NBER WORKING PAPER SERIES

ENDOGENOUS GROWTH AND THE ROLE OF HISTORY

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Working Paper No. 3151

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
October 1989

This research was supported by the LSE Financial Markets Group, the Suntory-Toyota International Centre for Economics and Related Disciplines, and the Inland Revenue. Any views expressed are those of the authors alone. We are very grateful to our colleagues in the FMG for continual comment and criticism, and to Bill Brainard, John Campbell, David Livesey, Andrei Shleifer and Peter Whittle for helpful comments. We thank Victor Hung for excellent research and computational assistance. This paper is part of NBER's research programs in Growth, Economic Fluctuations, and Taxation.

NBER Working Paper #3151
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ABSTRACT

This paper presents a model in which the realizations of stochastic tax and depreciation rates determine both the level and growth rate of output: externalities to investment - learning by watching - are characterized by diminishing returns, yielding a nonlinear "technical progress function". This results in multiple steady-state growth rates. History matters. It is possible that two economies with identical "deep" parameters and initial capital stocks may cycle around different trend growth rates, depending upon the historical path of fiscal shocks. Growth and cycles interact, and the nonlinearity means that output changes cannot be decomposed into a stochastic trend and a trend-stationary process.

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1. Introduction

Economists have been reluctant to embrace historical explanations of differences in growth rates. In part, this is because such explanations tend to be divorced from any analysis of underlying economic factors. In this paper we show that within the framework of an endogenous growth model the historical pattern of one particular type of government intervention - tax policy - determines the level and growth rate of output.

The recent literature on endogenous growth has tried to reconcile indefinite increases in incomes per head with diminishing returns to factors of production, without resort to exogenous technical progress that falls like manna from heaven. The new models generate growth endogenously by assuming positive externalities from private investment in physical or human capital to production possibilities as a whole (Romer 1986, 1988, Lucas 1988). The existence of a side-effect of private embodied investment on public disembodied knowledge leads to aggregate increasing returns to scale and hence the possibility of growth. We follow the Kaldor-Arrow-Romer tradition of modelling a link from investment to productivity growth, but we argue that it is important to put some structure on the size of the externalities that are created. In particular, we argue that there are diminishing returns in the production of the externality. This assumption yields a nonlinear "technical progress function" that results in multiple steady-state growth rates and a rich dynamic structure that determines both growth and cycles. Previous papers have worked either with a linear technology for the production of knowledge (Uzawa 1965, Lucas 1988, King and Rebelo 1988) or at a level of generality that precludes a systematic analysis of the effects of policy variables on growth (Romer, 1986).

Our aim is to show that economies with identical economic structures can display large dispersion of growth rates. Variations in growth rates result from differing realisations of government policies, even though the underlying process generating policy is the same in each country. Either deterministic or stochastic variations in the tax rate on capital income will generate both cyclical fluctuations around a trend growth rate and changes in the trend growth rate itself. The tax rate process is not modelled as the result of an optimisation by a benevolent government. Instead it is regarded as an exogenous stochastic process. But policy does affect both the level and growth rate of output, whereas as Romer (1989, p.51) has pointed out, "in models with exogenous technological change .... it never really mattered what the government did". We use an empirically estimated stochastic process for tax rates in the results reported below.

Many other types of shock could be modelled and would lead to similar results for the behaviour of the growth rate. We provide an example below in terms of stochastic depreciation.

These ideas are illustrated by extending the simplest possible neoclassical growth model to include stochastic shocks and endogenous growth in the form of a "technical progress function" that builds on the work of Kaldor (1957) and Arrow (1962). Growth and cycles are then shown to interact because the strength of the propagation mechanism depends upon agents' adjustments to shocks. Moreover, the model generates multiple steady-state growth paths and the realisation of the stochastic shocks determines towards which growth rate the economy tends. Two types of stochastic shocks are analysed. First, "fiscal policy shocks" to the tax wedge between the return on physical investment and the return on saving are assumed to follow an integrated stochastic process. Formally, the tax rate on comprehensive income is modelled as a Markov process. Secondly, the rate of economic depreciation is assumed to follow a stationary white noise process. The propagation mechanism derives from the assumption of a technical progress function which relates the rate of technical progress to the investment rate - defined as the ratio of net investment to gross domestic product. This represents the effect of "learning by watching" - the demonstration effect of new ideas on the efficiency of the existing capital stock. Learning by watching is assumed to take the form of a pure Marshallian externality so that a firm does not take into account the beneficial effect of its investment on productivity in the economy. A competitive equilibrium is, therefore, feasible.

We solve for the optimal policy response function of the representative agent and hence the implied stochastic path for output and the growth rate. We construct the nonlinear stochastic difference equation that describes the evolution of capital and output over time. Closed-form solutions are not available and we use numerical methods to compute the asymptotic distributions of the growth rate of output and other variables.

A distinctive feature of the model is the existence of multiple equilibria which result from the nonlinearities - in contrast to the linear approximations studied in the real business cycle literature - so that history matters, in two ways. First, the **level** of technical knowledge depends upon the past path of output. Second, the equilibrium **growth rate** itself also depends upon historical realisations of the underlying stochastic processes. Hence two economies with identical "deep" parameters **and** initial capital stocks can not only experience different time paths for output but may also cycle around different "natural" growth rates. The stochastic path for output is such that it is impossible to decompose the variance into two parts that may be attributed to stochastic shocks to the trend growth rate, on the one hand, and transitory fluctuations on the other. An economy can experience "premature maturity", in the sense that if it starts with too high a level of capital the competitive equilibrium may lead it to cycle around a zero growth rate path.

In two respects the results of this paper have a Kaldorian flavour. First, we assume a technical progress function in which investment drives technical progress. Kaldor introduced this idea in his 1957 paper and it was taken up by Arrow in his work on learning by doing (Arrow (1962)).[1] But the Kaldorian technical progress function relates the rate of growth of output per head to the rate of growth of investment per head, and so the only steady-state growth path that is feasible is given by the intersection of an exogenously given curve with the 45^0 line (Kaldor (1957), Kaldor and Mirrlees (1962)). In contrast, in our model the equilibrium growth rate is endogenous. Second, Kaldor (1966) argued that "premature maturity" was responsible for the "low" (compared with other OECD countries) growth rates observed in the UK. He attributed this to the fact that, having industrialised early, the UK could no longer benefit from the transfer of unproductive labour from agriculture to more productive manufacturing. No formal model was presented and it is difficult to reconcile his idea with the observation that countries with initially lower levels of productivity not only grow more rapidly but can also overtake levels of output per head in "mature" economies. In the model presented here a different explanation of the phenomenon is given by the existence of multiple steady-state growth rates.[2]

The technical progress function is motivated in section 2, and the full growth model is described in section 3. Steady-state growth paths are analysed in section 4 and the complete dynamic solution with stochastic tax rates provided in section 5. Section 6 analyses stochastic depreciation, and our conclusions are presented in section 7.

2. A Technical Progress Function

The idea used in this paper is that much technical progress takes the form of "learning by watching". There is a demonstration or contagion effect from observations of new ideas embodied in new investment projects to the level of output that can be produced from the existing stocks of capital and labour. Cohen and Levinthal (1989) cite a number of studies which show that many innovations in one firm or industry originated in developments from other firms and industries.[3] New investment projects in one sector of the economy have a demonstration effect on the efficiency of other sectors.

