NBER WORKING PAPER SERIES

TECHNOLOGY (AND POLICY) SHOCKS IN MODELS OF ENDOGENOUS GROWTH

Larry E. Jones
Rodolfo E. Manuelli
Ennio Stacchetti

Working Paper 7063
http://www.nber.org/papers/w7063

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
April 1999

Presented at the NBER Conference "The Patent System and Innovation" funded by the Alfred P. Sloan Foundation via a grant to the NBER Project on Industrial Technology and Productivity. We thank Larry Christiano, Martin Eichenbaum, Fernando Alvarez and Craig Burnside for their help, Henry Siu for assistance, and the National Science Foundation for financial support. The views expressed in this paper are those of the authors and do not reflect those of the National Bureau of Economic Research.

Technology (and Policy) Shocks in Models of Endogenous Growth
Larry E. Jones, Rodolfo E. Manuelli, and Ennio Stacchetti
NBER Working Paper No. 7063
April 1999

ABSTRACT

Is there a trade-off between fluctuations and growth? The empirical evidence is mixed, with some studies (Kormendi and Meguire (1985)) finding a positive relationship, while others (Ramey and Ramey (1995)) finding the a negative one. Our objective in this paper is to understand how fundamental uncertainty can affect the long run growth rate, and what are the factors that determine the nature (positive or negative) of the relationship.

Qualitatively, we show that the relationship between volatility in fundamentals and policies and mean growth can be either positive or negative. We identify the curvature of the utility function as a key parameter that determines the sign of the relationship. Quantitatively, we find that when we move from a world of perfect certainty to one with uncertainty that resembles the average uncertainty in a large sample of countries, growth rates increase somewhere between 0.17% and 0.80%, with 0.20% being a "reasonable" estimate. Even though these are nontrivial changes, they are not large enough be themselves to account for the large differences in mean growth rates observed in the data. However, we find that differences in the curvature of preferences have very substantial effects on the estimated variability of stationary objects like the consumption/output ratio and hours worked. For this reason, we expect that the models considered in this paper will provide the basis of sharp estimates of the curvature parameter.

Larry Jones
MEDS - Kellogg School
Northwestern University
Evanston, IL, 60208
and NBER
lej@nwu.edu

Rodolfo E. Manuelli
Department of Economics
University of Wisconsin
Madison, WI, 53706
and NBER
manuelli@ssc.wisc.edu

Ennio Stacchetti
Department of Economics
University of Michigan
Ann Arbor, MI, 48109 - 1220
ennio@econ.lsa.umich.edu

1. Introduction

In his celebrated 1987 book **Models of Business Cycles**, Robert Lucas presented some simple calculations to argue that the trade-off between fluctuations and growth is such that a representative agent's willingness to pay --in terms of growth rates-- for a more stable environment is almost zero. Even though Lucas' conclusion has been challenged by studying models that relax some of the details in his basic environment --ranging from the specification of preferences to the details of the market structure--[2] none of these analyses question a fundamental implicit assumption: that the factors that affect fluctuations do not affect long run growth.[3]

Is there any evidence that volatility of shocks --both policy and productivity shocks-- has an impact on long run growth? The results are mixed. In an early study, Kormendi and Meguire (1985) find that variability in output is positively related to mean growth in a cross section of countries. More recently, Ramey and Ramey (1995) find that higher volatility decreases growth, also in a cross section of countries. Empirical work that relates policy variability --mostly inflation-- and growth also seems to point to a negative relationship (Judson and Orphanides (1996)). Simple regressions of mean growth rates on measures of volatility of growth rates suggest a U-shape relationship, with an "upward sloping" segment only at very high levels of volatility.

Our objective in this paper is to study a class of models, in the neoclassical tradition, in which fundamental uncertainty can affect the long run growth rate.[4] Our main focus is to understand both qualitatively and quantitatively how important fluctuations are for growth. Our analysis includes both theoretical results and numerical evaluations.

Qualitatively, we show that the relationship between volatility in fundamentals and policies and mean growth can be either positive or negative. We identify the curvature of the utility function

---

[2] For the former see Manuelli and Sargent (1988), and for the latter Imrohoroğlu (1989) and Atkeson and Phelan (1994).

[3] The current standard in the real business cycle literature, is to view long run growth as exogenous and, hence, independent of the fundamental shocks. For an explicit discussion see Cooley and Prescott (1995).

[4] Although we emphasize a "technology-shock" interpretation of the type used in the real business cycle literature for the shocks in our model (e.g., see Cooley (1995) for an up to date survey of this literature), the shocks that we model can also be interpreted as random fiscal policies (for an equivalence result see Jones and Manuelli (1999)).

as a key parameter that determines the sign of the relationship: if preferences have curvature at least as high as the log, increased volatility increases growth. In the opposite case --large intertemporal elasticity of substitution-- theory does not provide a sharp prediction in all cases; if the driving shocks are i.i.d. and depreciation rates are high, increased uncertainty decreases growth. Overall, we expect a positive relationship between uncertainty and growth. Thus, for a class of simple specifications, elimination of the type of fluctuations that we study will decrease long run growth.

We also find that the decomposition of the variance of the fundamental shocks into its autocorrelation and innovation variance components matters: increases in the variability of the innovations to the fundamental shocks is likely to have a larger impact than increases in the serial correlation. Finally, we show that these models can generate positively autocorrelated growth rates but, for this to be the case, it is necessary that the driving shocks be positively autocorrelated themselves.

Even though our work follows the recent analyses of stochastic endogenous growth models in which the "source" of shocks is either technology,[5] policies,[6] or a combination of the two,[7] it has a different emphasis. Our main point is to highlight the critical roles played by the degree of risk aversion and the variability of the fundamental shocks on the distribution of growth rates.[8]

To explore the quantitative importance of uncertainty on growth, we simulate a general version of our model. The numerical exercise is revealing. The main result is that when we move from a world of perfect certainty to one with uncertainty that resembles the average uncertainty in a large sample of countries, growth rates increase somewhere between 0.17% and 0.80%, with 0.20% being a "reasonable" estimate. Even though these are nontrivial changes, they are not large enough be themselves to account for the large differences in mean growth rates observed in the data.

As expected, differences in the curvature of the utility function and the specification of the

---

[5] For example, see, King and Rebelo (1988), King, Plosser and Rebelo (1988).

[6] See Eaton (1981), Bean (1990), Aizenman and Marion (1993), Gomme (1993), Obstfeld (1994), Hopenhayn and Muniagurria (1996), and Dotsey and Sarte (1997)).

[7] See, for example, Kocherlakota and Yi (1994).

[8] Eaton (1981) also indicates how the results change with the degree of curvature of preferences. The case he studies is one in which shocks are i.i.d. and returns are exogenous, given the linearity of the technology.

fundamental uncertainty, have impacts upon the second order properties of growth rates. In general, increases in the coefficient of risk aversion decrease the standard deviation of the growth rate, and increases in the serial correlation of the exogenous shocks also increase the serial correlation of growth rates. Unlike exogenous growth models, the class of models that we study can generate positively autocorrelated growth rates.

Differences in the curvature of preferences have very substantial effects on the estimated variability of stationary objects like the consumption/output ratio and hours worked. For this reason, we expect that the models considered in this paper will provide the basis of sharp estimates of the curvature parameter. This is in contrast with the results in exogenous growth models --in which curvature has only a small effect-- and confirms the critical role that the shape of the utility function has in determining the long run growth rate in non-stochastic versions of models of endogenous growth.

Section 2 presents the basic theoretical results. Section 3 discusses a key property of endogenous growth models that makes them computationally tractable. Section 4 contains numerical results for calibrated versions of the model, and section 5 offers some concluding comments.

2. Stochastic Growth Models: Analytic Results

In this section we explore the theoretical implications of increases in uncertainty upon the distribution of growth and savings rates. Not surprisingly, an analytic characterization of the solutions to stochastic endogenous growth models is hard to come by. For this reason, we will restrict attention to two simple but revealing examples. In the first, we explore the effects of i.i.d. shocks with elastic labor supply. The second example is a version of the Ak model (alternatively, labor supply is exogenous), but allows for correlated shocks. Our results are extensions of the literature on optimal savings (e.g. Phelps (1962), Levhari and Srinivasan (1969), Rothschild and Stiglitz (1971)) expanded to incorporate general equilibrium effects, elastic labor supply, and serially correlated shocks.[9]

The equilibrium of the class of models that we study can be computed as the solution to the

---

[9] Eaton (1981) was the first to apply these ideas to growth models.

following planner's problem,

(P.1) $\text{Max } E\{\sum_t c_t^{1-\sigma} v(n_t)/(1-\sigma)\}$

subject to,

$$c_t + k_{t+1} + h_{t+1} \leq s_t A\, k_t^{\alpha}(n_t h_t)^{1-\alpha} + (1-\delta)k_t + (1-\delta)h_t$$

h_0 and k_0 given.

The shock process $\{s_t\}$ is assumed to be a linear Markov process given by,

$$s_{t+1}=1-\rho+\rho s_t+\epsilon_{t+1},$$

where ϵ_{t+1} is white noise. We assume that the distribution of ϵ is given by the measure μ_θ, where θ is an index of riskiness. More precisely, $\theta'>\theta$ means that $\mu_{\theta'}$ is dominated, in the sense of second order stochastic dominance, by μ_θ. Thus, a higher θ corresponds to higher volatility of the innovation to the technology shock. For this specification, the standard deviation of the $\{s_t\}$ process (using the invariant distribution) is given by $\sigma_s=\sigma_\epsilon/(1-\rho^2)^{½}$, where σ_ϵ is the standard deviation of the innovation. If we define increases in σ_s, holding the mean constant at one, as increases in risk, it follows that there are two ways to increase risk: increases in σ_ϵ, and increases in ρ. In our setting, increases in θ correspond to increases in the variability of the innovations, σ_ϵ. Even though we will take $\{s_t\}$ to be a productivity shock, it is possible to reinterpret $1-s_t$ as a tax shock, provided that income is used to buy a good that does not affect preferences for consumption and leisure.[10]

It is assumed that v is such that the utility function is concave, and that the marginal utility of working is negative.

2.1 The i.i.d. Case

In our first example, we consider i.i.d. shocks ($\rho = 0$), and full depreciation of both stocks ($\delta = 1$). To guarantee that an equilibrium exists, it must be the case that the economy is not too productive (for a discussion see Jones and Manuelli(1990)). For this example, the relevant condition --which we assume holds--is,

$$[\beta(A\,(1-\alpha)^{(1-\alpha)}\alpha^{\alpha})^{(1-\sigma)}]^{1/\sigma}\,(\textstyle\int_S (1+\epsilon)^{(1-\sigma)}\mu_\theta(d\epsilon))^{1/\sigma} < 1.$$

---

[10] For a more thorough discussion see Jones and Manuelli (1999). Dotsey (1997) also mentions the connection between monetary shocks and tax shocks.

To ensure an interior (in terms of n) solution, we need stronger conditions, namely,

(C.1) $[\beta(A\ (1-\alpha)^{(1-\alpha)}\alpha^{\alpha})^{(1-\sigma)}]^{1/\sigma}\ (\int_S (1+\epsilon)^{(1-\sigma)}\mu_{\theta}(d\epsilon))^{1/\sigma} < 1 - [(\sigma-1)(1-\alpha)v(1)/v'(1)]$,

and,

(C.2) If $0 < \sigma < 1$, $\lim_{n\to 0} 1 - [(\sigma-1)(1-\alpha)v(n)/(nv'(n))] < 0$.

