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Turning Box-Cox including Quadratic Forms in Regression

Marc Gaudry [1, 2, 3]
Ulrich Blum [4]
Tran Liem [1]

[1] Agora Jules Dupuit (AJD)
Centre de recherche sur les transports (CRT), Montréal
<www.crt.umontreal.ca/crt/AgoraJulesDupuit/>

[2] Bureau d'économie théorique et appliquée (BETA)
Université Louis Pasteur
Strasbourg

[3] Département de sciences économiques
Université de Montréal, Montréal
<gaudry@crt.umontreal.ca>

[4] Chair of Economic Policy Research
Dresden University of Technology
<blum@wipo.wiwi.tu-dresden.de>

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ABSTRACT

In a regression model where a Box-Cox transformation is used on a positive independent variable X which appears only once in the equation, the effect of X on the dependent variable Y is either strictly increasing or decreasing over the whole range of X, since the transformation is a monotonic function of X, increasing or decreasing depending on the Box-Cox parameter λ. This paper considers the case where the variable X appears twice in the regression with two different Box-Cox parameters λ_1 and λ_2, to allow a turning point in Y which can be a maximum or minimum. First and second-order conditions for the critical point are derived. This general specification includes as a special case the quadratic form in X where λ_1 and λ_2 are set equal to 1 and 2, respectively. If, instead of using the Box-Cox transformations, one uses simple powers of X, this form is equivalent to the Box-Cox form except that neither λ_1 nor λ_2 can be equal to zero, since in this case X^{λ_1} or X^{λ_2} reduces to a constant of value 1.

Keywords: Box-Cox Transformation, Quadratic Form, Asymmetric U-shaped Forms, Regression.

RÉSUMÉ

Dans un modèle de régression où l'on utilise une transformation de Box et Cox sur une variable indépendante positive X qui n'apparaît qu'une seule fois dans l'équation, l'effet de X sur la variable dépendante Y est soit strictement croissant ou décroissant sur tout l'intervalle de X, car la transformation est une fonction monotone croissante ou décroissante de X, selon le paramètre Box-Cox λ. Cet article considère le cas où la variable X apparaît deux fois dans la régression avec deux paramètres Box-Cox différents λ_1 et λ_2, pour permettre un point de retournement dans Y qui peut être un maximum ou minimum. On dérive les conditions du premier et du second ordre pour trouver le point critique. Cette spécification générale inclut comme cas particulier la forme quadratique en X où λ_1 et λ_2 sont fixés à 1 et 2, respectivement. Au lieu d'utiliser les transformations Box-Cox, on peut aussi employer de simples puissances de X. Cette forme est équivalente à celle de Box et Cox excepté que ni λ_1 ni λ_2 ne peut être égal à zéro, puisque dans ce cas X^{λ_1} ou X^{λ_2} se réduit à une constante de valeur 1.

Mots-clés: Transformation Box-Cox, Forme quadratique, Forme en U asymétrique, Régression.

ZUSAMMENFASSUNG

In einem Regressionsmodell, in dem eine Box-Cox-Transformation auf eine unabhängige Variable X angewendet wird, die nur einmal in der Gleichung vorkommt, ist der Einfluß von X auf die unabhängige Variable Y entweder streng monoton wachsend oder fallend über den gesamten Definitionsbereich von X, da die Transformation selber eine monotone Funktion darstellt, die nur vom Box-Cox-Parameter λ abhängt. In diesem Artikel wird der Fall betrachtet, daß die Variable X zweimal in der Regression mit zwei unterschiedlichen Box-Cox-Parametern λ_1 und λ_2 vorkommt, um eine Extremalstelle, die ein Maximum oder ein Minimum sein kann, zu ermöglichen. Die Bedingungen erster und zweiter Ordnung des kritischen Punkts werden abgeleitet. Die allgemeine Spezifikation enthält als einen Spezialfall die quadratische Form in X, in der λ_1 und λ_2 die Werte 1 und 2 annehmen. Wenn man anstelle der Box-Cox-Transformationen einfache Potenzen von X verwendet, ist diese Spezifikation äquivalent zu der Box-Cox-Transformation, nur können dann weder λ_1 noch λ_2 gleich null sein, da in diesem Fall X^{λ_1} oder X^{λ_2} einen konstanten Wert von eins annähmen.

Stichworte : Box-Cox-Transformation, quadratische Form, asymmetrische U-förmige Formen, Regression.

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TURNING BOX-COX INCLUDING QUADRATIC FORMS IN REGRESSION

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1 Model with two Box-Cox transformations on a same independent variable

Our problem[1], prompted by the increasing search for flexible turning functions, for instance in the explanation of how some transport costs might vary with distance, or of how fatal road accident frequency and severity fall might rise and then fall[2] with increased traffic (Tegnér and Loncar-Lucassi, 1997), or of how the impact of alcohol on fatal road accident frequency might be U-shaped[3], is to find the conditions under which the function $y(X)$ has a maximum or minimum over the positive region of X in the following model:

$$y^{(\lambda_y)} = \beta_0 + \beta_1 X^{(\lambda_1)} + \beta_2 X^{(\lambda_2)} + \ldots + u \tag{1}$$

where the positive independent variable X is transformed by two different Box-Cox parameters ($\lambda_1 \neq \lambda_2$) so that the model is identified in terms of the transformed variables.

