CHAPTER 28

PARETO'S LAW AND THE GENERAL PROBLEM OF MATHEMATICALLY DESCRIBING THE FREQUENCY DISTRIBUTION OF INCOME

The problem of formulating a mathematical expression which shall describe the frequency distribution of income in all places and at all times, not only closely, but also elegantly, and if possible rationally as opposed to empirically, has had great attractions for the mathematical economist and statistician. The most famous of all attempts at the solution of this fascinating problem are those which have been made by Vilfredo Pareto. Professor Pareto has been intensely interested in this subject for many years and the discussion of it runs through nearly all of his published work. The almost inevitable result is that "Pareto's Law" appears in a number of slightly different forms and Professor Pareto's feelings concerning the "law" run all the way from treating it as inevitable and immutable to speaking of it as "merely empirical."

In its best known, most famous, and most dogmatic form, Pareto's Law runs about as follows:

1. In all countries and at all times the distribution of income is such that the upper (income-tax) ranges of the income frequency distribution curve may be described as follows: If the logarithms of income sizes be charted on a horizontal scale and the logarithms of the numbers of persons having an income of a particular size or over be charted on a vertical scale, then the resulting observational points will lie approximately along a straight line. In other words, if

x = income size and
y = number of persons having that income or larger

$$\text{then } \log y = \log b + m \log x$$
$$\text{or } y = bx^m.$$[1]

2. In all countries and at all recent times the *slope* of this straight line fitted to the cumulative distribution, that is, the constant m in the equation $y = bx^m$, will be approximately 1.5.[2]

3. The rigidity and universality of the two preceding conclusions strongly

[1] If the cumulative distribution (cumulating from the higher towards the lower incomes as Pareto does) on a double log scale could be exactly described by the equation $y = bx^m$, the non-cumulative distribution could be described by the equation $Y = -mbx^{m-1}$.

[2] Strictly, *minus* 1.5, though Pareto neglects the sign.

suggest that the shape of the income frequency distribution curve on a double log scale is, for all countries and at all times, inevitably the same not only in the upper (income-tax) range but throughout its entire length.

4. If then the nature of the whole income frequency distribution is unchanging and unchangeable there is, of course, no possibility of economic welfare being increased through any change in the proportion of the total income going to the relatively poor. Economic welfare can be increased only through increased production. In other words, Pareto's Law in this extreme form constitutes a modern substitute for the Wages Fund Doctrine.

This is the most dogmatic form in which the "law" appears. In his later work Professor Pareto drew further and further away from the confidence of his first position. He had early stated that the straight line did not seem adequate to describe distributions from all times and places and had proposed more complicated equations.[1] He has held more strongly to the significance of the similarity of slopes but he has wavered in his faith that the lower income portions of the curve (below the income-tax minimum) were necessarily similar for all countries and all times. He has given up the suggestion that existing distributions are inevitable though still speaking of the law as true within certain definite ranges. To translate from his *Manuel* (p. 391): "Some persons would deduce from it a general law as to the only way in which the inequality of incomes can be diminished. But such a conclusion far transcends anything that can be derived from the premises. Empirical laws, like those with which we are here concerned, have little or no value outside the limits for which they were found experimentally to be true." Indeed Professor Pareto has himself drawn attention to so many difficulties inherent in the crude dogmatic form of the law that this chapter must not be taken as primarily a criticism of his work but rather as a note on the general problem of mathematically describing the frequency distribution of incomes.

Almost as soon as he had formulated his law Professor Pareto recognized the impossibility of extrapolating the straight line formula into the lower income ranges (outside of the income-tax data which he had been using). The straight line formula involves the absurdity of an infinite number of individuals having approximately zero incomes. Professor Pareto felt that this zero mode with an infinite ordinate was absurd. He believed that the curve must have a definite mode at an income size well above zero [2] and with a finite number of income recipients in the modal group.

[1] The inadequacy of these more complicated equations is discussed later. See pp. 348, 363 and 364.

[2] This is, of course, not absolutely necessary. It depends upon our definitions of *income* and *income recipient*. If we include the negligible money receipts of young children living at home we might possibly have a mode close to zero. There are few children who do not really earn a few pennies each year. Compare Chart 31A page 416.

Having come to the conclusion that the income frequency distribution curve must inevitably have a definite mode well above zero income and tail off in both directions from that mode, Professor Pareto was led to think of the possibilities of the simplest of all frequency curves, the normal curve of error. However, after examination and consideration, he felt strongly that the normal curve of error could not possibly be used. He became convinced that the normal curve was not the law of the data for the good and sufficient reason that the part of the data curve given by income-tax returns is of a radically different shape from any part of a normal curve.[1]

Professor Pareto finds a further argument against using the normal curve in the irrationality of such a curve outside the range of the data. The mode of the complete frequency curve for income distribution is at least as low as the minimum taxable income. Income-tax data prove this. However, a normal curve is symmetrical. Hence, if a normal curve could describe the upper ranges of the income curve as given by income-tax data then in the lower ranges it would cut the y axis and pass into the second quadrant, in other words show a large number of negative incomes.