These two conditions guarantee that the equilibrium labor supply is strictly between 0 and 1. We assume that both hold. From now on, we will describe the conditions for the case $\sigma \neq 1$.[11]

We next argue that the equilibrium decision rules display three properties: saving is a constant fraction, ϕ, of income, labor supply is constant, and the ex-post rates of return to physical and human capital are equal. First, if rates of return to the two forms of capital are equal (for each realization of s) then the stocks of human and physical capital must satisfy, $h_t = [(1-\alpha)/\alpha]\ k_t$. Given this, the saving rate, ϕ, and the level of employment, n, must solve,

(2.1) $\phi = 1 - [(\sigma-1)(1-\alpha)v(n)/(nv'(n))]$,

(2.2) $\phi = D\ \hat{s}^{1/\sigma}\ n^{(1-\alpha)(1-\sigma)/\sigma}$,

where A* is $A\ (1-\alpha)^{(1-\alpha)}\alpha^{\alpha}$, $D= [\beta(A^*)^{(1-\sigma)}]^{1/\sigma}$, and $\hat{s} = \int_S (1+\epsilon)^{(1-\sigma)}\mu_{\theta}(d\epsilon)$. Basically, (2.1) guarantees that, at the conjectured equilibrium, the marginal rate of substitution between consumption and income is equal to the real wage, while (2.2) is the Euler equation that ensures equality between the intertemporal marginal rate of substitution in consumption and the rate of return on capital. Let the solution to (2.1) and (2.2) be a pair (ϕ, n), which depends on the parameters (σ,μ_{θ}). An equilibrium is fully characterized by this pair. The growth rate associated with this equilibrium is given by,

(2.3) $y_{t+1}/y_t \equiv \gamma_{t+1} = s_{t+1}\ A^*\ n^{1-\alpha}\ \phi = s_{t+1}\ \gamma$,

where, since $Es_t=1$, γ is the mean growth rate.

---

[11] If $\sigma = 1$, the utility function is $\log(c) + v(n)$, with $v'(n) < 0$. In this case, conditions (C.1) and (C.2) take slightly different forms. We present the derivation in the proof of Proposition 1.

Proposition 1: Assume that condition C holds. Then an equilibrium of the conjectured form exists and is unique. Moreover, if $\theta' > \theta$, the equilibrium satisfies,

(a) The effects of increases in risk:
- ii) (ϕ, n, γ) increase with θ if $\sigma > 1$,
- iii) (ϕ, n, γ) decrease with θ if $0<\sigma<1$,
- iv) (ϕ, n, γ) are independent of θ if $\sigma = 1$.

b) Amplification: The ratio of the standard deviation of the growth rate and the standard deviation of the technology shock, σ_γ/σ_s, satisfies,
- i) Is greater than one ($\sigma_\gamma/\sigma_s > 1$) if the growth rate is positive ($\gamma > 1$),
- ii) σ_γ/σ_s increases with θ if $\sigma > 1$,
- iii) σ_γ/σ_s decreases with θ if $0<\sigma<1$,
- iv) σ_γ/σ_s is independent of θ if $\sigma = 1$.

Proof: See Appendix A.

Thus, in very simple economies, it is clear that even the sign of the effect of increased uncertainty upon average growth rates varies with preference parameters. The model can accommodate a positive relationship, consistent with the findings of Kormendi and Meguire (1985), if $\sigma>1$, or a negative relationship, as found by Ramey and Ramey (1995) if $0<\sigma<1$.[12] The i.i.d. (and full depreciation) version of the model has a sharp implication about a second moment of the distribution of growth rates: the first order serial correlation coefficient is zero. This statistic has been at the center of the arguments on the inability of the real business cycle model to display persistence (see Hall (1998), Cogley and Nason (1995)). The most cited evidence for non-zero serial correlation is described by Cogley and Nason (1995). These authors find, for the U.S., a small positive autocorrelation in annual data[13]. The international evidence --using the Summers-Heston data-- is less conclusive. In a sample of 148 countries, we find that in over 2/3 of the countries, the

---

[12] There is one borderline case in which changes in θ have no impact on growth rates. This case corresponds to $v(n) = n^{1+\chi}/(1+\chi)$, with $(1+\chi) > 0$, and $\sigma > 1$. For this specification $v'(n)n/v(n)$ is constant, and (2.1) pins down ϕ, which of course, is independent of the properties of $\{s_t\}$.

[13] However, when using the Summers-Heston data set for the U.S. it is not possible to reject the hypothesis that the annual first order correlation coefficient is zero.

point estimate of the first order autocorrelation coefficient is not significantly different from zero[14]. In the majority --but not all-- of the remaining countries the point estimate is greater than zero, and the average of the point estimates is close to 0.3. Thus, the evidence suggests that there is a fair amount of heterogeneity in the distribution of the serial correlation coefficients of the growth rate across countries.

2.2 The Markov Case

Our next step is to show how properties of the stochastic process $\{s_t\}$ and the rate of depreciation affect the distribution of growth rates. To this end, we study a version of problem (P.1) which abstracts from labor supply --we set v(n) equal to one-- but allows both for serially correlated shocks ($\rho \neq 0$) and less than full depreciation ($\delta < 1$). It turns out that both elements are important.

In order to guarantee that the solution to the planner's problem, (P.1), is well defined, it is necessary to bound how fast output can grow in this economy. A natural generalization of the condition discussed in Jones and Manuelli (1990) is,

(2.4) $\sup_s \beta \int (A(1-\rho+\rho s)+1-\delta+\epsilon)^{1-\sigma}\, \mu_\theta(d\epsilon) \leq \kappa < 1.$

In Proposition 2 we show that the optimal capital accumulation rule can be written as,

(2.5) $k_{t+1} = \phi(s_t;\theta,\rho)(s_t A + 1 - \delta)k_t.$

In this case, the growth rate is given by,

(2.6) $y_{t+1}/y_t \equiv \gamma(s_t,s_{t+1};\theta,\rho) = s_{t+1}\phi(s_t;\theta,\rho)[s_t A + 1-\delta]/s_t,$

and the conditional --on the current shock-- expected growth rate, $\bar{\gamma}(s_t;\theta,\rho)$, satisfies,

(2.7) $E_t\{y_{t+1}/y_t\} \equiv \bar{\gamma}(s_t;\theta,\rho) = \phi(s_t;\theta,\rho)\,(1-\rho+\rho s_t)\,(s_t A + 1-\delta)/s_t.$

First, we want to describe how changes in the riskiness of the innovation, that is, θ, affect the stochastic process for growth rates. To this end we first describe how $\phi(s_t;\theta,\rho)$, the fraction of broad income saved, depends on s and θ. Formally, we show,

---

[14] We used t-statistics in excess of 1.4 as indicating rejections of the null hypotheses of zero autocorrelation. Had we used the more stringent threshold of 2.5, we would have rejected the null for only ten countries, which is whithin the standard margin.

Proposition 2: Assume that (2.4) holds. Then, the optimal capital accumulation rule is of the form, $k_{t+1} = \phi(s_t,\theta)(s_tA + 1 - \delta)k_t$. Moreover, the function $\phi(s,\theta)$ is,

i) increasing and concave in s, and decreasing in θ, if $0<\sigma<1$,
ii) constant and equal to β if $\sigma = 1$,
iii) decreasing and convex in s, and increasing in θ, if $\sigma>1$.

Proof: See Appendix A.

Note that the conditional growth rate, $\bar{\gamma}(s_t;\theta,\rho)$, depends on θ only through $\phi(s_t;\theta,\rho)$. Thus, Proposition 2 shows that, in the case of small risk aversion ($0<\sigma<1$), increases in risk decrease the conditionally expected growth rate. If the utility function has more curvature than the log ($\sigma > 1$), risk increases --as measured by θ-- result in conditional growth rate increases. Thus, for conditional growth rates, the role of curvature identified in the i.i.d. case remains unchanged.

What are the effects of changes in risk on expected or average growth rates? The expected growth rate is given by,

(2.8) $\bar{\gamma}(\theta,\rho) = \int_S \phi(s;\theta,\rho)(1-\rho+\rho s)\,(sA + 1-\delta)/s\ \lambda_\theta(ds)$,

where λ_θ is the invariant measure of the process[15]. It follows from our model that $\theta'>\theta$ implies that λ_θ is dominated (in the second order stochastic dominance sense) by $\lambda_{\theta'}$.

Thus, if the integrand is convex an increase in θ will increase $\bar{\gamma}(\theta,\rho)$, while if it is concave the converse is true. We can now give a partial answer to the question of the effects of increased risk on average growth rates. In the case of high risk aversion ($\sigma > 1$), the function $\phi(s;\theta,\rho)$ is decreasing and convex, while the function $(1-\rho+\rho s)(sA + 1-\delta)/s$ is always convex. A sufficient condition for $\phi(s;\theta,\rho)(1-\rho+\rho s)(sA + 1-\delta)/s$ to be convex is that $(1-\rho+\rho s)(sA + 1-\delta)/s$ be decreasing for all values of s. This is equivalent to a restriction on the size of ρ. It is easy to check that, since s has support in [0,2], a sufficient condition for the expected growth rate to be a convex function of s is that $\rho \leq (1-\delta)/2(A + 1-\delta)$. Thus, for moderate serial correlation, increases in riskiness, as measured by θ, increase the expected growth rate. Of course, this condition is only sufficient, and in our numerical results we find that increases in the riskiness of the innovation process increase average growth rates

---

[15] In this case, the invariant distribution is the distribution of the random variable $\sum_{j=0}^{\infty} \rho^j \epsilon_{t-j}$, which inherits the changes in risk from the variables ϵ_t.

(if $\sigma > 1$), even for high values of the serial correlation coefficient, ρ.

When the utility function is logarithmic, $\phi(s;\theta,\rho)$ is constant, and the growth rate is a strictly convex function of s, if $\delta < 1$, and a linear function if $\delta = 1$. This shows the role played by the depreciation rate. If it is small --which is the more realistic case-- increases in the riskiness of the innovations increase average growth rates. Since this result is independent of the magnitude of the correlation coefficient, ρ, it applies to the i.i.d. case as well. Thus, the independence of the growth rate from the riskiness of the innovations that we found in Proposition 1 cannot, in general, survive -- even in an approximate sense-- whenever depreciation rates are small.[16]

We summarize this discussion in the following Proposition:

Proposition 3: Assume the conditions of Proposition 2 hold. Then, the expected --using the invariant measure-- growth rate, $\bar{\gamma}(\theta,\rho)$ is strictly increasing in θ, if

i) $\sigma = 1$, and $0 < \delta < 1$,
ii) $\sigma > 1$, and $\rho \leq (1-\delta)/2(A + 1-\delta)$.

The case of small risk aversion, $0 < \sigma < 1$, is more complicated because the function $\phi(s;\theta,\rho)(1-\rho+\rho s)(sA + 1-\delta)/s$ is the product of a concave and a convex function. Theory does not provide a prediction. Although we only explored "small" deviations from the log specification, we found that, in all our numerical exercises, increases in θ increase expected growth rates even in this case.

We next explore the effects of changes in the serial correlation coefficient, ρ. It turns out that in our highly non-linear model it is not possible to give general results. To make some progress we look at approximations (see Appendix B for derivations). Our basic finding is that:

> If the fraction of "broad" income saved is not very sensitive to small shocks, the expected growth rate, ã̄(è,ñ), does not vary too much with ñ.

If preferences are logarithmic, the standard deviation of the growth rate, σ_γ, can be approximated by,

---

[16] A continuity argument shows that if $\delta < 1$, the average growth rate increases with uncertainty, even when σ is less (but close to) one.

