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General equilibrium in an economy with exhaustible resources and an unbounded horizon

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We show the existence of a general competitive equilibrium in an economy with exhaustible resources and an unbounded horizon. The model is a generalization of several equilibrium models already known in the literature. The method of proof uses a classical idea due to Negishi, extended to economies with an infinite-dimensional commodity space, brought about by the infinite horizon.

Key words: Infinite-dimensional commodity space; General equilibrium; Negishi approach

JEL classification: D51; Q30

1. Introduction

A major issue in the economics of natural exhaustible resources is the timing of the rate of extraction. The decision maker, be it an individual resource owner or a central agency, therefore faces a dynamic problem. Although there are models where the horizon is finite, for example when the resource stock is assumed to become valueless after a finite time because of the supposed emergence of substitutes [see Dasgupta and Heal (1974, pp. 175–181)], the general formulation is one where the horizon is infinite or at least indeterminate. (Choosing for an indeterminate horizon allows for the optimal horizon to be infinite.) This observation raises serious questions with respect to the existence of optimal programmes, because most existence theorems relate to finite time problems. Moreover, existence theorems for unbounded horizons 'only' yield existence of measurable controls, whereas from an economic point of view this is

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not a very appealing function space. The complexity of the problem is increased to a large extent if one is not only interested in the optimal decisions of a single agent, but also in the question whether the actions of an arbitrary number of agents yield a general equilibrium or not. This is the issue we address in the present paper: we consider an economy with an arbitrary number of consumers, producers, and resource extractors, who all face an infinite horizon and who take the market prices as given, and we show the existence of a general equilibrium in such an economy.

There is a number of ways to tackle this problem. Since we are dealing with an economy having an infinite horizon, the commodity space is of an infinite dimension. One could therefore hope to apply directly the by-now well-known existence results for this type of economies, due to, for example, Bewley (1972), Mas-Colell (1986), Zame (1987), and others. Unfortunately, the model we have in mind does not allow for this application because we do not wish to make the assumption of boundedness of the production set of the economy nor assumptions like properness or boundedness of marginal efficiency, which seem crucial in their approach. An alternative line of attack which has proved successful in some circumstances is to consider first truncated economies (i.e., economies with a finite horizon), show then the uniform boundedness (uniform with respect to the horizon) of equilibrium allocations and some of the prices, and, finally, demonstrate that the limits of these allocations and prices constitute a general equilibrium of the infinite horizon economy. This approach is followed by van Geldrop et al. (1991) for a model in a discrete time setting, where the existence of an equilibrium in the finite time economy can be established using the standard Arrow–Debreu arguments. But it is not clear whether the approach simply carries over to continuous time models, since there even with a finite horizon the commodity space is infinite-dimensional. In the present paper we shall take a third route, which we do not claim to be superior to the ones outlined above, but which has in the case at hand the advantage of providing the desired result in a rather straightforward way. The basic idea is that a general equilibrium, if any, is Pareto-efficient so that it should be possible to find the general equilibrium from the set of Pareto efficient allocations. Clearly, this idea is not new: it has been introduced by Negishi (1960) and was fruitfully used by a.o. Arrow and Hahn (1971). More recently the idea was exploited by Mas-Colell (1986). Kehoe, Levine, and Romer (1990), Dana and Le Van (1991), and Hadji and Le Van (1992) use this concept in an explicit intertemporal setting with discrete time. We employ a continuous time framework.

The plan of the paper is as follows. The model and its assumptions are introduced and discussed in section 2. Section 3 states the problem of finding the set of Pareto-efficient allocations. Subsequently this problem is then solved by invoking an existence result from control theory, due to Toman (1985). Section 4 elaborates on the necessary conditions for an optimum. Section 5 uses a fixed point argument to show that our model allows for a general equilibrium. Section 6 discusses possible generalizations and concludes.

Finally, it should be stressed that we are presently interested in existence only. An important issue from an economic point of view is naturally also the characterization of the equilibrium. For this we refer to van Geldrop and Withagen (1991).

2. The model

In the economy there are $n+2$ physically distinguishable commodities. The first commodity will be called the composite commodity. As a stock it serves as an input in the production processes and as a store of value (in this capacity it will be called capital); as a flow it is a consumer good. There are n stocks of natural exhaustible resources which are distinguished according to the costs that have to be made to exploit them. Finally, there is the extracted raw material, which is homogeneous. So the stocks differ in quality but the extracted commodity is physically the same for all resources. There are l consumers (indexed by h); m firms (indexed by i) produce the composite commodity; and n firms (indexed by j) are engaged in extraction, where the jth firm is identified by the capacity to exploit the jth natural resource. In the following outline of the model it will be assumed that all flow variables and prices are Lebesgue-measurable on $[0, \infty)$, that all stock-trajectories are absolutely continuous, and that all integrals are well defined.

Consumer h is endowed with a stock $K^h > 0$ of the composite commodity, stocks $(S_1^h, \ldots, S_n^h) \geq 0$ of the exhaustible resources, and shares $(\vartheta_1^h, \ldots, \vartheta_{m+n}^h)$ in the profits of the production sectors. The instantaneous utility function of consumer h is denoted by U_h and depends only on his rate of consumption. The rate of time preference of the consumer is denoted by $\rho_h(>0)$. Given a consumption profile C_h: $[0, \infty) \to \mathbb{R}_+$ total welfare of the consumer is

$$W^h[C_h] := \int_0^\infty \mathrm{e}^{-\rho_h t} U_h(C_h(t))\,\mathrm{d}t. \tag{2.1}$$

About U_h we will assume

U^1) U_h is continuous on $\mathbb{R}_+$.

U^2) U_h is strictly concave.

U^3) U_h is strictly increasing on $\mathbb{R}_+$.

U^4) U_h is C^2 on $\mathbb{R}_{++}$.

$\eta_h(C) := U_h'' C/U_h' \leq \eta_h < 0$ for some constant η_h and all $C > 0$.

U^5) $U_h'(0) = \infty$.

Here

$$R_+ := \{x \in \mathbb{R} \,|\, x \geq 0), \qquad \mathbb{R}_{++} := \text{int}\, \mathbb{R}_+ .$$

The composite commodity serves as the numéraire. It will turn out below that this choice is warranted. Let $r:[0,\infty) \to \mathbb{R}_+$ denote the gross interest rate or the rental price of the composite commodity. Define

$$\pi(t) := \exp\left(- \int_0^t (r(\tau) - \mu)\, \mathrm{d}\tau \right),$$

where $\mu > 0$ is the constant depreciation rate of the capital stock. So, $\pi(t)$ gives the present value at time 0 of one unit of a numéraire commodity held at time t, which yields a gross interest of $r(\tau)$ at instant of time τ $(0 \leq \tau \leq t)$, but depreciates at a rate μ. With a perfect capital market the budget constraint of consumer h then reads

$$\int_0^\infty \pi(t) C_h(t)\, \mathrm{d}t \leq K^h + \sum_{j=1}^n p_{0j} S_j^h + \int_0^\infty \pi(t) P^h(t)\, \mathrm{d}t\,, \tag{2.2}$$

where $P^h(t)$ stands for the total profits accruing to the consumer h at instant of time t and $p_0 := (p_{01}, p_{02}, \ldots, p_{0n})$ are the initial prices of the resource stocks. So the budget constraint simply requires that total discounted income is sufficient to cover total discounted expenditures. Note that we do as if all resource stocks are sold at the outset. In view of the supposed existence of a perfect capital market and in the absence of uncertainty this is obviously warranted.