Solution

The first derivative of $y(X)$ with respect to X is:

$$\frac{\partial y}{\partial X} = \frac{\partial y}{\partial y^{(\lambda_y)}} \frac{\partial y^{(\lambda_y)}}{\partial X} = \frac{1}{y^{\lambda_y - 1}} \left[\beta_1 X^{\lambda_1 - 1} + \beta_2 X^{\lambda_2 - 1} \right] \quad . \tag{2}$$

Equating this derivative to zero and solving for a critical point of X give:

$$X^* = \left(-\frac{\beta_1}{\beta_2} \right)^{\frac{1}{\lambda_2 - \lambda_1}} = \left(-\frac{\beta_2}{\beta_1} \right)^{\frac{1}{\lambda_1 - \lambda_2}} \quad . \tag{3}$$

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[1] This paper is the final version of, and cancels, a manuscript, widely circulated in Canada, Germany, Sweden and the United States, usually called: Gaudry, M., "FIQ: Fractional and Integer Quadratic Forms Estimated with the LEVEL algorithm in TRIO", 5 pages, November 5, 1996, augmented on March 9, 1997, and on October 28, 1997. A fourth version, produced with Ulrich Blum on March 20, 1999, was not circulated.

[2] The SAAQ in Quebec City has established such relationships since December 1992 with symmetric forms. Since 1997, asymmetric forms are used: they are now part of the official model for Quebec (Fournier et Simard, 1999) and imply that traffic reaches a point where additional vehicles have no impact on accident frequency or even reduce accident frequency, i.e. confer a positive externality.

[3] For a summary, see in particular Chapter 1 in Gaudry and Lassarre (2000).

First-order conditions

1. Since the independent variable X is always positive, the critical value of X should also be positive. Hence the coefficients β_1 and β_2 should have opposite signs for any values of the λ 's ($\lambda_1 \neq \lambda_2$).
2. Conversely, if the coefficients have the same sign, then X^* does not exist, i.e. there is no maximum or minimum.

To determine that the critical point X^* corresponds to a maximum or a minimum, we have to analyze the sign of the second derivative of $y(X)$ at this point:

$$\begin{aligned}\left.\frac{\partial^2 y}{\partial X^2}\right|_{X=X^*} &= \frac{1}{y^{\lambda_y - 1}}\left[\beta_1(\lambda_1 - 1)X^{*\,\lambda_1-2} + \beta_2(\lambda_2 - 1)X^{*\,\lambda_2-2}\right] \\ &= \frac{X^{*\,\lambda_1-2}}{y^{\lambda_y-1}}\left[\beta_1(\lambda_1 - 1) + \beta_2(\lambda_2 - 1)X^{*\,\lambda_2-\lambda_1}\right] \\ &= \frac{X^{*\,\lambda_1-2}}{y^{\lambda_y-1}}\left[\beta_1(\lambda_1 - 1) + \beta_2(\lambda_2 - 1)\left(-\frac{\beta_1}{\beta_2}\right)\right] \\ &= \frac{X^{*\,\lambda_1-2}}{y^{\lambda_y-1}}\left[\beta_1(\lambda_1 - 1) - \beta_1(\lambda_2 - 1)\right] \\ &= \frac{X^{*\,\lambda_1-2}}{y^{\lambda_y-1}}\beta_1(\lambda_1 - \lambda_2) \quad .\end{aligned} \tag{4}$$

Since the terms $X^{*\,\lambda_1-2}$ and y^{λ_y-1} are always positive, the sign of the second derivative at X^* depends only on the sign of the product $\beta_1(\lambda_1 - \lambda_2)$. By factoring out $X^{*\,\lambda_2-2}$ instead of $X^{*\,\lambda_1-2}$ like above, an equivalent property can be derived:

$$\begin{aligned}\left.\frac{\partial^2 y}{\partial X^2}\right|_{X=X^*} &= \frac{1}{y^{\lambda_y - 1}}\left[\beta_1(\lambda_1 - 1)X^{*\,\lambda_1-2} + \beta_2(\lambda_2 - 1)X^{*\,\lambda_2-2}\right] \\ &= \frac{X^{*\,\lambda_2-2}}{y^{\lambda_y-1}}\left[\beta_1(\lambda_1 - 1)X^{*\,\lambda_1-\lambda_2} + \beta_2(\lambda_2 - 1)\right] \\ &= \frac{X^{*\,\lambda_2-2}}{y^{\lambda_y-1}}\left[\beta_1(\lambda_1 - 1)\left(-\frac{\beta_2}{\beta_1}\right) + \beta_2(\lambda_2 - 1)\right] \\ &= \frac{X^{*\,\lambda_2-2}}{y^{\lambda_y-1}}\left[-\beta_2(\lambda_1 - 1) + \beta_2(\lambda_2 - 1)\right] \\ &= \frac{X^{*\,\lambda_2-2}}{y^{\lambda_y-1}}\beta_2(\lambda_2 - \lambda_1) \quad .\end{aligned} \tag{5}$$

Since the terms $X^{*\,\lambda_2-2}$ and y^{λ_y-1} are always positive, the sign of the second derivative at X^* depends only on the sign of the product $\beta_2(\lambda_2 - \lambda_1)$.

Second-order conditions

1. X^* corresponds to a maximum if the second derivative at X^*, i.e. the product $\beta_1(\lambda_1-\lambda_2)$ or $\beta_2(\lambda_2-\lambda_1)$, is negative.
2. X^* corresponds to a minimum if the second derivative at X^*, i.e. the product $\beta_1(\lambda_1-\lambda_2)$ or $\beta_2(\lambda_2-\lambda_1)$, is positive.

Table 1 combines the first and second-order conditions to obtain a maximum or minimum at X^*. Due to the first-order condition that the two β 's should have opposite signs, the second-order conditions that the product $\beta_1(\lambda_1-\lambda_2)$ is negative for a maximum and positive for a minimum are equivalent to the ones with the product $\beta_2(\lambda_2-\lambda_1)$. Moreover, for a given set of specific values of $(\beta_1,\lambda_1,\beta_2,\lambda_2)$, Maximum1 and Maximum2 are equivalent, and so are Minimum1 and Minimum2, due to the interchangeability of the values of the pairs (β_1,λ_1) and (β_2,λ_2).