(2.9) $\sigma_\gamma = [2\upsilon/(1+\rho) + (1-\upsilon)^2/(1-\rho^2)]^{1/2}\sigma_\epsilon$,

where $\upsilon \equiv (1-\delta)/(A+1-\delta)$ is a number close to 0.9 for reasonable specifications. It follows that:

Increases in ñ, holding ó$_{s}$ constant, decrease ó$_{ã}$.
Increases in ó$_{å}$, holding ó$_{s}$ constant, increase ó$_{ã}$.

To study the autocorrelation of growth rates we approximate (2.6) around $s_t=s_{t+1}=1$, to get,

$$\rho_\gamma \approx \rho + (\varsigma - \upsilon)/[1+ (\varsigma - \upsilon)^2 + \rho(\varsigma - \upsilon)],$$

where $\varsigma \equiv \phi_s(1;\theta,\rho)/\phi(s;\theta,\rho)|_{s=1}$ is the elasticity of the (broad) saving rate with respect to the shock, and υ is as before.

First, this expression shows that, unless shocks are positively serially correlated, ρ_γ is negative. To see this, take the limit as ρ goes to zero and observe that, in that case, $\phi(s_t;\theta,\rho)$ is independent of s_t and, hence, that $\varsigma = 0$. Thus, in the i.i.d. case,

$$\rho_\gamma \approx -\upsilon/(1+\upsilon^2) \approx -0.5,$$

given that υ is close to one.[17]

In order to generate growth rates more persistent than the fundamental shocks ($\rho_\gamma > \rho$) it is necessary (but not sufficient) that $(\varsigma - \upsilon) > 0$. For this to be the case, investment must respond positively to the shock, that is, $\phi_s(1;\theta,\rho) > 0$. From Proposition 2 it follows that this can happen only if $0<\sigma<1$. It also follows that high depreciation rates (this corresponds to low υ) "help" the model get persistence. In summary then:

For realistic parameter values, ñ ≈ 0, then ñ$_{ã}$ ≈ -1/2.
For ñ$_{ã}$ > ñ, it is necessary that ö$_{s}$ > 0, and this requires ó < 1.
If ó > 1, then ñ$_{ã}$ < ñ.
Higher depreciation rates increase ñ$_{ã}$.

It is difficult to compare our results on the characterization of the effects of fundamental uncertainty upon the distribution of growth rates with those in the real business cycle literature. The major problem is that, in the RBC world, the curvature of the utility function plays a minor role, while in models of endogenous growth it turns out to be a major determinant of the distribution of

---

[17] Note that this result does not coincide with our findings in the previous model with i.i.d. shocks due to the difference in depreciation. In that model, $\delta = 1$, which implies, in our case, $\upsilon = 0$ and $\rho_\gamma = 0$, as showed above.

growth rates. The intuition for this is simple: differences in curvature play no role in the determination of the steady state of a standard Cass-Koopmans model, while the endogenous growth analog --a balanced growth path-- depends crucially on the degree of intertemporal substitution. This different role played by the curvature parameter in the non-stochastic version, still remains in stochastic analyses, with the added impact that it balances income and substitution effects in determining the impact of additional risk on consumption and saving.

Our findings in this section indicate that the relationship between fundamental shocks and the distribution of growth rates is highly non-linear. We find that for curvature levels greater than or equal to the log --low intertemporal elasticity of substitution-- increases in risk increase both investment over (properly defined) output ratios and average growth rates. Our results for curvature less than the log indicate that investment as a fraction of (properly defined) output decreases with risk. However, the implications for expected growth rates are less clear since only in the extreme case of i.i.d. shocks and full depreciation does the model give a robust prediction. Here, decreases in average growth follow from increases in risk.

The other two theoretical predictions of interest are first, the differential impact of serial correlation and variability of innovations upon the second moments of the growth rates, and, second, the possibility of serially correlated growth rates if fundamental shocks are also serially correlated.

3. Computing Equilibria of Linear Endogenous Growth Models

The actual solution of general versions of the models described in the previous sections does cause some problems. The natural choice of the state is the vector (k_t,h_t,s_t). The difficulty is that both k_t and h_t are converging to infinity (at least for versions of the model that exhibit growth on average). This renders numerical methods useless: they simply do not apply to this case. Despite this, the form of the value and policy functions have relatively simple characterizations under some additional assumptions about the form of the utility and production functions. The key property that we will exploit is that for general versions of the models of the type described in (P.1) to have a balanced growth path, both preferences and technology must be restricted in a specific way (see King, Plosser and Rebelo (1988), and Alvarez and Stokey (1995)). Specifically, utility functions must be of the form,

$$u(c,\ell) = \begin{cases} v(\ell)\, c^{1-\sigma}/(1-\sigma) & \text{with } \sigma \neq 1, \text{ but } \sigma > 0, \text{ or} \\ \log(c) + v(\ell) & \end{cases}$$

In our discussion, we will concentrate on the non-separable case. The same arguments apply to the separable case.

On the technology side we consider technology sets given by,

$$c_t + x_{zt} + x_{ht} + x_{kt} \leq F(k_t,z_t,s_t)$$

$$z_t \leq M(n_{zt},h_t,x_{zt})$$

$$k_{t+1} \leq (1-\delta_k)\, k_t + x_{kt}$$

$$h_{t+1} \leq (1-\delta_h)\, h_t + G(n_{ht},h_t,x_{ht})$$

$$\ell_t + n_{ht} + n_{zt} \leq 1,$$

h_0 and k_0 given.

Here $\{s_t\}$ is a stochastic process which we assume is Markov with transition probability function P(s,A) and c_t is consumption, x_{kt} is investment in physical capital, k_t is the stock of phsyical capital, x_{ht} is investment in human capital, h_t is stock of human capital, z_t is "effective labor," n_{zt} is hours spent working in the market, n_{ht} is hours spent augmenting human capital and ℓ_t is leisure. δ_k and δ_h are the depreciation rates on physical and human capital, respectively.

Thus, this is a fairly standard endogenous growth model in which effective labor is made up of a combination of hours and human capital which is supplied to the market. For specific choices of functional forms, many models in this literature are special cases of this formulation. For example, if $M = n_z h$ and $G = G_0 h n_h$, the model corresponds to Lucas (1988) in the absence of externalities. If $M = n_z h$ and $G = x_h$, this corresponds to the two capital goods version discussed in Jones, Manuelli and Rossi (1993). Finally, note that the standard one sector growth model with exogenous technological change is also a special case (but the s_t itself is not Markov in that case-- $G \equiv 0$, $M(\,) = n_z$). Given convexity of technologies and preferences, if markets are complete --as we assume-- the equilibrium allocation can be found solving a planner's problem of this form.

It can be shown that the essential property is that the technology set be linearly homogeneous in reproducible factors. This corresponds to,

(i) F is concave and homogeneous of degree one in (k,z)

(ii) M is concave and homogeneous of degree on in (h,x_z)

(iii) G is concave and homogeneous of degree one in (h,x_h).

These restrictions effectively imply that the choice set in this more general version of (P.1) is linearly homogeneous in the initial stocks and that preferences are homothetic, holding the non-reproducible choice variables, n in our application, fixed. This implies that knowledge of the current shock and the current human capital/physical capital ratio (the two relevant pseudo state variables) is sufficient to determine the optimal choices of employment and next period's human to physical capital ratio.

Let $\{e_t\}$ be the entire state/date contingent plan for the reproducible factors. The plan $\{e_t,n_t\}$ is feasible from initial state $e_0=(h_0, k_0)$, for a given s_0, if and only if $\{\lambda e_t,n_t\}$ is feasible from the initial state $\lambda e_0=(\lambda h_0, \lambda k_0)$ $(\lambda > 0)$. Moreover, utility (i.e., the entire expected discounted sum) realized from $\{\lambda e_t,n_t\}$ is $\lambda^{1-\sigma}$ times the utility of $\{e_t,n_t\}$. Formally, consider the maximization problem:

(P.2) $\quad \text{Max } U(e, n)$

subject to

$$(e,n) \in \Gamma(h_0, k_0, s_0),$$

where, as noted, (e,n) is interpreted as the entire date/state contingent path of the endogenous variables and vector of labor supplies and U is the resulting expected discounted sum of utilities. Let $V(h_0, k_0, s_0)$ denote the maximized value in this problem (assuming that it exists) and let $(e^*(h_0, k_0, s_0), n^*(h_0, k_0, s_0))$ denote the optimal plan. Then, it follows that:

Proposition 4: Assume that the utility function in (P.2) is homogeneous of degree $1-\sigma$ in e (holding n fixed) and that the feasible set, Γ, is linearly homogeneous in (h,k) (holding n and s fixed) and that a solution exists for all (h,k,s). Then, the value function, V, for the problem (P.2) satisfies $V(\lambda k,\lambda h,s) = \lambda^{(1-\sigma)} V(k,h,s)$, for all $\lambda>0$. Moreover, the optimal plans are homogeneous of degree one in z and zero in n-- $(e^*(\lambda k,\lambda h,s), n^*(\lambda k,\lambda h,s)) = (\lambda e^*(h,k,s), n^*(h,k,s))$.

Proof: See Appendix A.

From the point of view of a numerical approximation of problems like (P.2), this result implies that it is possible to estimate the optimal decision rules for c/k, x_j/k, j= h,k,z as functions of the bounded --within a reasonably large set-- variable h/k, and then calculate:

$$k' = (1-\delta_k)\, k + k(x_k/k) \text{ and}$$

$$h' = (1-\delta_h)\, h + h\, G(x_h/k,\, h/k,\, n_h)$$

to determine h'/k'. Thus, in this case, the Euler equations corresponding to (P.2) are solved by functions that depend on the stationary variables, h/k and s only.

Proposition 4 applies to any planning problem that has the required linearity and homogeneity properties. These include models with multiple sectors, preferences that depend on the state (e.g. human capital determines effective leisure), etc. A separate, but related question is under what conditions equilibrium allocations can be represented as solutions to planners' problems of the type described in (P.1). This class includes convex endogenous growth models with no external effects and the same class of models with proportional income taxes (see Jones and Manuelli (1999)) among others.

Proposition 4 does not apply to planner's problems in which the technology displays increasing (e.g. Romer (1986)) or decreasing (e.g Brock and Mirman (1972) and the real business cycle application) returns to scale in reproducible factors, or ones that have distortions with no planning representations (e.g different tax rates on capital and labor income).

4. Quantitative Effects of Uncertainty

In this section we rely on numerical methods to analyze the quantitative effects of variability in fundamentals upon the distribution of growth rates.

4.1 Model Specification and Calibration

We study a special case of the model of section 3. Specifically, we assume,

$$u(c,1-n) = [c(1-n)^{\psi}]^{1-\sigma}/(1-\sigma),\ F(k,z,s) = sAk^{\alpha}z^{1-\alpha},\ G(n_{ht},h_t,x_{ht}) = x_{ht},\ M(n,h) = nh.$$

$$s_{t+1}=1-\rho+\rho s_t+\epsilon_{t+1},\ \text{with } \epsilon_t \text{ i.i.d., and } \epsilon_t \sim U[-\zeta,\zeta].$$

The specification is standard. Our assumption that only x_{ht} enters in the production of new human capital (which is, of course, produced using labor, and both physical and human capital through the technology F) amounts to an aggregation assumption --namely that the technology used to produce human capital as a function of capital and effective labor is identical to that in the final goods sector. Finally, we specify $\delta_k = \delta_h$. This assumption greatly simplifies the solution since it implies a constant physical/human capital ratio (for details see Appendix B)

To calibrate the model, we assumed that α --capital's share-- is given by .36, and held β fixed at .95. We assumed that the common depreciation rate of human and physical capital is given by δ = .075, and in each case, calibrated the model so that labor supply in the non-stochastic steady state is given by n = .17 (see Jones, Manuelli and Rossi (1993)).[18] These restrictions still leave one degree of freedom in the selection of the parameters of preferences and technology. One way to solve this indeterminacy is to choose σ (the coefficient of risk aversion) and the calibrated growth rate, γ. In order to do this consistently, it is necessary to simultaneously adjust A (the average technology parameter) and ψ (a curvature parameter in the utility function) to keep n at 0.17. We chose as our 'base case' a value of $\sigma = 1.5$ and $\gamma = 1.02$, or 2% growth per year.