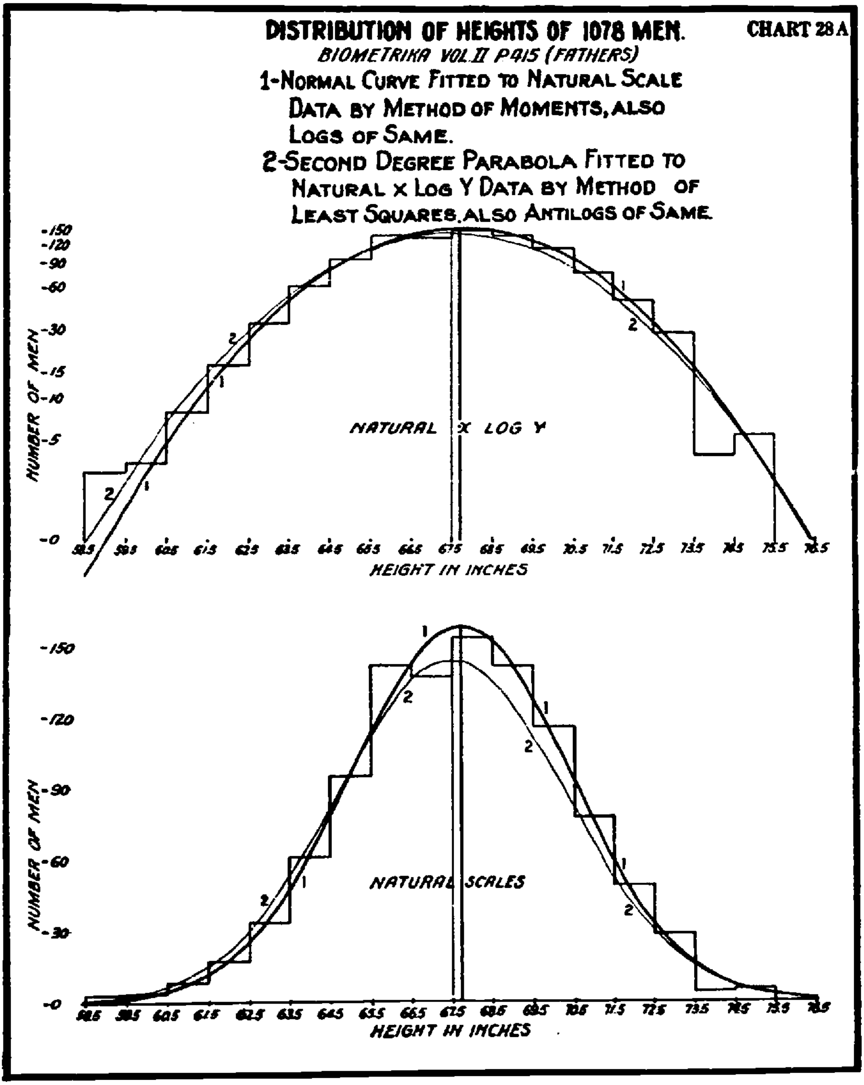
Now, aside from the fact that this whole argument is unnecessary if the data themselves cannot be described even approximately by a normal curve, Professor Pareto's discussion reveals a curious change in his middle term. If he had said that a symmetrical curve on a natural scale with a mode at least as low as the income-tax minimum would show *unbelievably large* negative incomes we could follow him but when he states that not only can there be no zero incomes but that there can be no incomes below "the minimum of existence" we realize that he has unconsciously changed the meaning of his middle term. Having examined a mass of income-tax data, all of which were concerned with *net money* income and from these data having formulated a law, he now apparently without realizing it, changes the meaning of the word income from *net money income* to *money value of commodities consumed*, and assumes that those who receive a *money* income less than a certain minimum must inevitably die of starvation.

[1] Though Pareto seems to have thoroughly understood this fact, his discussion is not altogether satisfactory. He states that the data for the higher incomes show a larger number of such incomes than the normal curve would indicate. This is hardly adequate. To have stated that the upper and lower ranges showed too many incomes *as compared with the middle range* would have been better. An easy way to realize clearly the impossibility of describing income-tax data by a normal curve is to plot a portion of the non-cumulative data on a *natural* x *log* y basis. When so charted the data present a concave shaped curve. However, if the data were describable by any part of a normal curve of error. they would show a convex appearance, or in the limiting case a straight line, as the equation of the normal curve of error $\left(y_x = y_0 e^{\frac{-x^2}{2\sigma^2}}\right)$ becomes, on a natural x log y scale, $\log_e y_x = \log_e y_0 - \frac{x^2}{2\sigma^2}$, or a second degree parabola whose axis is perpendicular to the x axis of coördinates.

The reader must note that the limiting straight line case mentioned above is on a *natural* x log y scale and not (as the Pareto straight line) on a *log* x log y scale. (Note concluded page 347.)

Children receive in general negligible *money* incomes. Many other persons in the community are in the same position. A business man may "lose money" in a given year, in other words he may have a negative money income. There seems no essential absurdity in assuming that a large number of persons receive money incomes much less than necessary to

(Note 1 page 346 concluded.)
Chart 28A showing curves fitted to observations on the heights of men illustrates the appearance of the normal curve on a natural scale and on a natural x log y scale. That chart also illustrates another fact of importance in this discussion, namely, that fitting to a different function of the variable gives a different fit.



support existence. When in 1915 Australia took a census of the incomes of all persons "possessed of property, or in receipt of income," over 14 per cent of the returns showed incomes "deficit and nil." [1]

Professor Pareto's realization of the impossibility of describing income distributions by means of normal curves led him to the curious conclusion that such distributions were somehow unique and could not be explained upon any "chance" hypothesis. "The shape of the curve which is furnished us by statistics, does not correspond at all to the curve of errors, *that is to say* [2] to the form which the curve would have if the acquisition and conservation of wealth depended only on chance." [3] Moreover, while Professor Pareto's further suggestion of possible heterogeneity in the data corresponds we believe to the facts, his reason for making such a suggestion, namely that the data cannot be adequately described by a normal curve, is irrelevant.[4] "Chance" data distributions are no longer thought of as necessarily in any way similar to the normal curve. Even error distributions commonly depart widely from the normal curve. The best known system of mathematical frequency curves, that of Karl Pearson, is intended to describe homogeneous material and is based upon a probability foundation, yet the normal curve is only one of the many and diverse forms yielded by his fundamental equation $\frac{d \log y}{dx} = \frac{x + a}{b_o + b_1 x + b_2 x^2}$.[5]

While Pareto's Law in its straight line form was at least an interesting suggestion, his efforts to amend the law have not been fruitful. His attempts to substitute $\log_e N = \log_e A - \alpha \log_e(x + a)$ or even $\log_e N = \log_e A - \alpha \log_e(x + a) - \beta x$ for the simpler $\log N = \log A - \alpha \log x$ have not materially advanced the subject.[6] The more complicated curves have the same fundamental drawbacks as the simpler one. Among other peculiarities they involve the same absurdity of an infinite number of persons in the modal interval and none below the mode. Along with the doubling of the number of constants, there comes of course the possibility of improving the fit within the range of the data. Such improvement is, however, purely artificial and empirical and without special significance, as can be easily appreciated by noticing the mathematical characteristics of the equation.