Production of the composite commodity requires the input of capital and the raw material. Production takes place according to neoclassical production functions F_i: $\mathbb{R}_+^2 \to \mathbb{R}_+$ $(i = 1, 2, \ldots, m)$, satisfying:

F^1) F_i is continuous on $\mathbb{R}_+^2$.

F^2) F_i is strictly concave on $\mathbb{R}_{++}^2$.

F^3) F_i is strictly increasing on $\mathbb{R}_{++}^2$.

$F_i(0, R) = F_i(K, 0) = 0$ for all $(K, R) \in \mathbb{R}_+^2$.

F^4) F_i is C^1 on $\mathbb{R}_{++}^2$.

Let $p:[0,\infty) \to \mathbb{R}_+$ denote a price trajectory of the raw material and K_i^y: $[0,\infty) \to \mathbb{R}_+$ and R_i: $[0,\infty) \to \mathbb{R}_+$ input trajectories of capital and the raw material, respectively. Then total discounted profits in sector i are

$$P_i := \int_0^\infty \pi(t)[F_i(K_i^y(t), R_i(t)) - r(t)K_i^y(t) - p(t)R_i(t)]\,dt. \tag{2.3}$$

Resource extraction is carried out by means of capital. The input of capital in resource sector j is denoted by K_j^e. Extraction is denoted by E_j. Total discounted profits in resource sector j are then

$$P_{m+j} := \int_0^\infty \pi(t)[p(t)E_j(t) - r(t)K_j^e(t)]\,dt - p_{0j}S_j^d, \tag{2.4}$$

with

$$K_j^e(t) = G_j(E_j(t)). \tag{2.5}$$

Here G_j describes the extraction technology and S_j^d is the amount of resource stock j the sector initially buys. We assume

G^1) G_j is continuous on $\mathbb{R}_+$.

G^2) G_j is strictly convex.

G^3) G_j is strictly increasing.
$G_j(0) = 0$, $G_j'(\infty) = \infty$.

G^4) G_j is C^1 on $\mathbb{R}_+$.

A condition that must be satisfied in resource sector j is that

$$\dot{S}_j(t) = -E_j(t), \quad E_j(t) \geq 0, \quad S_j(t) \geq 0, \quad S_j(0) = S_j^d. \tag{2.6}$$

We are aware of the fact that G^1 is superfluous in view of G^4, and that also some assumptions on the U_h's and F_i's intermingle. But, for some of our results differentiability is not needed. This is the reason for mentioning continuity and differentiability separately.

A general equilibrium is then a set of prices (p, r): $[0, \infty) \to \mathbb{R}_+^2$ and $p_0 \in \mathbb{R}_+^n$, a set of input–output functions in the production sectors of the economy $(K^y, R) := (K_1^y, \ldots, K_m^y, R_1, \ldots, R_m)$, $(K^e, E, S^d) := (K_1^e, \ldots, K_n^e, E_1, \ldots, E_n, S_1^d, \ldots, S_n^d)$, and consumption trajectories $C = (C_1, \ldots, C_l)$ such that

i) for all i, (K_i^y, R_i) maximizes (2.3),

ii) for all j, (K_j^e, E_j, S_j^d) maximizes (2.4) subject to (2.5) and (2.6),

iii) for all h, C_h maximizes (2.1) subject to (2.2) where P^h consists of the maximized profits,

iv) $$\sum_i R_i(t) \leq \sum_j E_j(t), \quad p(t)\left(\sum_j E_j(t) - \sum_i R_i(t)\right) = 0, \quad t \in [0, \infty), \tag{2.7}$$

$$S_j^d \leq \sum_h S_j^h, \quad j = 1, 2,, \ldots, n, \quad \sum_j p_{j0}\left(\sum_h S_j^h - S_j^d\right) = 0, \tag{2.8}$$

$$\sum_i F_i(K_i^y(t), R_i(t)) = \sum_h C_h(t) + \mu K(t) + \dot{K}(t), \quad t \in [0, \infty), \tag{2.9}$$

where

$$K(t) := \sum_i K_i^y(t) + \sum_j K_j^e(t), \quad t \in [0, \infty), \tag{2.10}$$

$$K(0) = \sum_h K^h. \tag{2.11}$$

The model presented here is a generalization of a number of models in the field of exhaustible resources, e.g., Dasgupta and Heal (1974), Chiarella (1980), Kemp and Long (1980), and Elbers and Withagen (1984). Toman (1986) deals explicitly with the existence problem in a similar but less general model. The generalization refers to the number of consumers (countries), extractors and producers of the composite commodity, the introduction of non-unilateral ownership of the exhaustible resources and the functional form of the technologies involved.

3. Pareto efficiency

The first step in the existence proof is to consider the set of Pareto-efficient allocations. To that end each of the consumers is attributed a nonnegative weight α_h. It will turn out to be convenient to take $\alpha = (\alpha_1, \ldots, \alpha_l)$ on the unit simplex Δ. Then the aim is to find the allocations which maximize the weighted sum of social welfare taking into account the technological and feasibility constraints. So, the Pareto problem can be stated as follows. Maximize

$$\int_0^\infty \sum \alpha_h e^{-\rho_h t} U_h(C_h(t))\, dt,$$

subject to (2.5)–(2.11) with the second parts in (2.7) and (2.8) omitted. Strictly speaking, feasibility does not require equalities in (2.9)–(2.11), but since capital is perfectly malleable with the consumer commodity and the U_h's are strictly increasing, there is no loss in generality to depart from (2.9)–(2.11) as they stand. Let us introduce the following additional notation. For $(\tilde{K}^y, \tilde{R}) \in \mathbb{R}^2_+$, F is defined as

$$F(\tilde{K}^y, \tilde{R}) := \max \sum_{i=1}^{m} F_i(K_i^y, R_i),$$

subject to

$$\sum_{i=1}^{m} K_i^y \leq \tilde{K}^y, \qquad \sum_{i=1}^{m} R_i \leq \tilde{R}.$$

So, the function F describes the maximal output of the composite commodity given the totally available inputs. By virtue of the same argument as used above, the left-hand side of (2.9) can be replaced by $F(\tilde{K}^y, \tilde{R})$, where

$$\tilde{R} := \sum_j E_j \quad \text{and} \quad \tilde{K}^y := K - \sum_j G_j(E_j).$$

In theorems on the existence of solutions of optimal control problems *boundedness of the state variables and instruments* plays an important role. We shall deal with that issue first. Boundedness of the state variables and the rates of extraction shows up quite naturally, but the rates of consumption present a difficulty.