Once an idea is embodied in an investment project the spill-over effect occurs up-front. Even though the project continues to operate there is no subsequent additional demonstration effect. This means that it is the investment that creates the spill-over, not the capital stock, and leads to the concept of the technical progress function. At low investment rates the probability of contact with a new idea is low. It increases with the fraction of investment in economic activity, but the demonstration effect declines as the investment rate increases beyond a certain point - either because of a saturation effect similar to the contagion models of consumer durables or because there is a limit on the rate at which new ideas can be absorbed. Negative investment, however, does not lead to technical regress. We assume that at all non-positive levels of net investment the demonstration effect is zero. The model does not - unlike many real business cycle models - require negative technology shocks to generate cycles. For simplicity we ignore exogenous technical progress altogether, although it is trivial to extend the model to include this. Our assumptions mean that a stationary state is feasible in which zero net investment is accompanied by a zero rate of technical progress.[4]

The level of output of the representative firm is given by a conventional production function which exhibits constant returns to scale in capital and labour. Output per head, y_t, as a function of capital per head, k_t, is given by

$$y_t = f(A_t, k_t) \tag{1}$$

Lower case variables refer to values at the level of the individual, upper case variables to aggregate values. The level of technical knowledge evolves over time according to

$$A_{t+1} = A_t \, e^{\phi(X_t)} \tag{2}$$

where we define the aggregate net investment rate, X_t, to be gross investment, I_t, less the nonstochastic rate of depreciation δ resulting from normal wear and tear, as a proportion of total gross domestic product

$$X_t = \frac{I_t - \delta K_t}{Y_t} \tag{3}$$

From our discussion above the technical progress function ϕ has the following properties

$$
\begin{aligned}
\phi(X) &= 0 & &\text{for } X \leq 0 \\
\phi'(X) &\geq 0 & &\forall\, X \\
\phi''(X) &\gtrless 0 & &0 < \hat{X} \gtrless X
\end{aligned}
\qquad (4)
$$

A functional form that meets these conditions is the truncated logistic function defined by[5]

$$
\begin{aligned}
\phi(X) &= \frac{\bar{n}e^{\mu X+b} - \underline{n}}{e^{\mu X+b} + 1} & &\text{for } X \geq 0, \qquad b = \ln(\underline{n}/\bar{n}) \\
&= 0 & &\text{for } X < 0
\end{aligned}
\qquad (5)
$$

The restriction on the value of b ensures that the curve passes through the origin. Figure 1 shows the general shape of the technical progress function.

3. A Stochastic Growth Model

In this section we introduce the technical progress function into a one sector neoclassical growth model with time-varying tax rates. Investment and saving behaviour are assumed to be determined by the optimal consumption programme of an infinitely-lived "representative" individual. This is clearly a strong assumption but it enables us to compare the time path for output implied by the competitive equilibrium of an economy that permits multiple steady-state growth paths with the predictions of conventional real business cycle models. The "representative" individual is an owner-manager who maximises expected utility subject to the production constraints, assuming that he or she cannot influence the overall level of technical knowledge next period. The impact of investment on the dissemination of knowledge is a pure externality. For simplicity labour supply is fixed and the size of the population is normalised to unity. Preferences are assumed to be described by an additively separable isoelastic function of consumption per head. For individual i utility in period t is[6]

$$U_t = E_t \sum_{j=0}^{\infty} \beta^j \frac{c_{t+j}^{1-\gamma}}{1-\gamma} \qquad \gamma \geq 0 \quad (6)$$

In order that steady-state growth paths are possible equilibria of the model we shall assume that technical progress is Harrod-neutral. For simplicity we assume also that the production opportunities of the representative individual are described by a Cobb-Douglas production function in which the elasticity of output with respect to capital is denoted by α. Technical progress is, therefore, both Hicks-and Harrod-neutral. Hence

$$y_t = A_t^{1-\alpha} k_t^{\alpha} \qquad (7)$$

The capital stock owned by the agent evolves according to the non-stochastic difference equation

$$k_{t+1} = k_t(1-\delta) + y_t - c_t - \tau(k_t) + l_t \qquad (8)$$

where $\tau(.)$ denotes the tax function describing the payment as a function of the capital stock and l is a lump-sum subsidy. We analyse a proportional tax on comprehensive income which implies

$$\tau(k_t) = \tau_t (y_t - \delta k_t) \qquad (9)$$

All tax revenues are assumed to be returned to agents as a lump-sum subsidy, and so $l_t = \tau(k_t)$. Similar results to those presented below follow from the assumption that revenues are spent on government consumption. The exogenous driving force of the model will be time-varying tax rates. The tax rate τ_t is assumed to follow a Markov process described by[7]

$$\text{Prob}\{\tau_{t+1} \leq \tau \mid \tau_t \leq \hat{\tau}\} = F(\tau, \hat{\tau}) \qquad (10)$$

A deterministic cycle in tax rates can be thought of as a special case of the stochastic process (10) with a degenerate Markov transition matrix. The owner-manager faces an infinite horizon stochastic programming problem that involves maximising (6) subject to (i) equations (7) and (8), (ii) the distribution of the stochastic shocks described by equations (9) and (10), and (iii) an assumption about the time path of A_t which is exogenous to the owner-manager. The assumption of rational expectations implies that technical knowledge is assumed to evolve according to (2) and (3) and the latter equation satisifies the condition that k = K, where K is the aggregate level of capital per head.

There is one state variable, k_t, and one control variable, c_t. The necessary conditions for an optimum of this programme are the following Euler equation and transversality condition:

$$c_t^{-\gamma} = \beta\, E_t\{c_{t+1}^{-\gamma}\exp(r_{t+1})\} \tag{11}$$

where the (stochastic) post-tax return to capital in period t+1, denoted by r_{t+1}, is given by

$$\exp(r_{t+1}) = 1 + (1-\tau_{t+1})\ (\alpha A_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} - \delta) \tag{12}$$

Note that the partial derivative of output with respect to capital is taken holding A constant. The transversality condition is

$$\lim_{j\to\infty} E_t\ \beta^{t+j}\ u'(c_{t+j})k_{t+j} = 0 \tag{13}$$

These conditions are also sufficient given that (i) the utility and production functions are concave and satisfy the Inada derivative conditions, and (ii) the stochastic process for τ_t is stationary and bounded.[8] Under these conditions there exists a unique continuous optimal policy response function (Brock and Mirman 1972, Lucas and Prescott 1971, Danthine and Donaldson 1981 (appendix 1)).

To study this problem it is helpful to transform the variables measuring output, capital and consumption into levels per efficiency unit of labour in order to arrive at a stationary system.[9] Define the transformed variables by

$$z_t^* \equiv \frac{z_t}{A_t} \qquad z = y,k,c \tag{14}$$

The transformed system may then be defined as follows

$$y_t^* = k_t^{*\alpha} \tag{15}$$

$$k_{t+1}^* e^{\phi(X_t)} = k_t^*(1-\delta) + y_t^* - c_t^* \tag{16}$$

where the government budget constraint has been used to obtain (16). The Euler equation becomes

$$\exp(\gamma\phi(X_t))\ c_t^{*-\gamma} = \beta\, E_t\left\{ c_{t+1}^{*-\gamma} \exp(r_{t+1}) \right\} \tag{17}$$

$$\exp(r_{t+1}) = 1 + (1-\tau_{t+1})\ (\alpha k_{t+1}^{*\alpha-1} - \delta) \tag{18}$$

The solution to the infinite horizon optimisation problem for the representative agent is described by a time-invariant policy rule that maps the relevant state variables in period t - the transformed value of his capital stock, the tax rate, and the aggregate investment rate - into the agent's investment rate (and hence into the level of the transformed capital stock in period t+1).

$$x_t = x(k_t^*, \tau_t; X_t) \tag{19}$$

In general, as we discuss in section 5, numerical methods must be used to solve for the optimal policy response function. These involve finding a fixed point in the space of continuous functions of the mapping defined implicitly by the Euler equation.

The value of the optimal investment rate chosen by the representative agent must be consistent with the aggregate level of the investment rate that is assumed when individual decisions are made. Hence in equilibrium

$$x_t = X_t \tag{20}$$

With endogenous growth the competitive equilibrium investment rate is the fixed point of the mapping given by (19) and (20), and may be written as[10]

$$x_t = \chi(k^*_t, \tau_t) \tag{21}$$

Equivalently, in state-space the equilibrium transformed value of the capital stock evolves according to a time-invariant function k (derived straightforwardly from the function x)

$$k^*_{t+1} = k(k^*_t, \tau_t; X_t) \tag{22}$$

From (3), (15) and (16) and (20) it also follows that

$$k^*_{t+1} = \frac{X_t k^{*\alpha}_t + k^*_t}{e^{\phi(X_t)}} \tag{23}$$

The competitive equilibrium of the economy is the intersection of these two surfaces. This may be written as

$$k^*_{t+1} = \kappa(k^*_t, \tau_t) \tag{24}$$

The dynamic behaviour of the economy is determined by the mapping described by equation (24). As we shall show, the optimal programme implies that a competitive equilibrium may exhibit multiple turnpikes - that is, the value of the initial capital stock determines to which of several steady-state growth paths the economy converges. The nonlinearity of the technical progress function means that the unique turnpike theorem (and the implied saddlepoint path) for the one-sector neoclassical growth model no longer holds.[11] Before providing a complete solution of the dynamic behaviour of output, it is helpful to discuss multiple equilibria by examining the existence of steady-state growth paths along which expectations are realised with perfect foresight.