To determine the stochastic process for the fundamental uncertainty, it is necessary to specify, ρ and ζ. For our base case we chose $\rho = 0.9$ and $\zeta = 0.08$. Unfortunately, there is no counterpart of s_t in the data. However, different $\{s_t\}$ processes imply different stochastic processes for the growth rate. We used the average (over countries) standard deviation of the growth rate and its first order serial correlation as the moments to match. In the Summers and Heston data, the average (across countries) standard deviation of the per capita growth rate is 0.0601 and its serial correlation is 0.1256. For our base case, the stochastic process given by $\rho = 0.9$ and $\zeta = 0.08$ comes close to replicating these values for the endogenously determined process for the growth rates.[19]

Even though our base case parameters are motivated by the desire to match observations, the principal aim of the paper is to understand how variability in fundamentals affects the distribution of growth rates more generally. Thus, we will study alternative parameter values to better understand the effects of volatility on growth. First, our theoretical results indicate that some parameters --e.g. σ, ρ, and the standard deviation of the innovation, σ_ϵ-- are important determinants of the transmission mechanism of exogenous shocks. Second, the available evidence on growth rates from the Summers and Heston data set shows large variation in average values across countries; even though the average over all countries is 1.98%, the first quartile is given by an average growth rate of 0.91%, while the third quartile is given by a rate of 3.23%. Since we want to explore the

---

[18] In earlier versions of the paper, we also tried calibrations so that the non-stochastic steady state labor supply was n = .3. This had only minor effects, and hence, the results are not included here.

[19] For our cross country data we use the PWT 5.6. See Summers and Heston (1991) and (1993)).

possibility that the heterogeneity in average growth is due to differences in the country specific $\{s_t\}$ processes we will also consider changes in ρ and σ_ϵ.

We study the following variations to our base case:

a) We varied σ from 0.9 to 3.0.

b) We varied the calibrated growth rate from 0% to 4% per year.

c) We varied ρ from 0.7 to 0.915.

d) We varied the standard deviation of the innovation from 0.035 to 0.081.[20]

Except in those exercises corresponding to the change in the non-stochastic mean (i.e. point b) above), every time that a parameter is changed the model is recalibrated so as to match the same moments as the base case.

To solve the model, we compute the optimal decision rules after we discretize the state space. We then draw a realization of $\{s_t\}$ of size 5000, and compute the moments using this realization. In those cases in which the stochastic process $\{s_t\}$ is not changed, we have used the same realization to facilitate comparisons.

4.2 Uncertainty, Risk Aversion and Growth Rates

In this section we study how changes in the curvature of the utility function, σ, affect the distribution of growth rates. The basic results, for several specifications, are presented in Table 1. We report the values of $E(\gamma)$, the average growth rate in the simulation, σ_γ, the standard deviation of the growth rate in the simulation and ρ_γ, the first order autocorrelation coefficient of the growth rate in the simulation. For reference we also present, in the last three rows, comparable statistics from the Summers and Heston (denoted SH) dataset. Thus, the SH mean of 1.98% means that the average across countries is 1.98% while the middle 50% of countries had average growth rates between 0.91% and 3.23%. Similarly, the average across countries of the standard deviation of the growth rate is 0.06, while the middle 50% of countries had standard deviations between 0.041 and 0.077. In this table we hold ρ and ζ fixed at their base case values of 0.9 and 0.08 respectively, and adjust σ

---

[20] It is easy to check that $\sigma_\epsilon = \zeta/3^{1/2}$. Thus, in terms of ζ we tried values from 0.06 to 0.14.

from 0.9 to 3.0.[21]

Since the non-stochastic version of all these versions is calibrated to grow at 2%, any difference between the E(γ) column and 2% is due to the increase in the standard deviation of the shock from 0 to 10% (as a percentage of the mean). The major findings are:

> a) Our base case, corresponding to case 4 in Table 1, matches the standard deviation and first order autocorrelation of the per capita growth rates fairly well. For this base case, the impact of increased uncertainty upon mean growth is small, and approximately equal to one fifth of one percent per year. The largest impact of uncertainty, occurs for preferences that are less concave than the log.
>
> b) As suggested by Proposition 3, the average growth rate in the simulations exceeds the calibrated value of 2% for each value of σ.
>
> c) As expected, increases in the curvature of the utility function, σ, result in decreases in the standard deviation of growth rates. Thus, for coefficients of relative risk aversion exceeding one, we find that increases in risk aversion increase mean growth and decrease its variability.
>
> d) For all these specifications, the model's prediction of the autocorrelation coefficient of growth rates is small, and often negative. It is clear that the growth rate of output --unlike the growth rate of capital-- does not inherit the serial correlation properties of the driving shock.
>
> e) As expected from the theoretical results, the smaller the curvature of the utility function the higher the autocorrelation coefficient. More curvature makes investment respond negatively to the current shock and this, in turn, implies that the growth rate is more negatively serially correlated.
>
> f) The effect of a given amount of uncertainty upon the expected growth rate varies with the curvature parameter σ, and it is not a monotone function of curvature. Figure 1 shows that the largest impact of uncertainty occurs for values just below log utility. Moreover, for $\sigma > 1$, increases in risk aversion increase E(γ). Overall, the relationship between σ and E(γ) has a U-shape.

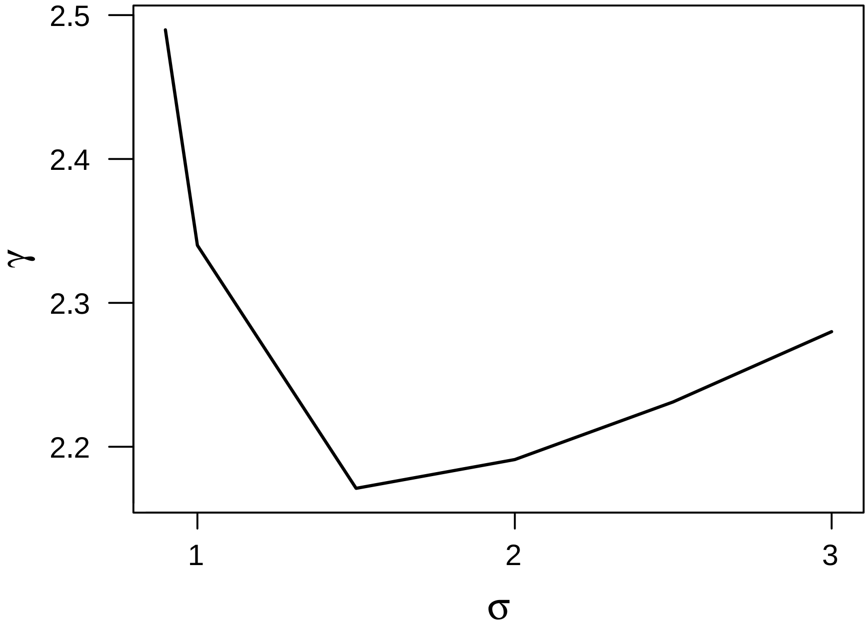
---

[21] Note that if $\sigma < 1$, concavity of the utility function puts some restrictions on what ψ can be. For each σ, we adjusted A and ψ to keep the growth rate for the non-stochastic version of the model fixed at 2% and labor supply equal to .17. Thus, we could equally well index the cases by either A, or ψ.

Table 1. The Effect of σ on E(γ), the average growth rate

Case	σ	ρ	ζ	σ_s	$E(\gamma)$	σ_γ	ρ_γ
1	0.9	0.9	0.07	0.093	2.49%	0.099	0.031
2	1.0	0.9	0.08	0.106	2.34%	0.090	0.012
3	1.5	0.9	0.08	0.106	2.17%	0.064	-0.002
4	2.0	0.9	0.08	0.106	2.19%	0.059	-0.005
5	2.5	0.9	0.08	0.106	2.23%	0.056	-0.007
6	3.0	0.9	0.08	0.106	2.28%	0.055	-0.008
SH mean	-	-	-		1.98%	0.060	0.123
SH Q1	-	-	-		0.91%	0.041	-0.039
SH Q3	-	-	-		3.23%	0.077	0.307

Figure 1: Curvature and Mean Growth Rates



Overall, we find that qualitatively uncertainty affects growth in the expected direction.

Quantitatively the results are more difficult to interpret. The changes in average growth due to uncertainty range from 1/5 to 1/2 of 1% per year. Although the observed differences in average growth rates across countries seen in the Summers-Heston data are substantially larger, it is not clear what fraction of these differences could potentially be due to differences in volatility.

4.3 The Nature of Uncertainty and its Effects on the Distribution of Growth Rates

For the linear stochastic Markov process $\{s_t\}$, the standard deviation is $\sigma_s=\sigma_\epsilon/(1-\rho^2)^{1/2}$. This moment depends on the magnitude of the standard deviation of the innovation, σ_ϵ, and the autocrrelation coefficient, ρ. In this section we study the effects of varying these two components, σ_ϵ and ρ, on the distribution of growth rates.

Our first set of experiments studies changes in σ_ϵ given a value of ρ.[22] In the context of the theory developed in section 2, and increase in σ_ϵ corresponds to an increase in risk, θ. In Figure 2 we present the $\sigma = 1.5$ case for two different values of ρ, 0.9 and 0.8. In this figure, we plot the effect of changes in σ_s due to changes in σ_ϵ upon the expected growth rate.[23] Our major findings are:

> a) As expected, increases in σ_ϵ increase the expected per capita growth rate. The impact is not linear, with larger effects for high levels of uncertainty. At the high end, when the standard deviation of the shock is 13.5%, the average growth rate is 2.8%, an increase of 0.8% over the deterministic benchmark.[24]
>
> b) The effect of a given change in σ_s can have substantially different impacts on mean growth rates depending on the serial correlation of the shock. For example, for the $\rho=0.9$ economy (dashed line in Figure 2) an increase in σ_s from 10% to 12% has a very small impact on mean growth rates, while the same increase for the $\rho=0.8$ economy results in a substantial change in mean growth rates close to ½ of one percent. Based on these examples, it seems that the

---

[22] Increases in ζ correspond to increases in σ_ϵ.

[23] See Appendix C, Table C.1 for the basic data.

