Strategic Knowledge Sharing in Bayesian Games: Applications *

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Abstract

This paper studies the properties of endogenous information structures in some classes of Bayesian games in which a first stage of strategic information revelation is added. Sufficient conditions for the existence of perfectly revealing or non-revealing equilibria are characterized. In particular, the existence of a perfectly revealing equilibrium is demonstrated for linear Bayesian games with an ordered information structure. Those games include, for example, Cournot games with incomplete information about the cost or the demand of industry, when firms may face any level of higher-order uncertainty. Several examples and different economic applications are examined to illustrate other results presented in the paper.

KEYWORDS: Strategic information revelation; Bayesian games; Endogenous information structure; Certifiability.

JEL CLASSIFICATION: C72; D82.

1 Introduction

In most interactive decision situations, agents may have strong incentives to modify the information structure by sharing some of their knowledge. That is, when distributed knowledge is not common knowledge, predicted outcomes should be biased by communication possibilities. In this paper we use the model of strategic knowledge sharing presented in Koessler (2002) to investigate different classes of Bayesian games where it is possible to characterize endogenous information structures generated by voluntary and direct communication. More precisely, we elaborate sufficient conditions for the initial game to become common knowledge or, on the

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contrary, for the information structure of the initial Bayesian game to remain unchanged, even if a first stage game of information revelation is added. Several examples and economic applications are also examined.

In general terms, the idea is that a set of players is involved in a game with uncertain payoffs, where each player has some piece of knowledge. A player might perfectly know the game, another might completely ignore the real state of Nature, another might know who has some piece of knowledge that can complement his own knowledge, another might not know if others have some relevant information, etc. For example, say that two players, 1 and 2, have to choose between several risky actions, their payoffs depending both on the real state of Nature and on the actions taken by both of them. Assume that there are four states of the world ω_1, ω_2, ω_3 and ω_4. The (initial) configuration of knowledge is given by an information structure represented by two partitions of the set of the states of the world, $H_1 = \{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}\}$ and $H_2 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$. These partitions (which are informally assumed common knowledge) describe, for each player, which states of the world they can distinguish and which they cannot. Here, player 1 cannot distinguish ω_2 from ω_3; player 2 cannot distinguish ω_1 from ω_2 and ω_3 from ω_4. First, assume that the same game is played at ω_1 and ω_3 and another game is played at ω_2 and ω_4. In this case, player 1 knows the game at ω_1 and ω_4 and he does not know it at ω_2 and ω_3. Furthermore, player 2 never knows the game and he never knows whether player 1 knows it. Alternatively, suppose that the same game is played at ω_2, ω_3 and ω_4 and another game is played at ω_1. At ω_1, player 1 knows the game and player 2 does not; at ω_3, they both know the game, player 2 knows that player 1 knows it but player 1 does not know that player 2 knows it; finally, at ω_4, they both know the game, they both know that they both know it, but player 2 does not know that player 1 knows that player 2 knows it.

In these two different situations, how can strategic communication take place and which behavior can we expect? More precisely, are agents in the first situation sometimes incited to reveal their knowledge about others' knowledge? In the second situation, is there any incentive for an agent to reveal his knowledge about the fundamentals if everybody knows the fundamentals and everybody knows that everybody knows them, as it is the case at ω_4? Answering these questions is one of the objectives of this paper.

Our study is conducted using the knowledge equilibrium developed in Koessler (2002). In short, knowledge equilibria of a given Bayesian game are particular sequential equilibria of a two-stage game in which the Bayesian game is preceded by a communication stage. In a knowledge equilibrium, i) each agent's payoff-relevant (second stage) strategy is optimal given others' payoff-relevant strategies and given the information structure generated by the first stage of information revelation; ii) no agent has an incentive to modify his communication strategy in the first stage game given the second stage equilibria and others' communication behaviors and inferences; iii) the information structure of the second stage game, expressed in terms of possibility correspondences, is consistent with the initial information structure, communicated messages, and Bayesian updating. The particularity of knowledge equilibria comparing to sequential equilibria concerns this consistency condition on second stage beliefs which is stronger than Kreps and Wilson's (1982) consistency condition.

Since uniqueness of equilibrium is rare in incomplete information games, we will mainly focus on the existence of perfectly revealing and non-revealing equilibria. The first types of equilibria are interesting because they indicate that if agents behave strategically with respect to their knowledge, information incompleteness may disappear and hence, the results and phenomena obtained with incomplete information may also disappear. The second are interesting because they indicate that information asymmetries still remain when agents are able to voluntarily share their knowledge. Hence, the existence of non-revealing equilibria indicates a strong robustness of the associated Bayesian equilibrium outcomes.

In Section 2 we briefly define the knowledge equilibrium and some definitions and properties characterizing it. A more detailed account of the model used here may be found in Koessler (2002). However, this preliminary section reports several definitions and propositions not explored in the previous paper.

In the following sections we consider different classes of games in which sufficient conditions for particular types of knowledge equilibria are characterized. More precisely, we study some circumstances in which information is perfectly revealed or, on the contrary, in which the information structure is not modified. Various examples and economic applications satisfying these conditions are examined throughout the exposition.

In Section 3 we consider information structures where only one player is informed about the state of the world. Such games are called *one side information games.* Knowledge equilibria are illustrated by several simple examples. Then, sufficient and easily verifiable conditions for the existence of perfectly revealing and non-revealing equilibria are characterized, and are checked in our examples. Our conditions are shown to apply directly to commonly analyzed persuasion games.

In Section 4 we consider *common interest games*, i.e., games in which players have a commonly preferred outcome at each state of the world. We show that only some conditions on the richness of the messages space are necessary to obtain a perfectly revealing equilibrium. Under these conditions, an efficient and perfectly revealing equilibrium exists for any information structure, and whatever the number of players. However, in settings with partial communication possibilities, it is shown that some information which can be transmitted is *never* strategically revealed, even in the subclass of common interest games which are perfectly symmetric.

Sufficient and general conditions for the existence of a perfectly revealing equilibrium are investigated in Section 5 when the information structure satisfies some ordering properties. Our conditions for complete knowledge sharing do not rely on the complexity of the information structure and on prior probabilities. In particular, Bayesian-Nash equilibria need *not* be computed. A class of Cournot games with incomplete information about the intercept of demand or about the cost in the industry, with possibly correlated information and high levels of uncertainty about others' information, satisfies our conditions. Section 6 concludes the paper. All proofs can be found in the Appendix.[1]

2 Model

In this section we briefly define an equilibrium concept close to the sequential equilibrium, called *knowledge equilibrium*, for Bayesian games in which we add a first stage of strategic information revelation. Several useful definitions and properties of such an equilibrium are also elaborated.

2.1 Preliminaries

We consider an *initial Bayesian game* $G \equiv \langle N, \Omega, p, h, A, (u_i)_{i\in N}\rangle$ where:

- $N = \{1, \ldots, n\}$ is a finite set of players;
- Ω is a finite set of states of the world;
- p is a full-support probability distribution on Ω;

[1] It is worth noticing that the assumptions made throughout the paper to obtain particular knowledge equilibria can be generalized, especially those concerning the set of available messages. However, our aim is rather to present tractable and easily verifiable conditions, which simplifies greatly the exposition.

- $h = (h_i)_{i \in N}$ is the *initial information structure* on Ω, where $h_i : \Omega \to 2^\Omega \backslash \{\emptyset\}$ is player i's *initial information function*, generating a partition $H_i = \{h_i(\omega) : \omega \in \Omega\}$ of Ω, where $\omega \in h_i(\omega)$ for all $\omega \in \Omega$. The *Join* (coarsest common refinement) of agents' partitions is denoted by $\mathbb{J} = \bigvee_{i \in N} H_i$, where $J(\omega) = \bigcap_{k \in N} h_k(\omega)$ is the element of $\mathbb{J}$ containing ω. Let $J_N = (J)_{i \in N}$. The *Meet* (finest common coarsening) of agents' partitions is denoted by $\mathbb{M} = \bigwedge_{i \in N} H_i$, and $M(\omega)$ is the element of $\mathbb{M}$ containing ω.

- A_i is player i's set of *effective* or *payoff-relevant actions*, and $A = \prod_{i \in N} A_i$;

- $u_i : A \times \Omega \to \mathbb{R}$ is player i's von Neumann Morgenstern utility function. The *payoff-relevant partition* $\mathbb{P}$ is the partition of Ω generated by the vector of utility functions $u = (u_i)_{i \in N}$. Write $P(\omega)$ for the element of $\mathbb{P}$ containing ω.

We will always assume that the payoff-relevant partition $\mathbb{P}$ is coarser than $\mathbb{J}$, i.e., $J(\omega) \subseteq P(\omega)$ for all $\omega \in \Omega$. This means that if all agents perfectly share their information, then the game which is played at every state becomes common knowledge.

For any information structure $h' = (h'_i)_{i \in N}$ on Ω, we denote by

$$G(h') \equiv \langle N, \Omega, p, h', A, (u_i)_{i \in N} \rangle$$

the Bayesian game which is the same as G except that the information structure is h' instead of h. With some abuse of notations we also denote by $G(\omega)$ the strategic form game (with complete information) associated with G at ω. That is, $G(\omega) = \langle N, A, (u_i^\omega)_{i \in N} \rangle$, where for all $i \in N$ and $\omega \in \Omega$, $u_i^\omega : A \to \mathbb{R}$ is the function satisfying $u_i^\omega(a) = u_i(a, \omega)$ for all $a \in A$. We denote by $\Delta(A_i)$ the set of probability distributions over A_i. A mixed strategy of player i in the Bayesian game G is given by a H_i measurable function $\phi_i : \Omega \to \Delta(A_i)$. A profile of mixed strategies of G is given by $\phi = (\phi_i)_{i \in N} \in \Phi = \prod_{i \in N} \Phi_i$, where Φ_i is player i's set of mixed strategies in G. We shall sometimes write $\Phi_i(h)$ and $\Phi(h)$ to specify the information structure we consider. A pure strategy of player i in G is denoted by $\varphi_i : \Omega \to A_i$. With some abuse of notations, utility functions u_i are extended to mixed strategies of G by $u_i(\phi, \omega) = \sum_{a \in A} \phi(a \mid \omega) u_i(a, \omega)$. Then, player i's expected utility when he is at his information set $h_i(\omega)$ and the strategy $\phi \in \Phi$ is used is given by

$$\begin{aligned} U_i(\phi \mid h_i(\omega)) \equiv E_p\big(u_i(\phi, \cdot) \mid h_i(\omega)\big) &= \sum_{\omega' \in \Omega} p\left(\omega' \mid h_i(\omega)\right) u_i(\phi, \omega') \\ &= \sum_{\omega' \in \Omega} p\left(\omega' \mid h_i(\omega)\right) \sum_{a \in A} \phi(a \mid \omega')\, u_i(a, \omega'). \end{aligned} \tag{1}$$

As usual, $\phi \in \Phi(h)$ is a *Bayesian–Nash equilibrium* of $G(h)$ if for all $i \in N$ and $\omega \in \Omega$ we have

$$U_i(\phi \mid h_i(\omega)) \geq U_i(a_i, \phi_{-i} \mid h_i(\omega)), \quad \forall\, a_i \in A_i. \tag{2}$$

We denote the set of equilibria of $G(h)$ by $\Phi^*(h) \subseteq \Phi(h)$. Before the initial Bayesian game G is played, players are able to publicly and simultaneously send a message containing some of their information. Formally, for all $i \in N$, let $\mathcal{Y}_i$ be the σ-algebra generated by H_i, minus the empty set, and let $\mathcal{X}_i \subseteq \mathcal{Y}_i$ be a set of events such that $\mathcal{X}_i \cup \{\emptyset\}$ is closed under intersection and $\Omega \in \mathcal{X}_i$. The condition $\Omega \in \mathcal{X}_i$ means that players are always allowed to underreport how much information they have, i.e., they always have the option to remain silent. Write $Y_i(\omega) = \{y_i \in \mathcal{Y}_i : \omega \in y_i\}$, $X_i(\omega) = \{x_i \in \mathcal{X}_i : \omega \in x_i\}$, $\mathcal{Y} = \prod_{i \in N} \mathcal{Y}_i$, $\mathcal{X} = \prod_{i \in N} \mathcal{X}_i$, $Y = (Y_i)_{i \in N}$, and $X = (X_i)_{i \in N}$. The function $X : \Omega \to (2^{\mathcal{X}_i})_{i \in N}$ is called the *certifiability level*. Accordingly, a *communication strategy* is a H_i measurable function $c_i : \Omega \to \mathcal{X}_i$ such that $c_i(\omega) \in X_i(\omega)$ for all $\omega \in \Omega$. Let $C = \prod_{i \in N} C_i$ be the set of communication strategy profiles.

After the communication stage, each player i has a mixed *effective strategy* $\sigma_i : \mathcal{X} \times \Omega \to \Delta(A_i)$ such that $\sigma_i(x, \omega) = \sigma_i(x, \omega')$ for all $x \in \mathcal{X}$, $\omega \in \Omega$ and $\omega' \in h_i(\omega)$. A profile of mixed effective strategies is given by $\sigma = (\sigma_i)_{i \in N} \in \Sigma = \prod_{i \in N} \Sigma_i$, where Σ_i is player i's set of mixed effective strategies. Similarly, a pure effective strategy of player i is a function $s_i : \mathcal{X} \times \Omega \to A_i$ such that $s_i(x, \omega) = s_i(x, \omega')$ for all $x \in \mathcal{X}$, $\omega \in \Omega$ and $\omega' \in h_i(\omega)$. A profile of pure effective strategies is denoted by $s = (s_i)_{i \in N} \in S = \prod_{i \in N} S_i$.