4. Deterministic Steady-state Growth Paths

In order to illustrate the existence of multiple turnpikes we consider, first, the existence of deterministic steady-state growth paths. These are characterised by the following four behavioural equations and one definitional identity. The natural growth rate - denoted by n - is given by the technical progress function

$$n = \phi(x) \tag{25}$$

If the expected growth rate of consumption is g then the Euler equation (11) becomes

$$g = \gamma^{-1} (r - \rho) \tag{26}$$

where ρ (= $-\ln\beta$) is the rate of pure time preference.

The investment criterion is that the private marginal product of capital net of depreciation is equal to the cost of capital which is the post-tax rate of return grossed up by the tax rate.

$$\frac{\alpha}{v} = \frac{r}{1-\tau} + \delta \tag{27}$$

where v denotes the steady-state capital-output ratio.

Equilibrium requires that the "warranted" - or, in modern parlance, rational expectations - growth rate g is equal to the natural rate of growth

$$g = n \tag{28}$$

Finally, by definition

$$x = gv \tag{29}$$

These five equations determine the steady-state equilibrium values of the five endogenous variables, n,g,v,x, and r.

The solution may be illustrated diagrammatically in the growth rate - investment rate space. The natural growth rate is given by the technical progress function. The rational expectations growth rate is given by the capital market equilibrium curve given from equations (26), (27) and (29) by

$$g = \frac{x(\rho+\delta(1-\tau))}{\alpha(1-\tau)-\gamma x} \tag{30}$$

Figure 2 shows the two curves. Steady-state growth paths exist when the natural rate equals the rational expectations growth rate. The shape of the technical progress function was motivated above. It is easy to show from equation (30) that the capital market equilibrium curve is a monotone increasing and convex function of x with a vertical asymptote at $x=\alpha(1-\tau)/\gamma$. By construction there always exists a stationary equilibrium because both curves pass through the origin. But depending upon the parameter configuration there may also exist up to two additional equilibrium growth paths at positive growth rates.[12] These are shown as n_L and n_H in Figure 2. The most interesting feature of these multiple steady-state growth rates is that, for a given tax rate, the growth rate is **inversely** related to the transformed steady-state level of capital per head. The savings equation (26) shows that the growth and interest rates are positively related, and the investment equation (27) that the interest rate and capital intensity are negatively related. It is this result which will underlie some of the unusual dynamics described below.

5. Growth and Cycles

The dynamic behaviour of the model is given by the nonlinear first-order difference equation for k^*, equation (24), which is given by the intersection of the surfaces defined by equations (22) and (23). Even when tax rates follow a deterministic path explicit solutions can be obtained only in very special cases. For example, when $\gamma = 1$, $\delta = 1$, and $\tau_t = 0\ \forall t$, the optimal policy response function is

$$k^*_{t+1} = e^{-\phi(x_t)} \alpha\beta\, k_t^{*\alpha} \tag{31}$$

Combining this with (23) gives the competitive equilibrium as

$$k^*_{t+1} = \exp\left\{ -\phi(\alpha\beta - k_t^{*1-\alpha}) \right\} \alpha\beta\, k_t^{*\alpha} \tag{32}$$

In general, however, the dynamic equilibrium must be solved by numerical methods. We search for a continuous function which is an approximation to (21) for the competitive equilibrium investment rate. On empirical grounds we constrain the tax rate to lie at percentile points of the range 0.00 to 0.99. Since (21), however, is continuous by construction, we solve initially for a discrete set of points in state space k^*_t and then interpolate using free cubic splines to estimate x_t as a continuous function of continuous k^*_t, for each discrete τ_t.[13]

Given an initial guess $x^0(k^*_t,\tau_t)$ at the function, an improved guess x^1 is obtained by applying Newton-Raphson to find the required root of the Euler equation (17) and (18) to a given tolerance. For a given τ_t, k^*_t determines x_t via the guess x^0, hence also k^*_{t+1} from (23) and c^*_t from (16); but in order to solve (17) it is necessary to determine c^*_{t+1} too, by advancing all these equations one period. The technique employed is therefore to solve, for each τ_t, for the value of $x^1(k^*_t)$ which satisfies (17) for k^*_{t+1} and c^*_t, conditional on the function $x^0(k^*_{t+1})$ used, with interpolation as necessary, to determine x_{t+1}, k^*_{t+2} and hence c^*_{t+1}. The method is clearly equivalent to the solution of a finite horizon problem, although in this model we do not find it particularly helpful to think of it in that way. Iteration over these functions is complete when x^{N+1} differs from x^N by less than a given tolerance at every point (k^*_t,τ_t), providing an approximation to the time-invariant competitive equilibrium function for k^*_{t+1}.

For the case of a constant tax rate the difference equation for k^* is plotted in figure 3. Its shape depends upon the number of equilibrium steady-state growth rates. Steady-state growth paths exist when $k^*_{t+1} = k^*_t$. Figure 3 is plotted for parameter values such that there are three steady-state growth equilibria. Because of the inverse relationship between the growth rate and capital intensity it can be seen that the high growth and the zero growth equilibria are stable, whereas the low growth equilibrium path is unstable.

When tax rates are stochastic - and follow a Markov process - there are as many curves as there are possible tax rates.[14] In each case the current tax rate and capital stock jointly form a sufficient statistic for next period's capital stock. With several curves the dynamics become more interesting, and we can show the interaction between growth and cycles. Consider, first, the simplest case in which there are only two tax rates, and hence a 2×2 transition matrix describing the stochastic process for taxes. There are two equilibrium curves relating k^*_{t+1} to k^*_t. Both can intersect the 45° line up to three times. The dynamic path for capital and output depends upon the relative positions of the two curves. The number of possible configurations depends upon the number and order of the intersections with the 45° line. One possible configuration is shown in figure 4. If the economy finds itself with a capital stock in the region AB then it will exhibit "growth cycles" around a stochastic trend which varies with the tax wedge. It is easy to show that the trend growth rate is inversely related to the tax rate. In the region EF there are stable cycles around a stationary level of output. The region CD is characterised by unstable growth cycles.

However, the case of two tax rates is very special. With $N > 2$ tax rates the dynamic behaviour of the economy depends upon the configuration of the possible 3N equilibria. To make it easier to analyse the different resulting configurations of the equilibrium k^*_{t+1} curves, we introduce an "indicator" function defined in the appendix. The indicator function can be used to show whether history matters, and how. The appendix demonstrates that as the number of tax rates increases, it becomes more likely that the range of values of k^*_t constitutes a closed and irreducible set of states. In that case two economies which have the same parameters and transition matrices, but different initial conditions and realisations of the stochastic variable, must have the same long run probability distribution of the **transformed** values of all relevant variables, such as the capital stock and output. The frequency distribution of growth rates in such a case is shown in figure 5. The distribution is bimodal, around the zero and high steady-state growth rates. The **actual** levels of output and consumption depend upon the entire past history of shocks. By contrast when there is more than one such closed and irreducible set (typically two, corresponding to zero and high growth rate cycles) then there are multiple asymptotic distributions for the transformed variables which depend on the initial conditions. The economy will be driven to either a zero or a high growth rate stochastic cycle and once there can never re-emerge. In this case history determines not only the current level of output but also the asymptotic distribution of growth rates. With additional sources of shocks, however, it is possible that the economy may jump from the zero growth to the high growth rate path, or vice versa. To illustrate this possibility we now introduce stochastic depreciation.

6. Stochastic Depreciation

In the model of section 5 a sequence of low tax rates can take the economy from a path of stochastic cycles around a zero trend growth rate to a region of high growth cycles. Other sources of stochastic shocks may achieve the same effect. In particular we show in this section that a partial destruction of the capital stock, such as that resulting from a major war, for example, may move the economy onto a high growth path. Suppose that at the beginning of each period there is a stochastic depreciation rate, denoted by ϵ_t, over and above the normal depreciation rate δ. The stochastic difference equation for k^* (formerly equation (16)) becomes

$$k^*_{t+1} e^{\phi(X_t)} = (1-\epsilon_{t+1}) \left[k^*_t (1-\delta) + y^*_t - c^*_t \right] \tag{33}$$

The idea of infrequent war destruction could be captured by a simple two-point distribution in which there was a very high probability that ϵ was zero, and a correspondingly small probability that ϵ was large. Solving for the competitive equilibrium in this case yields the result that two economies that are **completely** identical in all respects - the same values of the "deep" preference and technology parameters and the same initial capital stock - can experience very different paths for output and growth. Suppose both economies start with identical initial capital stocks. Then if one economy suffers the "high" depreciation rate and the other does not, but from then on only the normal depreciation rate occurs, the economy that "suffered" the high rate may shift from zero growth cycles to the regime of cycles around the high growth rate and the other undamaged economy will stagnate, cycling around the stationary state. For some parameter configurations the first economy can be better off - as measured by the expected utility of the representative agent - from the effects of a "war"; for others worse off. Of course, if the high depreciation state recurs then the loss of capital could offset the benefits of shifting to a high growth regime.