Table 1. Conditions for a maximum or minimum in the model with Box-Cox transformations

CASE	β_1	β_2	$\lambda_1-\lambda_2$	$\beta_1(\lambda_1-\lambda_2)$ or $\beta_2(\lambda_2-\lambda_1)$
Maximum1	+	−	−	−
Minimum1	+	−	+	+
Maximum2	−	+	+	−
Minimum2	−	+	−	+

Figures 1 and 2 illustrate respectively the first two cases where the dependent variable y is not transformed by a Box-Cox for simplicity reasons:

1. Maximum1: $y=5+3X^{(1.1)}-2X^{(2.5)}$, that is equivalent to Maximum2: $y=5-2X^{(2.5)}+3X^{(1.1)}$ when the values of the pairs (β_1,λ_1) and (β_2,λ_2) are permuted.
2. Minimum1: $y=3+2X^{(1.6)}-4X^{(-2.1)}$, that is equivalent to Minimum2: $y=3-4X^{(-2.1)}+2X^{(1.6)}$ when the values of the pairs (β_1,λ_1) and (β_2,λ_2) are permuted.

Figure 1. Graph of $y = 5 + 3X^{(1.1)} - 2X^{(2.5)}$, with Maximum1 at (1.3359, 5.173)

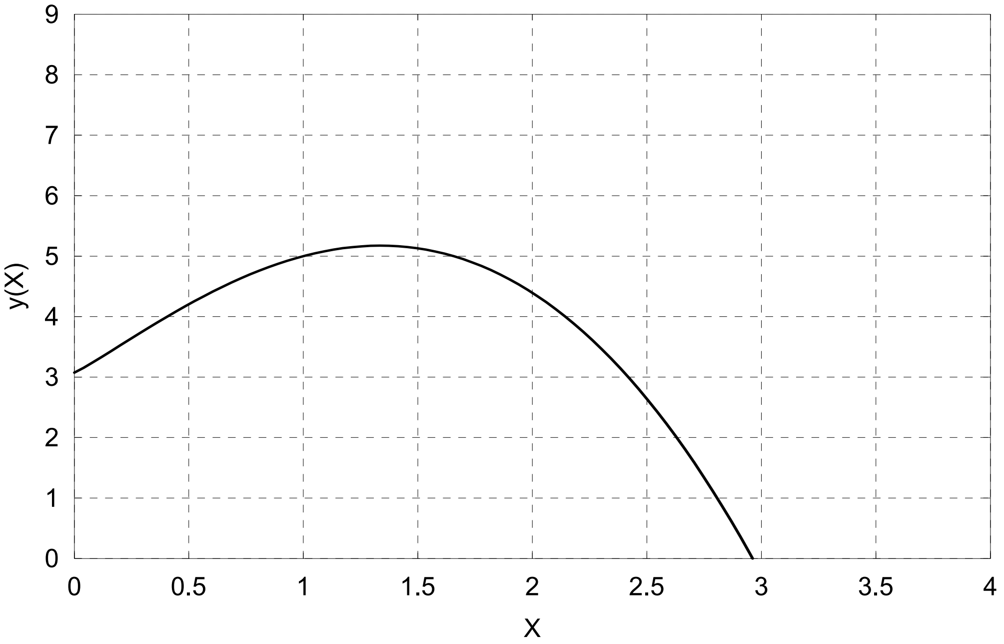
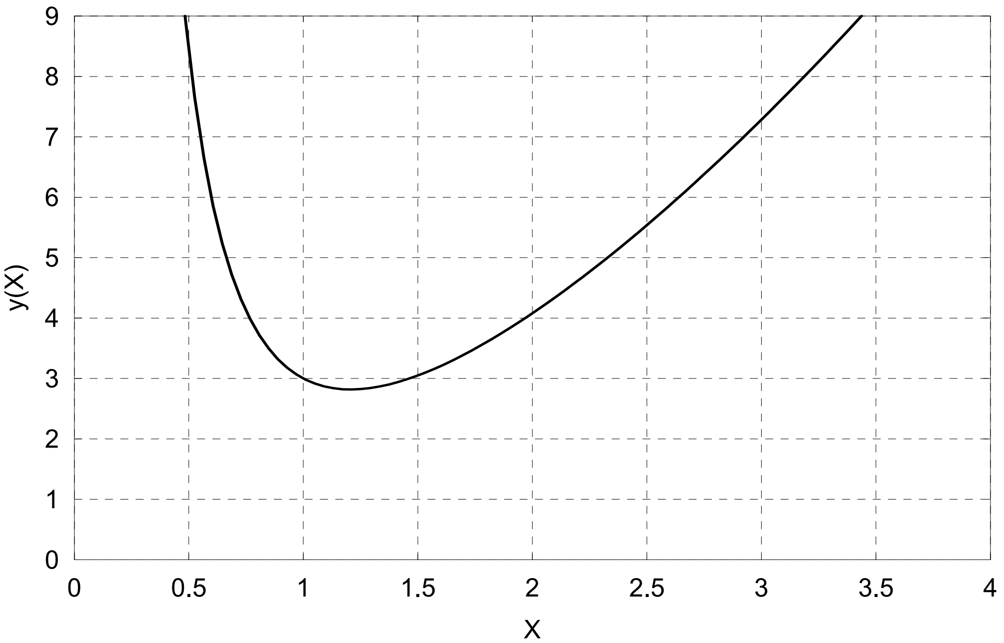


Figure 2. Graph of $y = 3 + 2X^{(1.6)} - 4X^{(-2.1)}$, with Minimum1 at (1.206, 2.8174)