Since the resources are not replenishable there obviously exists $\bar{S}$ such that

$$0 \leq S_j \leq \bar{S}, \qquad j = 1, 2, \ldots, n \quad \text{and} \quad \text{for all } t. \tag{3.3}$$

Since $\sum K_i^y \leq K$ and $K_j^e \leq K$ for all j and $R_i \leq \sum E_j$ for all i, we have from (2.9)

$$\dot{K} \leq F\left(K, \sum_{j=1}^{n} G_j^{-1}(K)\right) - \mu K.$$

For $K > 1$ we have from the concavity of the F_i's that

$$\dot{K}/K \leq F\left(1, \sum_j G_j^{-1}(K)/K\right) - \mu.$$

Since $G_j'(\infty) = \infty$ and by virtue of F^3 we have $\dot{K}/K \leq -\mu$ if $K \to \infty$ [cf. Hadji and Le Van (1992) who employ a discrete time setting and assume $K_{t+1} - K_t \leq -\mu K_t$ for large feasible capital stocks]. Hence there exists $\tilde{K}$ such that for $K > \tilde{K}$, $\dot{K} \leq 0$. Take $\bar{K} > \max[K(0), \tilde{K}]$. Then, along any feasible programme,

$$0 \leq K \leq \bar{K} \quad \text{for all } t. \tag{3.4}$$

As a consequence of the fact that $\bar{K} \geq K \geq K_j^e = G_j(E_j)$ for all j there exists $\bar{E}$ such that

$$0 \leq E_j \leq \bar{E}, \qquad j = 1, 2, \ldots, n \quad \text{and} \quad \text{for all } t. \tag{3.5}$$

In a discrete time analogue of our model, boundedness of the state variables is a sufficient condition for boundedness of the rates of consumption. With time considered continuous, this is obviously not the case. There is no *a priori* upper bound on the rates of consumption. Therefore, for the moment the rates of consumption will be forced to lie in a bounded set. Obviously $C_h \geq 0$ for all h. We take some $\bar{C} > 0$ and add as a condition

$$0 \leq C_h \leq \bar{C}, \qquad h = 1, 2, \ldots, l \quad \text{and} \quad \text{for all } t. \tag{3.6}$$

For the readers' convenience the optimal control problem is now cast into the format of an existence theorem due to Toman (1985). Define

$$x := (S_1, \ldots, S_n, K), \qquad u := (C, E),$$

$$f_0(x, u, t) := \sum_{h=1}^{l} \alpha_h \mathrm{e}^{-\rho_h t} U_h(C_h),$$

$$f_j(x, u, t) := -E_j, \qquad j = 1, 2, \ldots, n,$$

$$f_{n+1}(x, u, t) := F\left(K - \sum G_j(E_j), \sum E_j\right) - \mu K - \sum_{h=1}^{l} C_h,$$

$$A := \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+^{n+1} \mid K \leq \bar{K};\ S_j \leq \bar{S}, j = 1, 2, \ldots, n\},$$

$$U(t, x) := \left\{ u \mid 0 \leq C_h \leq \bar{C} \text{ all } h; 0 \leq E_j \leq \bar{E} \text{ all } j; K - \sum_{j=1}^{n} G_j(E_j) \geq 0 \right\},$$

$$B = \{(t, x) \in A \mid t = 0;\ x = x_0\},$$

where $U(t, x)$ is defined for $(t, x) \in A$ and $x_0 := (\sum S_1^h, \sum S_2^h, \ldots, \sum S_n^h, \sum K^h)$.

The constrained Pareto problem [$P(\bar{C})$ is then defined as follows:

$$\max \int_0^\infty f_0(x, u, t)\,dt\,,$$

subject to

$$\dot{x} = f(x, u, t),$$

$$u \in U(t, x), \quad (t, x(t)) \in A, \quad (0, x(0)) \in B\,.$$

It would go too far to outline in detail that Toman's Theorem 2 applies to this problem. The essential issues to note are that A and $U(t, x)$ are compact sets, that f is continuous, that all functions involved satisfy the concavity requirements, and that $U(t, x)$ is an upper semi-continuous correspondence. Therefore the following holds:

Theorem 3.1. Under assumptions $U^1 - U^3$, $F^1 - F^3$, and $G^1 - G^3$, there exist absolutely continuous $\hat{x}$ and measurable $\hat{u}$ which solve problem $P(\bar{C})$. ■

Note that the differentiability assumptions on U_h, F_i, and G_j are not needed in this theorem. They will however play a role in the sequel.

4. Characteristics of the solution of the Pareto problem

The objective in this section is twofold. First, it will be shown that the upper-bound $\bar{C}$ imposed on the rates of consumption can be chosen such that it is never binding, i.e., not binding for any $t \in [0, \infty)$ nor for any α in the unit simplex. Second, we will prove that the solutions satisfy some continuity properties.

Our first concern is the upper bound on the rates of consumption. The second part of assumption U^4 says that the elasticities of marginal utility are bounded from above. It covers a large class of instantaneous utility functions, including Bernoulli-type functions. We may then define

$$\gamma := \min_h \frac{\rho_h + \mu}{\eta_h}\,.$$

Define $\tilde{F}$ by

$$\tilde{F} := \max_{K,\, K^y,\, R} \left(F(K^y, R) - \mu K\right),$$

subject to

$$0 \le K^y \le \bar{K}, \quad 0 \le K \le \bar{K}, \quad 0 \le R \le n\bar{E}.$$

Now fix some $t_1 > 0$ and choose $\bar{C}$ such that

$$\int_0^{t_1} (\tilde{F} - \bar{C}e^{\gamma t})\,dt < -\bar{K}.$$

The idea behind this construction is as follows. Irrespective of the upper bound that is put on the rates of consumption in the constrained Pareto problem $P(\bar{C})$, the maximal net output of the composite commodity is $\tilde{F}$, which is finite. Therefore, along an optimum, $\bar{C}$ cannot be maintained forever. So, the rates of consumption will eventually decrease in view of the feasibility of optimal programmes. But, getting ahead of the story, the rates of decline are bounded: if a rate of consumption is interior it must satisfy the well-known Keynes–Ramsey rule

$$\eta_h(C_h)\dot{C}_h/C_h = \rho_h + \mu - F_K,$$

and, therefore, $\dot{C}_h/C_h \geq \gamma$. So it would take too long to get the rate of consumption to a sustainable level, if any. Now the Pareto problem is reconsidered with $\bar{C}$ defined above as the upper bound (with t_1 fixed throughout).

We wish to work within the framework employed by Cesari (1983). He provides necessary conditions for the case where the solution of an optimal control problem has measurable (rather than piece-wise continuous) instruments and absolutely continuous (rather than piece-wise differentiable) state variables. Cesari deals with a fixed control region, i.e., a control region U not depending on time or the state variables. It is also required that the optimal trajectory is interior. Finally, Cesari works within finite time. So, the problem we started with has to be modified in these respects.

Take $T > 0$ fixed and large enough and let $\hat{x}(T)$ be the optimal state corresponding with the Pareto problem $P(\bar{C})$ at instant of time T. Take some $\varepsilon > 0$ and redefine

$$A := \{(t,x)\in\mathbb{R}_+ \times \mathbb{R}^{n+1} \,|\, 0 \leq K \leq \bar{K} + \varepsilon; -\varepsilon \leq S_j \leq \bar{S} + \varepsilon, j = 1, \ldots, n\},$$

$$U := \{u \,|\, 0 \leq C_h \leq \bar{C} \text{ all } h;\ 0 \leq E_j \leq \bar{E} \text{ all } j\},$$

$$B := \{(t_1, x(t_1), t_2, x(t_2))\in\mathbb{R}^{2(n+1)+2} \,|\, t_1 = 0;$$
$$x(t_1) = x_0;\ t_2 = T;\ x(t_2) = \hat{x}(T)\}.$$

Consider now problem $P^T(\bar{C})$:

$$\max \int_0^T f_0(x, u, t)\,dt,$$

subject to

$$\dot{x} = f(x, u, t)$$

$$u(t) \in U, \quad (t, x) \in A, \quad (0, x(0), T, x(T)) \in B.$$

This problem obviously has a solution. It will be denoted by $(\hat{x}^T, \hat{u}^T)$. Evidently $(\hat{x}^T(t), \hat{u}^T(t)) = (\hat{x}(t), \hat{u}(t))$ for $0 \leqslant t \leqslant T$. We shall deal next with the necessary conditions.