Examples of the time path of output for these two economies are shown in Figure 6. The economy that experiences the adverse shock to depreciation (country A) not only catches up the country (B) that does not, but actually overtakes it. This is the sort of phenomenon that commentators seem to have in mind when they refer to the advantages that Japan and Germany experienced from losing the war.

7. Conclusions

In the model of endogenous growth with stochastic fiscal policy and depreciation shocks, it is clear that shocks of either type can alter both the trend growth rate and the transitory adjustment path. Innovations to the trend and autoregressive components of output are correlated. It is, therefore, impossible to decompose the variance of output into proportions that can be attributed to a random walk component on the one hand and stationary shocks on the other. A much richer nonlinear stochastic time series process for output, investment and growth rates emerges.

An important implication of the model is that history matters. The existence of multiple steady-state growth rates means that, even with identical structures and starting points, economies can experience very different growth patterns that have permanent effects on both the level and rate of growth of output. Within this simple framework it would make no sense to explain differences in growth rate and levels of output per head without an examination of the historical experience of the countries concerned. This turns on its head Hahn's (1971) phrase "The theory of growth is not a theory of economic history".

APPENDIX: Analysing Competitive Equilibrium Dynamics

In order to analyse the dynamic path of the competitive equilibrium we introduce an "indicator" function defined by $I(k^*_t) = \Sigma \, sgn(k^*_{t+1})$ where

$$sgn(k^*_{t+1}) = \begin{cases} +1 & k^*_{t+1} > k^*_t \\ 0 & k^*_{t+1} = k^*_t \\ -1 & k^*_{t+1} < k^*_t \end{cases}$$

and summation is over all N possible values of τ_t. It is assumed here that N is finite, but similar results hold if it is countably infinite. We shall call points at which $k^*_{t+1} = k^*_t$ "crossover points", and label them $K_i(\tau_t)$, i=h,l,z according as to whether they correspond to high, low or zero growth rates respectively. min K_h denotes the value of k^* at the high growth crossover point on that curve which crosses at the lowest k^*_t; corresponding definitions hold for min K_l and min K_z. The following properties of $I(k^*_t)$ follow directly:

$$\begin{array}{llll} I(k^*_t) & = & +N & k^*_t < \min K_h \\ |I(k^*_t)| & \leq & N & \min K_h \leq k^*_t \leq \max K_z \\ I(k^*_t) & = & -N & \max K_z < k^*_t \end{array}$$

and $I(k^*_t)$ is constant in any interval $[K-\epsilon, K+\epsilon]$ which contains no crossover points.

Intuitively $I(k^*_t)$ is the number of curves above the line $k^*_{t+1} = k^*_t$ minus the number of curves below the line, at the point k^*_t; equivalently, it can be thought of as the cumulative difference between the number of stable and unstable crossover points, counting over the half-line $k < k^*_t$. Then $I(k^*_t)$ is a step function which changes by ± 2 around each discrete crossover point, and does not change elsewhere. We can use this function to decompose the state space of k^*_t (the real line) into subsets which are either sets of transient states or closed and irreducible sets of persistent states, according to the standard definitions of Markov theory.[15]

The decomposition theorem for Markov chains states that any chain M can be decomposed as $M = T \cup C^1 \cup C^2 \cup \ldots$ where T is the set of transient states and the C^i are closed irreducible sets of persistent states.

All points at which $|I(k^*_t)| = N$ must be transient. For, to take one of the two values, whenever $I(k^*_t) = +N$ then $k^*_{t+1} > k^*_t$ for every value of τ_t, and conversely when $I(k^*_t) = -N$, $k^*_{t+1} < k^*_t$ for all τ_t. If therefore $|I(k^*_t)| = N$ for some values of k^*_t in the interior of the range of the extreme crossover points, then by monotonicity of all the functions $k^*_{t+1}(k^*_t, \tau_t)$ in k^*_t, the set of states in that range $[k^*_t: \min K_h \leq k^*_t \leq \max K_z]$ cannot be irreducible. For if $I(k^*_t) = +N$ then $k^*_{t+1} > k^*_t$ for all τ_t. If $I(k^*_{t+1}) = +N$, then $k^*_{t+2} > k^*_t$ and so on. After some time T, $|I(k^*_{t+T})| < N$, because at the highest crossover point, $\max K_z$, the indicator function must increase to $+N$ from $+N-2$. Any set of states for which $|I(k^*_{t+T})| = N$ must be left within a finite time and return is impossible.

So in addition to the transient states $[k^*_t: k^*_t < \min K_h]$ and $[k^*_t: k^*_t > \max K_z]$, there may be transient states interior to the range $[k^*_t: \min K_h \leq k^*_t \leq \max K_z]$. Moreover, these can be of two quite different types, depending on whether $|I(k^*_{t+T})| = N$. For in regions where this condition holds, k^*_t is either decreasing or increasing, but not both, for all values of τ_t. In regions where $|I(k^*_{t+T})| < N$, k^*_t may decrease and increase according to the realisation of τ_t, and such regions may be either transient (T) or closed and irreducible (C).

An example - of the case illustrated in figure 4 - will demonstrate the use of the indicator function, which is shown in figure 7. The maximum number of crossover points is 3N and, therefore, of regions is 3N+1. When the crossover points are grouped together by type, so that all the K_h precede all the K_l which in turn precede all the K_z, without any overlap, the nature of the regions in the two tax rate case illustrated in figure 4 is:

a)

Region	Indicator		State
→A	$I(k^*_t)$	$= +N$	T
AB	$\|I(k^*_t)\|$	$< N$	C
BC	$I(k^*_t)$	$= -N$	T
CD	$\|I(k^*_t)\|$	$< N$	T*
DE	$I(k^*_t)$	$= +N$	T
EF	$\|I(k^*_t)\|$	$< N$	C
F←	$I(k^*_t)$	$= -N$	T

Note that in general the two k^*_{t+1} curves may cross each other either twice, once, or not at all, between the extreme crossover points, min K_h and max K_z.

In region →A, k^*_t is always increasing and in region BC k^*_t is decreasing. The same applies for regions DE and F←. In region CD, k^*_t can be either increasing or decreasing, and in general this will be a transitive state. However, it is distinguished as T* because in some circumstances it might never be left, depending on the time path of the τ_t (for example, tax rates could follow a deterministic cycle so as to leave k^* in CD). With N=2, this case occurs if the curves corresponding to τ^0 and τ^1 both have 3 crossover points and the ordering is $K_h(\tau^i)$, $K_h(\tau^{1-i})$, $K_l(\tau^j)$, $K_l(\tau^{1-j})$, $K_z(\tau^k)$, $K_z(\tau^{1-k})$, i,j,k = 0,1. A different ordering of these points when N=2 provides different configurations of the indicator function.

In general, as N increases the likelihood of some overlap between the three sets of crossover points increases, and accordingly it becomes increasingly probable that there will be only three regions - two transient and only one closed and irreducible - instead of the seven shown in figure 4. To demonstrate this we have computed the competitive equilibrium for an economy with a large number of possible tax rates, and the Markov transition matrix is estimated from data on UK tax rates over the period 1913-87. Other parameter values used in this example are $\alpha=0.45$, $\beta=0.98$, $\gamma=1.0$, $\delta=0.085$, $\mu=50$, $\underline{n}=0.001$, and $\bar{n}=0.15$. Figure 8 shows the indicator function for this case with 51 values for τ_t and no overlapping of the groups of crossover points. In figure 9 there are 100 values for τ_t, the 51 of figure 3 and a further 49, and as a result of overlapping there are now clearly only three regions.

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FOOTNOTES

1. In Arrow's model (and its various elaborations, for example Sheshinski (1967)), the level of technical knowledge is a concave function of cumulative past investment and positive growth of output per head is impossible without population growth.