Special case: Quadratic form

The quadratic form can be obtained by setting $\lambda_1 = 1$ and $\lambda_2 = 2$ in model (1):

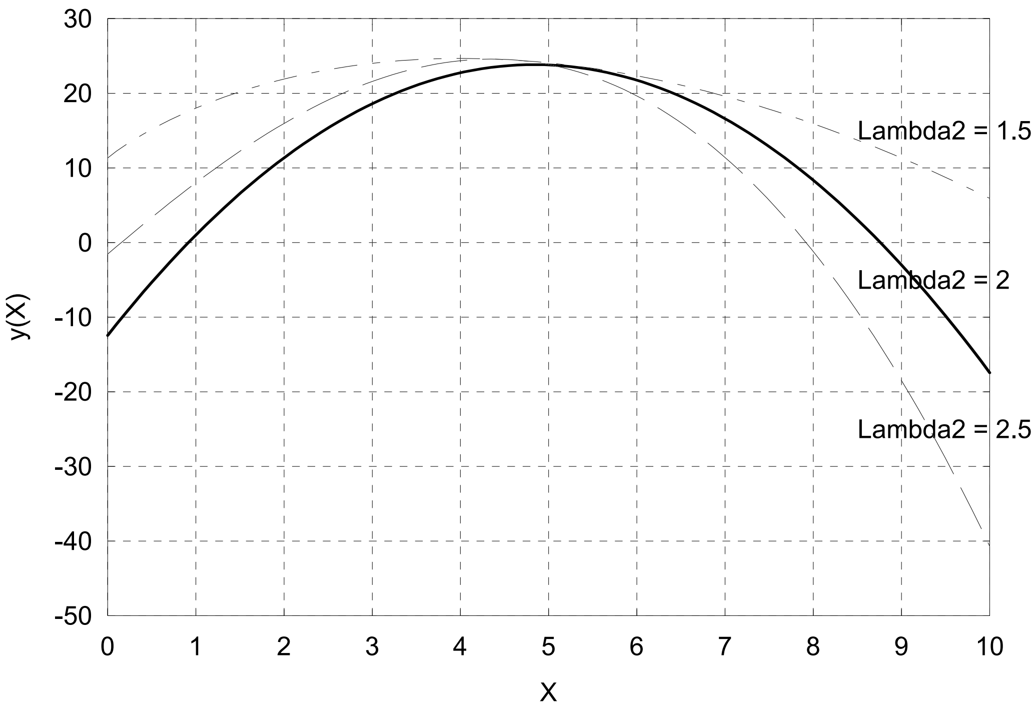
$$y^{(\lambda_y)} = \beta_0 + \beta_1 X^{(1)} + \beta_2 X^{(2)} + \ldots + u \quad . \tag{6}$$

From Table 1, we can only have two cases where $\lambda_1 - \lambda_2$ is negative:

1. Maximum1 if $\beta_1 > 0$ and $\beta_2 < 0$, that is equivalent to Maximum2 when the values of the pairs (β_1, λ_1) and (β_2, λ_2) are permuted.
2. Minimum2 if $\beta_1 < 0$ and $\beta_2 > 0$, that is equivalent to Minimum1 when the values of the pairs (β_1, λ_1) and (β_2, λ_2) are permuted.

If we consider only the portion $y(X) = \beta_0 + \beta_1 X^{(1)} + \beta_2 X^{(2)}$ in (6), it is a quadratic function of X which is symmetric with respect to a maximum or minimum point X^*. If λ_2 differs from 2, then $y(X)$ is a nonlinear function which is no longer symmetric. Figure 3 illustrates the symmetry/asymmetry property of the function $y(X)$ for λ_2 = 1.5, 2, 2.5.

Figure 3. Symmetric ($\lambda_2 = 2$) and Asymmetric ($\lambda_2 \neq 2$) Forms



2 Model with powers λ_1 and λ_2 only on a same independent variable

The model (1) can be rewritten in terms of powers λ_1 and λ_2 on X as:

$$y^{(\lambda_y)} = \beta_0^* + \beta_1^* X^{\lambda_1} + \beta_2^* X^{\lambda_2} + \ldots + u \qquad (\lambda_1 \text{ and } \lambda_2 \neq 0) \tag{7}$$

where the new β^*'s are related to the original β's as follows:

$$\beta_0^* = \beta_0 - \frac{\beta_1}{\lambda_1} - \frac{\beta_2}{\lambda_2}$$

$$\beta_1^* = \frac{\beta_1}{\lambda_1}$$

$$\beta_2^* = \frac{\beta_2}{\lambda_2} \quad .$$

This model is not equivalent to the model (1) with Box-Cox transformations for two reasons:

1. The simple power transformation X^{λ_1} or X^{λ_2} does not include the logarithmic form of the variable X.
2. The ordering of the data is not preserved when the parameter λ_1 or λ_2 changes its sign. For example, consider two values of X, say 10 and e where $10 > e$, then $10^{\lambda_1} < e^{\lambda_1}$ if $\lambda_1 < 0$, and $10^{\lambda_1} > e^{\lambda_1}$ if $\lambda_1 > 0$, whereas with the Box-Cox transformation, $(10^{\lambda_1} - 1)/\lambda_1 > (e^{\lambda_1} - 1)/\lambda_1$ for any value of λ_1.

Hence, in practice, we cannot estimate jointly the β^*'s and λ 's, but only the β^*'s with fixed values of the λ 's.