One of the prerequisites in Cesari's necessary conditions is that f_i $(i = 0, 1, \ldots, n+1)$ are defined and continuous in the set $A \times U$, as well as their partial derivatives with respect to t and x. In the model at hand this is obviously not the case. There is of course no problem with t and S_j $(j = 1, 2, \ldots, n)$ because for these variables the above conditions are trivially satisfied. It is the stock of the composite commodity that poses a problem. However, this problem can be dealt with as follows. $\hat{K}(t) > 0$ for all t, because otherwise consumption would be zero from some moment in time on, which cannot be optimal in view of U^5. Moreover, along a solution $(\hat{x}, \hat{u})$ of $P(\bar{C})$ it cannot be true that $\hat{K} - \sum G_j(\hat{E}_j) = 0$ for a set of t's with positive measure, because then a reduction of extraction along t's belonging to this set would increase total output and thereby consumption, which would then yield more aggregate welfare than in the presupposed optimum. The differentiability problem can then get rid of by making the assumption that F is C^1 on $\mathbb{R}^2_{++}$. It should be noted that merely assuming that F_i is C^1 on $\mathbb{R}^2_{++}$ does not entirely solve the differentiability problem because F may then still exhibit kinks where a transition takes place from producing according to one production function to producing according to another (this could actually occur when the individual production functions display constant returns to scale). However, at the cost of quite some cumbersome calculus it can be shown that also in that case the problem can be circumvented. F_i being C^1 on $\mathbb{R}^2_{++}$ (rather than on its entire domain of definition $\mathbb{R}^2_+$) suffices because $\hat{K} - \sum G_j(\hat{E}_j) > 0$ a.e. So F^4 virtually solves the differentiability problem. In view of these preliminary observations we can now proceed with the Cesari necessary conditions. Define the Hamiltonian

$$H(x, u, t, \lambda_0, \lambda, \varphi) := \lambda_0 \sum_{h=1}^{l} \alpha_h e^{-\rho_h t} U_h(C_h) + \sum_{j=1}^{n} \lambda_j(-E_j)$$
$$+ \varphi\left(F\left(K - \sum_{j=1}^{n} G_j(E_j), \sum_{j=1}^{n} E_j\right) - \mu K - \sum_{h=1}^{l} C_h\right).$$

Lemma 4.1. *Let $(\hat{x}^T, \hat{u}^T)$ solve $P^T(\bar{C})$. Then the following holds:*

1. *There exists an absolutely continuous vector function $(\hat{\lambda}_0^T, \hat{\lambda}^T, \hat{\varphi}^T)$ which is never zero on $[0, T]$ with $\hat{\lambda}_0^T$ (≥ 0) a constant and $\hat{\lambda}^T \geq 0$ such that*

$$-\dot{\hat{\lambda}}_j^T = \partial H/\partial \hat{S}_j^T = 0, \quad j = 1, 2, \ldots, n, \quad \text{a.e.}, \tag{4.1}$$

$$-\dot{\hat{\varphi}}^T = \partial H/\partial \hat{K}^T, \quad \text{a.e.}. \tag{4.2}$$

2. *H is maximized a.e. with respect to (C, E) in U.*

3. $$\hat{\lambda}_j^T \left[\hat{S}_j(T) - \int_0^T \hat{E}_j^T(s)\,\mathrm{d}s \right] = 0, \quad j = 1, 2, \ldots, n,$$

$$\hat{\varphi}^T(T)[\hat{K}(T) - \hat{K}^T(T)] = 0.$$

Proof. See Cesari (1983, pp. 196–198). ■

Formally, one has to deal with the possibility that $\hat{\lambda}_0^T = 0$. This is handled in:

Lemma 4.2. $\hat{\lambda}_0^T > 0$ *and* $\hat{\varphi}^T(t) > 0$ *for all* $t \in [0, T]$.

Proof. Suppose $\hat{\lambda}_0^T = 0$. If there exists $t \in [0, T]$ such that $\hat{\varphi}^T(t) = 0$, then $\hat{\varphi}^T(t) = 0$ for all $t \in [0, T]$, because it follows from (4.2) that $-\dot{\hat{\varphi}}^T(t) = \hat{\varphi}^T(t)(F_K - \mu)$.

Let J be the subset of $\{1, 2, \ldots, n\}$ such that $\hat{\lambda}_j^T > 0$ for $j \in J$. It follows from the maximization of the Hamiltonian (with $\hat{\varphi}^T = 0$) that, for $j \in J$, $\hat{E}_j^T(t) = 0$ a.e. Next consider problem $P^T(\bar{C})$ with resources $j \in J$ omitted. This problem has virtually the same solution as the original problem. But that implies that in the modified problem $\hat{\lambda}_j^T = 0$ for $j \notin J$ if $\hat{\lambda}_0^T(0) = \hat{\varphi}^T(0) = 0$. This is not allowed.

Therefore, if $\hat{\lambda}_0^T = 0$, then $\hat{\varphi}^T(t) > 0$ for all $t \in [0, T]$. But then the maximization of the Hamiltonian with respect to C yields $\hat{C}^T(t) = 0$ a.e., which cannot be optimal. Hence $\hat{\lambda}_0^T > 0$. Then $\hat{\varphi}^T(t) > 0$ follows immediately, because otherwise $\hat{C}_h^T(t) > \bar{C}$ for all h and all t. ■

As a consequence of this lemma we can safely put $\hat{\lambda}_0^T = 1$. Due to the structure of the problem at hand, it can be shown that there exist continuous controls that solve $P^T(\bar{C})$. This will turn out to be rather helpful in the sequel.

Define $C^T := (C_1^T, \ldots, C_l^T)$: $[0, T] \to \mathbb{R}_+^l$ as the solution of

$$\max_{C(t)} \sum \alpha_h e^{-\rho_h t} U_h(C_h(t)) - \hat{\varphi}^T(t) C_h(t),$$

subject to

$$0 \le C_h(t) \le \bar{C}, \qquad h = 1, 2, \ldots, l.$$

C^T is well defined on $[0, T]$ and continuous, because $\hat{\varphi}^T$ is absolutely continuous and the U_h's are strictly concave.

Define $E^T := (E_1^T, \ldots, E_n^T)\colon [0, T] \to \mathbb{R}_+^n$ as the solution of

$$\max_{E(t)} \hat{\varphi}^T(t) F\left(\hat{K}^T(t) - \sum_{j=1}^{n} G_j(E_j(t)), \sum_{j=1}^{n} E_j(t) \right) - \sum_{j=1}^{n} \hat{\lambda}_j^T E_j(t),$$

subject to

$$0 \le E_j(t) \le \bar{E}, \qquad j = 1, 2, \ldots, n.$$

E^T is well defined on $[0, T]$ and continuous, because $\hat{\varphi}^T$ is absolutely continuous and the F_i's and $-G_j$'s are strictly concave. Define also

$$S_j^T(t) := \sum_h S_j^h - \int_0^t E_j^T(s)\,\mathrm{d}s, \qquad j = 1, 2, \ldots, n,$$

and $K^T(t)$ by

$$K^T(t) = \int_0^t \left[F\left(\hat{K}^T(s) - \sum_{j=1}^{n} G_j(E_j^T(s)), \sum_{j=1}^{n} E_j^T(s) \right) - \mu \hat{K}^T(s) - \sum_{h=1}^{l} C_h^T(s) \right] \mathrm{d}s + K(0).$$

Then the following lemma holds:

Lemma 4.3. (i) $x^T(t) = \hat{x}(t)$ *for all* $t \in [0, T]$; $u^T(t) = \hat{u}(t)$ *for almost all* $t \in [0, T]$. (ii) (x^T, u^T) *solves* $P^T(\bar{C})$. (iii) x^T *is differentiable and* u^T *is continuous.*

Proof. (i) $\hat{u}^T(t) = \hat{u}(t)$ for almost all $t \in [0, T]$ and $u^T(t) = \hat{u}^T(t)$ for almost all $t \in [0, T]$; $x^T(t) = \hat{x}(t)$ follows from the construction of $x^T(t)$. (ii) This is evident view of (i). (iii) This is so by construction. ■

Now define $(x(t), u(t))$ for all $t \in [0, \infty)$ by

$$(x(t), u(t)) = (x^t(t), u^t(t)).$$

So, $(x(t), u(t))$ is the solution of $P^t(\bar{C})$ at instant of time t.

