p} B \hat{q} + \tfrac{1}{2}\varepsilon \leq \hat{p} B q^{**} + \varepsilon .$$

Then (4.11) and (4.12) imply: $(\hat{p}, q^{**}) \in E^{\varepsilon,\varepsilon}(A, B)$.

So we have proved that for each pair (ε, k):

$$E^{\varepsilon,\varepsilon}(A, B) \neq \emptyset \quad \text{or } E^{0,k}(A, B) \neq \emptyset .$$

In a similar way as at the end of the proof of theorem 4.5 it follows that (A, B) satisfies (D.1) or (D.3). Hence, (A, B) is *c*-determined.

The next theorem deals with games where each row of A is one-sided bounded and B is one-sided bounded.

THEOREM 4.9. – *Let (A, B) be an $(m\times\infty)$-bimatrix game with the properties*:

(i) *B is one-sided bounded*

(ii) *each row of A is upper bounded or lower bounded.*

(iii) *at least one row of A is lower bounded and the corresponding row of B is bounded.*

Then (A, B) is c-determined.

PROOF. – If B is upper bounded, theorem 4.2 implies the c-determinateness. Hence, suppose w.l.o.g. that $B\geqslant 0$.

Let

$$I_1 := \{i\in\{1, 2, \ldots, m\} : e^iA \text{ and } e^iB \text{ bounded}\} ,$$

$$I_2 := \{i\in\{1, 2, \ldots, m\} : e^iA \text{ upper bounded and not bounded, } e^iB \text{ bounded}\}$$

$$I_3 := \{i\in\{1, 2, \ldots, m\} : e^iA \text{ lower bounded and not bounded, } e^iB \text{ bounded}\} ,$$

$$I_4 := \{i\in\{1, 2, \ldots, m\} : e^iB \text{ unbounded}\} .$$

If $I_4 = \emptyset$, then the c-determinateness follows from theorem 4.2. If $I_4\neq\emptyset$ and $I_3\neq\emptyset$, the theorem follows from theorem 4.4. If $I_4\neq\emptyset$ and $I_3=\emptyset$, condition (iii) implies that $I_1\neq\emptyset$ and then theorem 4.8 implies the c-determinateness. This finishes the proof.

We do not know whether condition (iii) in theorem 4.5 is superfluous. One can prove that this condition can be dropped out if the answer to the following problem is yes.

PROBLEM 4.10. – Suppose (A, B) is an $(m\times\infty)$-bimatrix game with $A\leqslant 0$, $B\geqslant 0$ and B possesses bounded and unbounded rows while all rows of A are unbounded. Is (A, B) c-determined?

In the next section we will prove that all full mixed extensions of semi-infinite bimatrix games with one-sided bounded payoff matrices are determined, so the problem 4.10 is solved with full extensions.

No boundedness conditions are necessary to define c-mixed extensions. We found many sufficient conditions of c-determinateness. Almost all have to do with one-sided boundedness. Let us close this section with the problem, which we leave open.

PROBLEM 4.11. – Does there exists an $(m\times\infty)$-bimatrix game, which is not c-determined?

5. – Determinateness of full mixed extensions.

In this section we need extended real valued payoff functions. Furthermore, we take definition 3.4 as a definition of determinateness also for this kind of games.

In section 2 we noted already that the full mixed extension $\langle S^m, S^\infty, E_A, E_B\rangle$ can be defined for the $(m\times\infty)$-bimatrix game (A, B) if both payoff matrices are one-sided bounded. Much work about determinateness of full mixed extensions is in fact already done in section 4, if we note that the full mixed extension is determined if the c-mixed extension is determined. More explicitly we have

$$E^{\varepsilon_1,\varepsilon_2}(A, B)\subset \underline{E}^{\varepsilon_1,\varepsilon_2}(A, B)\,, \qquad E^{\varepsilon,k}(A, B)\subset \underline{E}^{\varepsilon,k}(A, B)\,,$$

$$E^{k,\varepsilon}(A, C)\subset \underline{E}^{k,\varepsilon}(A, B)\,, \qquad E^{k_1,k_2}(A, B)\subset \underline{E}^{k_1,k_2}(A, B)$$

where $\underline{E}^{\varepsilon_1,\varepsilon_2}(A, B)$ is the set of $(\varepsilon_1, \varepsilon_2)$-equilibria of $\langle S^m, S^\infty, E_A, E_B\rangle$, etc.

We are able to prove that all full mixed extensions of one-sided bounded $(m\times\infty)$-bimatrix games are determined.

We need the following lemma.

LEMMA 5.1. – *Let $x := (x_1, x_2, \ldots)\in \boldsymbol{R}^\infty$ be an unbounded row.*

(i) *If x is lower bounded, then there exists a $q\in S^\infty$ such that* $\sum_{j=1}^{\infty} x_j q_j = +\infty$.

(ii) *If x is upper bounded, then there exists a $q\in S^\infty$ such that* $\sum_{j=1} x_j q_j = -\infty$.