The first derivative of y with respect to X is:

$$\frac{\partial y}{\partial X} = \frac{\partial y}{\partial y^{(\lambda_y)}} \frac{\partial y^{(\lambda_y)}}{\partial X} = \frac{1}{y^{\lambda_y - 1}} \left[\beta_1^* \lambda_1 X^{\lambda_1 - 1} + \beta_2^* \lambda_2 X^{\lambda_2 - 1}\right] \quad . \tag{8}$$

Equating this derivative to zero and solving for a critical value of X give:

$$X^{**} = \left(-\frac{\beta_1^* \lambda_1}{\beta_2^* \lambda_2}\right)^{\frac{1}{\lambda_2 - \lambda_1}} = \left(-\frac{\beta_2^* \lambda_2}{\beta_1^* \lambda_1}\right)^{\frac{1}{\lambda_1 - \lambda_2}} \quad . \tag{9}$$

First-order conditions

1. Since the independent variable X is always positive, the critical value of X should also be positive. Hence the terms $\beta_1^* \lambda_1$ and $\beta_2^* \lambda_2$ should have opposite signs for any values of the λ 's ($\lambda_1 \neq \lambda_2$), implying that if the two β^*'s have the same sign, the two λ 's should have opposite signs and vice versa.
2. Conversely, if the terms have the same sign, then X^* does not exist, i.e. there is no maximum or minimum.

To determine that the critical point X^{**} corresponds to a maximum or a minimum, we should analyze the sign of the second derivative of $y(X)$ at this point:

$$\begin{aligned}
\left.\frac{\partial^2 y}{\partial X^2}\right|_{X=X^{**}} &= \frac{1}{y^{\lambda_y - 1}}\left[\beta_1^* \lambda_1(\lambda_1 - 1)X^{**^{\lambda_1 - 2}} + \beta_2^* \lambda_2(\lambda_2 - 1)X^{**^{\lambda_2 - 2}}\right] \\
&= \frac{X^{**^{\lambda_1 - 2}}}{y^{\lambda_y - 1}}\left[\beta_1^* \lambda_1(\lambda_1 - 1) + \beta_2^* \lambda_2(\lambda_2 - 1)X^{**^{\lambda_1 - \lambda_1}}\right] \\
&= \frac{X^{**^{\lambda_1 - 2}}}{y^{\lambda_y - 1}}\left[\beta_1^* \lambda_1(\lambda_1 - 1) + \beta_2^* \lambda_2(\lambda_2 - 1)\left(-\frac{\beta_1^* \lambda_1}{\beta_2^* \lambda_2}\right)\right] \\
&= \frac{X^{**^{\lambda_1 - 2}}}{y^{\lambda_y - 1}}\left[\beta_1^* \lambda_1(\lambda_1 - 1) - \beta_1^* \lambda_1(\lambda_2 - 1)\right] \\
&= \frac{X^{**^{\lambda_1 - 2}}}{y^{\lambda_y - 1}}\beta_1^* \lambda_1(\lambda_1 - \lambda_2) \quad .
\end{aligned} \tag{10}$$

Since the terms $X^{**^{\lambda_1 - 2}}$ and $y^{\lambda_y - 1}$ are always positive, the sign of the second derivative at X^{**} depends only on the sign of the term $\beta_1^* \lambda_1(\lambda_1 - \lambda_2)$.

By factoring out $X^{**^{\lambda_2 - 2}}$ instead of $X^{**^{\lambda_1 - 2}}$ like above, an equivalent property can be derived:

$$\begin{aligned}
\left.\frac{\partial^2 y}{\partial X^2}\right|_{X=X^{**}} &= \frac{1}{y^{\lambda_y - 1}}\left[\beta_1^* \lambda_1(\lambda_1 - 1)X^{**^{\lambda_1 - 2}} + \beta_2^* \lambda_2(\lambda_2 - 1)X^{**^{\lambda_2 - 2}}\right] \\
&= \frac{X^{**^{\lambda_2 - 2}}}{y^{\lambda_y - 1}}\left[\beta_1^* \lambda_1(\lambda_1 - 1)X^{**^{\lambda_1 - \lambda_2}} + \beta_2^* \lambda_2(\lambda_2 - 1)\right] \\
&= \frac{X^{**^{\lambda_2 - 2}}}{y^{\lambda_y - 1}}\left[\beta_1^* \lambda_1(\lambda_1 - 1)\left(-\frac{\beta_2^* \lambda_2}{\beta_1^* \lambda_1}\right) + \beta_2^* \lambda_2(\lambda_2 - 1)\right] \\
&= \frac{X^{**^{\lambda_2 - 2}}}{y^{\lambda_y - 1}}\left[-\beta_2^* \lambda_2(\lambda_1 - 1) + \beta_2^* \lambda_2(\lambda_2 - 1)\right] \\
&= \frac{X^{**^{\lambda_2 - 2}}}{y^{\lambda_y - 1}}\beta_2^* \lambda_2(\lambda_2 - \lambda_1) \quad .
\end{aligned} \tag{11}$$

Since the terms $X^{**^{\lambda_2 - 2}}$ and $y^{\lambda_y - 1}$ are always positive, the sign of the second derivative at X^{**} depends only on the sign of the term $\beta_2^* \lambda_2(\lambda_2 - \lambda_1)$.

Second-order conditions

1. X^{**} corresponds to a maximum if the second derivative at X^{**}, i.e. the term $\beta_1^* \lambda_1(\lambda_1 - \lambda_2)$ or $\beta_2^* \lambda_2(\lambda_2 - \lambda_1)$, is negative.
2. X^{**} corresponds to a minimum if the second derivative at X^{**}, i.e. the term $\beta_1^* \lambda_1(\lambda_1 - \lambda_2)$ or $\beta_2^* \lambda_2(\lambda_2 - \lambda_1)$, is positive.