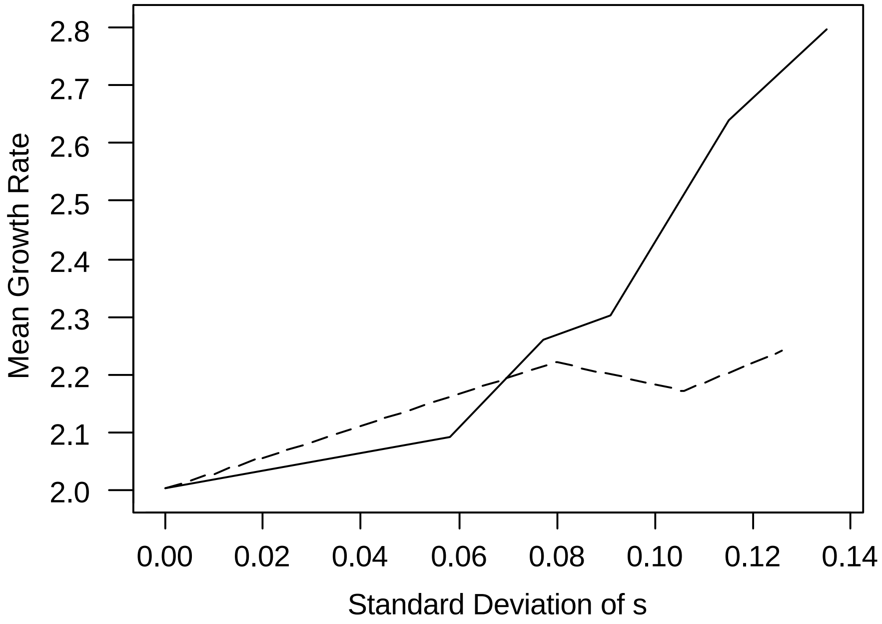
[24] This is a substantial impact, but it comes at a high cost: In this case, the model predicts the standard deviation of the per capita growth rate to be 0.12; twice the average value from the Summers and Heston data set, although still in the support of the distribution of standard deviations.

higher the level of serial correlation the smaller the impact of variability on average growth.[25]

c) Changes in σ_s given by changes in σ_ϵ have almost linear effects on the standard deviation of the growth rate, and very small effects on the autocorrelation of growth rates. (Table C.1)

Figure 2: Standard Deviation of Innovations and Mean Growth

Solid - correlation = .8, Dashed - correlation = .9



For our next set of experiments, we held σ_ϵ constant, and changed σ_s through changes in the correlation coefficient of the driving shocks, ρ. The major findings (the basic data are in Tables C.1 and C.2) are:

a) Increases in ρ (holding σ_ϵ constant) have small --but negative-- effects on the average growth rate (Table C.2 contains the data).

b) Increases in ρ have a small --but negative, as expected-- impact on the standard deviation

---

[25] This exercise also shows why we were forced to stay away from the standard linear-quadratic approximations used in the real business cycle literature. In the case of a linear approximation to the Euler equations, the theoretically predicted impact of changes in σ_ϵ upon the decision rules is zero. It is because of our interest in this higher order effect that we used a different numerical strategy.

of growth rates.

c) Increases in ρ, increase the serial correlation of the growth rate (see Table C.2).

Is it possible that uncertainty has a different effect for “high” growth and “low” growth countries?[26] To explore these issues we tried adjusting the non-stochastic steady state growth rate to which we calibrate the model. For our base case we tried several values from 0% to 4%. Our numerical results (see Table C.3) show that the non-stochastic steady state has no impact upon measured moments of the distribution of growth rates.

4.4 Volatility and Cyclical Behavior

Even though our primary interest in this paper is to begin the exploration of the effects of uncertainty upon growth, the model “delivers” implications for cyclical variables. However, unlike more standard real business cycle models, we are not free to detrend the data. Our theoretical model implies that the appropriate detrending procedure is to consider the ratio of each variable, except for hours worked (our variable n), to output. In the case of hours, the model implies that it is a stationary variable.

Before we confront the model's predictions with the data, it is necessary to match the notion of investment in human capital with observable quantities. In the model, the variable x_h corresponds to investment and is conceptually different from consumption. What is the counterpart in the data? One reason why this is a difficult question is that it is not clear what human capital is. Probably most economists would agree that it includes education and training, but it is also likely to encompass other activities like health care, investments in mobility and the like. Even for those items in which there is consensus (e.g. education and training) there are no good measures. To say the least, training is poorly measured and, depending on its nature, may not even be part of measured output. In the

---

[26] In the context of this paper the differences in growth rates could be due to distortionary taxes and/or differences in technology.

case of education, and some forms of training, gross investments appear in consumption[27]. Thus, in this paper, we assume that all of x_h is part of measured output, and we experiment with two notions of consumption: the "narrow" view that consumption in the data corresponds to consumption in the model, and the "broad" view that consumption in the data is the sum of consumption and investment in human capital, $c+x_h$.[28]

In Table 2 we report the results for our base case and for various levels of curvature. There are several interesting features:

> a) Uncertainty has a small effect upon the mean of consumption/output ratio, both in its narrow version, c/y, and its broad version, $(c+x_h)/y$. However, the choice of narrow versus broad consumption has a substantial effect on the mean value of c/y, roughly adding 30 percentage points.
>
> b) The model has the very sharp implication that higher values of relative risk aversion result in decreases in the standard deviation of the consumption/output ratio using either measure. This decrease is dramatic. The standard deviation increases six fold when moving from $\sigma=3.0$ to $\sigma=0.9$. For reference, the standard deviation of measured c/y in the U.S. is 0.019.[29] If we wanted the model to match the standard deviation of the consumption/output ratio, the best estimate of σ is slightly above one.
>
> c) The model implies that the amount of curvature in the utility function has sharp implications for the coefficient of variation of the number of hours worked. The basic data are in the last column of Table 2. As the coefficient of relative risk aversion moves from 3.0 to 0.9, the predicted coefficient of variation of the number of hours worked increases by a factor of eight. For comparative purposes, the analogous value of the coefficient of variation

---

[27] Of course, it is possible to net out educational expenditures, both private and public, however, other components like health care are much more difficult to allocate since not all expenditures probably qualify as investments in productive human capital.

[28] For an extended discussion of alternative strategies in terms of allocating x_h in different ways, see Jones, Manuelli, Siu and Stacchetti (1998).

[29] Let $\sigma((c+x_h)/y)$ be the standard deviation of a "broad" measure of consumption that includes investment in human capital, x_h, as part of consumption, and let $\sigma(c/y)$ be analog for a "narrow" notion of consumption. It follows that $\sigma((c+x_h)/y) = \alpha\sigma(c/y)$. Thus, "broad" consumption is less variable than "narrow" measure because the former includes x_h which is an investment good and, as such, its ratio to output increases in good times and decreases in bad times. The curvature in the utility function implies that the c/y ratio decreases in good times and increases in bad times. Thus, roughly, c/y and x_h/y are negatively correlated. Hence, their sum exhibits lower variability than either of the components.

of hours worked in the U.S. is 0.034[30]. Thus, in this case the "best" value of σ is something close to 2.

d) In the cases presented to this point, n(s) is strictly increasing as a function of s. However, it is possible to modify the model a get a non-linear n(s) function. Our results (not presented here) suggest that cases in which the mean growth rate is small (say less than 1.4%) and the serial correlation of the shock is large (exceeding 0.95) are consistent with an increasing response of hours worked to productivity shocks when the shock is small, and a decreasing response when the shock is large. Whether that asymmetric response can account for puzzles like the productivity slowdown and the behavior of hours worked over the cycle we do not know at this point.

Table 2. Properties of Consumption and Labor Supply

Case	σ	ρ	ζ	E(c/y)	E((c+x_h)/y)	σ(c/y)	σ((c+x_h)/y)	σ(n)/E(n)
1	0.9	0.9	0.07	0.37	0.77	0.089	0.032	0.186
2	1.0	0.9	0.08	0.37	0.77	0.068	0.024	0.137
3	1.5	0.9	0.08	0.41	0.79	0.027	0.010	0.054
4	2.0	0.9	0.08	0.44	0.80	0.019	0.007	0.035
5	2.5	0.9	0.08	0.47	0.81	0.016	0.006	0.027
6	3.0	0.9	0.08	0.50	0.82	0.014	0.005	0.022

5. Conclusion

For the class of neoclassical models that we study changes in the variability of fundamentals also results in changes in average growth rates. For levels of risk aversion at least as high as the log, eliminating cycles completely would result in lower growth rates. The size of this effect ranges from .2% per year to .5% per year, depending on the parameters of preferences. Of course, this only reinforces Lucas' conclusions that the payoff from eliminating cycles is not too large.

Theoretically, we show that it is possible for increased uncertainty to decrease average growth. However, this requires parameter values that lie outside the usual range --high intertemporal

---

[30] For the U.S. data we use the Burnside and Eichenbaum (1994) data. To calculate the coefficient of variation of hours worked we did not detrend the per capita number of hours.

substitution, no correlation of major shocks and very short lived capital.

We also identify, for reasonable parameter values, changes in the variability of the innovations to fundamental shocks as having a larger impact upon average growth rates than changes in the serial correlation of the shocks.

From a quantitative point of view there are two major findings. First, for reasonable values of exogenous uncertainty, variability in fundamentals is not large enough to be the only reason why average growth rates differ so much across countries. Second, uncertainty in fundamentals has a large impact on the predicted standard deviation of “cyclical” variables (e.g. consumption-output ratio), and the size of the impact is very sensitive to the degree of curvature of preferences.

Our finding that increased uncertainty increases average growth seems at odd with the empirical work of Ramey and Ramey (1995). However, since it is possible to interpret the shocks in our model as shocks to tax rates, our results imply that --holding average tax rates fixed-- increases in the variance of tax rates increases average growth. Of course, if growth inhibiting policies (on average) are associated with volatile policies, the model could deliver a negative correlation between volatility and average growth. However, in this case, it is not the high volatility that is causing growth to be low, but the high average tax rates.[31]

Our preliminary conclusion is that, even though there is a trade-off between fluctuations and growth, bringing stochastic elements to the class of endogenous growth models that we studied does not radically improve its ability to explain “growth” facts. However, it delivers very sharp implications about the effect of curvature in preferences on the variability of cyclical variables and, hence, it can use data to pin down preference parameters. The version of the model that we studied is too simple to proceed with this program. One manifestation of this is the difficulty in matching growth and cyclical observations simultaneously. In ongoing work (see Jones, Manuelli, Siu and Stacchetti (1998)) we study versions of these models that allow for variable capital/human capital ratios and different specifications of the human capital augmentation technology.

---

[31] In their work, Ramey and Ramey (1995) find that policy variability is associated with residual uncertainty.

REFERENCES

Aizenman, J. and N. Marion, 1993, “Policy Uncertainty, Persistence and Growth,” **Review of International Economics**, 1. pp: 145-163

Alvarez, F. and N.L. Stokey, 1995, “Dynamic Programming with Homogeneous Functions,” Working paper, University of Chicago.

Atkeson, A. and C. Phelan, 1994, “Reconsidering the Costs of Business Cycles with Incomplete Markets,” **NBER Macreconomics Annual**, pp: 187-207.

Barro, R. and X. Sala-i-Martin, 1995, **Economic Growth**, McGraw-Hill, New York, Saint Louis.

Bean, C.I., 1990, “Endogenous Growth and the Procyclical Behavior of Productivity,” **European Economic Review**, 34, pp:355-363.

Becker, R., 1985, “Capital Income Taxation and Perfect Foresight,” **Journal of Public Economics**, 26, 147-167.

Burnside, C. and M. Eichenbaum, 1994, “Factor Hoarding and Propagation of Business Cycle Shocks,” NBER working paper No. 4675.

Christiano. L., 1988, “Why Does Inventory Investment Fluctuate So Much,” **Journal of Monetary Economics**, Vol 21, pp: 247-280.

Cooley, T.F. (ed), 1995, **Frontiers in Business Cycle Research**, Princeton University Press, Princeton, New Jersey.

Colley, T.F and, E. C. Prescott, 1995, “Economic Growth and Business Cycles,” in Cooley, T.F. (ed), **Frontiers in Business Cycle Research**, Princeton University Press, Princeton, New Jersey, pp: 1-38.

Danthine, J.P. and Donaldson, J., 1985, “A Note on the Effects of Capital Income Taxation on the Dynamics of a Competitive Economy,” **Journal of Public Economics**, 28, 255-265.

Dotsey, M and P-D Sarte, 1997, “Inflation Uncertainty and Growth in a Simple Monetary Model,” Federal Reserve Bank of Richmond, May

Eaton, J., 1981, “Fiscal Policy, Inflation and the Accumulation of Risky Capital,” **Review of Economic Studies**, XLVIII, 435-445.