2. In this paper we model only pure externalities arising from investment, thus retaining the ability to model outcomes as a competitive equilibrium. Kaldor (for example, 1975) would have had more sympathy with a model in which some of the increasing returns were internal to the firm, thus requiring a model of monopolistic competition (Shell (1973), Canning (1988), and Romer (1988)).

3. To give one example, "in his study of twenty-five major discoveries introduced into the United States by DuPont, Mueller (1962) indicated that, despite the company's reputation for path-breaking research, fifteen originated with work done outside the company" (Cohen and Levinthal fn.2, p.570).

4. The technical progress function is defined here in terms of net investment so that for expositional purposes a stationary state is a feasible solution to the non-stochastic version of the model. The qualitative nature of the results is unchanged if the argument of the technical progress function is instead gross investment.

5. Other functional forms that satisfy (4) include linear transforms of the hyperbolic tangent and piecewise linear functions.

6. When $\gamma = 1$ (6) becomes the discounted sum of the natural logarithm of consumption.

7. We do not discuss here the sources of stochastic tax policy. For a full discussion of the motivation for modelling taxes in this way see King and Robson (forthcoming) which also presents estimates of the Markov transition matrix for tax rates in the US and UK.. Bizer and Judd (1989) also examine random taxation.

8. In the deterministic case conditions were provided by Mirrlees (1967).

9. See Mirrlees (1967), Sheshinski (1967) and King, Plosser and Rebelo (1988).

10. Existence of a competitive equilibrium - the mapping described by equation by (21) - is assured by the assumption that the technical progress function is bounded.

11. Multiple turnpikes were obtained by Kurz (1968), who assumed that capital was an argument of the utility function, and by Liviatan and Samuelson (1969), who studied joint production. Our model exhibits some of the characteristics of joint production in that investment produces both capital and knowledge. Deterministic turnpike theorems in infinite horizon models are surveyed by McKenzie (1986).

12. For three non-negative growth rate equilibria to exist a necessary condition is that the capital market equilibrium curve be steeper at the origin than the technical progress function. This holds for all tax rates when

$$\frac{\rho + \delta}{\alpha} > \frac{\mu \, \underline{n} \, \hat{n}}{\underline{n} + \hat{n}}$$

In the simulations reported below the parameter values used satisfy this condition. For high enough values of the tax rate only one steady-state growth rate - namely zero - exists.

13. Full details of the algorithm are given in King and Robson (forthcoming).

14. The same is true when tax rates follow a stationary deterministic time path - a nonlinear cycle - because, as noted above, the deterministic case corresponds to a degenerate Markov transition matrix.

15. A state k^p is <u>persistent</u> if, given that $k^*_t = k^p$, the probability that $k^*_{t+T} = k^p$ for some $T>0$ is 1. A state k^t is <u>transient</u> if it is not persistent. A set S of persistent states is <u>closed</u> if, given any $k^1 \in S$, $k^2 \notin S$, the probability that if $k^*_t = k^1$ then $k^*_{t+1} = k^2$ is 0. S is <u>irreducible</u> if, given any k^1, $k^3 \in S$, the probability that if $k^*_t = k^1$ then $k^*_{t+T} = k^3$ for some $T>0$ is strictly positive.