Lemma 4.4. (i) $x(t) = \hat{x}(t)$ *for all* $t \in [0, \infty)$; $u(t) = \hat{u}(t)$ *for almost all* $t \in [0, \infty)$. (ii) (x, u) *solves* $P^{\infty}(\bar{C})$, *with* $0 \leq C_h \leq \bar{C}$ *omitted.* (iii) x *is differentiable and* u *is continuous.*

Proof. The proof of (i) and (iii) is straightforward and will not be given here. With regard to (ii) it has to be shown only that the upper bound $\bar{C}$ will never be binding. This follows from the construction of $\bar{C}$ and the continuity of the vector function C. ■

Summarizing thus far, we started from the Pareto problem for an arbitrary $\bar{C}$, needed to establish the existence of a solution. Next it has been shown that for any $\bar{C}$ the Pareto problem has continuous instruments as a solution. This continuity property has been used to show that there can be found a $\bar{C}$ which turns out to be never binding. Let us now return to the original Pareto problem with disaggregated production. Define $(K^y(t), R(t))$ as the solution of

$$\max \sum_{i=1}^{m} F_i(K_i^y(t), R_i(t)),$$

subject to

$$\sum_{i=1}^{m} K_i^y(t) + \sum_{j=1}^{n} G_j(E_j(t)) \leq K(t), \tag{4.3}$$

$$\sum_{i=1}^{m} R_i(t) \leq \sum_{j=1}^{n} E_j(t), \tag{4.4}$$

where K and E_j $(j = 1, 2, \ldots, n)$ are optimal. It is immediate that this problem has a piece-wise continuous solution. So the following theorem can be stated.

Theorem 4.1. *Define* $u := (C, K^y, R, E)$. *Then*:

(i) (x, u) *is a Pareto-efficient allocation in the economy described in section 2.*
(ii) x *is differentiable and* u *is piece-wise continuous.* ■

5. General equilibrium

The final step is to consider the set of Pareto-efficient allocations and to search for the one that constitutes a general competitive equilibrium. The set of Pareto-efficient allocations can be found by solving the Pareto problem for all weights α in the unit simplex Δ. Consequently $z(t; \alpha)$ will henceforth denote the optimal value of variable z at instant of time t when the vector of weights is α.

For $\alpha \in \Delta$ we define the excess of shadow income of consumer h over his shadow expenditures along the Pareto trajectory corresponding with α:

$$J^h(\alpha) := -\int_0^\infty \varphi C_h \,\mathrm{d}t + \varphi(0)K^h + \sum_{j=1}^{n} \lambda_j S_j^h$$

$$+ \sum_{i=1}^{m} \vartheta_i^h \int_0^\infty \varphi[F_i(K_i^y, R_i) - rK_i^y - pR_i]\,\mathrm{d}t$$

$$+ \sum_{j=1}^{n} \vartheta_{m+j}^h \left\{ \int_0^\infty \varphi[pE_j - rG_j(E_j)]\,\mathrm{d}t - \lambda_j \sum_{h=1}^{l} S_j^h \right\}.$$

Here the arguments t and α have been omitted in the right-hand side since there is no danger of confusion. Furthermore, φ and λ_j ($j = 1, 2, \ldots, n$) are the co-state variables arising from the necessary conditions and r and p are the Lagrangean multipliers associated with the constraints (4.3) and (4.4), respectively. It is easily seen that $J^h(\alpha)$ is defined for all $\alpha \in \Delta$. Of course $C_h(t;\alpha) = 0$ if $\alpha_h = 0$. The interpretation of $J^h(\alpha)$ is as follows. The first part is the present shadow value of expenditures. The second and third term represent the shadow revenues from selling the initially held stocks of capital and resources respectively. The final parts are the shadow profits accruing to the consumer from the composite commodity producing sectors and the raw material sectors respectively. J^h will play a crucial part in the sequel because it will be shown that there exists $\hat{\alpha} \in \Delta$ such that $J^h(\hat{\alpha}) = 0$ for all h. To that end we construct a fixed point mapping. Before doing so, some properties of $J^h(\alpha)$ will be listed.

Lemma 5.1. (i) $\alpha_h = 0 \Rightarrow J^h(\alpha) > 0$, $h = 1, 2, \ldots, l$. (ii) $\sum J^h(\alpha) = 0$ *for all* $\alpha \in \Delta$. (iii) J^h *is continuous on* Δ.

Proof. (i) is trivial, because $\alpha_h = 0$ implies $C_h(t) = 0$ for all t, $\varphi(0)K^h > 0$, and all other terms in the right-hand side are nonnegative. (ii) and (iii) are proven in the appendices. ■

Now consider the mapping $g\colon \Delta \to \Delta$ defined by

$$g_h(\alpha) = \frac{\max\{J^h(\alpha), 0\} + \alpha_h}{\sum_h \max\{J^h(\alpha), 0\} + 1}.$$

The mapping g is the fixed point mapping alluded to above. See also Negishi (1972). By standard arguments we obtain:

Lemma 5.2. *There exists* $\hat{\alpha} \in \Delta$ *such that* $J^h(\hat{\alpha}) = 0$ *for all* h. ■

We are now ready to state the main theorem of this paper:

Theorem 5.1. *Let the economy satisfy* U^1–U^5, F^1–F^4, G^1–G^3. *Then the economy has a general competitive equilibrium.*

Proof. Let $(x(\hat{\alpha}), u(\hat{\alpha}))$ correspond with $\hat{\alpha}$ defined above. Define in a similar way $r(\hat{\alpha})$ and $p(\hat{\alpha})$. Define also $p_{0j}(\hat{\alpha}) := \lambda_j(\hat{\alpha})/\varphi(0; \hat{\alpha})$.

1) It is trivially true that the allocation corresponding with $\hat{\alpha}$ is feasible.