A number of other statisticians have at various times fitted different types of frequency curves to distributions of income, wages, rents, wealth,

[1] Compare Table 29A.

[2] My italics.

[3] *Manuel*. p. 385. See also *Cours*, pp. 416 and 417.

[4] Vid. *Cours*, pp. 416 and 417.

[5] Professor A. W. Flux in a review of Pareto's *Cours d'Economic Politique* (*Economic Journal*, March, 1897) drew attention to the inadequacy of Pareto's conception of what were and what were not "chance" data.

[6] Cf. *Cours*, vol. II. p. 305. note.

or allied data.[1] However, no one has advanced such claims for a "law" of *income*[2] distribution as were at one time made by Professor Pareto. When considering the possibility of helpfully describing the distribution of income by any simple mathematical expression, one inevitably begins by examining "Pareto's Law." It is so outstanding. Let us therefore examine Pareto's Law.

1. Do income distributions, when plotted on a double log scale, approximate straight lines closely enough to give such approximation much significance?

Before attempting to answer this question it is of course necessary to decide how we shall obtain the *straight line* with which comparisons are to be made.

Professor Pareto fitted straight lines directly by the method of least squares to the *cumulative* distribution plotted on a double log scale. The disadvantage of this procedure is that, though one may obtain the straight line which best fits the *cumulative* distribution, such a straight line may be anything but an admirable fit to the *non-cumulative* figures. For example, if a straight line be fitted by the method of least squares to Prussian returns for 1886 (as given by Professor Pareto) the total number of income recipients within the range of the data is, according to the fitted straight line, only 5,399,000 while the actual number of returns was 5,557,000, notwithstanding the fact that Prussia, 1886, is a sample which runs much more nearly straight than is usual. How bad the discrepancy may be where the data do not even approximate a straight line is seen in Professor Pareto's Oldenburg material. There the least-squares straight line fitted to the cumulative distribution on a double log scale gives 91,222 persons having incomes over 300 marks per annum while the data give only 54,309.

[1] Among others. Karl Pearson. F. Y. Edgeworth. Henry L. Moore. A. L. Bowley. Lucien March. J. C. Kapteyn. C. Bresciani. C. Gini. F. Savorgnan.

[2] Professor H. L. Moore. in his *Laws of Wages*, is concerned primarily with *wages* not *income*.

Professor J. C. Kapteyn has presented a pretty but somewhat hypothetical argument suggesting that the skewness in the income frequency curve should be such that plotting on a log x basis would eliminate it.

"In several cases we feel at once that the effect of the causes of deviation cannot be independent of the dimension of the quantities observed. In such cases we may conclude at once that the frequency curve will be a skew one. To take a single example:

"Suppose 1000 men to begin trading. each with the same capital: in order to see how their wealth will be distributed after the lapse of 10 years, consider first what will be their condition at some earlier epoch. say at the end of the fifth year.

"We may admit that a certain trader A will then only possess a capital of £100. while another may possess £100.000.

"Now if a certain cause of gain or loss comes to operate. what will happen?

"For instance: Let the price of an article in which both A and B have invested their capital. rise or fall. Then it will be evident that if the gain or loss of A be £10. that of B will not be £10. but £10.000: that is to say. the effect of this cause will not be independent of the capital, but proportional to it."

J. C. Kapteyn. *Skew Frequency Curves in Biology and Statistics*. p. 13.

The reason for this peculiarity of the fit to the cumulative distribution becomes clear when we remember that the least-squares straight line may easily deviate widely from the first datum point while a straight line giving the same number of income recipients as the data must necessarily pass *through* the first datum point.[1]

A straight line fitted in such a manner that the total number of persons and total amount of income correspond to the data for these items gives what seems a much more intelligible fit. Charts 28B to 28G show cumulative United States frequency distributions from the income-tax returns for the years 1914 to 1919 on a double log scale (Professor Pareto's suggestion). Two straight lines are fitted to each distribution—one a solid least-squares line fitted to the cumulative data points and the other a dotted line so fitted that the total number of persons and total amount of income correspond to the data figures. While the least-squares line may appear much the better fit to these cumulative data, a mere glance at Tables 28B to 28G will reveal the fact that such a line is, to say the least, a less interpretable fit to the non-cumulative distribution.[2] It is, of course, evident that neither line is in any year a sufficiently good fit to the actual non-cumulative distribution to have much significance. No mathematics is necessary to demonstrate this.[3]

[1] e. g. in the case of Prussia, 1886, the first datum point is x = "over 300M" and y = 54.309 persons.

[2] Professor Warren M. Persons discussed the fit of the *least-squares* straight line to Professor Pareto's Prussian data for 1892 and 1902 in the *Quarterly Journal of Economics*, May, 1909, and demonstrated the badness of fit of that line to those data.

[3] The income returned for the years 1914 and 1915 was estimated from the *number* of returns. *Income* is not given in the reports for those years.

In fitting straight lines to the data of Tables 28B to 28G the lowest income interval (in which married persons making a joint return are exempt) has always been omitted. To have included in our calculations these lowest intervals would have increased still further the badness of the fit in the other intervals.