Technical progress function

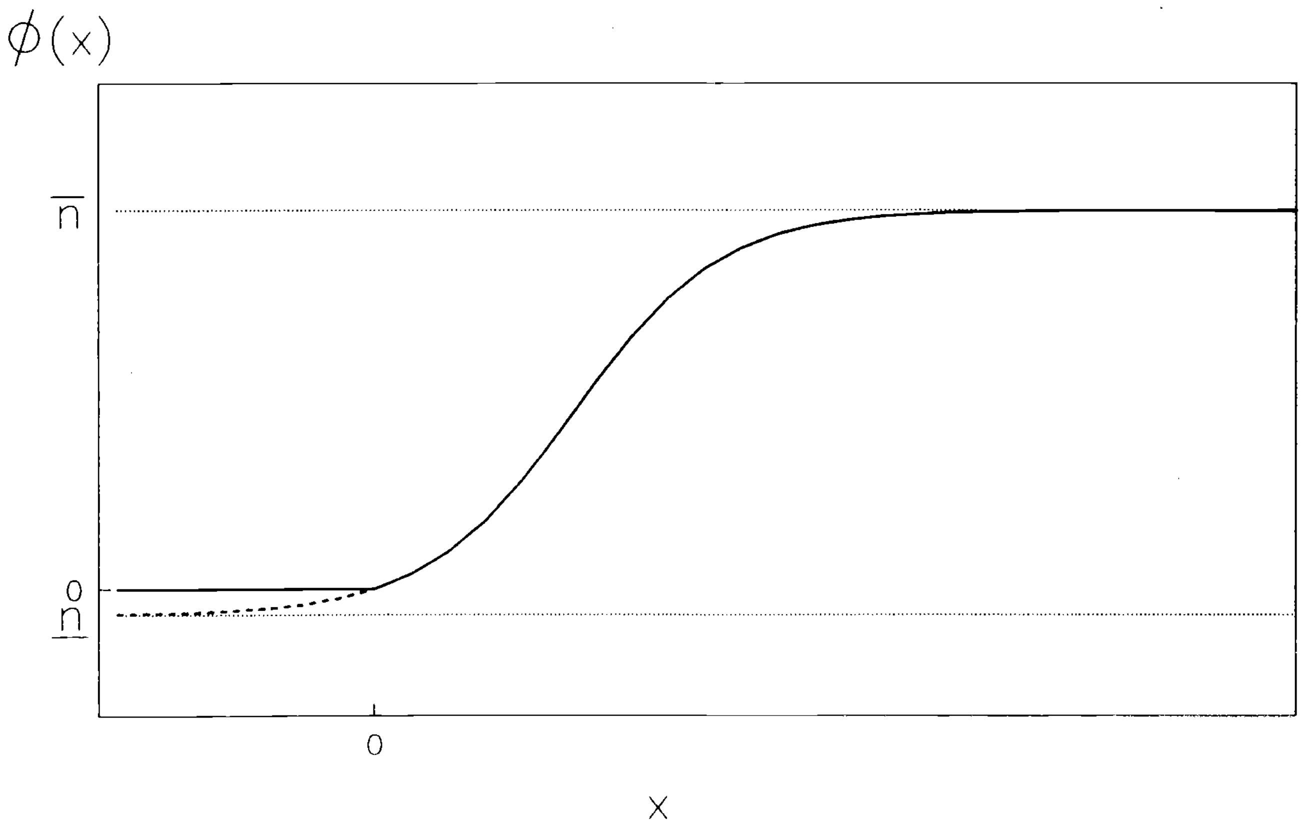

Figure 1

Steady state growth solution

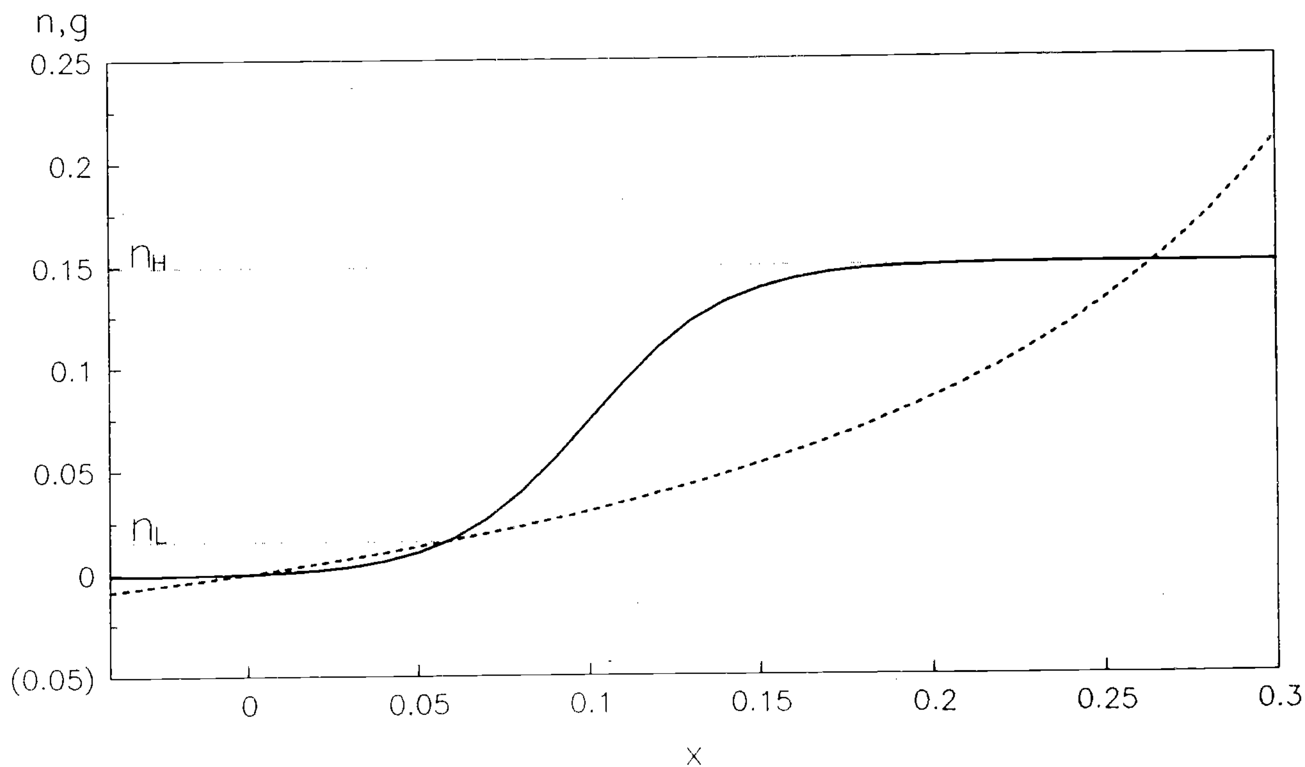

Figure 2

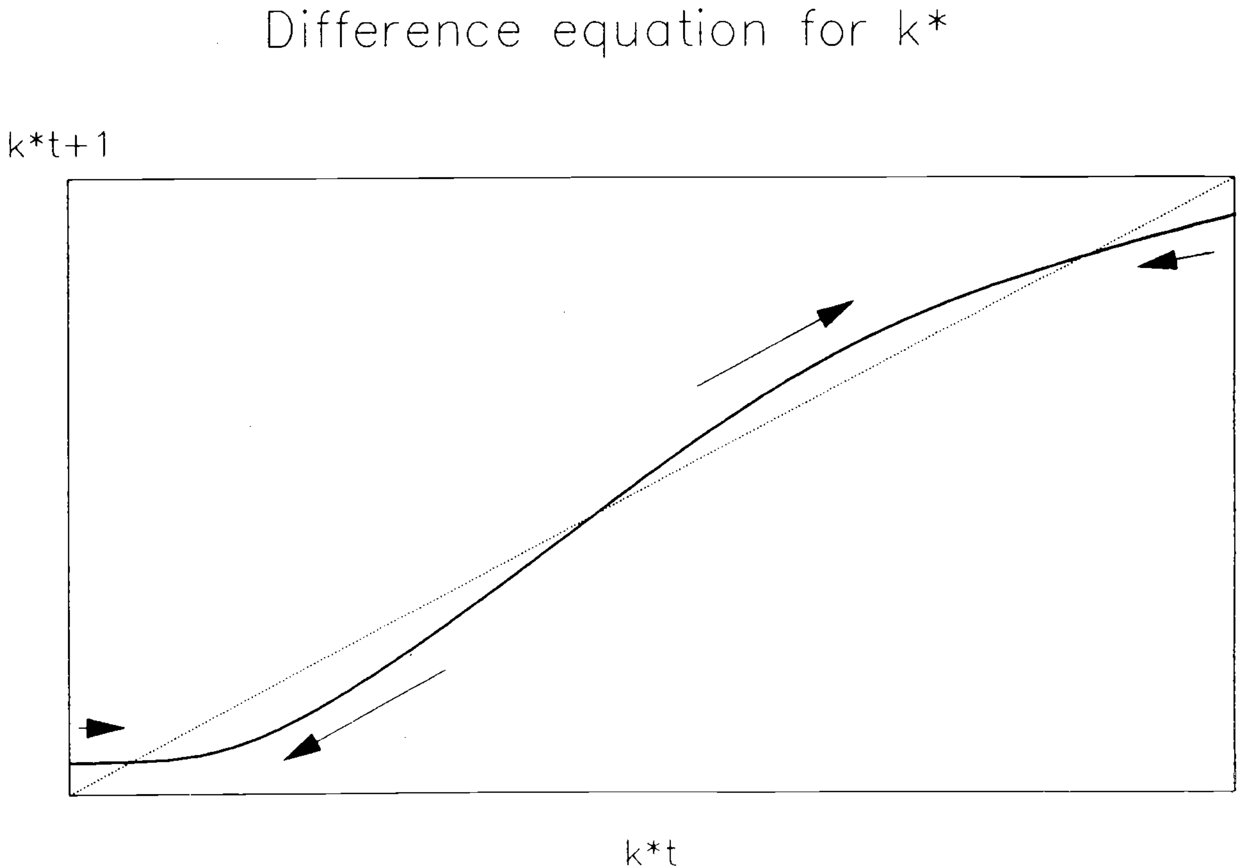

Figure 3
(not to scale)

Difference equation for k*
with two tax rates

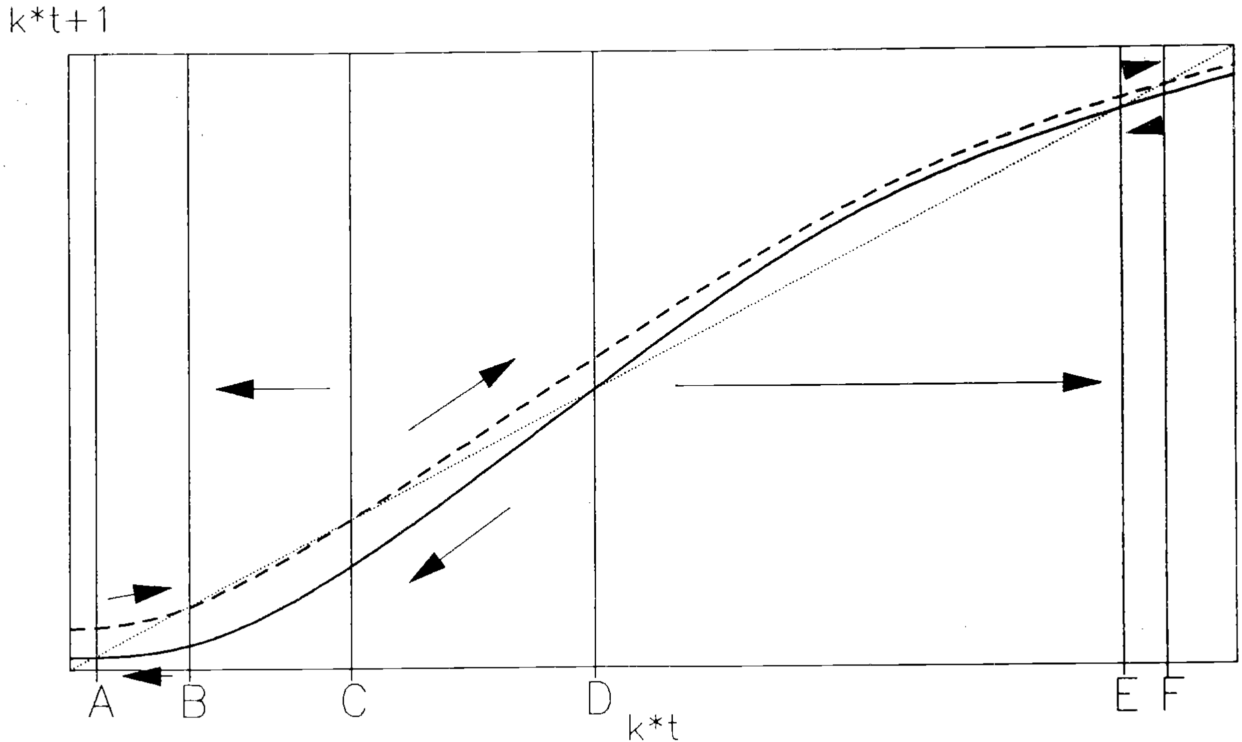

Figure 4
(not to scale)