2) Take some $h \in \{1, 2, \ldots, l\}$. Since $J^h(\hat{\alpha}) = 0$ and $\varphi(0; \hat{\alpha}) > 0$, we have, omitting t and $\hat{\alpha}$ where there is no danger of confusion,

$$\int_0^\infty \frac{\varphi(t)}{\varphi(0)} C_h \,\mathrm{d}t = K^h + \sum_{j=1}^{n} \frac{\lambda_j}{\varphi(0)} S_j^h$$

$$+ \sum_{i=1}^{m} \vartheta_i^h \int_0^\infty \frac{\varphi(t)}{\varphi(0)} [F_i(K_i^y, R_i) - rK_i^y - pR_i] \,\mathrm{d}t$$

$$+ \sum_{j=1}^{n} \vartheta_{m+j}^h \left\{ \int_0^\infty \frac{\varphi(t)}{\varphi(0)} [pE_j - rG_j(E_j)] \,\mathrm{d}t - \frac{\lambda_j}{\varphi(0)} \sum_{h=1}^{l} S_j^h \right\}.$$

Now recall that r and p are the shadow prices corresponding with (4.3) and (4.4), respectively, in the problem of maximizing total output given the available inputs. Therefore $r = \partial F_i/\partial K_i^y$ if $F_i > 0$. Moreover, there exists i with $K_i^y > 0$ and $r = \partial F/\partial K^y$, where F is defined in section 3. So, we have from (4.2) that

$$\varphi(t)/\varphi(0) = \exp\left(\int_0^t (\mu - r(\tau)) \,\mathrm{d}\tau \right) = \pi(t).$$

From the definition of r and p it is clear that (K_i^y, R_i) maximizes profits in nonresource sector i. It is also clear that E_j maximizes total discounted profits in resource sector j. That C_h maximizes utility subject to the budget constraint easily follows from the concavity of U_h, $h = 1, 2, \ldots, l$. ■

6. Discussion and extensions

In summary, the analysis has been conducted along the following line. Continuity, concavity/convexity, and monotonicity of the functions describing consumers' tastes and the economy's technology together with an (artificially) imposed upper bound on the rates of consumption were sufficient to guarantee the existence of a restricted Pareto-efficient allocation in the infinite horizon economy. Since these assumptions are commonly made in the growth literature including exhaustible resources and are plausible indeed, no further discussion is required.

One might ask, however, how important it is to assume a positive rate of depreciation. It is well-known that with a positive rate of depreciation, no extraction costs, and Cobb–Douglas specification of the aggregate composite commodity production function, consumption will necessarily approach zero eventually [see Stiglitz (1974)]. And, indeed, also in the model at hand a positive rate of depreciation makes it easy to find upper bounds on the stock of capital and the rates of extraction. However, it may be shown that this is not essential for the existence of a restricted Pareto-efficient allocation, because the capital stock will be exponentially bounded anyhow. It is our strong conjecture that a redefinition of all variables involved would then imply that the analysis needs no substantial alteration.

One might also wonder how the analysis would change when the remaining resource stocks enter into the extraction technologies, so as to cope with the widely accepted view that marginal extraction costs are larger the smaller the stocks are. Obviously this would not cause any problem in the proof of the existence of Pareto-efficient allocations, because it would enhance boundedness. One is tempted to argue that consequently the existence of a general equilibrium poses no problem. However, a formal analysis of this issue should be subject to further research.

The second step has been the characterization of the set of Pareto-efficient allocations. Theorems on the necessary conditions for optimal control problems with an infinite horizon generally depart from the existence of piece-wise continuous controls, whereas existence theorems 'only' provide us with measurable controls. Therefore we have resorted to necessary conditions for a finite horizon economy with the final values of the state variables equal to the corresponding values in the infinite time Pareto problem. In order to state the necessary conditions we had to make some differentiability assumptions. However, in our opinion it would be too restrictive to assume differentiability over the entire domain of the functions involved. For that reason we have limited ourselves to the assumption of differentiability on the interior of the domain. The purpose of assuming unbounded marginal utility at zero was to prevent the stock of capital from becoming zero in finite time. However, with bounded marginal utilities only a slight modification of the analysis is needed to reach the same goal in terms of the characterization of the set of Pareto-efficient allocations, i.e., piece-wise continuous controls. One simply observes that in that case the stock of capital might become zero within finite time, implying that the economy 'ends' at the moment where capital becomes zero. But then we have just an ordinary optimal control problem with a finite horizon, with capital strictly positive before doomsday. If the stock of capital is not becoming zero within finite time, the analysis naturally remains unchanged. Therefore the assumption of unbounded marginal utility is only made for expository purposes.

The third step was to define the pseudo-budget constraints of the consumers, i.e., the value of excess supply of each consumer. Without any additional assumptions we have shown a.o. the continuity of this function in the unit simplex from which the weights in the Pareto problem were taken. This allowed for the construction of a mapping having a fixed point, which then gave us the general equilibrium.

Which conclusions can be drawn? First of all, under mild assumptions we have established the existence of a general equilibrium for a rather broad class of models with exhaustible resources, including models known from the literature. Perhaps more importantly, we have presented a rigorous application of the original idea due to Negishi of searching for a general equilibrium in the set of Pareto-efficient allocations to an infinite horizon, continuous time economy. This might open perspectives also for a fruitful equilibrium analysis of other types of models, for example taking into account environmental aspects together with exhaustible resources. So it is likely that the analysis can be extended to a set of presently actual models.

Finally, one could argue that it is intuitively clear that in the economy under consideration a general equilibrium exists, so that it is not worthwhile to put so much effort into the analysis. Apart from the fact that such a statement is at variance with accepted methodology, the line of attack has some merits on its own. Moreover, there are numerous examples of economies which have an equilibrium in finite time but not in infinite time [see, e.g., Zame (1987)]. Therefore, one has to be very careful with intuitive reasoning here and a formal approach, however tedious, is required.

Appendix A

Define $x := (S_1, S_2, \ldots, S_n, K)$, $u := (C_1, \ldots, C_l, K_1^y, \ldots, K_m^y, R_1, \ldots, R_m, E_1, \ldots, E_n)$. Fix some $\alpha \in \Delta$. It has already been shown that there exist differentiable x^α and piece-wise continuous u^α solving the following problem:

$$\max \int_0^\infty \sum_{h=1}^{l} \alpha_h \mathrm{e}^{-\rho_h t} U_h(C_h)\,\mathrm{d}t\,,$$

subject to

$$\dot{S}_j = -E_j, \quad j = 1, 2, \ldots, n, \qquad S_j(0) = \sum_h S_j^h\,,$$

$$\dot{K} = \sum_{i=1}^{m} F_i(K_i^y, R_i) - \mu K - \sum_{h=1}^{l} C_h, \qquad K(0) = \sum_h K^h\,,$$

$$\sum_{i=1}^{m} R_i \le \sum_{j=1}^{n} E_j\,, \qquad \sum_{i=1}^{m} K_i^y + \sum_{j=1}^{n} G_j(E_j) \le K\,.$$

Define $\lambda := (\lambda_1, \ldots, \lambda_n)$ and the Lagrangean:

$$\begin{aligned} &L(x, u, \lambda, \varphi, \tilde{p}, \tilde{r}, t) \\ &:= \sum_{h=1}^{l} \alpha_h e^{-\rho_h t} U_h(C_h) + \sum_{j=1}^{n} \lambda_j(-E_j) \\ &\quad + \varphi\left(\sum_{i=1}^{m} F_i(K_i^y, R_i) - \mu K - \sum_{h=1}^{l} C_h \right) \\ &\quad + \tilde{p}\left(\sum_{j=1}^{n} E_j - \sum_{i=1}^{m} R_i \right) + \tilde{r}\left(K - \sum_{i=1}^{m} K_i^y - \sum_{j=1}^{n} G_j(E_j) \right). \end{aligned}$$

Now the following holds (the suffix α is omitted when there is no danger of confusion). There exist λ which is constant a.e., φ which is absolutely continuous, and $(\tilde{p}, \tilde{r})$ which are continuous except possibly where u is discontinuous, such that

$$-\dot{\varphi} = -\varphi\mu + \tilde{r}\,, \tag{A.1}$$

$$L \text{ is maximized a.e. with respect to } u\,, \tag{A.2}$$

$$\lim_{t\to\infty} \varphi(t)K(t) + \sum_{j=1}^{n} \lambda_j S_j(t) = 0\,. \tag{A.3}$$

(A.1) and (A.2) are straightforward applications of the Pontryagin maximum principle; (A.3) is proven in appendix B. It has been shown in the main text that $\varphi(t) > 0$. We define $\tilde{p} = p\varphi$, $\tilde{r} = r\varphi$.