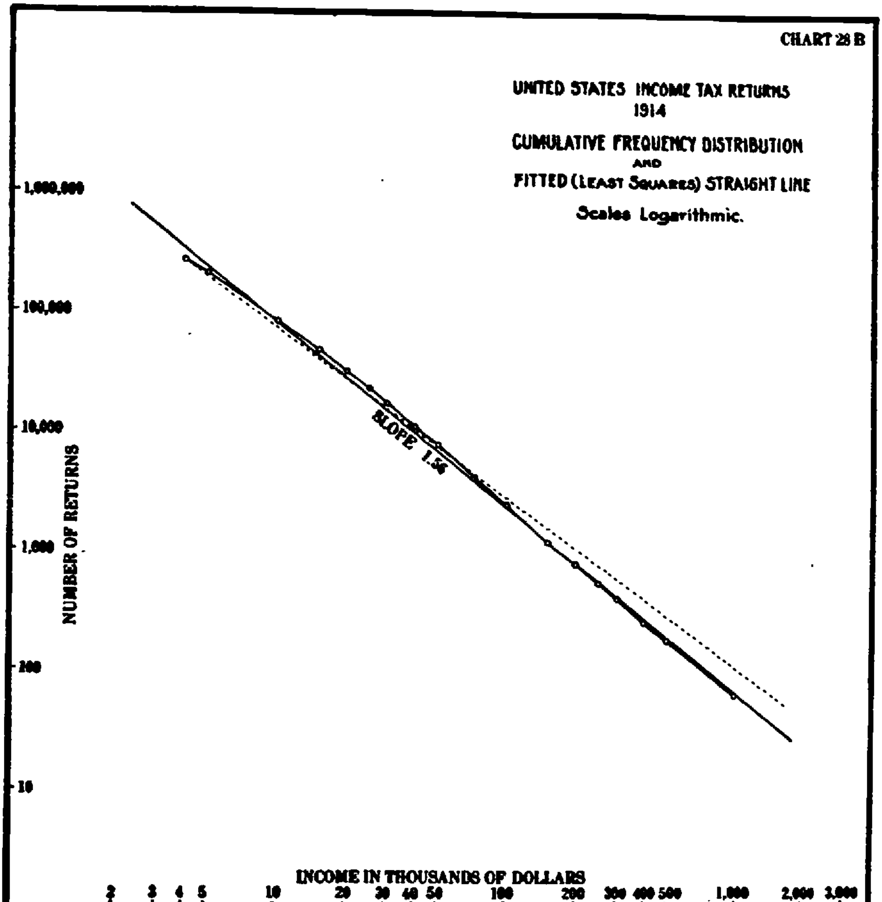


CHART 28 B
UNITED STATES INCOME TAX RETURNS
1914
CUMULATIVE FREQUENCY DISTRIBUTION
AND
FITTED (LEAST SQUARES) STRAIGHT LINE
Scales Logarithmic.
SLOPE 1.56
NUMBER OF RETURNS
1,000,000
100,000
10,000
1,000
100
10
INCOME IN THOUSANDS OF DOLLARS
2
3
4
5
10
20
30
40
50
100
200
300
400
500
1,000
2,000
3,000