Players' second stage information is characterized by *possibility correspondences*. Player i's possibility correspondence is a function $\mathcal{P}_i : \mathcal{X} \times \Omega \to 2^{\Omega} \backslash \{\emptyset\}$. The second stage information structure is denoted by $\mathcal{P} = (\mathcal{P}_i)_{i \in N}$. According to this information structure, player i's expected utility at $\omega \in \Omega$ in the second stage game, when the vector of messages $x \in X(\omega)$ has been sent during the communication stage, is given by

$$U_i(\sigma, x, \mathcal{P}_i, \omega) \equiv \sum_{\omega' \in \Omega} p(\omega' \mid \mathcal{P}_i(x, \omega)) \sum_{a \in A} \sigma(a \mid x, \omega')\, u_i(a, \omega').$$

In pure strategies we have

$$U_i(s, x, \mathcal{P}_i, \omega) \equiv \sum_{\omega' \in \Omega} p(\omega' \mid \mathcal{P}_i(x, \omega))\, u_i(s(x, \omega'), \omega').$$

Player i's expected utility at $\omega \in \Omega$ at the beginning of the first stage game is given by

$$EU_i(\sigma, c, \mathcal{P}_i \mid h_i(\omega)) \equiv \sum_{\omega' \in \Omega} p(\omega' \mid h_i(\omega))\, U_i(\sigma, c(\omega'), \mathcal{P}_i, \omega').$$

Given a complete, reflexive, and transitive ordering $\succeq_j$ over a partition H_j of Ω, define for all $E \subseteq \Omega$,

$$\begin{aligned} \text{Maxi}\{E \mid H_j, \succeq_j\} &\equiv \{\omega \in E : h_j(\omega) \succeq_j h_j(\omega') \text{ for all } \omega' \in E\} \\ &\equiv E \cap \{h_j \in H_j : h_j \succeq_j h'_j \text{ for all } h'_j \in H_j,\ h'_j \cap E \neq \emptyset\}. \end{aligned}$$

This application will be used to characterize players' interpretations of outside equilibrium messages. We denote by $\mathcal{I}_j$ the partition of Ω generated by the equivalence relation $\sim_j$ associated with the ordering $\succeq_j$ over the partition H_j.

2.2 Knowledge Equilibrium

For all $x \in \mathcal{X}$, let $\Sigma^*(\mathcal{P}, x) \subseteq \Sigma$ be the set of effective strategy profiles σ such that for all $i \in N$ and $\omega \in \bigcap_{k \in N} x_k$ we have,

$$U_i(\sigma, x, \mathcal{P}_i, \omega) \geq U_i(a_i, \sigma_{-i}, x, \mathcal{P}_i, \omega), \quad \forall\, a_i \in A_i.$$

Let $\Sigma^*(\mathcal{P}) = \bigcap_{x \in \mathcal{X}} \Sigma^*(\mathcal{P}, x)$ be the set of equilibrium effective strategies when the second stage information structure is $\mathcal{P}$, and let $S^*(\mathcal{P}, x)$ and $S^*(\mathcal{P}) = \bigcap_{x \in \mathcal{X}} S^*(\mathcal{P}, x)$ be the associated pure effective strategies.

For any communication strategy profile $c \in C$ and any state $\omega \in \Omega$, we say that a vector of certifiable events $x \in X(\omega)$ at ω is an *observable deviation* by player i at ω if $h_i(\omega) \cap c^{-1}(x) = \emptyset$. When a deviation is observable, let $N_i(c, x, \omega) \equiv \{j \in N : h_i(\omega) \cap c_{-j}^{-1}(x_{-j}) \cap x_j \neq \emptyset\}$ be the set of *potential deviants* for player i at ω. Let $X(c, \omega)$ be the set of unilateral deviations from c at ω, i.e., $X(c, \omega) = \{x \in X(\omega) : \exists\ i \in N,\ x = (x_i, c_{-i}(\omega))\}$. By assuming that x is a unilateral deviation from c at ω, we necessarily have $N_i(c,x,\omega) \neq \emptyset$. If $N_i(c,x,\omega) = \{j\}$ for some $j \in N$, then the deviation x is said *j-identifiable* by player i at ω.

For any bijection $\rho : N \to N$, let

$$\overline{N}_i(c,x,\omega \mid \rho) \in \arg\max_{k \in N_i(c,x,\omega)} \rho(k).$$

The rationale for the following consistency condition and equilibrium definition is explained and illustrated in Koessler (2002) and is therefore not discussed again in this paper.

Definition 1 A second stage information structure $\mathcal{P}$ is *consistent* with (c,X) if there exists a system of complete, reflexive, and transitive orderings $(H_k, \succeq_k)$ and a bijection $\rho : N \to N$ such that for all $\omega \in \Omega$, $i \in N$, and $x \in X(c,\omega)$ we have

$$\mathcal{P}_i(x,\omega) = \begin{cases} h_i(\omega) \cap c^{-1}(x) & \text{if } h_i(\omega) \cap c^{-1}(x) \neq \emptyset \\ \text{Maxi}\{h_i(\omega) \cap c_{-j}^{-1}(x_{-j}) \cap x_j \mid H_j, \succeq_j\} & \text{otherwise,} \end{cases}$$

where $j = \overline{N}_i(c,x,\omega \mid \rho)$.

Definition 2 A *knowledge equilibrium* of the game (G,X) is a profile of effective strategies $\sigma \in \Sigma$, a profile of communication strategies $c \in C$, and a second stage information structure $\mathcal{P} = (\mathcal{P}_i)_{i \in N}$ satisfying the following conditions:

1. *Second Stage Rationality*: $\sigma \in \Sigma^*(\mathcal{P})$;

2. *Rational Communication*: For all $i \in N$, $\omega \in \Omega$, and $x_i \in X_i(\omega)$,

$$EU_i(\sigma, c, \mathcal{P}_i \mid h_i(\omega)) \geq EU_i(\sigma, x_i, c_{-i}, \mathcal{P}_i \mid h_i(\omega));$$

3. *Consistent Knowledge*: $\mathcal{P}$ is consistent with (c,X).

Given a knowledge equilibrium $(\sigma, c, \mathcal{P})$, let $h_i^c(\omega) \equiv \mathcal{P}_i(c(\omega),\omega) = h_i(\omega) \cap c^{-1}(c(\omega))$ be the *second stage equilibrium information set* of player i at ω, and let $H_i^c \equiv \{h_i^c(\omega) : \omega \in \Omega\}$ be his *second stage equilibrium partition*. By denoting $W(c)$ the partition generated by c, player i's second stage equilibrium partition can be rewritten as $H_i^c = H_i \vee W(c)$. When the equilibrium second stage information structure $H^c = (H_i^c)_{i \in N}$ and effective strategies σ are the same for different knowledge equilibria we will speak about the same equilibrium although communication strategies may differ. When not specified, the certifiability level considered in our examples is assumed to be perfect, i.e., $\mathcal{X} = \mathcal{Y}$, which means that players are able to reveal any piece of information they possess. Another interesting certifiability level is the *payoff-relevant* or *fundamental certifiability level.* With such certifiability possibilities, players are only able to reveal their knowledge about the fundamentals. Formally, by denoting $\mathcal{C}_0$ the σ-algebra generated by the payoff-relevant partition $\mathbb{P}$, we define, for all $i \in N$,

$$\mathcal{X}_i^{\mathbb{P}} \equiv \{E \in \mathcal{Y}_i : \exists\, F \in \mathcal{C}_0 \text{ s.t. } h_i(\omega) \subseteq F \;\forall\, \omega \in E \text{ and } h_i(\omega) \nsubseteq F \;\forall\, \omega \notin E\}.$$

Hence, for all $E \in \mathcal{X}_i^{\mathbb{P}}$, there exists a fundamental event $F \in \mathcal{C}_0$ such that player i knows F iff the real state of the world belongs to E. Using player i's knowledge operator $K_i : 2^\Omega \to 2^\Omega$, $\mathcal{X}_i^{\mathbb{P}}$ can be rewritten as

$$\mathcal{X}_i^{\mathbb{P}} = \{K_i F : F \in \mathcal{C}_0\} \backslash \{\emptyset\}.$$

Of course, if $\mathbb{P}$ is the degenerate partition of Ω, then $\mathcal{X}_i^{\mathbb{P}} = \mathcal{Y}_i$.

2.3 Definitions

In the next definition we define a taxonomy for different kinds of knowledge equilibria. This taxonomy is based on the equilibrium information structure obtained after the communication stage.

Definition 3 A knowledge equilibrium $(\sigma, c, \mathcal{P})$ of the communication game (G, X) is said

- *Perfectly revealing*, if $h_i^c(\omega) \subseteq P(\omega)$ for all $i \in N$ and $\omega \in \Omega$;
- *Non-revealing*, if $h^c = h$;
- *Partially revealing*, if $h^c \neq h$ and there exists $i \in N$ and $\omega \in \Omega$ such that $h_i^c(\omega) \nsubseteq P(\omega)$;
- *Perfectly communicating*, if for all $\omega \in \Omega$ and $i \in N$ there is no $x_i \in X_i(\omega)$ such that $x_i \subsetneq c_i(\omega)$.

Hence, a knowledge equilibrium is perfectly revealing when H_i^c is finer than $\mathbb{P}$ for all $i \in N$, i.e., the fundamentals are common knowledge. In particular, if $h_i^c(\omega) = J(\omega)$ for all $\omega \in \Omega$ and $i \in N$, then the equilibrium is perfectly revealing since we assumed that $\mathbb{J}$ is finer than $\mathbb{P}$. In that case, $H_i^c \equiv \{h_i^c(\omega) : \omega \in \Omega\} = \mathbb{J}$ for all $i \in N$, i.e., players have shared all distributed knowledge. Therefore, distributed knowledge is common knowledge after the communication stage because $H_i^c = \mathbb{J}$ for all $i \in N$ implies $\bigwedge_{i \in N} H_i^c = \mathbb{J}$. Inversely, a knowledge equilibrium is non-revealing if $H_i^c = H_i$ for all $i \in N$, i.e., if the information structure after the communication stage is the same as the initial information structure. An equilibrium is partially revealing if the information structure is modified by communication, but the fundamentals are not common knowledge. Finally, it is perfectly communicating if there is no certifiable event an agent initially knew which is strictly more informative than his current report. Under perfect certifiability, this condition simplifies to $c_i(\omega) = h_i(\omega)$ for all $i \in N$ and $\omega \in \Omega$, i.e., each player reveals exactly his actual information set in all states of the world. Hence, at a perfectly communicating equilibrium with perfect certifiability we obviously have $h_i^c(\omega) = J(\omega)$ for all $i \in N$ and $\omega \in \Omega$. Then, such an equilibrium is perfectly revealing. It is worth mentioning that, if certifiability is partial, a perfectly communicating equilibrium can be partially but also perfectly revealing, even if some knowledge is never certifiable. Equilibrium effective strategies profiles $\phi^* \in \Phi^*(J_N)$ of the game $G(J_N)$ will be called *full-information outcomes* because for each $\phi^* \in \Phi^*(J_N)$, $\phi^*(\omega)$ is a Nash equilibrium of the strategic form game $G(\omega)$ with complete information.

2.4 General Properties

In this subsection we present several general and useful properties concerning knowledge equilibria. We first remind that a knowledge equilibrium is a Kreps and Wilson's (1982) sequential equilibrium.

Proposition 1 *If a profile of communication and effective strategies forms a knowledge equilibrium of the communication game (G, X), then it also forms a sequential equilibrium.*

Proof. See Koessler (2002). □

The following proposition shows some properties of consistent information structures.

Proposition 2 *If the second stage information structure $\mathcal{P}$ is consistent with (c, X), then the following properties (revision rules) are satisfied for all $i \in N$, $\omega \in \Omega$ and $x \in X(c, \omega)$:*

(RR0) Perfect Recall: $\mathcal{P}_i(x,\omega) \subseteq h_i(\omega)$.

(RR1) Certifiability Constraint: $\mathcal{P}_i(x,\omega) \subseteq \bigcap_{k\in N} x_k$.

(RR2) Bayesian Updating: If $h_i(\omega) \cap c^{-1}(x) \neq \emptyset$, *then* $\mathcal{P}_i(x,\omega) = h_i(\omega) \cap c^{-1}(x)$.

(RR3) Admissible Revision: If $\omega' \in h_i(\omega)$, *then* $\mathcal{P}_i(x,\omega) = \mathcal{P}_i(x,\omega')$.

(RR4) Admissible Interpretation: $\mathcal{P}_i(x,\omega) = h_i(\omega) \cap \bigcap_{k\in N} y_k$ *for some* $y \in \mathcal{Y}$.

(RR5) Unilateral Deviations: If $h_i(\omega) \cap c^{-1}(x) = \emptyset$, *then* $\mathcal{P}_i(x,\omega) = h_i(\omega) \cap c_{-j}^{-1}(x_{-j}) \cap x_j \cap y_j$ *for some* $j \in N_i(c,x,\omega)$ *and* $y_j \in \mathcal{Y}_j$.

Proof. See Koessler (2002). □

The following proposition simplifies greatly the analysis of perfectly communicating and non-revealing equilibria.

Proposition 3 *If* $c \in C$ *is a perfectly communicating or a non-revealing communication strategy, then any unilateral deviation* $x \in X(c,\omega)$ *from* c *at* ω *is observable and identifiable by all players at* ω.

Many games of interest and almost all games studied in the literature on strategic information revelation are *two-player games* (e.g., duopoly games or persuasion games with one decision maker and one interested party) and *one side information games* (i.e., games in which only one player possesses some information). In the framework considered here, a one side information game is defined as follows.