Gomme, P. J., 1993, “Money and Growth Revisited: Measuring the Costs of Inflation in an

Endogenous Growth Model," **Journal of Monetary Economics**, 32, August, pp: 51-77.

Hopenhayn, H. and M. Muniagurria, 1996, "Policy Variability and Economic Growth," **Review of Economic Studies**, 63, pp: 611-625.

Imrohoroğlu, A,, 1989, "Costs of Business Cycles with Indivisibilities and Liquidity Constraints," **Journal of Political Economy**, 97(6), pp: 1364-1383.

Jones, L. E. and R. E. Manuelli, 1990, "A Convex Model of Equilibrium Growth: Theory and Policy Implications," **Journal of Political Economy**, 98, 1008-1038.

Jones, L. E. and R. E. Manuelli, 1999, "The Equivalence Between Productivity and Tax Shocks," working paper.

Jones, L. E., R. E. Manuelli and P. E. Rossi, 1993, "Optimal Taxation in Models of Endogenous Growth," **Journal of Political Economy**, Vol. 101, No. 3, 485-517.

Jones, L. E. and R. E. Manuelli, 1997, "The Sources of Growth" **Journal of Economic Dynamics and Control**, 27, pp: 75-114.

Jones, L. E., R. E. Manuelli, H. Siu and E. Stacchetti, 1998, "The Business Cycle Frequency Properties of Models of Endogenous Growth," working paper.

Judd, K., 1987, "Useful Planning Equivalents of Taxed Economies," working paper.

Judson, R. and A. Orphanides, 1996, "Inflation, Volatility and Growth," Finance and Economics Discussion series 96-19, Federal Reserve Board, May.

Kehoe, T., D. Levine and P. Romer, 1992, "On Characterizing Equilibria of Economies with Externalities and Taxes as Solutions to Optimization Problems," **Economic Theory**, 2, 43-68.

King, R. G., C. Plosser and S. Rebelo, 1988, "Production, Growth and Business Cycles, II: New Directions," **Journal of Monetary Economics**, 21, 309-341.

King, R. G. and S. Rebelo, 1988, "Business Cycles with Endogenous Growth," Rochester Working Paper.

Kocherlakota, N. and K.M. Yi, 1994, "Is There Endogenous Long Run Growth? Evidence From the U.S. and the U.K.?," working paper, September.

Kormendi, R. L. and P.G. Meguire, 1985, "Macroecomic Determinants of Growth: Cross-Country Evidence," **Journal of Monetary Economics**, Vol. 16, September, pp: 141-163.

Levhari D. and T.N. Srinivasan, 1969, “Optimal Savings Under Uncertainty,” **Review of Economic Studies**, Vol XXXVI, No. 106, April, pp: 153-163.

Lucas, R. E., Jr., 1988, “On the Mechanics of Economic Development,” **Journal of Monetary Economics**, 22, 3-42.

Manuelli, R. and T. J. Sargent, 1988, “Models of Business Cycles: A Review Essay,” in **Journal of Monetary Economics**, 22, pp: 523-542

Obstfeld, M., 1994, “Risk-Taking, Global Diversification and Growth,” **American Economic Review**, , , (December).

Phelps, E.S., 1962, “The Accumulation of Risky Capital: A Sequential Utility Analysis,” **Econometrica**, 30, pp: 729-743.

Ramey, G. and V. Ramey, 1995, “Cross-Country Evidence on the Link Between Volatility and Growth,” **American Economic Review**, 85, pp:1138-1151.

Rebelo, S., 1991, “Long Run Policy Analysis and Long Run Growth,” **Journal of Political Economy**, 99, 500-521.

Rothschild, M. and J. Stiglitz, 1971, “Increasing Risk II: Its Economic Consequences,” **Journal of Economic Theory**, 3, 66-84.

Stokey, N.L. and R.E. Lucas (with the collaboration of E.C. Prescott), 1989, **Recursive Methods in Economic Dynamics**, Harvard University Press.

Summers, R. and A. Heston, 1991, “The Penn World Table (Mark 5): An Expanded Set of International Comparisons, 1950-1988,” **Quarterly Journal of Economics**, 106, 2, May, pp: 327-368.

Summers, R. and A. Heston, 1993, “Penn World Tables, Version 5.5,” available on diskette from N.B.E.R.

Zhu, X., 1992, “Optimal Fiscal Policy in a Stochastic Growth Model,” **Journal of Economic Theory**, Vol. 58, No.2, 250-290.

Appendix A: Proofs

Proof of Proposition 1: We first consider the case $\sigma \neq 1$. The first order conditions for problem (P.1) are,

Consider the first model of section 2. The first order conditions are:

$$(A.1.1)\quad c_t\, v'(n_t) = (\sigma-1)\,(1-\alpha) y_t v(n_t)/n_t$$

$$(A.1.2)\quad c_t^{-\sigma} v(n_t) = \beta \int_S [c_{t+1}^{-\sigma} v(n_{t+1})][A\alpha k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha} n_{t+1}^{1-\alpha}(1+\epsilon_{t+1})]\mu_\theta(d\epsilon_{t+1}) \quad ,$$

$$(A.1.3)\quad c_t^{-\sigma} v(n_t) = \beta \int_S [c_{t+1}^{-\sigma} v(n_{t+1})][A(1-\alpha) k_{t+1}^{\alpha} h_{t+1}^{-\alpha} n_{t+1}^{1-\alpha}(1+\epsilon_{t+1})]\mu_\theta(d\epsilon_{t+1}) \quad ,$$

and the feasibility constraints at equality. In order to find the solution to the planner's problem, we first hypothesize that (A.1.2) and (A.1.3) are satisfied by having the terms in square brackets inside the integral operator equal in each state. Second, we conjecture that consumption is a constant fraction of income. Finally, we guess that the fraction of the time allocated to working is constant as well. These conjectures imply that the solution must satisfy,

$$(A.1.4)\quad h_t = [(1-\alpha)/\alpha]\, k_t,$$

$$(A.1.5)\quad (1-\phi)\, v'(n) = (\sigma-1)\,(1-\alpha) v(n)/n_t,$$

$$(A.1.6)\quad \phi^\sigma = \beta (A^*)^{1-\sigma} n^{(1-\alpha)(1-\sigma)} \int_S (1+\epsilon_{t+1})^{1-\sigma} \mu_\theta(d\epsilon_{t+1}) \quad ,$$

where ϕ is the fraction of income, y, which is saved (of course, $1-\phi$ is consumed), and A* is $A\,(1-\alpha)^{(1-\alpha)}\alpha^\alpha$. The solution to equations (A.1.5) and (A.1.6) can be used to construct an equilibrium by letting investment in physical capital, x_k, be given $\alpha\phi y$, while x_h is $(1-\alpha)\phi y$. To simplify notation, let $D = [\beta (A^*)^{(1-\sigma)}]^{1/\sigma}$, and let $\hat{s} = \int_S (1+\epsilon)^{(1-\sigma)} \mu_\theta(d\epsilon)$. Then, (A.1.5) and (A.1.6), imply that the equilibrium values of ϕ and n solve,

$$\phi = H(n) \equiv 1 - [(\sigma-1)(1-\alpha) v(n)/(n v'(n))],$$

$$\phi = G(n) \equiv D\, \hat{s}^{1/\sigma}\, n^{(1-\alpha)(1-\sigma)/\sigma},$$

which correspond to equations (2.1) and (2.2) in the text.

Note that the function G(n) is upward sloping if $0<\sigma<1$, and downward sloping if $\sigma>1$. Moreover, increases in $\hat{s}$ increase G(n). The properties of H(n) depend on v. However, concavity of the utility function imposes some restrictions. The nature of these restrictions depends on σ. It is straightforward to verify that positive marginal utility of leisure and concavity imply that $v'(n)/(1-\sigma)$ and $v''(n)/(1-\sigma)$ must both be negative. In addition, concavity requires that $(\sigma/(\sigma-1)) v''(n) v(n) - (v'(n))^2 > 0$. To ensure that these conditions hold for all values of σ, we will assume that $v''(n) v(n) - (v'(n))^2 > 0$. These restrictions imply that H(n) is an increasing function of n. Finally note that (C.1) simply states that $H(1) > G(1)$.

We first discuss existence and uniqueness for the two possible ranges of σ. Consider the case $\sigma > 1$. It follows that,

$\lim_{n\to 0} G(n) = \infty$, $G(1) = D\,\hat{s}^{1/\sigma}$, and $G'(n) < 0$,

and,

$\lim_{n\to 0} H(n) < \infty$, $H(1) > G(1)$, and $H'(n) > 0$.

It follows that there is a unique intersection. An example is shown in Figure A.1.

Consider next the case $0 < \sigma < 1$. In this case, we have,

$\lim_{n\to 0} G(n) = 0$, $G(1) = D\,\hat{s}^{1/\sigma}$, and $G'(n) > 0$,

and,

$\lim_{n\to 0} H(n) < 0$, $H(1) > G(1)$, and $H'(n) > 0$,

where the first inequality corresponds to (C.2). The “problem” here is that both H(n) and G(n) are upward sloping, and establishing uniqueness requires a separate argument. It is possible to show (details available from the authors) that if n* satisfies G(n*) = H(n*), then H'(n*) > G'(n*). Thus, the function H can intersect the function G only from below. This, of course, suffices for uniqueness. Possible H(n) and G(n) functions are displayed in Figure A.2.

Figure A.1

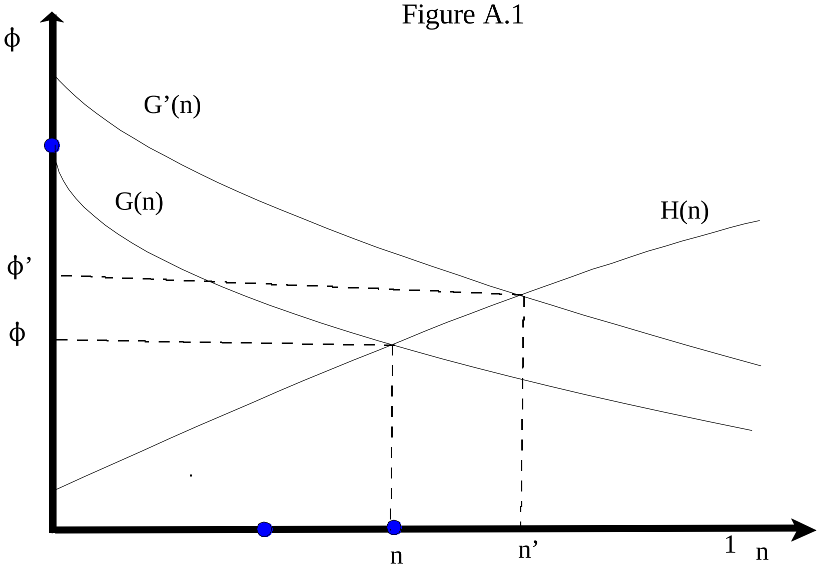
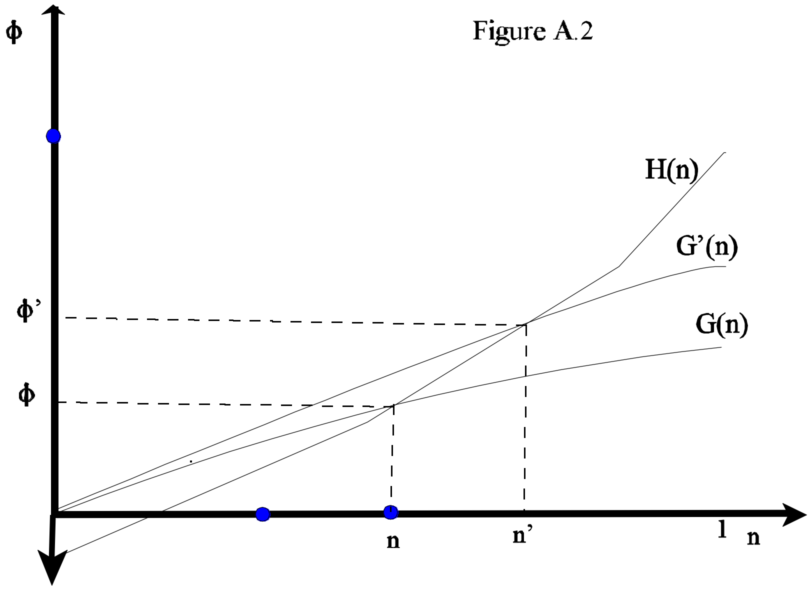