Now let $(\bar{x}, \bar{u})$ solve the problem when the weights are given by $\bar{\alpha}$. Let $(\bar{\lambda}, \bar{\varphi}, \bar{p}, \bar{r})$ be the corresponding co-state variables and (modified) multipliers. It follows from the maximization of the Lagrangeans that, for all h, all i, all j, and almost all t,

$$(\alpha_h - \bar{\alpha}_h)e^{-\rho_h t}(U_h(C_h) - U_h(\bar{C}_h)) - (\varphi - \bar{\varphi})(C_h - \bar{C}_h) \ge 0\,, \tag{A.4}$$

$$(\bar{\varphi}\bar{r} - \varphi r)(K_i^y - \bar{K}_i^y) + (\bar{\varphi}\bar{p} - \varphi p)(R_i - \bar{R}_i) \ge 0\,, \tag{A.5}$$

$$\begin{aligned} &(\varphi p - \bar{\varphi}\bar{p})(E_j - \bar{E}_j) - (\varphi r - \bar{\varphi}\bar{r})(G_j(E_j) - G_j(\bar{E}_j)) \\ &\quad - (\lambda_j - \bar{\lambda}_j)(E_j - \bar{E}_j) \ge 0\,. \end{aligned} \tag{A.6}$$

By virtue of the necessary conditions and the properties (i.e., concavity/convexity) of the functions involved, we also have

$$V^{\alpha}_{\bar{\alpha}} := \int_0^\infty [\sum \alpha_h e^{-\rho_h t} U_h(C_h) - \alpha_h e^{-\rho_h t} U_h(\bar{C}_h)]\,dt$$

$$\geq \int_0^\infty \sum \alpha_h e^{-\rho_h t} U_h'(C_h)(C_h - \bar{C}_h)\,dt$$

$$\geq \int_0^\infty \varphi \sum (C_h - \bar{C}_h)\,dt$$

$$= \int_0^\infty \varphi [\sum F_i(K_i^y, R_i) - \mu K - \dot{K} - \sum F_i(\bar{K}_i^y, \bar{R}_i) + \mu \bar{K} + \dot{\bar{K}}]\,dt$$

$$\geq \int_0^\infty \varphi [r \sum (K_i^y - \bar{K}_i^y) + p \sum (R_i - \bar{R}_i) + \mu(\bar{K} - K) + (\dot{\bar{K}} - \dot{K})]\,dt$$

$$= \int_0^\infty \varphi [r \sum (K_i^y - \bar{K}_i^y) + p \sum (R_i - \bar{R}_i) + r(\bar{K} - K)]\,dt$$

$$\geq \int_0^\infty \varphi [p \sum (R_i - \bar{R}_i) + r \sum (K_j^e - \bar{K}_j^e)]\,dt$$

$$\geq \int_0^\infty \varphi [p \sum (E_j - \bar{E}_j) - r \sum G_j'(E_j)(E_j - \bar{E}_j)]\,dt$$

$$= \int_0^\infty \sum \lambda_j (E_j - \bar{E}_j)\,dt$$

$$= \sum \lambda_j (\bar{S}_j(\infty) - S_j(\infty)) = 0\,. \qquad \text{(A.7)}$$

The same inequalities can be written for $V^{\bar{\alpha}}_{\alpha}$.

A.1. C_h and φ

It follows from (A.7) that

$$\int_0^\infty (\varphi - \bar{\varphi}) \sum (C_h - \bar{C}_h)\,dt \geq 0\,.$$

Combining this with (A.4) yields

$$0 = \lim_{\alpha \to \bar{\alpha}} \int_0^\infty \sum (\alpha_h - \bar{\alpha}_h) e^{-\rho_h t} (U_h(C_h) - U_h(\bar{C}_h))\,dt$$

$$= \lim_{\alpha \to \bar{\alpha}} \int_0^\infty (\varphi - \bar{\varphi}) \sum (C_h - \bar{C}_h)\,dt\,.$$

Take some $\varepsilon > 0$ and define

$$T_\varepsilon := \{t \in [0, \infty) \mid \limsup_{\alpha \to \bar{\alpha}} |\sum (C_h - \bar{C}_h)| > \varepsilon\}\,.$$

If the measure of T_ε is larger than zero, then we obtain a contradiction because $(\varphi - \bar{\varphi})(C_h - \bar{C}_h) \geq 0$. Therefore, for almost all t and all h,

$$\lim_{\alpha \to \bar{\alpha}} C_h(t) = \bar{C}_h(t), \qquad \lim_{\alpha \to \bar{\alpha}} \varphi(t) = \varphi(t)\,. \tag{A.8}$$

A.2. K_i^y, R_i, p, and r

Sum (A.5) over i, sum (A.6) over j, add the resulting inequalities, and integrate to obtain

$$\int_0^\infty (\bar{\varphi}\bar{r} - \varphi r)(K - \bar{K})\,dt + \int_0^\infty \sum (\lambda_j - \bar{\lambda}_j)(E_j - \bar{E}_j)\,dt$$

$$= \int_0^\infty (\bar{\varphi}\bar{r} - \varphi r)(K - \bar{K})\,dt \geq 0\,, \tag{A.9}$$

in view of (A.7). By virtue of (A.8) we have

$$\lim_{\alpha \to \bar{\alpha}} V_{\bar{\alpha}}^{\alpha} = \lim_{\alpha \to \bar{\alpha}} V_{\alpha}^{\bar{\alpha}} = 0\,.$$

This, together with (A.5) and (A.9), implies

$$\lim_{\alpha \to \bar{\alpha}} \int_0^\infty [(\varphi r - \bar{\varphi}\bar{r}) \sum (K_i^y - \bar{K}_i^y) + (\varphi p - \bar{\varphi}\bar{p}) \sum (R_i - \bar{R}_i)]\,dt = 0\,. \tag{A.10}$$

Hence, by (A.5),

$$\lim_{\alpha \to \bar{\alpha}} (\bar{\varphi}\bar{r} - \varphi r) \sum (K_i^y - \bar{K}_i^y) + (\bar{\varphi}\bar{p} - \varphi p) \sum (R_i - \bar{R}_i) = 0 \quad \text{a.e.} \tag{A.11}$$

Define, for some $\varepsilon > 0$,

$$T_\varepsilon^1 := \left\{ t\in[0,\infty) \,\middle|\, \limsup_{\alpha\to\bar\alpha} \varphi r \geq \bar\varphi\bar r + \varepsilon,\ \limsup_{\alpha\to\bar\alpha} \varphi p \geq \bar\varphi\bar p + \varepsilon \right\},$$

$$T_\varepsilon^2 := \left\{ t\in[0,\infty) \,\middle|\, \liminf_{\alpha\to\bar\alpha} \varphi r \leq \bar\varphi\bar r - \varepsilon,\ \liminf_{\alpha\to\bar\alpha} \varphi p \leq \bar\varphi\bar p - \varepsilon \right\},$$

$$T_\varepsilon^3 := \left\{ t\in[0,\infty) \,\middle|\, \limsup_{\alpha\to\bar\alpha} \varphi r \geq \bar\varphi\bar r + \varepsilon,\ \liminf_{\alpha\to\bar\alpha} \varphi p \leq \bar\varphi\bar p - \varepsilon \right\},$$

$$T_\varepsilon^4 := \left\{ t\in[0,\infty) \,\middle|\, \liminf_{\alpha\to\bar\alpha} \varphi r \leq \bar\varphi\bar r - \varepsilon,\ \limsup_{\alpha\to\bar\alpha} \varphi p \geq \bar\varphi\bar p + \varepsilon \right\}.$$

The strict concavity of the F_i's implies that higher (lower) input prices call for smaller (larger) inputs. Therefore the measures of T_ε^1 and T_ε^2 are zero.