Stationary distribution of growth rates
100000 observations

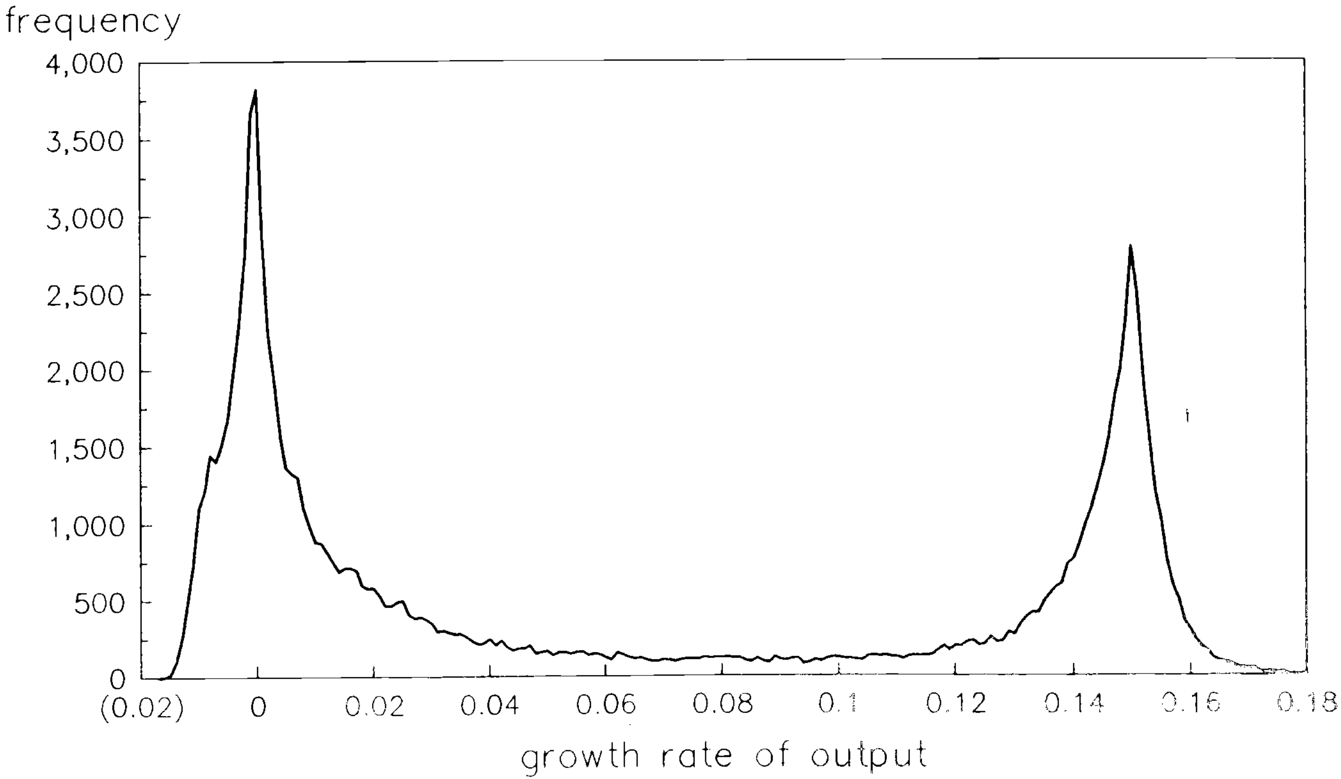

Figure 5

Effect of "high" depreciation on output

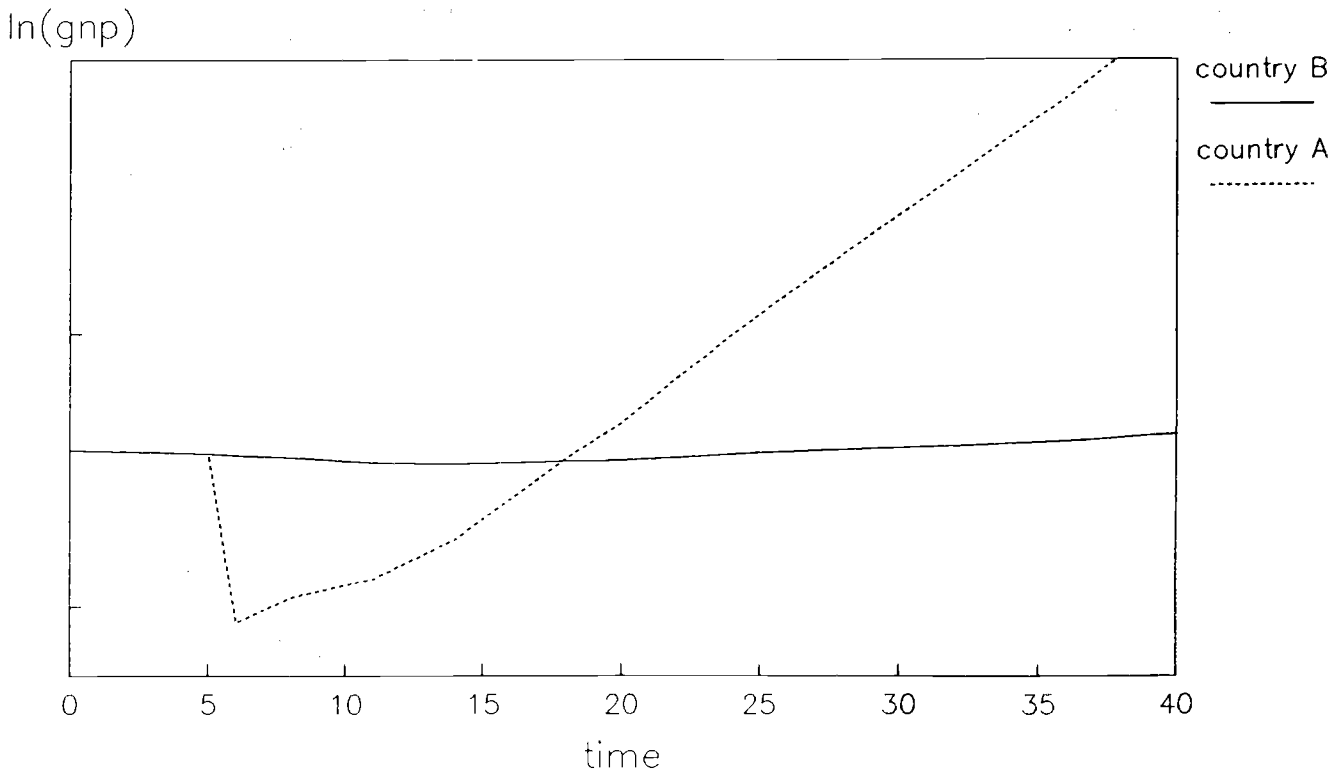

Figure 6

Indicator function for figure 4