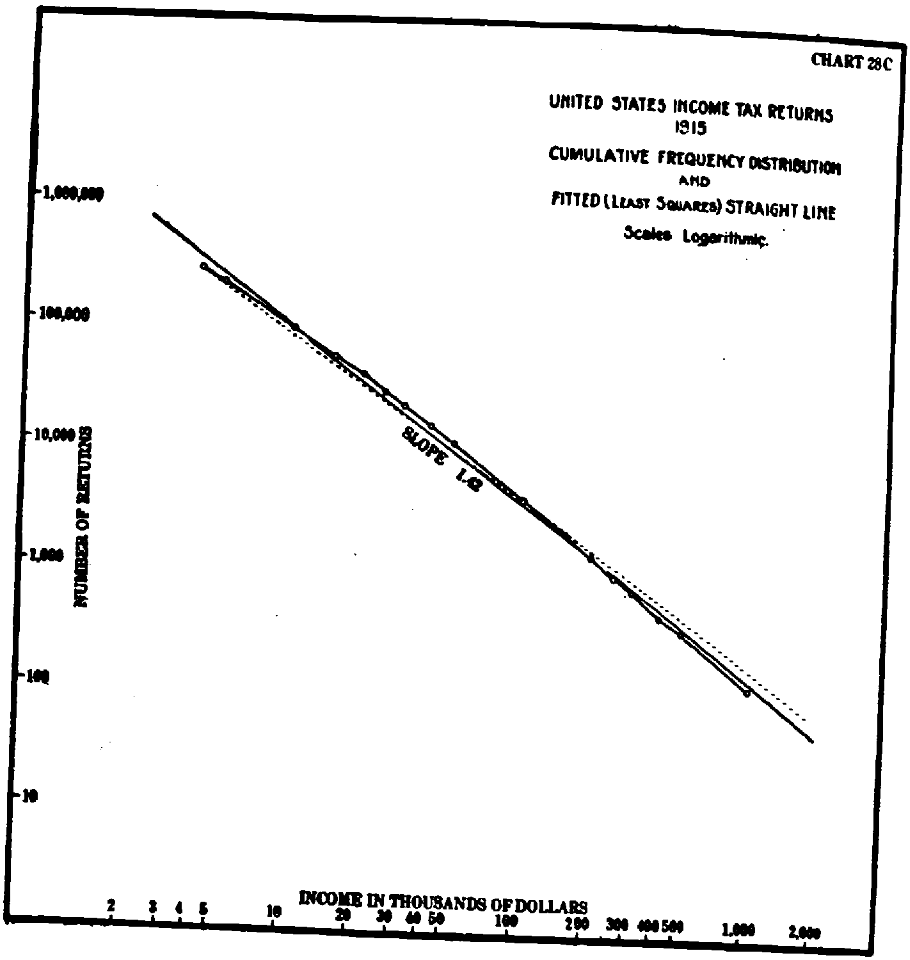


CHART 28C
UNITED STATES INCOME TAX RETURNS
1915
CUMULATIVE FREQUENCY DISTRIBUTION
AND
FITTED (LEAST SQUARES) STRAIGHT LINE
Scales Logarithmic.
SLOPE 1.42
NUMBER OF RETURNS
1,000,000
100,000
10,000
1,000
100
10
INCOME IN THOUSANDS OF DOLLARS
2
3
4
5
10
20
30
40
50
100
200
300
400
500
1,000
2,000

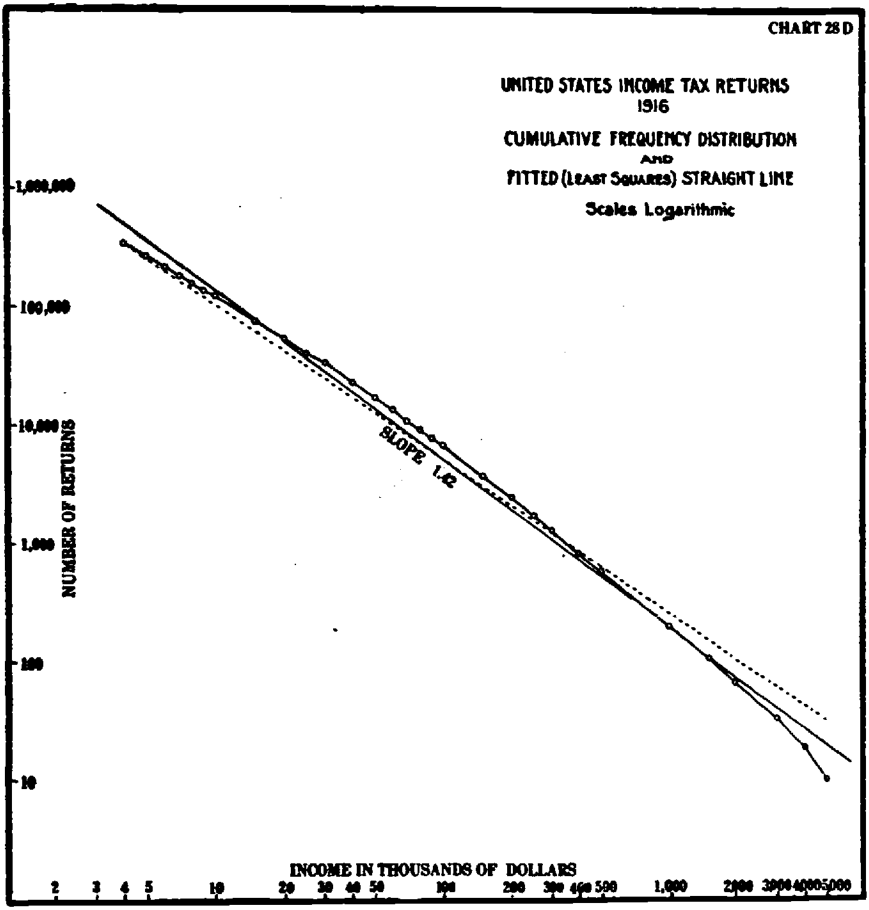


CHART 28 D
UNITED STATES INCOME TAX RETURNS
1916
CUMULATIVE FREQUENCY DISTRIBUTION
AND
FITTED (LEAST SQUARES) STRAIGHT LINE
Scales Logarithmic
NUMBER OF RETURNS
1,000,000
100,000
10,000
1,000
100
10
SLOPE 1.42
INCOME IN THOUSANDS OF DOLLARS
2
3
4
5
10
20
30
40
50
100
200
300
400
500
1,000
2000
3000
4000
5000