Figure A.2



In both Figures, we use G' to denote the function G corresponding to a higher value of $\hat{s}^{1/\sigma}$. Thus, it follows that increases in $\hat{s}^{1/\sigma}$ increase both the number of hours allocated to working (the utilization rate of human capital), n, and the fraction of income saved, ϕ. It is straightforward to calculate the growth rate of output. It is given by,

$$y_{t+1}/y_t \equiv \gamma_{t+1} = s_{t+1} A^* n^{1-\alpha} \phi = s_{t+1}\gamma.$$

Thus, the average growth rate, γ, is simply $A^* n^{1-\alpha} \phi$. It follows that growth rates are increasing in $\hat{s}$. Let $\hat{s}(\theta)$ be given by $\hat{s}(\theta) = \int_S (1+\epsilon)^{(1-\sigma)}\mu_\theta(d\epsilon)$. Since the function $(1+\epsilon)^{(1-\sigma)}$ is concave for $0<\sigma<1$ and convex for $\sigma>1$, it follows that, $0<\sigma<1$, $\hat{s}(\theta)$ is increasing, and if $\sigma>1$, $\hat{s}(\theta)$ is decreasing. This, in turn, implies that $[\phi,n,\gamma]$ are decreasing in θ whenever $0 < \sigma < 1$, and increasing otherwise.

From, $\gamma_{t+1}= s_{t+1}\gamma$, it follows that,

$$\sigma_\gamma = \gamma\sigma_s,$$

where σ_s is the standard deviation of the shock, s_t. Thus,

$$\sigma_\gamma/\sigma_s = \gamma,$$

and our claims follow from the properties of γ.

Now consider the case $\sigma = 1$. The first order conditions are satisfied with $\phi = \beta$, and n as the unique solution to $nv'(n) = (\alpha - 1)/(1-\beta)$. It is clear that, in this case, the key elements of the equilibrium are independent of θ. ■

Proof of Proposition 2 : Here we present the proof for the case $\sigma > 1$. The same type of arguments (with the signs suitably reversed) apply to the $0< \sigma < 1$ case. The relevant first order condition for the planner's problem is,

$$c_t^{-\sigma} = \beta\int c_{t+1}^{-\sigma}[As_{t+1} + 1-\delta]P_\theta(s_t,ds_{t+1}).$$

Under the guess $k_{t+1}= \phi(s_t,\theta)(s_tA+ 1- \delta)k_t$, this first order condition can be written as,

$$[1-\phi(s,\theta)]^{-\sigma} = \beta\int[1-\phi(z,\theta)]^{-\sigma} [\phi(s,\theta)]^{-\sigma}[Ax+1-\delta]^{1-\sigma} P_\theta(s,dx),$$

or,

$$\text{(A.2.1)} \qquad f(s,\theta) = 1 + (\beta\int[f(1-\rho+\rho s+\epsilon,\theta)(A(1-\rho+\rho s)+1-\delta+A\epsilon)^{1-\sigma/\sigma}]^\sigma \mu_\theta(d\epsilon))^{1/\sigma},$$

where $f(s,\theta) = [1-\phi(s,\theta)]^{-1}$.

Let the right hand side of (A.2.1) define the mapping T_θ. We want to show that T_θ is a contraction mapping. To this end, we will show that T_θ maps a set, $\mathscr{F}$, of continuous, bounded and convex functions into itself, and that T_θ satisfies Blackwell's conditions for a contraction.

Let $\mathscr{F} \equiv \{ f:S\rightarrow\Re$, such that $1 \leq f(s) \leq \bar{f}$, $f(s)$ decreasing, continuous and convex$\}$. Let $\bar{f}$ be given by,

$$\bar{f} = \sup_s 1/[1-(\beta\int(A(1-\rho+\rho s)+1-\delta+A\epsilon)^{1-\sigma}\mu_\theta(d\epsilon))^{1/\sigma}],$$

which is finite given our assumptions. It follows that if $f(s) \leq \bar{f}$, then $T_\theta f(s) \leq \bar{f}$. It follows that if $f(s)$ is decreasing and continuous so is $T_\theta f(s)$. We next show that T_θ maps convex functions into convex functions. Let $s^\lambda = \lambda s^1 + (1-\lambda)s^2$, for some $0<\lambda<1$. Let f be convex and decreasing. Define,

$$m(\epsilon) \equiv \lambda\, f(1-\rho+\rho s^1+\epsilon)(A(1-\rho+\rho s^1)+1-\delta+A\epsilon)^{1-\sigma/\sigma},$$

$$g(\epsilon) \equiv (1-\lambda)\, f(1-\rho+\rho s^2+\epsilon)(A(1-\rho+\rho s^2)+1-\delta+A\epsilon)^{1-\sigma/\sigma}.$$

For any ϵ, the function $f(1-\rho+\rho s+\epsilon)(A(1-\rho+\rho s)+1-\delta+A\epsilon)^{1-\sigma/\sigma}$ is a convex function of s. Thus,

$$[f(1-\rho+\rho s^\lambda+\epsilon)(A(1-\rho+\rho s^\lambda)+1-\delta+A\epsilon)^{1-\sigma/\sigma}]^\sigma \leq [m(\epsilon)+g(\epsilon)]^\sigma,$$

and,

$$\text{(A.2.2)}\ (\beta\int[f(1-\rho+\rho s^\lambda+\epsilon)(A(1-\rho+\rho s^\lambda)+1-\delta+A\epsilon)^{1-\sigma/\sigma}]^\sigma \mu_\theta(d\epsilon))^{1/\sigma} \leq (\beta\int[m(\epsilon)+g(\epsilon)]^\sigma\mu_\theta(d\epsilon))^{1/\sigma}.$$

From Minkowski's inequality (see, Rudin (1974), Theorem 3.5, p. 65), it follows that,

$$\text{(A.2.3)}\ (\beta\int[m(\epsilon)+g(\epsilon)]^\sigma\mu_\theta(d\epsilon))^{1/\sigma} \leq (\beta\int m(\epsilon)^\sigma\mu_\theta(d\epsilon))^{1/\sigma} + (\beta\int g(\epsilon)^\sigma\mu_\theta(d\epsilon))^{1/\sigma}.$$

Using (A.2.2) and (A.2.3) and adding one to both sides, we get,

$$1+(\beta\int[f(1-\rho+\rho s^\lambda+\epsilon)(A(1-\rho+\rho s^\lambda)+1-\delta+A\epsilon)^{1-\sigma/\sigma}]^\sigma \mu_\theta(d\epsilon))^{1/\sigma} \leq \lambda[1+(\beta\int[f(1-\rho+\rho s^1+\epsilon)(A(1-\rho+\rho s^1)+1-\delta+A\epsilon)^{1-\sigma/\sigma}]^\sigma \mu_\theta(d\epsilon))^{1/\sigma}] + (1-\lambda)[1+(\beta\int[f(1-\rho+\rho s^2+\epsilon)(A(1-\rho+\rho s^2)+1-\delta+A\epsilon)^{1-\sigma/\sigma}]^\sigma \mu_\theta(d\epsilon))^{1/\sigma}],$$

or,

(A.2.4) $T_\theta f(s^\lambda) \leq \lambda T_\theta f(s^1) + (1-\lambda)T_\theta f(s^2)$.

Thus, T_θ preserves convexity. To show that T_θ is a contraction mapping, it suffices to show that it satisfies the conditions of Blackwell's Theorem (see Lucas and Stokey (1989)). These conditions are

(A.i) (monotonicity) $f \geq g \rightarrow T_\theta f \geq T_\theta g$,

(A.ii) (discounting) There exists a υ satisfiying $0<\upsilon<1$, such that for all $a \in \Re_+$, $T_\theta(f+a) \leq T_\theta f + \upsilon a$.

It is immediate that (A.i) is satisfied. To prove (A.ii) use Minkowski's inequality to show that,

$$1+(\beta\int[f(1-\rho+\rho s+\epsilon)(A(1-\rho+\rho s)+1-\delta+A\epsilon)^{1-\sigma/\sigma} + a(A(1-\rho+\rho s)+1-\delta+A\epsilon)^{1-\sigma/\sigma}]^\sigma \mu_\theta(d\epsilon))^{1/\sigma}$$
$$\leq 1+(\beta\int[f(1-\rho+\rho s+\epsilon)(A(1-\rho+\rho s)+1-\delta+A\epsilon)^{1-\sigma/\sigma}]^\sigma \mu_\theta(d\epsilon))^{1/\sigma} +(\beta\int[a(A(1-\rho+\rho s^2)+1-\delta+A\epsilon)^{1-\sigma/\sigma}]^\sigma \mu_\theta(d\epsilon))^{1/\sigma},$$

or, equivalently,

$$T_\theta(f+a) \leq T_\theta f + a\,(\beta\int[(A(1-\rho+\rho s^2)+1-\delta+A\epsilon)^{1-\sigma/\sigma}]^\sigma \mu_\theta(d\epsilon))^{1/\sigma} \leq T_\theta f + a\kappa^{1/\sigma},$$

where κ is defined in (2.4) and it was assumed to be less than one.

It then follows that T_θ is a contraction mapping and it has a unique fixed point $f(s,\theta)$. Moreover, this fixed point is decreasing and convex in s.

We now show that if $\theta'>\theta$ then $f(s,\theta') > f(s,\theta)$. To see this note that,

$$T_{\theta'}f(s,\theta) \geq T_\theta f(s,\theta) = f(s,\theta),$$

and,

$$T^2_{\theta'}f(s,\theta) \geq T_{\theta'}T_\theta f(s,\theta) \geq T^2_\theta f(s,\theta) = f(s,\theta),$$

where the first inequality follows from (A.i) and the second because $T_\theta f(.,\theta)$ is a convex function. Similar arguments show that,

$$f(s,\theta') = \lim_{n\rightarrow\infty} T^n_{\theta'}f(.,\theta) \geq f(s,\theta).$$

It is easy to check that Minkowski's inequality cannot hold as an equality (see Rudin (1974)) and, hence, that the above inequality is strict.

Since $\phi(s,\theta) = 1 - f(s,\theta)^{-1}$, it follows that $\phi(s,\theta)$ inherits the properties of $f(s,\theta)$, namely it is decreasing and convex. ■

Proof of Proposition 4: Fix an arbitrary initial state, (h, k, s) and let (z*(h, k, s), n* (h, k, s)) denote the solution to (P.2) from this state. Now consider the same problem when the initial state is $(\lambda k,\lambda h,s)$. It follows immediately from the linear homogeneity of Γ that $(\lambda z^*(h, k, s), n^*(h, k, s))$ is feasible for the problem with initial state $(\lambda k,\lambda h,s)$. Contrary to the conclusion of the proposition, assume that $(\lambda z^*(h, k, s), n^*(h, k, s))$ is not optimal. Then, take some alternative plan, (z, n) that is feasible and gives higher utility--

(A.4.1) $U(z, n) > U(\lambda z^*(h, k, s), n^*(h, k, s))$.