Definition 4 A Bayesian game G is a *one side information game* if $H_1 = \{\{\omega_1\},\ldots,\{\omega_m\}\}$ and $H_i = \{\Omega\}$ for all $i \neq 1$.

The following proposition will be useful for the study of one side information games and for the characterization of knowledge equilibria in two-player games.

Proposition 4 *If* G *is a two-player or a one side information game, then for all* $\omega \in \Omega$, *any observable deviation* $x \in X(c,\omega)$ *at* ω *by player* i *is identifiable at* ω *by player* i.

2.5 Games with Trivial Knowledge Equilibria

Since only unilateral deviations from a profile of communication are allowed, we can formulate a trivial but practically interesting and general result on the existence of a perfectly revealing knowledge equilibrium for *any* Bayesian game with an information structure satisfying *non-exclusivity of information*.[2] Under this condition, any group of $n-1$ players collectively has knowledge which is distributed among all n players. Said differently, any piece of knowledge held by one player can be deduced by pooling the knowledge of all the other players. Formally, this condition can be written $\bigvee_{k\neq i} H_k = \mathbb{J}$ for all $i \in N$.

Example 1 This example satisfies non-exclusivity of information, but only the trivial event $\Omega = \{\omega_1,\omega_2,\omega_3,\omega_4\}$ is common knowledge at any state:

[2] This terminology stems from Postelwaite and Schmeidler (1986). Palfrey and Srivastave (1986) called this condition public information condition.

$$H_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$$
$$H_2 = \{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}$$
$$H_3 = \{\{\omega_2, \omega_3\}, \{\omega_1, \omega_4\}\}.$$

Indeed, we have $\mathbb{M} = H_1 \wedge H_2 \wedge H_3 = \{\Omega\}$ and $\bigvee_{k \neq i} H_k = \mathbb{J} = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}$ for all $i \in \{1, 2, 3\}$.

The following theorem asserts that under non-exclusivity of information, i.e., when *no* agent *alone* has an information which is not distributed among the others, there exists a perfectly revealing knowledge equilibrium as long as certifiability possibilities are sufficient.

Theorem 1 *If $\bigvee_{k \neq i} H_k = \mathbb{J}$ and if $h_i(\omega) \in X_i(\omega)$ for all $i \in N$ and $\omega \in \Omega$, then the communication game (G, X) has a perfectly revealing knowledge equilibrium.*

A consequence of Theorem 1 is that a replicated game of *any* Bayesian game admits a perfectly revealing knowledge equilibrium. More precisely, a *r-fold replicated game* of a Bayesian game G, with $r \geq 2$, denoted by rG is defined by

$$rG = \langle rN, \Omega, p, (h_{(i,j)}, A_{(i,j)}, u_{(i,j)})_{(i,j) \in rN} \rangle,$$

where $rN = \{(i, j) : i \in N,\ 1 \leq j \leq r\}$, $h_{(i,j)} = h_i$, $A_{(i,j)} = A_i$, and $u_{(i,j)} : \prod_{(i,j) \in rN} A_{(i,j)} \times \Omega \to \mathbb{R}$ is an utility function related to u_i (we do not need to specify it here). Hence, each replica of each player i has the same information and available actions as player i. Of course, for all $(i, j) \in rN$, we have $\bigcap_{(i',j') \neq (i,j)} h_{(i',j')}(\omega) = J(\omega)$ for all $\omega \in \Omega$, whenever $r \geq 2$. Hence, the information structure of rG satisfies the condition of Theorem 1, which implies that any replicated Bayesian game admits a perfectly revealing knowledge equilibrium.

Corollary 1 *If rG is a replicated game of G with $r \geq 2$ and if $h_i(\omega) \in X_i(\omega)$ for all $i \in rN$ and $\omega \in \Omega$, then the communication game (rG, X) has a perfectly revealing knowledge equilibrium.*

Replicated Bayesian games naturally arise in strategic market games with differential information. The following example gives an illustration inspired by Shin (1996).

Example 2 (Trading Systems with Higher-Order Uncertainty)
Shin (1996) compares the allocations which different institutions (trading systems) bring about in equilibrium. More precisely, the performance of a *decentralized market* is compared with that of a *dealership market* in presence of differential information. In particular, Shin analyzes how these trading institutions fare in the face of higher-order uncertainty.[3] In the decentralized (order-driven) market, traders play a Shapley-Shubik market game. That is, all traders submit, simultaneously, quantity orders to an auctioneer, who then sets a price to clear the market. In the dealership market, sellers (producers) decide how much to produce and post prices in anticipation of buyers' demand; then, each buyer, placed in a queue, ranks the sellers in order of preference and visits them in sequence until demand is satisfied.

Shin shows that when the fundamentals are common knowledge, the two trading systems deliver approximately the same outcomes, and the difference decreases with the number of agents (traders). However, when fundamentals are only mutually known (even at a high level),

[3]The study is done using original observations and contributions of Rubinstein (1989) and Carlson and van Damme (1993) on the effects of higher-order uncertainty in two-player games with finite action set, but in more conventional market settings.

it is shown that outcomes of the two trading systems diverge sharply. More precisely, the dealership market leads to the efficient allocations (those obtained under common knowledge) in most states, while the decentralized market has strictly lower trading volume in *all* states.

We will not present the model in detail since we are only interested in the information structure. In short, the model involves $n \geq 2$ coastal rice growers and n rice growers who live in the mountain. Only rice growers in the mountain have access to a technology which converts rice into rice pudding. Coastal traders regard rice and rice pudding as perfect substitutes, whereas mountain traders place zero value on the consumption of rice pudding. Traders' endowments depend on rice harvest in their region. If there is any rain in the early growing season (which lasts for exactly m days) of a region, then the rice harvest of this region yields one unit of rice for every trader. Otherwise, harvest fails, and thus endowments are zero for every trader of that region. The set of states of the world, which corresponds to the days of rain in each region, is given by

$$\Omega = \{(q, r) \in \{0, 1, \dots, m\}^2 : q = r \text{ or } q = r - 1\},$$

where q is the number of days of rain on the coast and r is the number of days of rain in the mountains. Note the $|\Omega| = 2m + 1$. Assuming a uniform probability distribution, the probability of $(q, r) \in \Omega$ is

$$p(q, r) = 1/(2m + 1).$$

Traders only observe the number of days of rain in their own region. Thus, a coastal trader's information set at $(q, r) \in \Omega$ is given by

$$h_C(q, r) = \{(q', r') \in \Omega : q' = q\},$$

and a mountain trader's information set at $(q, r) \in \Omega$ is given by

$$h_M(q, r) = \{(q', r') \in \Omega : r' = r\}.$$

This information structure can be represented as in Figure 1 on the following page, where coastal growers' information sets are represented by dashed boxes and mountain growers' information sets by solid boxes.

It is clear that the event in which harvests of both regions are good, denoted by $E^R = \{(q, r) \in \Omega : (q, r) \geq (1, 1)\}$, is never common knowledge since the *Meet* of traders' partitions is $\mathbb{M} = \{\Omega\}$. However, since the information structure satisfies non-exclusivity of information whenever $n \geq 2$, we know from Theorem 1 that there exists a (knowledge) equilibrium where all traders reveal their information. Consequently, when certifiable communication is allowed, the allocations obtained with common knowledge can be endogenously achieved at equilibrium.

Another immediate result concerning knowledge equilibria is obtained in *independent games*. In such games, there are no payoff externalities, which implies that all communication strategies constitute a knowledge equilibrium. More precisely, a Bayesian game G is an independent game if players' payoffs are only affected by their own actions. That is, $u_i(a_i, a_{-i}, \omega) = u_i(a_i, a'_{-i}, \omega)$ for all $i \in N$, $\omega \in \Omega$, $a_i \in A_i$ and a_{-i}, $a'_{-i} \in A_{-i}$.

Proposition 5 *If G is an independent game, then all profiles of communication strategies $c \in C$ form a knowledge equilibrium.*

Hence, independent games admit non-revealing equilibria and, depending on the certifiability level, partially and perfectly revealing equilibria. In particular, if for all $i \in N$, $X_i(\omega) \neq X_i(\omega')$ for all $\omega \neq \omega'$, then the game has a perfectly revealing equilibrium because in that case, there exists $c \in C$ such that $W(c) = \mathbb{J}$.[4]

[4]Remind that $W(c)$ is the partition generated by c.

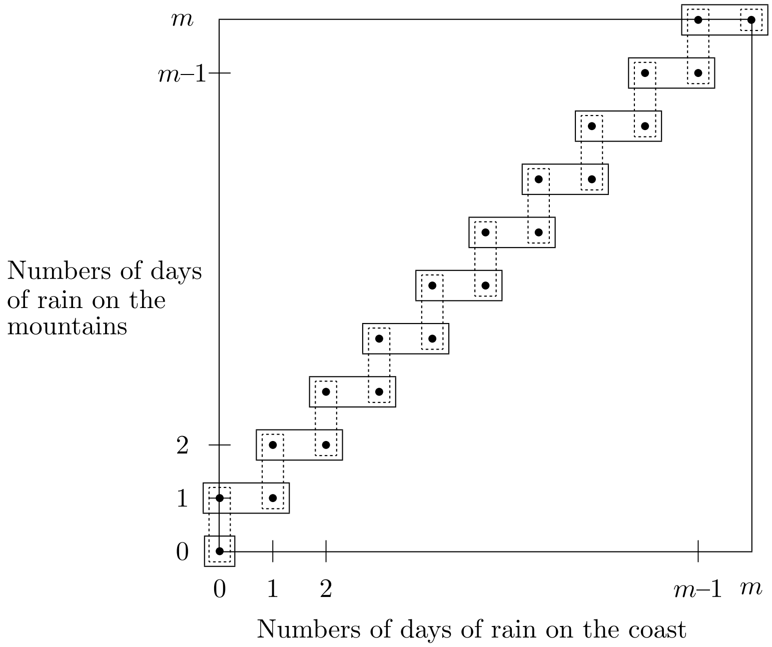

Figure 1: Information Structure in Shin's (1996) Model.

3 One Side Information Games

In this section we assume that only one player (player 1) has preliminary information about the states of the world. That is, $H_i = \{\Omega\}$ for all $i \neq 1$. We also assume for simplicity that $H_1 = \{\{\omega_1\}, \ldots, \{\omega_m\}\}$, i.e., player 1 exactly knows the real state of the world. Since only player 1 has some information to certify, let $\mathcal{X} = \mathcal{X}_1$, $X = X_1$, $c = c_1$, and $\succeq_1 = \succeq$ throughout this section.

In a one side information game, uninformed players (players $i \neq 1$) will have the same second stage beliefs. These players have no a priori reason to make different updating, given that they start exactly with the same information and that they observe the same messages. Therefore, $\mathcal{P}_i(x, \omega) = \mathcal{P}_j(x, \omega)$ for all $\omega \in \Omega$, $x \in X(\omega)$ and $i, j \neq 1$. Such a property is formally due to the fact that orderings over the informed player's partition are common to all players. We denote by $\mathcal{P}_r$ the common possibility correspondence of uninformed players (the "receivers"). Of course, $\mathcal{P}_1(x, \omega) = h_1(\omega) = \{\omega\}$ for all $\omega \in \Omega$ and $x \in X(\omega)$. Since $h_i(\omega) = \Omega$ for all $i \neq 1$, $\mathcal{P}_r(x, \omega)$ does not depend on ω. Hence, write $\mathcal{P}_r(x)$ the set of possible states for uninformed players when player 1 has reported the event $x \in \mathcal{X}$ in the communication stage, and let $\succeq$ be the associated ordering over H_1. Write $s_i(x)$ and $\sigma_i(x)$ the realization of receivers' effective strategies, for $i \neq 1$. Let c be the anticipated equilibrium communication strategy of player 1. If the informed player reveals $x \in X(\omega)$ at ω, with $x = c(\omega') \neq c(\omega)$ for some $\omega' \in \Omega$, then uninformed players do not observe the deviation and thus $\mathcal{P}_r(x) = c^{-1}(x)$. Otherwise, if they observe a deviation (i.e., $c^{-1}(x) = \emptyset$), then this deviation is 1-identifiable (see Proposition 3) and thus $\mathcal{P}_r(x) = \text{Maxi}\{x \mid H_1, \succeq\}$. To simplify the exposition we consider perfect certifiability, i.e., $\mathcal{X} = \mathcal{Y}$ (unless otherwise specified).

Before providing general results for one side information games, we begin by studying knowledge equilibria in several one side information games.

3.1 Examples

In this subsection we consider two players and two payoff-relevant states of the world. We assume that $p(\omega_1) = p(\omega_2) = 1/2$, $H_1 = \{\{\omega_1\}, \{\omega_2\}\}$, and $H_2 = \{\{\omega_1, \omega_2\}\}$. We also restrict our attention to pure strategies.

As a first example we consider a very simple game and we represent the two-stage game in extensive form. In this example, there are perfectly revealing and non-revealing knowledge equilibria, and they coincide with the sequential equilibria.[5] The perfectly revealing outcome is, however, quite non-intuitive.

Example 3 Consider the game of Figure 2 (actions are only available to player 2). The associated extensive form with information revelation possibilities is represented in Figure 3.

ω_1	A_2	B_2	C_2
	$(1,2)$	$(3,1)$	$(-3,-3)$

ω_2	A_2	B_2	C_2
	$(-3,-3)$	$(6,1)$	$(2,2)$

Figure 2: Bayesian Game of Example 3.

Without communication, the only Bayesian equilibrium entails player 2 playing B_2 at every state. Player 1 gets his maximum payoff at each state of the world. Then, he has intuitively no incentive to communicate, i.e., he should not voluntarily modify the information structure. In fact, no revelation is a knowledge equilibrium, but perfect revelation is also an equilibrium.