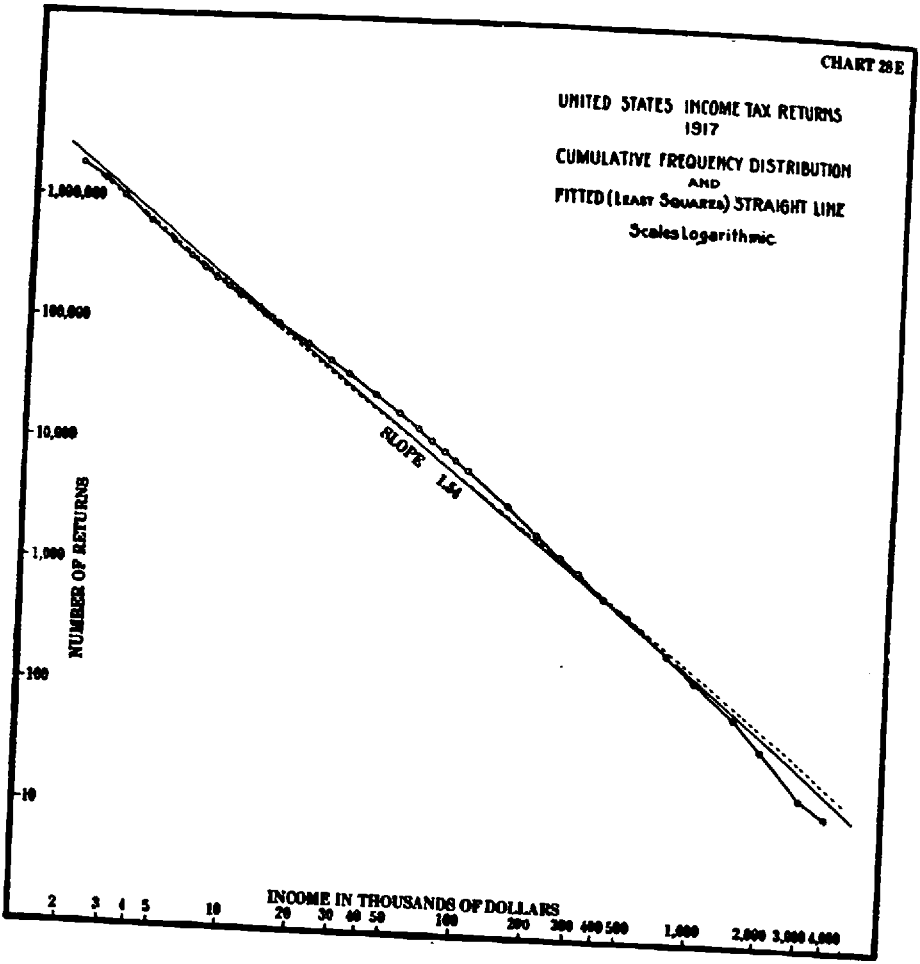


CHART 28 E
UNITED STATES INCOME TAX RETURNS
1917
CUMULATIVE FREQUENCY DISTRIBUTION
AND
FITTED (LEAST SQUARES) STRAIGHT LINE
Scales Logarithmic
SLOPE 1.54
NUMBER OF RETURNS
1,000,000
100,000
10,000
1,000
100
10
INCOME IN THOUSANDS OF DOLLARS
2
3
4
5
10
20
30
40
50
100
200
300
400
500
1,000
2,000
3,000
4,000

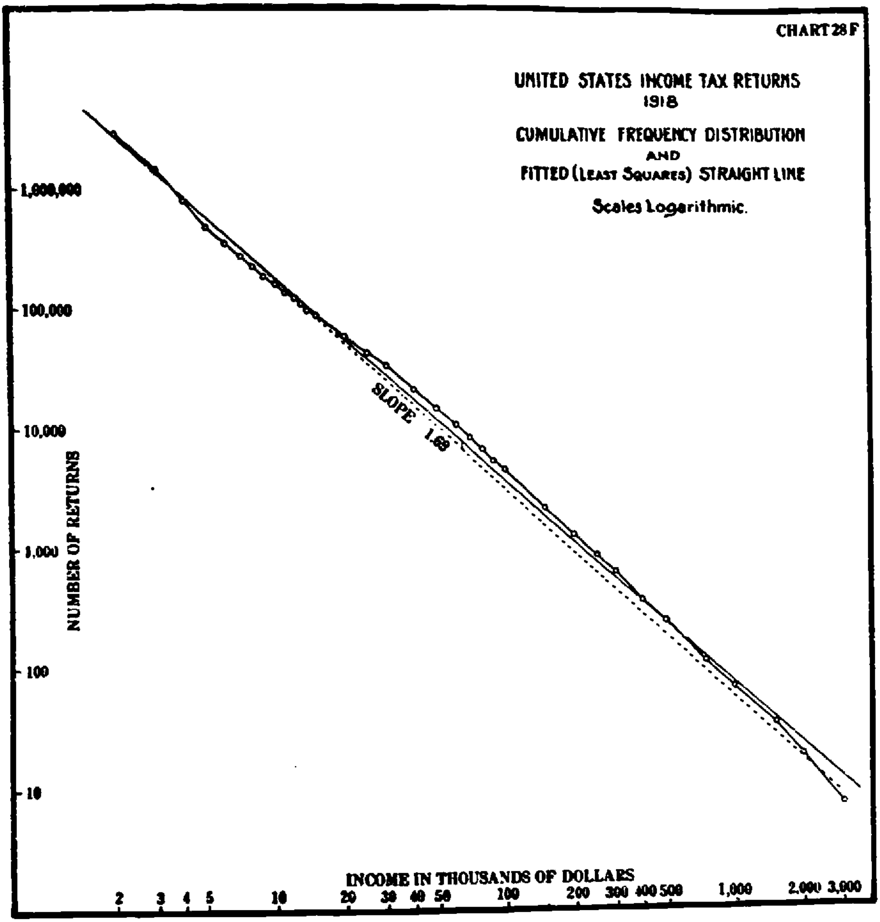


CHART 28 F
UNITED STATES INCOME TAX RETURNS
1918
CUMULATIVE FREQUENCY DISTRIBUTION
AND
FITTED (LEAST SQUARES) STRAIGHT LINE
Scales Logarithmic.
NUMBER OF RETURNS
1,000,000
100,000
10,000
1,000
100
10
SLOPE 1.69
INCOME IN THOUSANDS OF DOLLARS
2
3
4
5
10
20
30
40
50
100
200
300
400
500
1,000
2,000
3,000