with two tax rates

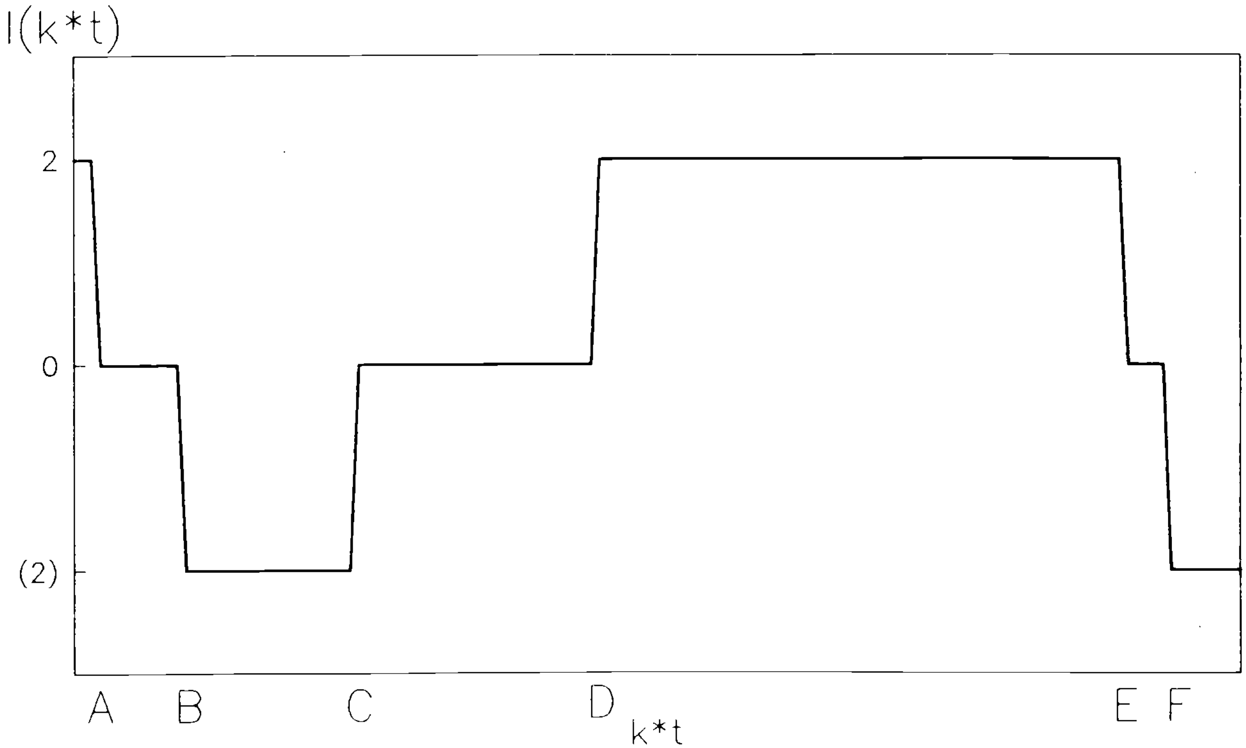

Figure 7

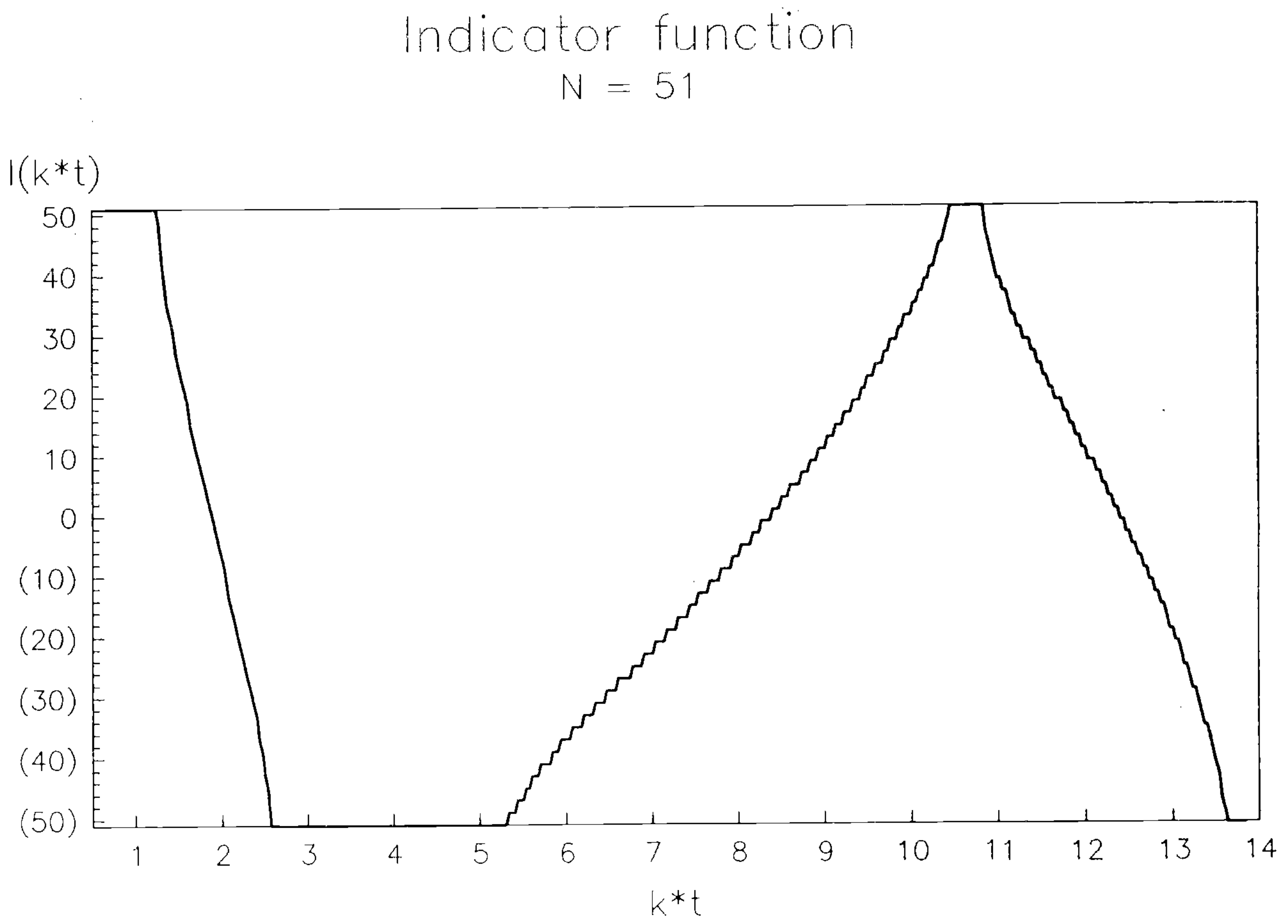

Figure 8
tax rates from 0.00 to 0.50

Indicator function
N=100

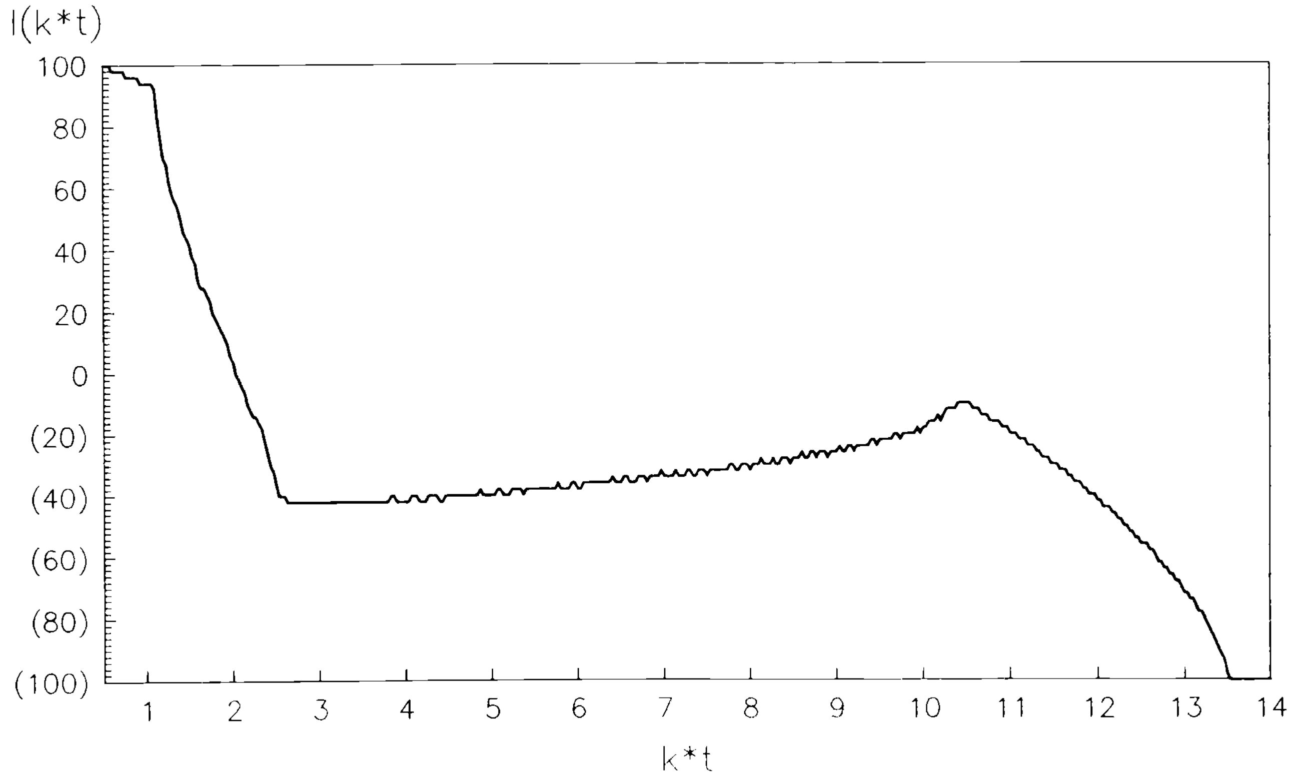

Figure 9
tax rates from 0.00 to 0.99