Table 2 combines the first and second-order conditions to obtain a maximum or minimum at X^{**}. Due to the first-order condition that the terms $\beta_1^*\lambda_1$ and $\beta_2^*\lambda_2$ should have opposite signs, the second-order conditions that the term $\beta_1^*\lambda_1(\lambda_1 - \lambda_2)$ is negative for a maximum and positive for a minimum are equivalent to the ones with the term $\beta_2^*\lambda_2(\lambda_2 - \lambda_1)$. The first eight cases correspond to $\beta_1^*\lambda_1 > 0$ and $\beta_2^*\lambda_2 < 0$, whereas the last eight correspond to $\beta_1^*\lambda_1 < 0$ and $\beta_2^*\lambda_2 > 0$.

Table 2. Conditions for a maximum or minimum in the model with powers λ_1 and λ_2 only

CASE	β_1^*	λ_1	$\beta_1^*\lambda_1$	β_2^*	λ_2	$\beta_2^*\lambda_2$	$\lambda_1 - \lambda_2$	$\beta_1^*\lambda_1(\lambda_1 - \lambda_2)$ or $\beta_2^*\lambda_2(\lambda_2 - \lambda_1)$
Maximum1.1	−	−	+	−	+	−	−	−
NoMin1.1							+	Since $\lambda_1 < 0$ and $\lambda_2 > 0$, the condition $\lambda_1 - \lambda_2 > 0$ cannot be satisfied for a minimum.
Maximum1.2	−	−	+	+	−	−	−	−
Minimum1.2							+	+
Maximum1.3	+	+	+	−	+	−	−	−
Minimum1.3							+	+
NoMax1.4	+	+	+	+	−	−	−	Since $\lambda_1 > 0$ and $\lambda_2 < 0$, the condition $\lambda_1 - \lambda_2 < 0$ cannot be satisfied for maximum.
Minimum1.4							+	+
Maximum2.1	−	+	−	−	−	+	+	−
NoMin2.1							−	Since $\lambda_1 > 0$ and $\lambda_2 < 0$, the condition $\lambda_1 - \lambda_2 < 0$ cannot be satisfied for a minimum.
Maximum2.2	+	−	−	−	−	+	+	−
Minimum2.2							−	+
Maximum2.3	−	+	−	+	+	+	+	−
Minimum2.3							−	+
NoMax2.4	+	−	−	+	+	+	+	Since $\lambda_1 < 0$ and $\lambda_2 > 0$, the condition $\lambda_1 - \lambda_2 > 0$ cannot be satisfied for a maximum.
Minimum2.4							−	+

Due to the interchangeability of the values of the pairs (β_1^*,λ_1) and (β_2^*,λ_2), the first eight cases are equivalent to the last eight for a given set of specific values of ($\beta_1^*,\lambda_1,,\beta_2^*,\lambda_2$). For example, Maximum1.1 and Maximum2.1 are equivalent, and so are NoMin1.1 and NoMin2.1.

Figures 4 and 5 illustrate respectively the two cases Maximum1.1 and Minimum1.4 where the dependent variable y is not transformed by a Box-Cox for simplicity reasons:

1. Maximum1.1: $y = 15 - 4X^{-1.4} - 2X^{2.3}$, that is equivalent to Maximum2.1: $y = 15 - 2X^{2.3} - 4X^{-1.4}$ when the values of the pairs (β_1^*,λ_1) and (β_2^*,λ_2) are permuted.
2. Minimum1.4: $y = -6 + 3X^{1.3} + 5X^{-0.7}$, that is equivalent to Minimum2.4: $y = -6 + 5X^{-0.7} + 3X^{1.3}$ when the values of the pairs (β_1^*,λ_1) and (β_2^*,λ_2) are permuted.

Special case: Quadratic form

The quadratic form can be obtained by setting $\lambda_1 = 1$ and $\lambda_2 = 2$ in model (7):

$$y^{(\lambda_y)} = \beta_0^* + \beta_1^* X^1 + \beta_2^* X^2 + \ldots + u \quad . \qquad (12)$$

From Table 2, there are three maxima and three minima that satisfy the condition $\lambda_1 - \lambda_2 < 0$:

1. Maximum1.1, Maximum1.2 and Maximum1.3, that are respectively equivalent to Maximum2.1, Maximum2.2 and Maximum2.3 when the values of the pairs (β_1^*,λ_1) and (β_2^*,λ_2) are permuted.
2. Minimum2.2, Minimum2.3 and Minimum2.4, that are respectively equivalent to Minimum1.2, Minimum1.3 and Minimum1.4 when the values of the pairs (β_1^*,λ_1) and (β_2^*,λ_2) are permuted.

3 Two-step transformations on a same independent variable

In TRIO estimation procedures (Gaudry *et al.*, 1993-1997), the user frequently adopts a two-step procedure to make transformations on the independent variable X. Having done previous tests with a single monotonic transformation, the user then searches for turning points, e.g.:

1. In the first step, a quadratic form in X is estimated using the equation (12).
2. In the second step, a same Box-Cox transformation on the two independent variables X and X^2 involving in the quadratic form of the previous step is estimated:

$$y^{(\lambda_y)} = \beta_0^{**} + \beta_1^{**} X^{(\lambda_1)} + \beta_2^{**} (X^2)^{(\lambda_1)} + \ldots + u \quad . \qquad (13)$$

This model can be reduced to the model (1) as follows:

Figure 4. Graph of $y = 15 - 4X^{-1.4} - 2X^{2.3}$, with Maximum1.1 at (1.0546, 9.0268)