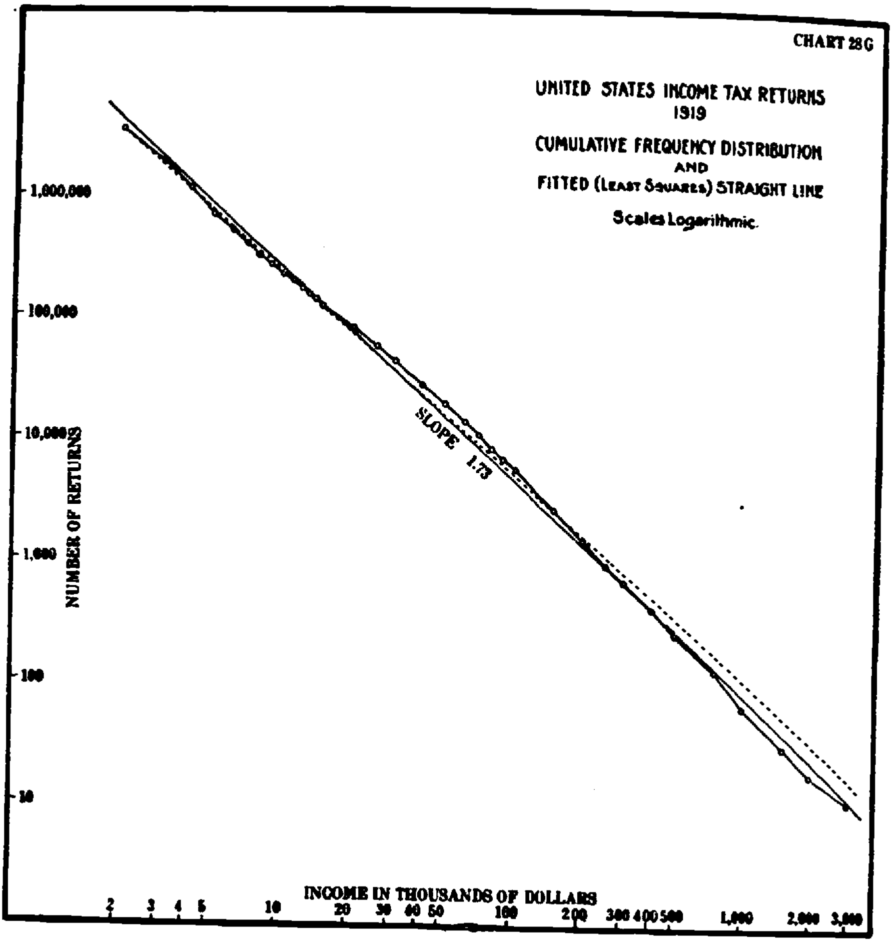


CHART 28 G
UNITED STATES INCOME TAX RETURNS
1919
CUMULATIVE FREQUENCY DISTRIBUTION
AND
FITTED (LEAST SQUARES) STRAIGHT LINE
Scales Logarithmic.
SLOPE 1.73
NUMBER OF RETURNS
1,000,000
100,000
10,000
1,000
100
10
INCOME IN THOUSANDS OF DOLLARS
2
3
4
5
10
20
30
40
50
100
200
300
400
500
1,000
2,000
3,000

TABLE 28B

UNITED STATES INCOME-TAX RETURNS, 1914

	A	B	C		
Income class	U. S. income-tax returns	Least-squares straight line	Straight line giving correct total returns and income	Per cent A is of B	Per cent A is of C
\$ 3,000-\$ 4,000	(82,754)				
4,000- 5,000	66,525	101,241	84,683	65.7	78.6
5,000- 10,000	127,448	160,545	115,347	79.4	110.5
10,000- 15,000	34,141	38,630	32,716	88.4	104.4
15,000- 20,000	15,790	15,853	14,102	99.6	112.0
20,000- 25,000	8,672	8,230	7,589	105.4	114.3
25,000- 30,000	5,483	4,879	4,631	112.4	118.4
30,000- 40,000	6,008	5,380	5,267	111.7	114.1
40,000- 50,000	3,185	2,793	2,835	114.0	112.3
50,000- 100,000	5,161	4,430	4,756	116.5	108.5
100,000- 150,000	1,189	1,065.5	1,241	111.6	95.8
150,000- 200,000	406	437.3	535	92.8	75.9
200,000- 250,000	233	227.1	288.1	102.6	80.9
250,000- 300,000	130	134.6	175.5	96.6	74.1
300,000- 400,000	147	148.46	199.9	99.0	73.5
400,000- 500,000	69	77.06	107.6	89.5	64.1
500,000-1,000,000	114	122.20	180.4	93.3	63.2
1,000,000 and over	60	62.78	107.5	95.6	55.8
Total (over \$4,000)	274,761	344,256.00	274,761.0		

TABLE 28C

UNITED STATES INCOME-TAX RETURNS, 1915

Income class	A U. S. income-tax returns	B Least-squares straight line	C Straight line giving correct total returns and income	Per cent A is of B	Per cent A is of C
$ 3,000–$ 4,000	(69,045)				
4,000– 5,000	58,949	92,064	68,540	64.0	86.0
5,000– 10,000	120,402	154,507	119,634	77.9	100.6
10,000– 15,000	34,102	40,358	33,013	84.5	103.3
15,000– 20,000	16,475	17,406	14,724	94.7	111.9
20,000– 25,000	9,707	9,372	8,124	103.6	119.5
25,000– 30,000	6,196	5,716	5,050	108.4	122.7
30,000– 40,000	7,005	6,508	5,875	107.6	119.2
40,000– 50,000	4,100	3,503	3,241	117.0	126.5
50,000– 100,000	6,847	5,880	5,653	116.4	121.1
100,000– 150,000	1,793	1,536	1,560	116.7	114.9
150,000– 200,000	724	662.5	695.4	109.3	104.1
200,000– 250,000	386	356.6	383.8	108.2	100.6
250,000– 300,000	216	217.5	238.6	99.3	90.5
300,000– 400,000	254	247.7	277.6	102.5	91.5
400,000– 500,000	122	133.3	153.2	91.5	79.6
500,000–1,000,000	209	223.8	267.1	93.4	78.2
1,000,000 and over	120	133.6	177.3	89.8	67.7
Total (over $4,000)	267,607	338,825.0	267,607.0		