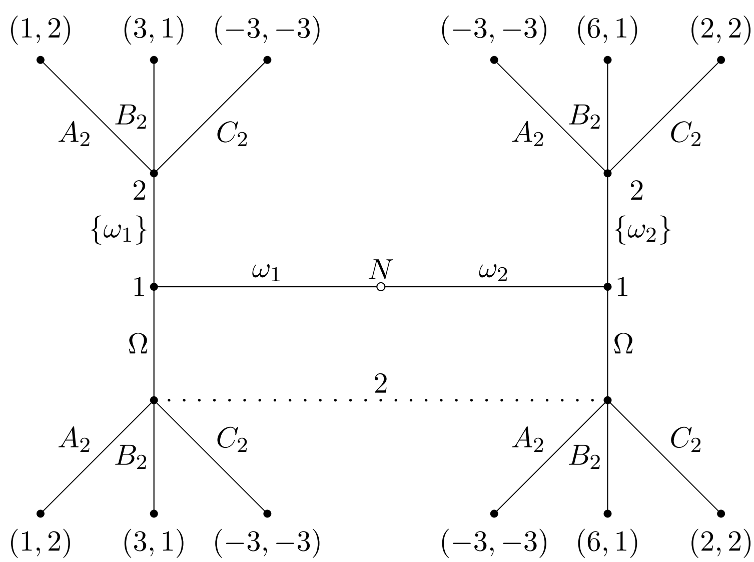

Figure 3: Complete Communication Game of Example 3.

To see that there is a perfectly revealing equilibrium in which player 1 always reveals his information and player 2 chooses action C_2 if he gets the message $x = \{\omega_2\}$ and chooses action A_2 elsewhere, let $c(\omega) = \{\omega\}$ for all $\omega \in \Omega$. We get $\mathcal{P}_2(c(\omega)) = h_2^c(\omega) = \{\omega\}$ for

[5]Detailed examples where the two types of equilibria differ can be found in Koessler (2002).

all $\omega \in \Omega$. Furthermore, assume that $\mathcal{P}_2(\Omega) = \{\omega_1\}$. This possibility correspondence is consistent. We obtain $s_2(\Omega) = A_2$, and thus $u_1(s_2(\Omega), \omega_1) = 1$ and $u_1(s_2(\Omega), \omega_2) = -3$. These payoffs are (weakly) smaller than those obtained by player 1 with full communication since $u_1(s_2(\{\omega_1\}), \omega_1) = 1$ and $u_1(s_2(\{\omega_2\}), \omega_2) = 2$. Thus he has no incentive to deviate.

Remark 1 This perfectly revealing equilibrium involves some kind of "threat by beliefs": player 1 discloses his information because, if not, player 2 will believe that player 1 is at his information set $\{\omega_1\}$ and thus player 2 will play A_2 instead of B_2. We find this interpretation "unreasonable" because player 1 is incited to deviate in both states, and thus player 2 should keep his prior beliefs when receiving the message Ω. This equilibrium is also a sequential equilibrium since all knowledge equilibria can be associated with sequential equilibria.[6]

To see that there is a non-revealing equilibrium, let $c(\omega) = \Omega$ for all $\omega \in \Omega$. In this case, the second stage possibility correspondence is uniquely defined by the certifiability constraint: $\mathcal{P}_2(\Omega) = \Omega$ and $\mathcal{P}_2(\{\omega\}) = \{\omega\}$ for all $\omega \in \Omega$. We obtain $u_1(s_2(\Omega), \omega) = u_1(B_2, \omega) > u_1(s(\{\omega\}), \omega)$ for all $\omega \in \Omega$.

In the following example we consider two decision makers. The unique knowledge equilibrium is perfectly revealing and the possibility to communicate and to certify player 1's knowledge in at least one state increases every player's payoff.

Example 4 Consider the game of Figure 4.[7]

ω_1	A_2	B_2
A_1	$(0, 0)$	$(6, -3)$
B_1	$(-3, 6)$	$(5, 5)$

ω_2	A_2	B_2
A_1	$(-20, -20)$	$(-7, -16)$
B_1	$(-16, -7)$	$(-5, -5)$

Figure 4: Bayesian Game of Example 4.

Without communication, i.e., with the initial information structure, there is only one Bayesian equilibrium where player 2 chooses A_2 at every state and player 1 chooses A_1 at ω_1 and B_1 at ω_2. Associated payoffs at each state are respectively $(0, 0)$ and $(-16, -7)$. If agent 1 reveals his information in at least one state, then the equilibrium information structure of the second stage becomes $H_1^c = H_2^c = H_1$. In this case, Player 2 chooses A_2 at ω_1 and B_2 at ω_2 and player 1 keeps the preceding strategy. Payoffs become respectively $(0, 0)$ and $(-5, -5)$. So, the unique knowledge equilibrium is perfectly revealing because if $c(\omega) = \Omega$ for all $\omega \in \Omega$, then player 1 deviates and reveals $x = \{\omega_2\}$ at ω_2. However, notice that the communication strategy c satisfying $c(\omega_1) = \{\omega_1\}$ and $c(\omega_2) = \Omega$ does not form an equilibrium (even if it is also perfectly revealing) because player 1 will deviate at ω_1 by revealing Ω. If certifiability is partial this result will not change as long as $\{\omega_2\} \in X_1(\omega_2)$. If not, the only knowledge equilibrium is non-revealing.

In the following example we show that an agent can be worse off, ex ante, if he can *freely* certify some of his information. The only knowledge equilibrium is perfectly revealing. The uninformed player is better off than without communication, but the player who communicates is, on average, worse off. This example shows the difference between communication at the ex ante stage and communication at the interim stage. It also surprisingly shows that if certifiability increases, then players who can reveal freely some information can be worse off.

[6] In this example, it suffices to assign an outside equilibrium belief about ω_1 greater than $4/5$ to player 2 when he receives the message Ω. Another perfectly revealing equilibrium exists where player 2 chooses action A_2 if he receives the message $\{\omega_1\}$ and chooses action C_2 otherwise.

[7] This game is taken from Bassan, Scarsini, and Zamir (1997).

Example 5 (Ex Ante vs Interim Communication) Consider the game of Figure 5 (actions are only available to player 2).

ω_1	A_2	B_2	C_2
	$(3,3)$	$(1,0)$	$(2,2)$

ω_2	A_2	B_2	C_2
	$(1,0)$	$(0,3)$	$(2,2)$

Figure 5: Bayesian Game of Example 5.

At the unique Bayesian equilibrium, player 2 chooses C_2 at every state. If player 1 reveals his information, player 2 chooses A_2 at ω_1 and B_2 at ω_2. Player 1's utility increases from 2 to 3 at ω_1 and decreases from 2 to 0 at ω_2. Thus, the only knowledge equilibrium is perfectly revealing because, if not, player 1 deviates by revealing $\{\omega_1\}$ at ω_1. At this equilibrium, if the real state of the world is ω_2 player 1's utility decreases: the fact that he reveals nothing proves to player 2 that the state of the world is not ω_1. Otherwise, player 2 would have received a message saying that the real state is ω_1. Then, if the real state is ω_2 it is the possibility to certify the event $\{\omega_1\}$ ($\{\omega_1\} \in X_1(\omega_1)$) that enables player 2 to know $\{\omega_2\}$. An ex ante maximization by player 1 (i.e., before private information is received) would not give him any incentive to communicate and to share his information, because it would give him an expected utility of 1.5 instead of 2. In the same way, if player 1's information can be acquired strategically before the communication stage, he would not have any incentive to acquire it.

Remark 2 The difference between communication behavior at the ex ante stage and communication behavior at the interim stage was already noted by Okuno-Fujiwara, Postlewaite, and Suzumura (1990, pp. 26–27). In the situation described by Example 5, a contract to prevent ex post communication at ω_1 is clearly impossible to enforce. This phenomenon cannot occur in cheap talk games where the set of available messages of a player is the same at all states of the world.

In the following example there are perfectly revealing and non-revealing knowledge equilibria.

Example 6 Consider the game of Figure 6, where $\alpha > \beta > 0$.[8]

ω_1	A_2	B_2
A_1	(α,α)	$(0,0)$
B_1	$(0,0)$	(β,β)

ω_2	A_2	B_2
A_1	(β,β)	$(0,0)$
B_1	$(0,0)$	(α,α)

Figure 6: Bayesian Game of Example 6.

The strategy profile $((A_1 \mid \{\omega_1\}, A_1 \mid \{\omega_2\}), (A_2 \mid \{\omega_1, \omega_2\}))$ is a Bayesian equilibrium of this game. If player 1 reveals his information we get two perfect information games, one at each state. In this case, the strategy profile $((B_1 \mid \{\omega_1\}, A_1 \mid \{\omega_2\}), (B_2 \mid \{\omega_1\}, A_2 \mid \{\omega_2\}))$ is a Bayesian equilibrium. Thus, there exists a non-revealing equilibrium because with this strategy profile player 1 is indifferent to communicate at ω_2, and prefers to reveal nothing at ω_1. This is verified even with perfect certifiability. The efficient knowledge equilibrium of this

[8] This game is also a common interest game, as defined in Section 4.

game consists in perfect revelation and in the strategy profile $((A_1 \mid \{\omega_1\}, B_1 \mid \{\omega_2\}), (A_2 \mid \{\omega_1\}, B_2 \mid \{\omega_2\})$.

In the final example of this subsection it is shown that stochastic communication is sometimes necessary to ensure the existence of an equilibrium, even in a very simple one side information persuasion game. In this case, the knowledge equilibrium cannot be directly applied since our consistency condition relies on pure communication strategies. Hence, beliefs are defined as in the sequential equilibrium (See Appendix A in Koessler, 2002).

Example 7 (Mixed Communication) Consider the game of Figure 7 (where actions are only available to player 2).

ω_1	A_2	B_2	C_2
	$(3,-4)$	$(2,2)$	$(1,1)$

ω_2	A_2	B_2	C_2
	$(1,4)$	$(2,-1)$	$(3,1)$

Figure 7: Bayesian Game of Example 7.

One can easily verify that there is no pure strategy equilibrium. The only equilibrium is perfectly mixed. By denoting $\pi(x \mid \omega)$ the probability that player 1 reports $x \in X(\omega)$ at ω, the unique equilibrium is $\pi(\Omega \mid \omega_1) = 3/5$, $\pi(\{\omega_1\} \mid \omega_1) = 2/5$ and $\pi(\Omega \mid \omega_2) = 1$. Hence, player 2's beliefs are given by $\mu_2(\omega_1 \mid \Omega, \omega) = 3/8$ and $\mu_2(\omega_2 \mid \Omega, \omega) = 5/8$ for all $\omega \in \Omega$. In the second stage game, player 2 chooses A_2 and C_2 with equal probability when player 1 reports Ω, he plays B_2 with probability one when player 1 reports $\{\omega_1\}$, and he plays A_2 with probability one when player 1 makes the outside equilibrium report $\{\omega_2\}$. Mixed communication strategies are not developed further in this paper because we are primarily interested in non-revealing and perfectly revealing equilibria, which are necessarily in pure communication strategies. Nevertheless, we do not exclude mixed effective strategies.

3.2 Sufficient Conditions for Revealing and Non-Revealing Equilibria

In this subsection we give sufficient conditions for perfectly revealing and non-revealing equilibria in one side information games. It turns out that these conditions are sufficient to characterize equilibria of the games analyzed in the previous subsection.

The following assumption will be sufficient for the existence of a perfectly revealing equilibrium.

Assumption 1 There exists a strict, complete, and transitive ordering $\succ$ over H_1 and a Bayesian equilibrium $\phi^* = (\phi^*_i)_{i \in N} \in \Phi^*(J_N)$ of the game $G(J_N)$ such that

$$\sum_{a \in A} \phi^*(a \mid \omega)\, u_1(a, \omega) \geq \max_{a_1 \in A_1} \sum_{a_{-1} \in A_{-1}} \phi^*_{-1}(a_{-1} \mid \omega')\, u_1(a_1, a_{-1}, \omega), \tag{3}$$

whenever $\{\omega'\} \succeq \{\omega\}$.

Assumption 1 is easy to check because only outcomes of complete information decisions must be compared. In particular, for a Bayesian equilibrium $\varphi^* = (\varphi^*_i)_{i \in N}$ of $G(J_N)$ in pure strategies, Equation (3) simplifies to

$$u_1(\varphi^*(\omega),\omega) \geq \max_{a_1 \in A_1} u_1(a_1, \varphi^*_{-1}(\omega'),\omega), \tag{4}$$

whenever $\{\omega'\} \succeq \{\omega\}$, where for all $\omega \in \Omega$, $\varphi^*(\omega)$ is a pure strategy Nash equilibrium of the strategic form game $G(\omega)$.

For instance, the game of Figure 8 satisfies Assumption 1, with $\{\omega_5\} \succ \{\omega_4\} \succ \cdots \succ \{\omega_1\}$, and by considering the (unique) full information Bayesian equilibrium ϕ^* satisfying $\phi^*_2(A_2 \mid \omega_1) = \phi^*_2(B_2 \mid \omega_2) = \phi^*_2(C_2 \mid \omega_3) = \phi^*_2(D_2 \mid \omega_4) = \phi^*_2(E_2 \mid \omega_5) = 1$.

	A_2	B_2	C_2	D_2	E_2
ω_1	1,5	0,3	0,4	0,1	0,1
ω_2	5,2	2,3	1,1	0,0	1,2
ω_3	8,4	4,4	1,5	0,3	0,0
ω_4	5,4	4,4	8,3	0,5	0,0
ω_5	5,4	8,4	3,3	2,1	0,5

Figure 8: A Bayesian Game in which Assumption 1 is Satisfied.