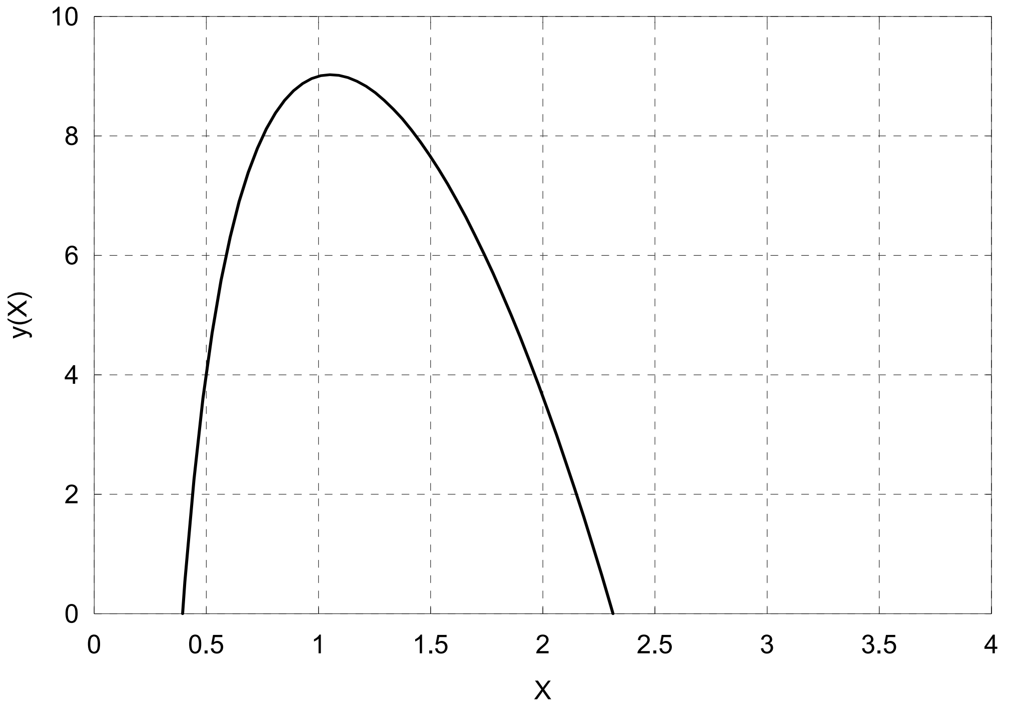
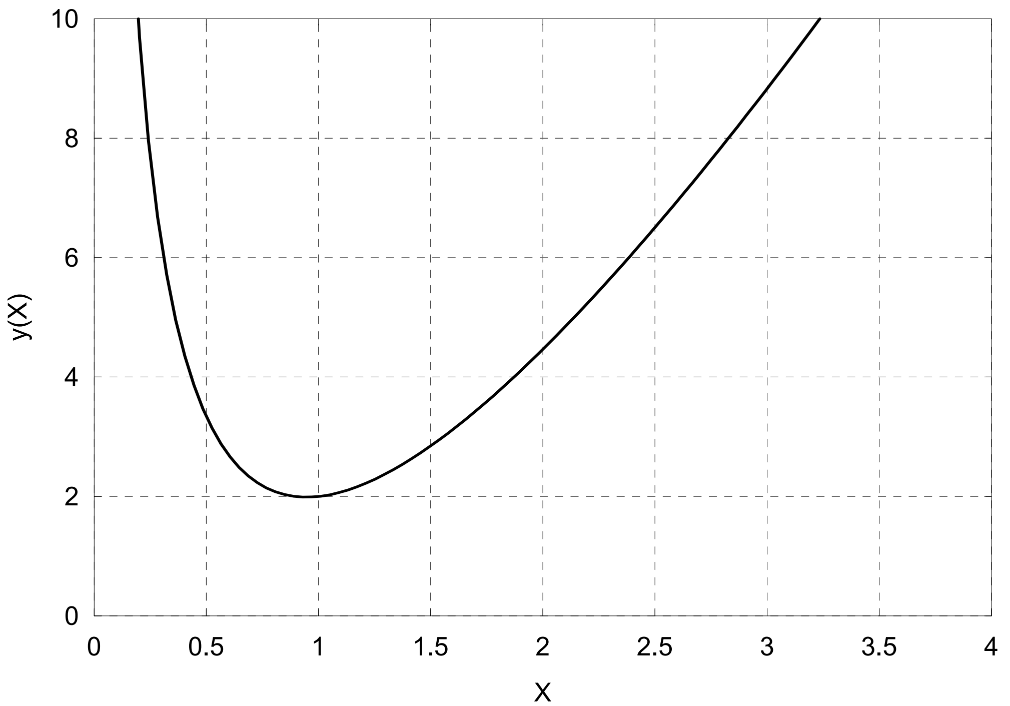


Figure 5. Graph of $y = -6 + 3X^{1.3} + 5X^{-0.7}$, with Minimum1.4 at (0.9473,1.9892)



$$
\begin{aligned}
y^{(\lambda_y)} &= \beta_0^{**} + \beta_1^{**} X^{(\lambda_1)} + 2\beta_2^{**} \frac{X^{2\lambda_1} - 1}{2\lambda_1} + \ldots + u \\
&= \beta_0^{**} + \beta_1^{**} X^{(\lambda_1)} + 2\beta_2^{**} X^{(2\lambda_1)} + \ldots + u \\
&= \beta_0 + \beta_1 X^{(\lambda_1)} + \beta_2 X^{(\lambda_2)} + \ldots + u \quad .
\end{aligned}
\tag{14}
$$

where $\beta_0 = \beta_0^{**}$, $\beta_1 = \beta_1^{**}$, $\beta_2 = 2\beta_2^{**}$ and $\lambda_2 = 2\lambda_1$.

This is not surprising in view of the property that the combination of a simple power and a Box-Cox transformation gives an equivalent Box-Cox transformation (Gaudry and Laferrière, 1989) with a rescaling effect for the coefficient β_2^{**}. In the first step, generalizing the form in X gives:

$$
y^{(\lambda_y)} = \beta_0^{*} + \beta_1^{*} X^{1} + \beta_2^{*} X^{m} + \ldots + u \quad . \tag{15}
$$

where m is a real number.