PROOF. – We need only to prove (i), because upper boundedness of x implies lower boundedness of $-x$. Hence, suppose x lower bounded. Then there is for each $t\in \boldsymbol{N}$ a $j(t)$ such that $x_{j(t)}\geqslant 2^t$, where we may suppose $j(1)<j(2)<\ldots$. Let $q := \sum_{t=1}^{\infty} 2^{-t} e^{j(t)}$. Then $q\in S^\infty$ and $\sum_{j=1}^{\infty} x_j q_j = +\infty$.

THEOREM 5.2. – *Let A and B be one-sided bounded $(m\times\infty)$-matrices. Then the full mixed extension $\langle S^m, S^\infty, E_A, E_B\rangle$ of (A, B) is determined.*

PROOF. – We consider 4 cases. In the first 3 cases the determinateness follows from the corresponding c-determinateness of (A, B).

Case 1. – Let B be upper bounded. Then theorem 4.2 does the work.

Case 2. – Let A and B be lower bounded. The theorem 4.3 does the work.

Case 3. – Let B be unbounded and lower bounded, A upper bounded and suppose that condition (iii) of theorem 4.9 holds.

Then theorem 4.9 does the work.

Case 4. – Let B be unbounded and lower bounded, A upper bounded and condition (iii) of theorem 4.9 does not hold. Then all rows of A are unbounded. By lemma 5.1, for each $i \in \{1, 2, \ldots, m\}$ we can take $q^i \in S^\infty$ such that $e^i A q^i = -\infty$. Let $I := \{i \in \in \{1, 2, \ldots, m\} : e^i B \text{ unbounded}\}$.

Then $I \neq \emptyset$ and by lemma 5.1, for each $i \in I$ we can take $r^i \in S^\infty$ such that $e^i B r^i = +\infty$. Now let

$$\hat{q} := (|I| + m)^{-1}\left(\sum_{i=1}^{m} q^i + \sum_{i \in I} r^i\right) \in S^\infty .$$

Then $p A \hat{q} = -\infty$ for all $p \in S^m$ and $e_i B \hat{q} = +\infty$ for each $i \in I$. This implies that $(e^i, \hat{q})$ is an equilibrium point for every $i \in I$.

With the aid of Lemma 5.1, for many cases we can obtain stronger results for full mixed extensions than for c-mixed extensions. As an example we give here the stronger version of theorem 4.5 (i) for full mixed extensions.

THEOREM 5.3. – *Let (A, B) be a one-sided bounded bimatrix game. Suppose that B is lower bounded and each row of B unbounded. Then $(S^m, S^\infty, E_A, E_B)$ is completely determined.*

PROOF. – With the aid of lemma 5.1 we find for each $i \in \{1, 2, \ldots, m\}$ a $q^i \in S^\infty$ with $e^i B q^i = +\infty$.

Then $(e^k, \hat{q})$ is an equilibrium point of $\langle S^m, S^\infty, E_A, E_B \rangle$, where $\hat{q} := m^{-1} \sum_{i=1}^{m} q^i$ and where k is such that $e^k A \hat{q} = \sup_{p \in S^m} p A \hat{q}$.

Although all one sided bimatrix games (A, B) have a determined full extension, not all such extensions are completely determined as the following simple example learns.

EXAMPLE 5.4. – Let $A = [0, 0, \ldots]$ and $B = [-1, -\frac{1}{2}, --, \ldots]$. Then $e^1, e^n)$ is a $(0, 1/n)$-equilibrium point for each $n \in \boldsymbol{N}$. But $E(S^m, S^c, E_A, E_B) = \emptyset$, $E(S^m, S^\infty, A_A, E_B) = \emptyset$.

6. – Conclusions and remarks.

In section 3 we have defined (almost) determinateness of general two-person games. That this definition is suitable illustrates theorem 3.3, where the existence of the value of a zero-sum game turns out to be equivalent to determinateness in the sense of definition 3.4. Furthermore, we found in section 5, theorem 5.2, an affirmative answer with respect to the determinatness of full mixed extensions of all semiinfinite bimatrix games with one-sided bounded payoff matrices. In section 4 the c-determinateness for many classes of semi-infinite games was proved. In problem 4.10 we described the only class of one-sided bounded games for which we do not know the answer. We concentrated in this paper on $(m\times\infty)$-bimatrix games, but of course, similar theorems hold for $(\infty\times n)$-bimatrix games. Also, many results with respect to c-determinateness of bimatrix games can be extended in a straightforward way to c-mixed extensions of two-person games (X, Y, K_1, K_2) where X is finite and Y is arbitrary.

In a similar way as in [8] (and [10]) also results about c-determinateness can be obtained for two-person games (and n-person games), where one of the strategy spaces ($n-1$ of the strategy spaces) is (are) topologically small, if we put (equi-) continuity conditions on payoff functions, The example of Wald [11] learns that there exist non-determined games if both strategy spaces are large.

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