In view of (A.11) there cannot be a subset of T_ε^3 with nonzero measure where $\sum \bar K_i^y > \sum K_i^y$ and $\sum R_i > \sum \bar R_i$. Nor can there be a subset of T_ε^3 with nonzero measure for which $\sum K_i^y > \sum \bar K_i^y$ and $\sum \bar R_i > \sum R_i$. Now assume that there exists a subset $\tilde T_\varepsilon^3 \subset T_\varepsilon^3$ with nonzero measure such that $\sum \bar K_i^y > \sum K_i^y$ and $\sum \bar R_i > \sum R_i$ for all $t\in\tilde T_\varepsilon^3$. Then $\sum \bar E_j > \sum E_j$ for almost all $t\in\tilde T_\varepsilon^3$, and there exists $j^*\in\{1, 2, \ldots, n\}$ such that

$$\int_{\tilde T_\varepsilon^3} \bar E_{j*}\,\mathrm{d}t > \int_{\tilde T_\varepsilon^3} E_{j*}\,\mathrm{d}t\,.$$

Then it must be the case that $\bar\lambda_{j*} < \lambda_{j*}$. And, in fact,

$$\int_{T_\varepsilon^3} \bar E_{j*}\,\mathrm{d}t > \int_{T_\varepsilon^3} E_{j*}\,\mathrm{d}t\,.$$

Since $\lambda_{j*} > \bar\lambda_{j*}$, we must have

$$\int_{T_\varepsilon^4} \bar E_{j*}\,\mathrm{d}t < \int_{T_\varepsilon^4} E_{j*}\,\mathrm{d}t\,.$$

But this is ruled out by (A.6). Therefore the measure of $\tilde T_\varepsilon^3$ equals zero. Along the same lines it can be shown that the subset of T_ε^3, for which $\sum \bar K_i^y > \sum K_i^y$ and $\sum \bar R_i < \sum R_i$, has zero measure as well. In view of the strict concavity of the F_i's there exists no subset of T_ε^3 with nonzero measure such that $\sum \bar K_i^y = \sum K_i^y$ or $\sum \bar R_i = \sum R_i$. Therefore the measure of T_ε^3 equals zero.

The same argument applies to show that the measure of T_ε^4 equals zero. This proves that

$$\lim_{\alpha\to\bar{\alpha}} \varphi(t)r(t) = \bar{\varphi}(t)\bar{r}(t), \quad \lim_{\alpha\to\bar{\alpha}} \bar{\varphi}(t)r(t) = \bar{\varphi}(t)\bar{r}(t) \quad \text{a.e.}$$

It then follows immediately from the maximization of $F_i - rK_i^y - pR_i$ that

$$\lim_{\alpha\to\bar{\alpha}} R_i(t) = \bar{R}_i(t) \quad \text{a.e.,} \qquad \lim_{\alpha\to\bar{\alpha}} K_i^y(t) = \bar{K}_i^y(t) \quad \text{a.e.}$$

A.3. K_j^e, E_j, and λ_j

Since $K_j^e = G_j(E_j)$, we confine ourselves to E_j and λ_j. It follows from (A.6) and the fact that φr and φp are continuous a.e. that

$$\limsup_{\alpha\to\bar{\alpha}} (\bar{\lambda}_i - \lambda_i)(E_j - \bar{E}_j) \geq 0 \quad \text{a.e.}$$

On the other hand, we have from (A.7)

$$\int_0^\infty (\bar{\lambda}_i - \lambda_i)(E_j - \bar{E}_j)\,\mathrm{d}t = 0\,.$$

If $\lim_{\alpha\to\bar{\alpha}} \lambda_j > \bar{\lambda}_j$, then $\lim_{\alpha\to\bar{\alpha}} E_j > \bar{E}_j$, which yields a contradiction. The other way around the proof is similar. Hence,

$$\lim_{\alpha\to\bar{\alpha}} \lambda_j = \bar{\lambda}_j, \quad \lim_{\alpha\to\bar{\alpha}} E_j(t) = \bar{E}_j(t) \quad \text{a.e.}$$

A.4. K

$$\lim_{\alpha\to\bar{\alpha}} K(t) = \bar{K}(t)\,,$$

because $K(t) = \sum K_i^y(t) + \sum K_j^e(t)$.

Taking A.1–A.4 together and using the Lebesgue dominated convergence theorem, we have shown the continuity of $J^h(\alpha)$ in α for all h. ■

Appendix B

It will be shown first that $\varphi(t)K(t) \to 0$ as $t \to \infty$.

Take some $\alpha \in \Delta$ fixed. Assume, without loss of generality, that $\alpha_1 > 0$. Define, for $t \in [0, \infty)$,

$$g(t) := \sum_{h=2}^{l} C_h(t)\,.$$

Define, for $K \geq 0$ and $t \in [0, \infty)$,

$$H(K,t) := F\left(K - \sum_{j=1}^{n} G_j(E_j(t)), \sum E_j(t) \right) - \mu K \quad \text{if} \quad K \geq \sum_{j=1}^{n} G_j(E_j(t))$$

$$:= -\mu K \quad \text{otherwise.}$$

Define

$$T := \{(K, \dot{K}) \mid K \in \mathbb{R}_+, \dot{K} \in \mathbb{R}\}.$$

Define, for $(K, \dot{K}) \in T$ and $t \in [0, \infty)$,

$$v(K, \dot{K}, t) := e^{-\rho_1 t} U_1(H(K,t) - g(t) - \dot{K}) \quad \text{if} \quad H(K,t) - g(t) - \dot{K} \geq 0$$

$$:= -\infty \text{ otherwise.}$$

Consider the following problem:

$$\max \int_0^\infty v(K, \dot{K}, t)\,dt,$$

subject to

$$(K, \dot{K}) \in T, K(0) = K_0.$$

Economically the problem is to maximize agent 1's welfare given the optimal consumption profiles of the other agents and given the optimal extraction rates, where the expression 'optimal' refers to the Pareto-efficient allocation corresponding with the fixed α we started with. This problem has a solution, namely $(K(t), \dot{K}(t))$, which of course coincides with the overall Pareto problem. It is now easily checked that the conditions of Theorem 3.A of Benveniste and Scheinkman (1982) are satisfied, so that

$$\lim_{t \to \infty} \varphi(t) K(t) = 0$$

is a necessary condition.

To show that $\lim_{t\to\infty} \lambda_j S_j(t) = 0$ for all j, the same type of argument can be used. The optimal control problem to be considered is then

$$\max \int_0^\infty \varphi(pE_j - rG_j(E_j))\,dt,$$

with $\dot{S}_j = -E_j$, and (φ, p, r) are the optimal values arising from the necessary conditions for the problem stated in appendix A.

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