In the second step, the model with the generalized form in X can be rewritten in terms of model (1):

$$
\begin{aligned}
y^{(\lambda_y)} &= \beta_0^{**} + \beta_1^{**} X^{(\lambda_1)} + \beta_2^{**} (X^{m})^{(\lambda_1)} + \ldots + u \\
&= \beta_0^{**} + \beta_1^{**} X^{(\lambda_1)} + m\beta_2^{**} \frac{X^{m\lambda_1} - 1}{m\lambda_1} + \ldots + u \\
&= \beta_0^{**} + \beta_1^{**} X^{(\lambda_1)} + m\beta_2^{**} X^{(m\lambda_1)} + \ldots + u \\
&= \beta_0 + \beta_1 X^{(\lambda_1)} + \beta_2 X^{(\lambda_2)} + \ldots + u \quad .
\end{aligned}
\tag{16}
$$

where $\beta_0 = \beta_0^{**}$, $\beta_1 = \beta_1^{**}$, $\beta_2 = m\beta_2^{**}$ and $\lambda_2 = m\lambda_1$.

Note that when computing the elasticity of y with respect to X at the sample means, for the model (13), the program TRIO considers the two independent variables X and X^2 as distinct variables not related to each other and gives two distinct elasticities, namely $\eta(y, X)\big|_{\bar{y},\bar{X}} = \beta_1^{**} \bar{X}^{\lambda_1} / \bar{y}^{\lambda_y}$ and $\eta(y, X^2)\big|_{\bar{y},\overline{X^2}} = \beta_2^{**} \overline{X^2}^{\lambda_1} / \bar{y}^{\lambda_y}$. If the second variable X^2 is considered as a function of X as it should be, the total elasticity of y with respect to X at the sample means is given by:

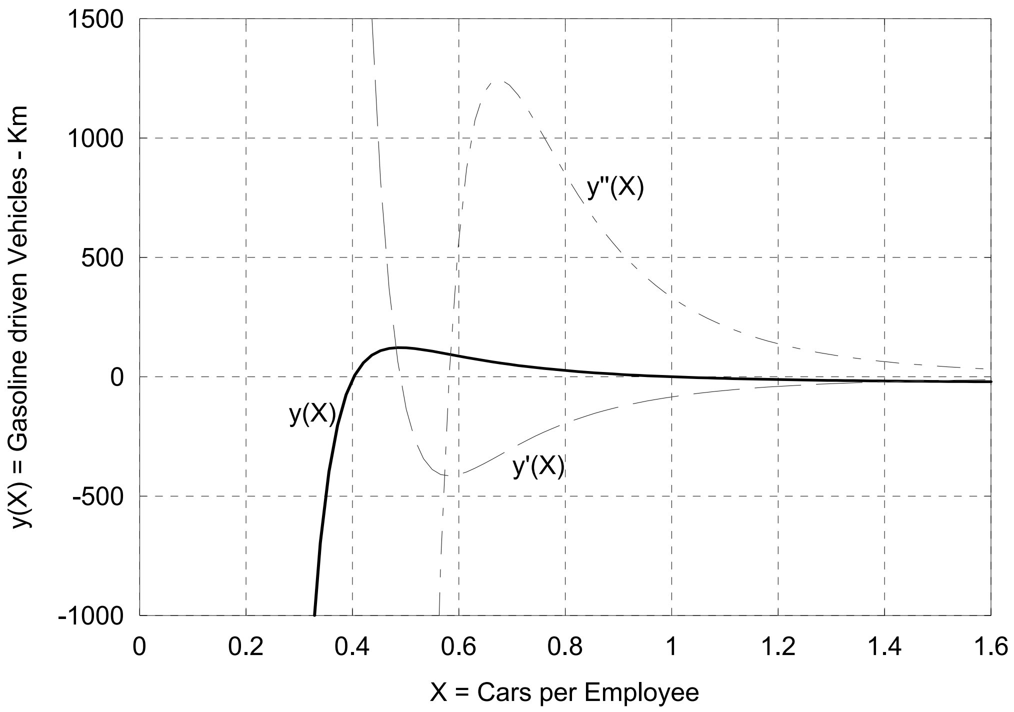
$$
\tilde{\eta}(y, X)\big|_{\bar{y},\bar{X}} = \beta_1^{**} \frac{\bar{X}^{\lambda_1}}{\bar{y}^{\lambda_y}} + 2\beta_2^{**} \frac{\bar{X}^{2\lambda_1}}{\bar{y}^{\lambda_y}} \tag{17}
$$

where the second component of the elasticity can be computed from the elasticity $\eta(y, X^2)\big|_{\bar{y},\overline{X^2}}$ given by TRIO as follows:

$$
2\beta_2^{**} \frac{\bar{X}^{2\lambda_1}}{\bar{y}^{\lambda_y}} = 2\eta(y, X^2)\big|_{\bar{y},\overline{X^2}} \frac{\bar{X}^{2\lambda_1}}{\overline{X^2}^{\lambda_1}} \quad . \tag{18}
$$

An example of the two-step procedure comes from the SNUS-2.5 Model (Blum and Gaudry, 2000), where the demand for road use with gasoline cars (y) is explained by the stock of cars per employee (X), among other things. Figure 6 gives the graph of the portion of the equation (13) where only X is involved, namely $y(X)=\beta_1^{**}X^{(\lambda_1)}+\beta_2^{**}(X^2)^{(\lambda_1)}$ where $\beta_1^{**}=-93$, $\beta_2^{**}=4.4$ and $\lambda_1=-3.3$. The first and second derivatives of $y(X)$, $y'(X)$ and $y''(X)$, are also plotted.

Figure 6. Result from SNUS-2.5 Model



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