The following theorem shows that if Assumption 1 is satisfied, then there exists a perfectly revealing equilibrium, whatever the prior probabilities.

Theorem 2 *If G is a one side information game, $h_i(\omega) \in X_i(\omega)$ for all $\omega \in \Omega$ and $i \in N$, and assumption 1 is satisfied, then there exists a perfectly revealing knowledge equilibrium.*

From the previous theorem we know that the game of Figure 8 has a perfectly revealing equilibrium. Assumption 1 is also satisfied in Example 3 whatever the ordering over H_1; in Example 4 with $\{\omega_1\} \succ \{\omega_2\}$; in Example 5 with $\{\omega_2\} \succ \{\omega_1\}$; and in Example 6 whatever the ordering. In Example 7 it is not satisfied, and the game does not admit, as seen, a perfectly revealing equilibrium.[9]

In the game of Figure 8, if the states of the world are uniformly distributed, there is also a non-revealing equilibrium where player 2 chooses action A_2 when nothing has been revealed. Presumably, this non-revealing equilibrium is more intuitive than the perfectly revealing one. We now give sufficient conditions for the existence of non-revealing equilibria.

Assumption 2 There exists a strict, complete, and transitive ordering $\succ$ over H_1, a Bayesian equilibrium $\phi \in \Phi^*(h)$ of the initial Bayesian game $G(h)$, and a Bayesian equilibrium $\phi^* \in \Phi^*(J_N)$ of the game $G(J_N)$ such that

$$\sum_{a \in A} \phi(a \mid \omega) u_1(a,\omega) \geq \max_{a_1 \in A_1} \sum_{a_{-1} \in A_{-1}} \phi^*_{-1}(a_{-1} \mid \omega') u_1(a_1, a_{-1}, \omega), \tag{5}$$

whenever $\{\omega'\} \succeq \{\omega\}$.

Note that in pure strategies, Equation (5) simplifies to

$$u_1(\varphi(\omega),\omega) \geq \max_{a_1 \in A_1} u_1(a_1, \varphi^*_{-1}(\omega'),\omega),$$

whenever $\{\omega'\} \succeq \{\omega\}$.

[9]Example 5 in Koessler (2002) does not satisfy Assumption 1 and, as shown, the Bayesian game does not admit a perfectly revealing equilibrium.

Theorem 3 *If G is a one side information game and if assumption 2 is satisfied, then there exists a non-revealing knowledge equilibrium, whatever the certifiability level.*

Example of Figure 8 satisfies Assumption 2 if, for example, states of the world are uniformly distributed ($p(\omega) = 1/5$ for all $\omega \in \Omega$). Thus, in this case, the communication game admits a non-revealing equilibrium. Likewise, Examples 3 and 6 satisfy Assumption 2 whatever the ordering over the informed player's partition. Finally, Assumption 2 is not satisfied in Examples 4, 5, and 7 and we have seen that these games do not have non-revealing equilibria.

Direct Application to Persuasion Games Theorem 2 directly applies to standard persuasion games considered in the literature dealing with strategic information revelation (see, e.g., Milgrom, 1981). Consider, for example, persuasion in seller-buyer relationships, although the problem described below fits many similar persuasion situations as well. The buyer has to purchase $q \in \mathbb{R}_+$ units of a commodity of quality $\omega \in \{\omega_1, \ldots, \omega_m\} = \Omega \subseteq \mathbb{R}$, where $\omega_1 < \cdots < \omega_m$. The quality is known by the seller but not by the buyer, and this configuration of information is common knowledge. The larger ω is, the better is the quality. If the quantity purchased by the buyer when he knows the quality, $q(\omega) = \arg\max_{q \in \mathbb{R}_+} u_B(q, \omega)$, is unique and increasing with the quality, and if the utility of the seller, $u_S(q, \omega)$, is increasing with sales (i.e., the seller maximize sales), then assumption 1 is satisfied, and a perfectly revealing equilibrium exists. Notice that this result does not depend on the buyer's prior beliefs about the quality.

We conclude this section with an example showing an interesting phenomenon which might appear in a very simple persuasion game. In this example, when certifiability possibilities increase, the perfectly revealing equilibrium disappears. More precisely, there exists a perfectly revealing equilibrium under a partial certifiability level called radical certifiability level, but under perfect certifiability there is no perfectly revealing equilibrium. We remind that X is a radical certifiability level if for all $\omega \in \Omega$ and $i \in N$ we have either $X_i(\omega) = \{\Omega\}$ or $X_i(\omega) = \{\Omega, h_i(\omega)\}$. This means that each player can either reveal nothing or all what he knows at every state of the world.

Example 8 We can verify that under the perfect certifiability level the Bayesian game of Figure 9 has no perfectly revealing equilibrium. Nonetheless, if $X_1(\omega) = \{\Omega, \{\omega\}\}$ for all $\omega \in \Omega$, then there is a perfectly revealing equilibrium with the outside equilibrium possibility correspondence $\mathcal{P}_2(\Omega) = \{\omega_3\}$.

ω_1	A_2	B_2	C_2
	$(0, 6)$	$(3, 7)$	$(2, 8)$

ω_2	A_2	B_2	C_2
	$(0, 6)$	$(1, 7)$	$(2, 4)$

ω_3	A_2	B_2	C_2
	$(0, 6)$	$(1, 3)$	$(2, 0)$

Figure 9: Bayesian Game of Example 8.

From this example we see that we can weaken conditions for the existence of a perfectly revealing equilibrium when the certifiability level is radical. The intuition is relatively simple:

allowing more vagueness allows an informed party to manipulate more easily the information structure.

Proposition 6 *Consider a one side information game G and assume that $X_1(\omega) = \{\Omega, \{\omega\}\}$ for all $\omega \in \Omega$. If there exists a Bayesian equilibrium $\phi^* = (\phi_i^*)_{i\in N} \in \Phi^*(J_N)$ of $G(J_N)$ and a state $\omega' \in \Omega$ such that*

$$\sum_{a\in A} \phi^*(a \mid \omega)\, u_1(a,\omega) \geq \max_{a_1\in A_1} \sum_{a_{-1}\in A_{-1}} \phi^*_{-1}(a_{-1} \mid \omega')\, u_1(a_1, a_{-1}, \omega), \tag{6}$$

for all $\omega \in \Omega$, then the communication game (G, X) has a perfectly revealing equilibrium.

4 Common Interest Games

In this section we consider *common interest games*, i.e., Bayesian games which possess at every state of the world an outcome (a profile of actions) such that every player's payoff attains a global maximum in this state of the world. Such an outcome is obviously a Nash equilibrium of the corresponding game under complete information, which Pareto dominates (weakly) all other outcomes. In practice, in these games people are often observed to be able to coordinate on the Pareto dominant equilibrium, especially when there are no complicating issues of risk dominance. However, under incomplete information, the Pareto optimal outcome of the game played at each state of the world is usually not achievable. The aim of this section is to analyze under which conditions full information occurs in common interest games. Contrary to the previous section, all types of information structures are allowed. Consequently, we are able to study some effects of higher-order uncertainty on information sharing by discriminating states that differ in some payoff-relevant way (in the fundamentals) and states that differ from the fact that agents' knowledge differs.

Definition 5 A Bayesian game $G = \langle N, \Omega, p, h, A, (u_i)_{i\in N}\rangle$ is a *common interest game* if for all $\omega \in \Omega$ there exists a profile of actions $a^* \in A$, called the *cooperative level* at ω, such that $u_i(a^*, \omega) \geq u_i(a, \omega)$ for all $i \in N$ and $a \in A$.

Note that in common interest games we do not necessarily have $u_i(a, \omega) = u_j(a, \omega)$ for $i \neq j$, $a \in A$ and $\omega \in \Omega$. Team games are particular common interest games in which $u_i = u$ for all $i \in N$. As in common interest games, agents in a team receive different signals about the underlying uncertainty, but they are assumed further to have exactly the same preferences.[10] In the next subsection we first illustrate, in two common interest games, some simple effects of higher-order uncertainty on communication behavior.

4.1 Examples

In Example 9, we consider the possibility to reveal events concerning agents' knowledge about the fundamentals *and* about others' knowledge. It is shown why in some circumstances players necessarily tend to disclose, if possible, higher-order information.

Example 9 (Strategic Communication of Non-Fundamental Events) Let

$$\Omega = \{\omega_1, \ldots, \omega_m\},$$

where $m \geq 4$ is an even number, $p(\omega) = 1/m$ for all $\omega \in \Omega$,

[10] Team games with incomplete information are analyzed in detail by Marschak and Radner (1972).

$$H_1 = \{\{\omega_1\}, \{\omega_2, \omega_3\}, \ldots, \{\omega_{m-2}, \omega_{m-1}\}, \{\omega_m\}\},$$

and

$$H_2 = \{\{\omega_1, \omega_2\}, \ldots, \{\omega_{m-1}, \omega_m\}\}.$$

Information structures with such overlapping information sets arise naturally in many classes of problems where there is a lack of common knowledge. Assume that player 2 has no action and that player 1 has two actions, A_1 and B_1. Players' common payoffs are given in Figure 10, where $\alpha > 1$.

	ω_k, k odd	ω_k, k even
A_1	$-\alpha$	1
B_1	0	0

Figure 10: Bayesian Game of Example 9.

The payoff-relevant partition is $\mathbb{P} = \{\{\omega_{2k-1}\}_{k\geq 1}, \{\omega_{2k}\}_{k\geq 1}\}$. Without communication, the unique Bayesian equilibrium entails player 1 playing A_1 if and only if the real state is ω_m. If only payoff-relevant events can be certified, then $X_2(\omega) = \mathcal{X}_2^{\mathbb{P}} = \{\Omega\}$ for all $\omega \in \Omega$. This means that player 2 cannot certify anything to player 1 because his knowledge about the game played is the same at each of his information sets. In other words, if player 2 "speaks" about his knowledge of the game, then he is not able to reveal anything. Hence, the only knowledge equilibrium is non-revealing. However, if certifiability is perfect, one can easily verify that the unique equilibrium is perfectly revealing, leading player 1 to choose A_1 whenever it yields a payoff equal to 1.

To sum up, with perfect certifiability the only knowledge equilibrium is perfectly revealing, but with the payoff-relevant certifiability level no information is revealed and the full information outcome is not achievable, leading to a very inefficient outcome. Therefore, letting player 2 reveal an event which does not characterize the fundamentals is relevant. For example, assume that $m = 4$. In this case, when it is revealed by player 2, the event $\{\omega_1, \omega_2\}$ means "I know that you do not know that the fundamental event is $\{\omega_2, \omega_4\}$" because $K_2\overline{K}_1\{\omega_2, \omega_4\} = K_2\{\omega_1, \omega_2, \omega_3\} = \{\omega_1, \omega_2\}$. Note that if player 2 can reveal his *beliefs* (as opposed to his knowledge) about the fundamentals, then the information structure will also not be modified since his beliefs about the fundamentals are always equal to 1/2. Thus, revealing them does not permit player 1 to learn anything.

Remark 3 The last example illustrates how interesting it would be to extend our model to repeated communication. Indeed, in this example, even if only payoff-relevant information can be revealed, player 1 might first reveal $\{\omega_1\}$ at ω_1, Ω at ω_2 and ω_3, and $\{\omega_4\}$ at ω_4. In that case, player 2 will be able to distinguish all states of the world and might reveal, in a second communication stage, $\{\omega_1, \omega_3\}$ at ω_1 and ω_3, and $\{\omega_2, \omega_4\}$ at ω_2 and ω_4, yielding a perfectly revealing equilibrium.

In the following example we show how an agent is strategically incited to communicate in order to achieve an outcome which requires common knowledge, i.e., an outcome which is not achievable with a finite (even very high) level of mutual knowledge.

Example 10 (Strategic Communication when Fundamentals are Known up to an Arbitrary Level) We consider the same information structure as in the preceding example, i.e., $\Omega = \{\omega_1, \ldots, \omega_m\}$, where $m \geq 4$ is an even number, $p(\omega) = 1/m$ for all $\omega \in \Omega$,

$H_1 = \{\{\omega_1\}, \{\omega_2, \omega_3\}, \ldots, \{\omega_{m-2}, \omega_{m-1}\}, \{\omega_m\}\}$ and $H_2 = \{\{\omega_1, \omega_2\}, \ldots, \{\omega_{m-1}, \omega_m\}\}$. Each player $i \in \{1, 2\}$ has two possible actions: action B_i is an investment decision in a risky project, and A_i is a default action (not to invest). The Bayesian game is given by Figure 11, where $\alpha > \beta > 0$. A player who does not invest is sure to get zero payoff, and a player who invests alone gets $-\alpha$. If both invest, their payoff depends on the state of the world. If the real state of the world is $\omega = \omega_1$, then their payoff is $-\beta$. On the contrary, if $\omega \neq \omega_1$, then they get a payoff equals to β. In this game, the payoff-relevant partition is $\mathbb{P} = \{\{\omega_1\}, \{\omega_2, \omega_3, \ldots, \omega_m\}\}$. Hence, player 1 always knows the fundamentals and player 2 knows the fundamentals only at states $\omega_3, \ldots, \omega_m$.

ω_1	A_2	B_2
A_1	$(0, 0)$	$(0, -\alpha)$
B_1	$(-\alpha, 0)$	$(-\beta, -\beta)$

$\omega \neq \omega_1$	A_2	B_2
A_1	$(0, 0)$	$(0, -\alpha)$
B_1	$(-\alpha, 0)$	(β, β)

Figure 11: Bayesian Game of Example 10.