Since (z, n) is feasible given initial state $(\lambda k, \lambda h, s)$, it follows from the the linear homogeneity of Γ that $(z/\lambda, n)$ is feasible when the initial state is $(\lambda k/\lambda, \lambda h/\lambda, s) = (h, k, s)$. Moreover, the utility of $(z/\lambda, n)$ is given by $U(z/\lambda, n) = U(z, n)/\lambda^{1-\sigma}$. Using this and (A.4.1) we have that

$$U(z/\lambda, n) = U(z, n)/\lambda^{1-\sigma} > U(\lambda z^*, n^*)/\lambda^{1-\sigma} = \lambda^{1-\sigma} U(z^*, n^*)/\lambda^{1-\sigma} = U(z^*, n^*).$$

That is, $(z/\lambda, n)$ is feasible when the initial state is (h, k, s) and it gives higher utility than (z^*, n^*), a contradiction.

That the value function is homogeneous of degree $1-\sigma$ in z (holding n fixed) follows immediately from the fact that the policy rules have the property that they do. ■

Appendix B: Derivations

a) *Approximations for the Second Moments of Growth Rates*

First, we study the effects of changing ρ upon the mean growth rate. From the proof of Proposition 2 it follows that $\phi(s;\theta,\rho)$ solves,

$$[1-\phi(s;\theta,\rho)]^{-\sigma} = [1+(\beta\int[1-\phi(1-\rho+\rho s+\epsilon;\theta,\rho)]^{-\sigma}[A(1-\rho+\rho s+\epsilon)+1-\delta]^{1-\sigma}\mu_\theta(d\epsilon))^{1/\sigma}]^{\sigma}.$$

Differentiating with respect to ρ, and evaluating at $s=1$ and if we get,

$$\phi_\rho(1;\theta,\rho)/(1-\phi(1;\theta,\rho)) = [1+(\beta\int[1-\phi(1+\epsilon;\theta,\rho)]^{-\sigma}[A(1+\epsilon)+1-\delta]^{1-\sigma}\mu_\theta(d\epsilon))^{-1/\sigma}]^{-1}$$
$$\{\int\phi_\rho(1+\epsilon;\theta,\rho)/(1-\phi(1+\epsilon;\theta,\rho))\psi(d\epsilon)\},$$

where $\psi(d\epsilon)$ is just $[1-\phi(1+\epsilon;\theta,\rho)]^{-\sigma}[A(1+\epsilon)+1-\delta]^{1-\sigma}\mu_\theta(d\epsilon)/\int[1-\phi(1+\epsilon;\theta,\rho)]^{-\sigma}[A(1+\epsilon)+1-\delta]^{1-\sigma}\mu_\theta(d\epsilon))$. It follows that if the fraction saved does respond very strongly to a small shock, $\phi_\rho(1+\epsilon;\theta,\rho)/(1-\phi(1+\epsilon;\theta,\rho)$ is approximately constant and, hence, $\phi_\rho(1;\theta,\rho)/(1-\phi(1;\theta,\rho)) \approx 0$.

To study the serial correlation properties of growth rates, we linearize the function that defines the growth rate. We can then approximate (2.6) around $s_t=s_{t+1}=1$, by,

$$\gamma(s_t,s_{t+1};\theta,\rho) \approx \omega^0(\theta,\rho) + \omega^1(\theta,\rho)(s_t-1) + \omega^2(\theta,\rho)(s_{t+1}-1),$$

where $\omega^0(\theta,\rho) = \omega^2(\theta,\rho) = \phi(1;\theta,\rho)[A + 1-\delta]$ is an approximation to the mean growth rate, $\bar{\gamma}(\theta,\rho)$, and $\omega^1(\theta,\rho) = \phi_s(1;\theta,\rho)(A + 1-\delta) - \phi(1;\theta,\rho)(1-\delta)$, where $\phi_s(1;\theta,\rho) = \partial\phi(s;\theta,\rho)/\partial s|_{s=1}$. It is more instructive to express these coefficients as,

$$\omega^0(\theta,\rho) = \omega^2(\theta,\rho) = \phi(1;\theta,\rho)[A + 1-\delta]= \bar{\gamma}(\theta,\rho), \qquad \omega^1(\theta,\rho) =\bar{\gamma}(\theta,\rho)(\varsigma - \upsilon),$$

where $\varsigma\equiv\phi_s(1;\theta,\rho)/\phi(s;\theta,\rho)|_{s=1}$ is the elasticity of the (broad) saving rate with respect to the shock, and υ is as before.

It is straightforward to compute the first order autocorrelation coefficient for the growth rate, ρ_γ . It follows that,

$$\rho_\gamma \approx \rho + \omega^1(\theta,\rho)\omega^2(\theta,\rho)/[(\omega^1(\theta,\rho))^2 + (\omega^2(\theta,\rho))^2 +\rho\omega^1(\theta,\rho)\omega^2(\theta,\rho)],$$
$$\rho_\gamma \approx \rho + (\varsigma - \upsilon)/[1+ (\varsigma - \upsilon)^2 +\rho(\varsigma - \upsilon)].$$

b) *Derivation of the First Order Conditions for the Model of Section 4*

The Euler equations for an interior solution are given by:

$$u_c(t) = E_t\{u_c(t+1)[1- \delta + F_k(t+1)]\}, \text{ and} \tag{B.1}$$
$$u_c(t) = E_t\{u_c(t+1)[1- \delta + n_{t+1} F_z(t+1)]\}, \tag{B.2}$$

where u_c is the partial derivative of u with respect to c and F_k and F_z are the partial derivatives of F

with respect to capital and effective labor.

For the Cobb-Douglas form, (B.1) and (B.2) can be combined to yield:

(B.3) $$E_t \{u_c(t+1)[\alpha F(t+1)/k_{t+1} - (1- \alpha) F(t+1)/h_{t+1}]\}= 0.$$

It follows that in any interior equilibrium, we must have that $h_t/k_t = (1-\alpha)/\alpha$ for all t. This is an important property of the specification of a Cobb-Douglas production function with equal depreciation rates: the human/physical capital ratio is independent of the level of employment and the productivity shock.

Given this, and setting $A^*= A(1-\alpha)^{1-\alpha}\alpha^{\alpha}$, it follows that

(B.4) $$c_t = k_t [s_t A^* n_t^{1-\alpha} ((1-n_t)/n_t) ((1-\alpha)/\alpha\psi)] \equiv k_t g_1(s_t,n_t).$$

Using this, we obtain,

(B.5) $$k_{t+1} = k_t \left[s_t A^* n_t^{1-\alpha} \left(1 - \frac{1-\alpha}{\psi} \frac{1-n_t}{n_t} \right) + 1 - \delta \right] \equiv k_t g_2(s_t,n_t) \quad .$$

Finally, after substitution, the relevant Euler equation becomes:

(B.6) $$[g_1(s_t,n_t) (1-n_t)^{\psi}]^{-\sigma} (1-n_t)^{\psi} = \beta \int_S [g_2(s_t,n_t)g_1(s_{t+1},n_{t+1})(1-n_{t+1})^{\psi}]^{-\sigma} \times(1-n_{t+1})^{\psi}[1-\delta+s_{t+1}A^*(n_{t+1})^{1-\alpha}]P(s_t,ds_{t+1})\}.$$

A solution to this equation is a function $n^*:S \rightarrow [0,1]$ with $n_t=n^*(s_t)$. Note that given n*, the optimal solution to the planner's problem is given by:

(B.7) $$\begin{aligned} n_t &= n^*(s_t) \\ k_{t+1} &= k_t\, g_2(s_t,n^*(s_t)) \\ h_{t+1} &= ((1-\alpha)/\alpha)\, k_t\, g_2(s_t,n^*(s_t)) \\ c_t &= k_t\, g_1(s_t,n^*(s_t)), \end{aligned}$$

which correspond to the equations calculated in section 4.

Appendix C: Basic Data from Simulations

Table C.1. The Effect of ζ on $E(\gamma)$, Selected Values of σ

Case	σ	ρ	ζ	σ_s	$E(\gamma)$	σ_γ	ρ_γ
1	1.0	0.9	0.095	0.126	2.48%	0.108	0.013
2	1.0	0.9	0.08	0.106	2.34%	0.091	0.012
3	1.0	0.9	0.06	0.080	2.38%	0.066	0.045
4	1.5	0.9	0.095	0.126	2.24%	0.077	-0.001
5	1.5	0.9	0.08	0.106	2.17%	0.064	-0.002
6	1.5	0.9	0.06	0.080	2.22%	0.047	0.029
7	1.5	0.8	0.14	0.135	2.80%	0.120	-0.096
8	1.5	0.8	0.12	0.115	2.64%	0.101	-0.056
9	1.5	0.8	0.095	0.091	2.30%	0.080	-0.082
10	1.5	0.8	0.08	0.077	2.26%	0.067	-0.079
11	1.5	0.8	0.06	0.058	2.09%	0.051	-0.071
12	2.0	0.9	0.095	0.126	2.27%	0.070	-0.004
13	2.0	0.9	0.08	0.106	2.19%	0.059	-0.005
14	2.0	0.9	0.06	0.080	2.22%	0.043	0.024
15	2.5	0.9	0.095	0.126	2.32%	0.067	-0.006
16	2.5	0.9	0.08	0.106	2.23%	0.056	-0.007
17	2.5	0.9	0.06	0.080	2.23%	0.041	0.022
SH mean	-	-	-		2.04%	0.062	0.123
SH Q1	-	-	-		0.91%	0.041	-0.039
SH Q3	-	-	-		3.25%	0.076	0.307

Table C.2. The Effect of ρ, $\sigma = 1.5$

Case	σ	ρ	ζ	$E(\gamma)$	σ_γ	ρ_γ	σ_s
1	1.5	0.7	0.08	2.28%	0.070	-0.131	0.065
2	1.5	0.8	0.08	2.26%	0.067	-0.079	0.077
3	1.5	0.9	0.08	2.17%	0.064	-0.002	0.106
4	1.5	0.915	0.08	1.94%	0.065	0.004	0.115
SH mean	-	-	-	2.04%	0.062	0.123	
SH Q1	-	-	-	0.91%	0.041	-0.039	
SH Q3	-	-	-	3.25%	0.076	0.307	

Table C.3. The Effect of γ_{SS} on the Effects of Uncertainty, $\sigma = 1.5$

Case	γ_{SS}	ρ	ζ	$E(\gamma)$	σ_γ	ρ_γ	σ_s
1	0.0%	0.9	0.095	0.26%	0.078	-0.023	0.126
2	0.0%	0.9	0.08	0.18%	0.0649	-0.023	0.106
3	0.0%	0.9	0.06	0.21%	0.048	0.008	0.080
4	2.0%	0.9	0.095	2.27%	0.077	-0.001	0.126
5	2.0%	0.9	0.08	2.19%	0.064	-0.002	0.106
6	2.0%	0.9	0.06	2.22%	0.047	0.029	0.080
7	4.0%	0.9	0.095	4.22%	0.077	0.024	0.126
8	4.0%	0.9	0.08	4.16%	0.064	0.021	0.106
9	4.0%	0.9	0.06	4.24%	0.047	0.051	0.080
SH mean	-	-	-	2.04%	0.062	0.123	
SH Q1	-	-	-	0.91%	0.041	-0.039	
SH Q3	-	-	-	3.25%	0.076	0.307	