It is easy to see that without communication the unique Bayesian equilibrium involves player 1 choosing A_1 and player 2 choosing A_2 at every state.[11] Hence, neither player invests. We now introduce the communication stage and we consider the payoff-relevant certifiability level, i.e., $X_1(\omega_1) = \{\{\omega_1\}, \Omega\}$, $X_1(\omega) = \{\{\omega_2, \ldots, \omega_m\}, \Omega\}$ for all $\omega \neq \omega_1$, $X_2(\omega_1) = X_2(\omega_2) = \{\Omega\}$, and $X_2(\omega) = \{\{\omega_3, \ldots, \omega_m\}, \Omega\}$ for all $\omega \in \{\omega_3, \ldots, \omega_m\}$. We also assume that when the game played at $\omega \neq \omega_1$ is common knowledge, then players coordinate to the Pareto optimal Nash equilibrium (B_1, B_2) in which both players invest. That is, we consider an effective strategy profile $\sigma \in \Sigma^*(\mathcal{P})$ such that $\sigma_1(B_1 \mid c(\omega), \omega) = \sigma_2(B_2 \mid c(\omega), \omega) = 1$ for all $\omega \neq \omega_1$, whenever the communication strategy profile $c \in C$ leads to an equilibrium information structure $h^c = (h_1^c, h_2^c)$ satisfying $h_i^c(\omega) \subseteq \{\omega_2, \ldots, \omega_m\}$ for all $\omega \neq \omega_1$ and $i \in N$. We can easily verify that, in this case, all knowledge equilibria lead to such an information structure, i.e., lead to the full information outcome. For example, it suffices that player 1 reveals $\{\omega_1\}$ at ω_1, which leads to common knowledge of the game played at every state of the world. Indeed, when $\{\omega_1\}$ will not be disclosed at $\omega \neq \omega_1$, it will be common knowledge that the game on the right hand of Figure 11 is played, and the outcome (B_1, B_2) will be achievable at all $\omega \neq \omega_1$. It is interesting to see that this is true even if nothing can be certified at $\omega \neq \omega_1$, i.e., at all states where the game on the right hand of Figure 11 is effectively played.

4.2 Existence of Perfectly Revealing and Efficient Equilibria

At first glance, it seems natural that if communication can be done at no cost, then all private information will spread in a game of common interests. That is, it is intuitive that no agent will be reluctant to disclose his information, as it was the case in the preceding examples. In fact, under some certifiability conditions, we show in the next theorem that with any initial information structure all common interest games admit a perfectly revealing equilibrium. The proof of this result is simple. It suffices to consider a full-information outcome in which all players play the cooperative level. In this case, no player has an incentive to change his effective strategy because they all attain their maximum payoff. Furthermore, they do not have incentives to withhold their information because less information may not enable them to achieve the cooperative level. Thus, in common interest games, the problem is

[11] Actually, this equilibrium can be obtained by iterated dominance (iterated deletion of dominated strategies). The propagation effect observed in this example is similar to the one observed in Rubinstein's (1989) electronic mail game and in global games (see Carlson and van Damme, 1993).

similar to a decision problem.[12] Nevertheless, we will show by a simple example that without sufficient certifiability, some certifiable knowledge is never transmitted, in any equilibrium. Therefore, the certifiability level may influence incentives to communicate: even in common interest games, it may be preferable for some agents to reveal nothing than to reveal partial information.

Let N^* be a set of players such that $\bigvee_{i \in N^*} H_i = \mathbb{J}$. The set N^* is a set of players having all the knowledge distributed among players of N.

Theorem 4 *If G is a common interest game and $h_i(\omega) \in X_i(\omega)$ for all $\omega \in \Omega$ and $i \in N^*$, then a Pareto efficient and perfectly revealing knowledge equilibrium exists.*

From this theorem, we have sufficient (albeit not necessary) conditions on certifiability for a decentralized team (i.e., a team with heterogeneous information) to become endogenously a centralized team (i.e., a team with homogeneous information) if communication is costless and voluntary. The next example shows that when certifiability is not perfect, some knowledge is *never* transmitted, *even if it could be certified*. In other words, in some situations, players never communicate all what they can, i.e., the perfectly communicating strategy is not an equilibrium.

Example 11 Let $H_1 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$ and $H_2 = \{\omega_1, \omega_2, \omega_3\}$ be an information structure, with $p(\omega) = 1/3$ for all $\omega \in \Omega = \{\omega_1, \omega_2, \omega_3\}$. Assume that players have a common utility and consider the game of Figure 12, where player 1 has no action.

	A_2	B_2
ω_1	0	4
ω_2	3	0
ω_3	3	-1

Figure 12: Bayesian Game of Example 11.

Suppose that the certifiability of player 1's knowledge is partial, given by $X_1(\omega_1) = X_1(\omega_2) = \{\{\omega_1, \omega_2\}, \Omega\}$ (i.e., $\mathcal{X}_1 = \{\Omega, \{\omega_1, \omega_2\}\}$). We can easily verify that the unique equilibrium communication strategy is $c_1(\omega_1) = \{\omega_1, \omega_2\}$ and $c_1(\omega_2) = c_1(\omega_3) = \Omega$. We obtain $H_2^c = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$. The effective equilibrium strategy satisfies $\sigma_2(B_2 \mid \{\omega_1, \omega_2\}, \omega) = \sigma_2(A_2 \mid \Omega, \omega) = 1$ for all $\omega \in \Omega$. At ω_2, player 2 does not know the event $\{\omega_1, \omega_2\}$ after the communication stage because $h_2^c(\omega_2) \nsubseteq \{\omega_1, \omega_2\}$. However, this event is certifiable at ω_2 by player 1 because $\{\omega_1, \omega_2\} \in X_1(\omega_2)$.

Example 11 has illustrated that even in an organization without conflicts of interest, but with certifiability limits, it can be suboptimal to share all certifiable knowledge. In other words, if the certifiability level is partial, then it is not always true that agents with the same preferences will share all their knowledge to reach superior decisions, contrary to the case in which some certifiability conditions are satisfied.

[12]Non-revealing and inefficient equilibria are not excluded. A first trivial case for no revelation appears when effective equilibrium strategies are the same for different information structures. Then, players cannot be strictly encouraged to communicate. But these cases are irrelevant since the outcome of the game will be the same as with full disclosure. More interesting types of "uncoordinated" equilibria with endogenous information asymmetries can arise, where communicating can be an irrational equilibrium deviation (see, e.g., Example 6). The reason is that if there are several equilibria for different information structures, an inefficient equilibrium may be played in the case of perfect revelation (even in perfectly symmetric common interest games, no criteria based on individual rationality may select a unique solution; for more details see, e.g., van Damme, 1995). Thus, there are equilibrium outcomes which differ from full-information outcomes and which are inefficient.

This observation conforms with many practical examples in which we observe only partial information homogenization even if there is no incentive problems between agents. Here, by assuming away technical communication costs and conflicts of interest, we have seen that some information specialization can be readily explained by coordination problems (as in Example 6) or by partial certifiability (as in Example 11).

5 Full Revelation with Ordered Information Structures

The result on the existence of perfectly revealing equilibria obtained in the previous section is relatively weak since the certifiability assumption should not be required to get complete knowledge sharing in common interest games. Indeed, we should expect that cheap talk communication with a sufficiently large set of messages is sufficient for the existence of a perfectly revealing equilibrium. In Section 3 we have seen, however, that conflicting interests can lead to perfectly revealing equilibria, but we made the assumption that only one player is informed. In this section, we generalize the conditions obtained in one side information Bayesian games to any Bayesian game with an ordered information structure, i.e., with an information structure where each player's partition is a set of ordered intervals of the state space. Such information structures include a wide class of possible uncertainties and allow players' information to be highly correlated.

After having characterized sufficient conditions for the existence of a perfectly revealing equilibrium in Bayesian games with an ordered information structure, we give a class of utility functions satisfying those conditions. Thereafter, we show that our conditions directly apply to Cournot games (with any number of firms) with uncertainty concerning either the intercept of demand (with possibly heterogeneous costs), or the common cost of the industry.

5.1 A Sufficient and General Condition for Complete Knowledge Sharing

We assume for sake of simplicity that $J(\omega) = \bigcap_{i \in N} h_i(\omega) = \{\omega\}$ for all $\omega \in \Omega$, i.e., the *Join*, $\mathbb{J}$, is the degenerated partition of Ω. Let $h^{-i}(\omega) = \bigcap_{k \in N \setminus \{i\}} h_k(\omega)$ be the set of states representing distributed knowledge at $\omega \in \Omega$ of players other than i. Assume, w.l.o.g., that states are real numbers and that $\omega_1 < \cdots < \omega_m$. In this section we restrict the analysis to ordered information structures according to the following definition.

Definition 6 An information structure H is called an *ordered information structure* if for each player $i \in N$, H_i is a set of ordered intervals of Ω, i.e., $\omega_k < \omega_{k'}$ and $h_i(\omega_k) \neq h_i(\omega_{k'})$ imply $\omega < \omega'$ for all $\omega \in h_i(\omega_k)$ and $\omega' \in h_i(\omega_{k'})$.

If the information structure of a Bayesian game is ordered in the sense of the previous definition, then a sufficient condition for the existence of a perfectly revealing equilibrium is given by the following theorem. Of course, this condition is satisfied in common interest games, and ensures Assumption 1 in one side information games.

Theorem 5 *Consider a Bayesian game G with an ordered information structure and assume that $h_i(\omega) \in X_i(\omega)$ for all $\omega \in \Omega$ and $i \in N$. If there exists a function $a^* : \Omega \to A$, where $a^*(\omega)$ is a Nash equilibrium of the strategic form game $G(\omega)$, such that for all $i \in N$,*

$$u_i(a^*(\omega), \omega) \geq u_i(a_i, a^*_{-i}(\omega'), \omega), \tag{7}$$

for all $\omega \in \Omega$, $\omega' \in h^{-i}(\omega)$ such that $\omega' > \omega$, and $a_i \in A_i$, then the communication game (G, X) has a perfectly revealing knowledge equilibrium.

From Theorem 5 the following result is immediate.

Corollary 2 *Consider a Bayesian game G with an ordered information structure and assume that $h_i(\omega) \in X_i(\omega)$ for all $i \in N$ and $\omega \in \Omega$. If there exists a profile of actions $a^* \in A$ such that a^* is a Nash equilibrium of $G(\omega)$ for all $\omega \in \Omega$, then the communication game (G, X) has a perfectly revealing knowledge equilibrium.*[13]

Example 12 The non-trivial Bayesian game of Figure 12, where $\Omega = \Omega_1 \cup \Omega_2$, with $\Omega_1 \cap \Omega_2 = \emptyset$, has a perfectly revealing knowledge equilibrium whatever the (ordered) information structure, and whatever the prior probabilities.

$\omega \in \Omega_1$	A_2	B_2
A_1	$(1,1)$	$(9,0)$
B_1	$(0,9)$	$(6,6)$

$\omega \in \Omega_2$	A_2	B_2
A_1	$(1,3)$	$(8,2)$
B_1	$(0,4)$	$(7,7)$

Figure 13: Bayesian Game of Example 12.

Example 13 In the Bayesian game of Figure 14, conditions of Corollary 2 are not satisfied, albeit condition (7) is satisfied, at thus the Bayesian game admits a perfectly revealing equilibrium whatever the information structure.[14]

ω_1	A_2	B_2
A_1	$(1,1)$	$(3,2)$
B_1	$(2,3)$	$(0,0)$

ω_2	A_2	B_2
A_1	$(1,1)$	$(0,-1)$
B_1	$(-1,0)$	$(-3,-3)$

Figure 14: Bayesian Game of Example 13.

Example 14 The Bayesian game of Figure 15 also satisfies condition (7), but the Nash equilibria also differ at each state of the world. A perfectly revealing equilibrium results whatever the (ordered) information structure, with $a^*(\omega_1) = (A_1, C_2)$, $a^*(\omega_2) = (B_1, B_2)$, and $a^*(\omega_3) = (C_1, A_2)$. Of course, we do not claim that such a perfectly revealing equilibrium is necessarily reasonable (and unique), but that it constitutes a sequential equilibrium of the communication game.

ω_1	A_2	B_2	C_2
A_1	$(0,0)$	$(3,2)$	$(3,3)$
B_1	$(2,1)$	$(0,0)$	$(2,3)$
C_1	$(1,1)$	$(1,2)$	$(0,0)$

ω_2	A_2	B_2	C_2
A_1	$(0,9)$	$(1,8)$	$(4,6)$
B_1	$(1,1)$	$(2,2)$	$(3,1)$
C_1	$(0,0)$	$(1,1)$	$(1,1)$

ω_3	A_2	B_2	C_2
A_1	$(0,5)$	$(2,9)$	$(6,6)$
B_1	$(0,5)$	$(8,8)$	$(9,2)$
C_1	$(1,1)$	$(3,0)$	$(5,0)$

Figure 15: Bayesian Game of Example 14.

In the next subsection we give sufficient conditions for condition (7) to be satisfied. These conditions are shown to apply directly to Cournot Bayesian games with incomplete information about the intercept of demand or about common costs, whatever the (ordered) information structure and prior probabilities. These conditions also generalize the conditions

[13] In particular, all (weighted) potential Bayesian games such that the potential at ω is maximized at $a^* \in A$ for all $\omega \in \Omega$ admit a perfectly revealing equilibrium (see Monderer and Shapley, 1996, for potential games).

[14] In this example, since there are only two states, the only non-trivial information structures correspond to the case in which either player 1 or player 2 is informed about the state of the world.

of Okuno-Fujiwara et al. (1990) to information structures where players' private signals are correlated. From our point of view, when uncertainties concern common values, ordered information structures are more natural than information structures with un-correlated types used by Okuno-Fujiwara et al. (1990). Indeed, when uncertainties concern a common value (as parameters of the demand or the cost of the industry) it is natural that firms have information on intervals and that their information sources are highly correlated. In particular, contrary to information structures considered by Okuno-Fujiwara et al., 1990, ordered information structures allow firms to be uncertain about other firms' uncertainty about other firms' uncertainty … about the fundamentals.

5.2 A Class of Linear Utility Functions

Let $\tau : \Omega \to \mathbb{R}$ be a function which assigns a fundamental real value $\tau(\omega)$ to each state of the world. Assume that τ is (weakly) monotone, i.e., either $\tau(\omega_1) \leq \tau(\omega_2) \leq \cdots \leq \tau(\omega_m)$ or $\tau(\omega_1) \geq \tau(\omega_2) \geq \cdots \geq \tau(\omega_m)$. We consider the class of Bayesian games where the utility function of each player i can be written in the following form:

$$u_i(a, \omega) = \alpha_i a_i \Big(\tau(\omega) + \gamma_i - \beta \sum_{j \neq i} a_j - a_i\Big), \tag{8}$$

where α_i, β and γ_i are parameters satisfying $\alpha_i > 0$, $\beta \in]0, 2[$ and $\gamma_i \in \mathbb{R}$ for all $i \in N$. Call such games *linear Bayesian games.*

Theorem 6 *If G is a linear Bayesian game with an ordered information structure and if $h_i(\omega) \in X_i(\omega)$ for all $\omega \in \Omega$ and $i \in N$, then the communication game (G, X) has a perfectly revealing equilibrium.*

5.3 Cournot Competition with Demand or Cost Uncertainty

In this final subsection, relying on the result of Theorem 5, we provide conditions under which firms in a Cournot game have an incentive to share their knowledge about a stochastic cost or a stochastic demand. More precisely, consider a market with $n \geq 2$ firms producing identical products. The inverse demand is given by $p(Q) = \alpha - \beta Q$, where Q denotes total market output and α, $\beta > 0$ are parameters. The constant marginal cost of firm i is given by $\lambda_i \geq 0$, and its output is denoted by $q_i \in \mathbb{R}_+$. Hence, $Q = \sum_{i \in N} q_i$. We consider either an unknown intercept of demand $\alpha(\omega)$ or an unknown common and constant marginal cost $\lambda(\omega) = \lambda_i(\omega)$ for all $i \in N$, where $\omega \in \Omega$ is some state of the world and $\lambda_i(\omega)$ and $\alpha(\omega)$ are (weakly) monotone. Firm i's profit (utility) at ω is

$$u_i(q, \omega) = q_i\Big(p(Q, \omega) - \lambda_i(\omega)\Big) = q_i\Big(\alpha(\omega) - \lambda_i(\omega) - \beta \sum_{j \in N} q_j\Big).$$

Unknown Demand Intercept Assume that $\lambda_i(\omega) = \lambda_i$ for all $\omega \in \Omega$ and $i \in N$, and let $\tau(\omega) = \alpha(\omega)$. The game has the form of a linear Bayesian game, and hence it admits a perfectly revealing equilibrium whatever the ordered information structure.

Unknown Common and Constant Marginal Cost Assume that $\alpha(\omega) = \alpha$ for all $\omega \in \Omega$, and let $\tau(\omega) = \lambda_i(\omega) = \lambda(\omega)$ for all $\omega \in \Omega$ and $i \in N$. The game is also a linear Bayesian game, and thus it admits a perfectly revealing equilibrium whatever the ordered information structure.

6 Conclusion

In some circumstances, the possibility to exchange knowledge and interactive knowledge is necessary to achieve specific outcomes. Depending on the strategic interactions under consideration, these outcomes can be obtained endogenously if agents communicate strategically with each other. Otherwise, the initial configuration of knowledge may not change even if agents can voluntarily reveal their information, and thus the outcomes obtained under the initial configuration of knowledge will remain unchanged. In this paper, by relying on payoff interdependencies as well as on the initial configuration of knowledge of different classes of games, we have provided several conditions ensuring particular properties of information structures generated by voluntary communication. We have also presented numerous examples which illustrate interesting phenomena arising in games with strategic information revelation. Our study relied on the model of knowledge sharing of Koessler (2002). This model was fruitful to conduct our different applications because it allows to determine explicitly players' inferences without considering Kreps and Wilson's (1982) sequences of perturbed games.

We first showed that with an information structure satisfying non-exclusivity of information, a perfectly revealing equilibrium always exists. While this result is relatively obvious, it is very general because it does not rely on players' preferences and on the complexity of the information structure. Moreover, it was shown that non-exclusivity of information shall arise in various economic contexts and in any replicated game. Another general result on the existence of a perfectly revealing equilibrium was provided in games with independent payoffs and in games with common interests.

Thereafter, we have analyzed incentives for information revelation in games with more complex and possibly conflicting interests. We began to perform this analysis in one side information games, and then we have extended our conditions to general games endowed with an ordered information structure. A class of games, called linear Bayesian games, was shown to satisfy our sufficient conditions for the existence of perfectly revealing equilibrium. These games include, for example, linear Cournot games.

Apart from our sufficient conditions for the existence of a perfectly revealing or a non-revealing equilibrium, we have also analyzed the effects of certifiability possibilities in various examples. Among the phenomenon illustrated, we have shown that even in some games without conflicts of interests (in particular, in team games), all certifiable knowledge will never be learned if certifiability possibilities are only partial. Communicating can even correspond to an irrational behavior. Another example has shown that a perfectly revealing equilibrium may fail to exist when certifiability possibilities increase. We have also shown that an agent may be worse off when he is able to voluntarily reveal his information. Finally, we have presented examples where strategic information revelation about the fundamentals is not sufficient to achieve the full information outcome, whereas larger communication possibilities concerning players' knowledge and interactive knowledge might be sufficient.

Appendix

Proof of Proposition 3. Let $x_i \neq c_i(\omega)$. If $c(\omega) = (\Omega, \ldots, \Omega)$ (i.e., c leads to a non-revealing equilibrium), then it is clear that $c_i^{-1}(x_i) = \emptyset$. On the other hand, if c is perfectly communicating, then, for all $\omega \in \Omega$, $c_i(\omega)$ is the smallest set of $\mathcal{X}_i$ containing ω. Since $\omega \in x_i$, we also have $c_i^{-1}(x_i) = \emptyset$. Therefore, a deviation from a non-revealing or perfectly communicating equilibrium is identifiable and observable by all players at ω. □

Proof of Proposition 4. The result is obvious for one side information games since only one player is able to communicate. Therefore, the definitions of an identifiable and an observable deviation are equivalent. If a deviation $x = (x_1, x_2) \in X(c, \omega)$ is observable at ω by player 1 in a two-player game, then $h_1(\omega) \cap c_1^{-1}(x_1) \cap c_2^{-1}(x_2) = \emptyset$. If player 1 is the deviant player at ω (i.e., $x_1 \neq c_1(\omega)$), then $h_1(\omega) \cap c_1^{-1}(x_1) = \emptyset$ because player 1's communication strategies are H_1 measurable. Thus, $h_1(\omega) \cap c_1^{-1}(x_1) \cap x_2 = \emptyset$. Moreover, since x is a unilateral deviation, we have $x_2 = c_2(\omega)$, and thus $\omega \in h_1(\omega) \cap x_1 \cap c_2^{-1}(c_2(\omega)) \neq \emptyset$, which implies that the deviation is 1-identifiable by player 1.[15] If player 2 is the deviant player at ω (i.e., $x_2 \neq c_2(\omega)$), then $h_1(\omega) \cap c_1^{-1}(c_1(\omega)) \cap c_2^{-1}(x_2) = h_1(\omega) \cap x_1 \cap c_2^{-1}(x_2) = \emptyset$ and $\omega \in h_1(\omega) \cap c_1^{-1}(c_1(\omega)) \cap x_2 \neq \emptyset$. Thus, the deviation is 2-identifiable by player 1. □

Proof of Theorem 1. It suffices to show that the information structure cannot be modified by any unilateral deviation from a perfectly communicating equilibrium, i.e.,

$$\mathcal{P}_i(x_j, c_{-j}(\omega), \omega) = h_i^c(\omega) = J(\omega),$$

for all $i, j \in N$, $\omega, \omega' \in \Omega$, and $x_j \in X_j(c, \omega)$. Let $c_i(\omega) = h_i(\omega)$ for all $i \in N$ and $\omega \in \Omega$. We get $c_i^{-1}(c_i(\omega)) = c_i(\omega) = h_i(\omega)$ and thus $h_i^c(\omega) = \bigcap_{k \in N} h_k(\omega) = J(\omega)$ for all $i \in N$ and $\omega \in \Omega$. Moreover, the condition $\bigvee_{k \neq j} H_k = \mathbb{J}$ is equivalent to $\bigcap_{k \neq j} h_k(\omega) = J(\omega)$ for all $\omega \in \Omega$. By the certifiability constraint we obtain $\mathcal{P}_i(x_j, c_{-j}(\omega), \omega) \subseteq \bigcap_{k \neq j} c_k(\omega) = \bigcap_{k \neq j} h_k(\omega) = J(\omega) = h_i^c(\omega)$ for all i, $j \in N$, ω, $\omega' \in \Omega$, and $x_j \in X_j(c, \omega)$. Moreover, admissible interpretation gives $J(\omega) \subseteq \mathcal{P}_i(x_j, c_{-j}(\omega), \omega)$. Hence, $\mathcal{P}_i(x_j, c_{-j}(\omega), \omega) = h_i^c(\omega)$. □

Proof of Corollary 1. Directly from Theorem 1. □

Proof of Proposition 5. Obvious. □

Proof of Theorem 2. Let ϕ^* be a full information outcome such that Assumption 1 is satisfied, and let $\succ$ be the associated strict ordering. Let $c(\omega) = \{\omega\}$ for all $\omega \in \Omega$, and let $\mathcal{P}_r(x) = \text{Maxi}\{x \mid H_1, \succeq\}$ for all $x \in \mathcal{X}$. By construction, the second stage information structure $\mathcal{P}$ is consistent with (c, X). We have to show that player 1 has no incentive to deviate from c. Let $\sigma \in \Sigma$ be an effective strategy profile such that for all $\omega \in \Omega$ and $x \in X(\omega)$,

$$\sigma(\{\omega\}, \omega) = \phi^*(\omega) \tag{9}$$

$$\sigma_{-1}(x, \omega) = \phi^*_{-1}(\overline{\omega}) \tag{10}$$

$$\sigma_1(x, \overline{\omega}) = \phi^*_1(\overline{\omega}), \tag{11}$$

where $\overline{\omega} \in \text{Maxi}\{x \mid H_1, \succeq\}$. If $\omega \neq \overline{\omega}$, then $\sigma_1(x, \omega)$ is a mixed strategy which assigns probability one to some action in $\arg\max_{a_1 \in A_1} u_1(a_1, \phi^*_{-1}(\overline{\omega}), \omega)$. Note that $\text{Maxi}\{x \mid H_1, \succeq\}$ is always reduced to a singleton since we consider a strict ordering.

We first show that $\sigma \in \Sigma^*(\mathcal{P})$, i.e.,

$$\begin{aligned} &\sum_{\omega' \in \Omega} p(\omega' \mid \mathcal{P}_i(x, \omega)) \sum_{a \in A} \sigma(a \mid x, \omega')\, u_i(a, \omega') \\ &\geq \sum_{\omega' \in \Omega} p(\omega' \mid \mathcal{P}_i(x, \omega)) \sum_{a_{-i} \in A_{-i}} \sigma_{-i}(a_{-i} \mid x, \omega')\, u_i(a_i, a_{-i}, \omega'), \end{aligned} \tag{12}$$

for all $i \in N$, $\omega \in \Omega$, $x \in X(\omega)$, and $a_i \in A_i$.

By assumption ϕ^* satisfies the following inequality for all $i \in N$, $\omega \in \Omega$ and $a_i \in A_i$:

$$\sum_{a \in A} \phi^*(a \mid \omega)\, u_i(a, \omega) \geq \sum_{a_{-i} \in A_{-i}} \phi^*_{-i}(a_{-i} \mid \omega)\, u_i(a_i, a_{-i}, \omega). \tag{13}$$

[15] Actually, a deviation by a player is always identifiable by this player.

If $x = \{\omega\}$, then Equations (9) and (13) give immediately (12). Now, let $x \neq \{\omega\}$, and let $\overline{\omega} \in \mathrm{Maxi}\{x \mid H_1, \succeq\}$. For player $i = 1$, Equation (12) is clearly satisfied. For players $i \neq 1$, since $\mathcal{P}_r(x) = \{\overline{\omega}\}$, Equation (12) is equivalent to

$$\sum_{a \in A} \sigma(a \mid x, \overline{\omega})\, u_i(a, \overline{\omega}) \geq \sum_{a_{-i} \in A_{-i}} \sigma_{-i}(a_{-i} \mid x, \overline{\omega})\, u_i(a_i, a_{-i}, \overline{\omega}),$$

i.e., from Equation (10) and (11),

$$\sum_{a \in A} \phi^*(a \mid \overline{\omega})\, u_i(a, \overline{\omega}) \geq \sum_{a_{-i} \in A_{-i}} \phi^*_{-i}(a_{-i} \mid \overline{\omega})\, u_i(a_i, a_{-i}, \overline{\omega}),$$

which is satisfied from Equation (13).

It remains to show that player 1 has no incentive to deviate from full communication given the second stage information structure $\mathcal{P}$ and the profile of effective strategies σ described above. That is, we must show that the condition for rational communication is satisfied: For all $\omega \in \Omega$ and $x \in X(\omega)$,

$$\sum_{a \in A} \sigma(a \mid \{\omega\}, \omega)\, u_1(a, \omega) \geq \sum_{a \in A} \sigma(a \mid x, \omega)\, u_1(a, \omega).$$

This inequality is equivalent to

$$\sum_{a \in A} \phi^*(a \mid \omega)\, u_1(a, \omega) \geq \max_{a_1 \in A_1} \sum_{a_{-1} \in A_{-1}} \phi^*_{-1}(a_{-1} \mid \overline{\omega})\, u_1(a_1, a_{-1}, \omega),$$

which is satisfied by Assumption (Equation (3)) since $x \in X(\omega) \Rightarrow \omega \in x$, and thus $\{\overline{\omega}\} \succeq \{\omega\}$. The condition of rational communication is obtained. This completes the proof. □

Proof of Theorem 3. The proof is similar to the Proof of Theorem 2. Let ϕ and ϕ^* be some Bayesian equilibria such that assumption 2 is satisfied, and let $\succ$ be the associated ordering. Let $c(\omega) = \Omega$ and $\mathcal{P}_r(x) = \mathrm{Maxi}\{x \mid H_1, \succeq\}$ for all $x \in \mathcal{X}$. By construction, this second information structure is consistent with (c, X). We have to show that player 1 has no incentive to deviate from c, i.e., has no incentive to reveal any event $x \subsetneq \Omega$. Let $\sigma \in \Sigma$ be an effective strategy profile such that for all $\omega \in \Omega$ and $x \in X(\omega)$, $x \neq \Omega$,

$$\sigma_{-1}(x, \omega) = \phi^*_{-1}(\overline{\omega}) \tag{14}$$
$$\sigma_1(x, \overline{\omega}) = \phi^*_1(\overline{\omega}) \tag{15}$$
$$\sigma(\Omega, \omega) = \phi(\omega), \tag{16}$$

where $\overline{\omega} \in \mathrm{Maxi}\{x \mid H_1, \succeq\}$. If $\omega \neq \overline{\omega}$, then $\sigma_1(x, \omega)$ is a mixed strategy which assigns probability one to some action in $\arg\max_{a_1 \in A_1} u_1(a_1, \phi^*_{-1}(\overline{\omega}), \omega)$. As in the previous proof, it is easy to check that $\sigma \in \Sigma^*(\mathcal{P})$. Accordingly, rational communication is equivalent to

$$\sum_{a \in A} \sigma(a \mid \Omega, \omega)\, u_1(a, \omega) \geq \sum_{a \in A} \sigma(a \mid x, \omega)\, u_1(a, \omega), \quad \forall\, \omega \in \Omega,\ x \in X(\omega).$$

From (14) and (16), this is equivalent to

$$\sum_{a \in A} \phi(a \mid \omega)\, u_1(a, \omega) \geq \max_{a_1 \in A_1} \sum_{a_{-1} \in A_{-1}} \phi^*_{-1}(a_{-1} \mid \overline{\omega})\, u_1(a_1, a_{-1}, \omega), \quad \forall\, \omega \in \Omega,\ x \in X(\omega) \backslash \{\Omega\}.$$

Since $\omega \in x$, we have $\{\overline{\omega}\} \succeq \{\omega\}$, and thus the inequality follows from Assumption 2. This completes the proof. $\square$

Proof of Proposition 6. It suffices to apply the reasoning of the Proof of Theorem 2 with $\{\omega'\} \succ \{\omega\}$ for all $\omega \neq \omega'$. In that case, $\mathcal{P}_r(\Omega) = \{\omega'\}$. $\square$

Proof of Theorem 4. Consider a communication strategy profile c satisfying

$$\begin{aligned} c_i(\omega) &= h_i(\omega), && \forall\, i \in N^* \\ c_j(\omega) &= \Omega, && \forall\, j \notin N^*. \end{aligned}$$

By the definition of N^*, we have $h_i^c(\omega) = \bigcap_{k \in N^*} c_k(\omega) = \bigcap_{k \in N^*} h_k(\omega) = J(\omega)$ for all $i \in N$ and $\omega \in \Omega$. For all $\omega \in \Omega$, let $a^*(\omega)$ be the cooperative level at ω. Consider any second stage information structure $\mathcal{P}$ consistent with (c, X). Consider an effective strategy profile $\sigma \in \Sigma$ satisfying

$$\sigma_i(a^*(\omega) \mid c(\omega), \omega) = 1, \quad \forall\, i \in N,\ \omega \in \Omega.$$

Since $a^*(\omega) = a^*(\omega')$ for all $\omega \in \Omega$ and $\omega' \in P(\omega)$, and since $\mathcal{P}_i(c(\omega), \omega) = h_i^c(\omega) = J(\omega) \subseteq P(\omega)$ for all $\omega \in \Omega$ and $i \in N$, we have

$$U_i(\sigma, c(\omega), \mathcal{P}_i, \omega') = u_i(a^*(\omega), \omega), \tag{17}$$

for all $i \in N$, $\omega \in \Omega$, and $\omega' \in J(\omega)$. By assumption, since $a^*(\omega)$ is the cooperative level at ω,

$$u_i(a^*(\omega), \omega) \geq u_i(a, \omega), \tag{18}$$

for all $i \in N$, $\omega \in \Omega$ and $a \in A$. Hence,

$$U_i(\sigma, c(\omega), \mathcal{P}_i, \omega') \geq U_i(a_i, \sigma_{-i}, c(\omega), \mathcal{P}_i, \omega'),$$

for all $i \in N$, $\omega \in \Omega$, and $\omega' \in J(\omega)$. Therefore, $\sigma \in \Sigma^*(\mathcal{P}, c(\omega))$ for all $\omega \in \Omega$. For $x \neq c(\omega)$ for all $\omega \in \Omega$, assume further that $\sigma \in \Sigma^*(\mathcal{P}, x)$. We now check for the condition of rational communication. From Equations (17) and (18) we have

$$U_i(\sigma, c(\omega), \mathcal{P}_i, \omega) \geq U_i(\sigma, x_i, c_{-i}(\omega), \mathcal{P}_i, \omega),$$

for all $i \in N$, $\omega \in \Omega$ and $x_i \in X_i(\omega)$, which implies

$$EU_i(\sigma, c, \mathcal{P}_i \mid h_i(\omega)) \geq EU_i(\sigma, x_i, c_{-i}, \mathcal{P}_i \mid h_i(\omega)).$$

Thus, $(\sigma, c, \mathcal{P})$ is a knowledge equilibrium. Pareto efficiency is obvious by construction. This completes the proof. $\square$

Proof of Theorem 5. For all $\omega \in \Omega$, let $a^*(\omega)$ be a Nash equilibrium of the game $G(\omega)$. Consider the perfectly communicating strategy profile c, i.e., $c_i(\omega) = h_i(\omega)$ for all $\omega \in \Omega$ and $i \in N$. Let $\mathcal{P}$ be a second stage information structure consistent with (c, X). Of course, $\mathcal{P}_i(c(\omega), \omega) = h_i^c(\omega) = \{\omega\}$ for all $\omega \in \Omega$ and $i \in N$ (from Bayesian updating or from the certifiability constraint). From Proposition 3 we have for $j \neq i$:

$$\mathcal{P}_j(x_i, c_{-i}(\omega), \omega) = \text{Maxi}\{h^{-i}(\omega) \cap x_i \mid H_i, \succeq_i\},$$

and, of course,

$$\mathcal{P}_i(x_i, c_{-i}(\omega), \omega) = \{\omega\}.$$

Consider an ordering $\succeq_i$ such that

$$h_i(\omega') \succ_i h_i(\omega) \quad \Leftrightarrow \quad \omega' > \omega \text{ and } h_i(\omega') \neq h_i(\omega).$$

We obtain, for all $j \neq i$,

$$\mathcal{P}_j(x_i, c_{-i}(\omega), \omega) = \max\{\omega' \in \Omega : \omega' \in h^{-i}(\omega) \cap x_i\},$$

which is clearly a singleton.

For all $i \in N$, let $s_j(x_i, c_{-i}(\omega), \omega) = a_j^*(\max\{\omega' \in \Omega : \omega' \in h^{-i}(\omega) \cap x_i\})$ for all $j \neq i$. It is easy to verify that this effective strategy satisfies sequential rationality given the previous second stage information structure. Impose further that $s_i(c(\omega), \omega) = a_i^*(\{\omega\})$. It remains to check for rational communication. For all $i \in N$, $\omega \in \Omega$ we have

$$\begin{aligned}
& EU_i(s, c, \mathcal{P}_i \mid h_i(\omega)) \geq EU_i(s, x_i, c_{-i}, \mathcal{P}_i \mid h_i(\omega)) \\
\Leftrightarrow \quad & \sum_{\omega' \in \Omega} p(\omega' \mid h_i(\omega))\, U_i(s, c(\omega'), \mathcal{P}_i, \omega') \geq \sum_{\omega' \in \Omega} p(\omega' \mid h_i(\omega))\, U_i(s, x_i, c_{-i}(\omega'), \mathcal{P}_i, \omega') \\
\Leftarrow \quad & U_i(s, c(\omega), \mathcal{P}_i, \omega) \geq U_i(s, x_i, c_{-i}(\omega), \mathcal{P}_i, \omega), \quad \forall\, \omega \in \Omega \\
\Leftrightarrow \quad & \sum_{\omega' \in \Omega} p(\omega' \mid \mathcal{P}_i(c(\omega), \omega))\, u_i(s(c(\omega), \omega'), \omega') \geq \\
& \sum_{\omega' \in \Omega} p(\omega' \mid \mathcal{P}_i(x_i, c_{-i}(\omega), \omega))\, u_i(s(x_i, c_{-i}(\omega), \omega'), \omega'), \quad \forall\, \omega \in \Omega \\
\Leftrightarrow \quad & u_i(s(c(\omega), \omega), \omega) \geq u_i(s(x_i, c_{-i}(\omega), \omega), \omega), \quad \forall\, \omega \in \Omega.
\end{aligned}$$

Given the effective strategies considered before, the last inequality is equivalent to

$$u_i(a^*(\omega), \omega) \geq u_i(a_i, a_{-i}^*(\overline{\omega}), \omega), \quad \forall\, \omega \in \Omega,\ a_i \in A_i,$$

where $\overline{\omega} = \max\{\omega' \in \Omega : \omega' \in h^{-i}(\omega) \cap x_i\}$. This completes the proof. □

Proof of Theorem 6. We show that conditions of Theorem 5 are satisfied. First, let us determine the (unique) Nash equilibrium of $G(\omega)$ for all $\omega \in \Omega$. For all $i \in N$ we have

$$\frac{\partial u_i(a, \omega)}{\partial a_i} = \alpha_i\Big(\tau(\omega) + \gamma_i - \beta \sum_{j \neq i} a_j - 2a_i\Big) = 0. \tag{19}$$

Hence, the best response of player i against a_{-i} at $\omega \in \Omega$ is

$$BR_i(a_{-i}, \omega) = \frac{\tau(\omega) + \gamma_i - \beta \sum_{j \neq i} a_j}{2}.$$

Equation (19) is satisfied for all $i \in N$ if and only if

$$2a = \tau(\omega)e + \gamma - \beta\big(e\,{}^t e - Id\big)a,$$

where ${}^t e = (1, \ldots, 1)$ and Id is the identity matrix. Equivalently,

$$\begin{aligned}
& a\Big(2Id + \beta(e\,{}^t e - Id)a\Big) = \tau(\omega)e + \gamma \\
\Leftrightarrow \quad & a\Big(Id + \frac{\beta}{2 - \beta} e\,{}^t e\Big) = \frac{\tau(\omega)e + \gamma}{2 - \beta}.
\end{aligned} \tag{20}$$

Let $M = Id + \frac{\beta}{2-\beta} e\,{}^t e$ and $k = \frac{\beta}{2-\beta}$. We compute the inverse of M. We have

$$\begin{aligned} & MM^{-1} = \left(Id + ke\,{}^t e\right)\left(Id - k'e\,{}^t e\right) = Id \\ \Leftrightarrow \quad & Id - k'e\,{}^t e + ke\,{}^t e - kk'ne\,{}^t e = Id \\ \Leftrightarrow \quad & k - k'(1+kn) = 0 \\ \Leftrightarrow \quad & k' = \frac{k}{1+kn} = \frac{\beta}{2+\beta(n-1)}. \end{aligned}$$

Hence, $M^{-1} = \left(Id - \frac{\beta}{2+\beta(n-1)} e\,{}^t e\right)$. Equation (20) is then equivalent to

$$a = \frac{1}{2-\beta}\left(Id - \frac{\beta}{2+\beta(n-1)} e\,{}^t e\right)\left(\tau(\omega)e + \gamma\right).$$

We get

$$a_i^*(\omega) = \frac{\tau(\omega)}{2+\beta(n-1)} + \frac{\gamma_i(2+\beta(n-2)) - \beta\sum_{j\neq i}\gamma_j}{(2-\beta)(2+\beta(n-1))}. \tag{21}$$

Thereafter, remark that if $a_i = BR_i(a_{-i}, \omega)$, then

$$u_i(a_i, a_{-i}, \omega) = \alpha_i\, {a_i}^2.$$

Since $a_j^*(\omega)$ is increasing with ω for all $j \in N$, we have $BR_i(a_{-i}^*(\omega), \omega) \leq BR_i(a_{-i}^*(\omega'), \omega')$ whenever $\omega' \geq \omega$, which implies that $u_i(a_i, a_{-i}^*(\omega), \omega) \leq u_i(a_i, a_{-i}^*(\omega'), \omega')$ for all $i \in N$ and $a_i \in A_i$. Hence, conditions of Theorem 5 are satisfied. □

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