TABLE 28D

UNITED STATES INCOME-TAX RETURNS, 1916

Income class	A U. S. income-tax returns	B Least-squares straight line	C Straight line giving correct total returns and income	Per cent A is of B	Per cent A is of C
$ 3,000-$ 4,000	(85,122)				
4,000- 5,000	72,027	139,096	86,588	51.8	83.2
5,000- 6,000	52,029	84,759	54,221	61.4	96.0
6,000- 7,000	36,470	56,533	36,899	64.5	98.8
7,000- 8,000	26,444	39,846	26,516	66.4	99.7
8,000- 9,000	19,959	29,292	19,801	68.1	100.8
9,000- 10,000	15,651	22,529	15,445	69.5	101.3
10,000- 15,000	45,309	60,668	42,879	74.7	105.7
15,000- 20,000	22,618	26,120	19,311	86.6	117.1
20,000- 25,000	12,953	14,044	10,726	92.2	120.8
25,000- 30,000	8,055	8,558	6,705	94.1	120.1
30,000- 40,000	10,068	9,731	7,854	103.5	128.2
40,000- 50,000	5,611	5,232	4,362	107.2	128.6
50,000- 60,000	3,621	3,189	2,730	113.5	132.6
60,000- 70,000	2,548	2,126	1,857	119.8	137.2
70,000- 80,000	1,787	1,499	1,334.8	119.2	133.9
80,000- 90,000	1,422	1,102	996.8	129.0	142.7
90,000- 100,000	1,074	847	777.5	126.8	138.1
100,000- 150,000	2,900	2,282.1	2,158.4	127.1	134.4
150,000- 200,000	1,284	982.6	972.1	130.7	132.1
200,000- 250,000	726	528.2	539.9	137.4	134.5
250,000- 300,000	427	321.9	337.6	132.6	126.5
300,000- 400,000	469	366.1	395.3	128.1	118.6
400,000- 500,000	245	196.8	219.6	124.5	111.6
500,000-1,000,000	376	329.6	387.4	114.1	97.1
1,000,000-1,500,000	97	85.83	108.7	113.0	89.2
1,500,000-2,000,000	42	36.96	48.88	113.6	85.9
2,000,000-3,000,000	34	31.98	44.19	106.3	76.9
3,000,000-4,000,000	14	13.77	19.91	101.7	70.3
4,000,000-5,000,000	9	7.40	11.05	121.6	81.4
5,000,000 and over	10	19.76	32.87	50.6	30.4
Total (over $4,000)	344,279	510,374.00	344,279.00		

TABLE 28E

UNITED STATES INCOME-TAX RETURNS, 1917

	A	B	C		
Income class	U. S. income-tax returns	Least-squares straight line	Straight line giving correct total returns and income	Per cent A is of B	Per cent A is of C
\$ 1,000–\$ 2,000	(1,640,758)				
2,000– 2,500	480,486	618,069	517,512	77.7	92.8
2,500– 3,000	358,221	367,835	284,620	97.4	125.9
3,000– 4,000	374,958	407,366	376,117	92.0	99.7
4,000– 5,000	185,805	212,569	184,854	87.4	100.5
5,000– 6,000	105,988	126,507	111,097	83.8	95.4
6,000– 7,000	64,010	82,746	73,355	77.4	87.3
7,000– 8,000	44,363	57,357	51,285	77.3	86.5
8,000– 9,000	31,769	41,556	37,362	76.4	85.0
9,000– 10,000	24,536	31,551	28,551	77.8	85.9
10,000– 11,000	19,221	24,097	21,900	79.8	87.8
11,000– 12,000	15,035	19,412	17,747	77.5	84.7
12,000– 13,000	12,328	15,707	14,440	78.5	85.4
13,000– 14,000	10,427	12,751	11,761	81.8	88.7
14,000– 15,000	8,789	10,709	9,909	82.1	88.7
15,000– 20,000	29,896	34,161	31,891	87.5	93.7
20,000– 25,000	16,806	17,825	16,876	94.3	99.6
25,000– 30,000	10,571	10,609	10,159	99.6	104.1
30,000– 40,000	12,733	11,749	11,385	108.4	111.8
40,000– 50,000	7,087	6,130	6,021	115.6	117.7
50,000– 60,000	4,541	3,649	3,622	124.4	125.4
60,000– 70,000	2,954	2,387	2,391	123.8	123.5
70,000– 80,000	2,222	1,653.5	1,672	134.4	132.9
80,000– 90,000	1,539	1,198.5	1,217.9	128.4	126.4
90,000– 100,000	1,183	910.0	930.8	130.0	127.1
100,000– 150,000	3,302	2,384.4	2,469.5	138.5	133.7
150,000– 200,000	1,302	985.2	1,039.6	132.2	125.2
200,000– 250,000	703	514.1	550.5	136.7	127.7
250,000– 300,000	342	305.9	330.8	111.8	103.4
300,000– 400,000	380	338.9	371.2	112.1	102.4
400,000– 500,000	179	176.8	196.3	101.2	91.2
500,000– 750,000	225	199.96	225.56	112.5	99.8
750,000–1,000,000	90	82.61	94.97	108.9	94.8
1,000,000–1,500,000	67	68.77	80.51	97.4	83.2
1,500,000–2,000,000	33	28.42	33.90	116.1	97.3
2,000,000–3,000,000	24	23.65	28.71	101.5	83.6
3,000,000–4,000,000	5	9.77	12.10	51.2	41.3
4,000,000–5,000,000	8	5.10	6.40	156.9	125.0
5,000,000 and over	4	12.42	16.25	32.2	24.6
Total (over \$2,000)	1,832,132	2,123,640.00	1,832,132.00		

TABLE 28F

UNITED STATES INCOME-TAX RETURNS, 1918

	A	B	C		
Income class	U. S. income-tax returns	Least-squares straight line	Straight line giving correct total returns and income	Per cent A is of B	Per cent A is of C
$ 1,000-$ 2,000	(1,516,938)				
2,000- 3,000	1,496,878	1,375,372	1,470,366	108.8	101.8
3,000- 4,000	610,095	537,892	566,044	113.4	107.8
4,000- 5,000	322,241	269,674	280,477	119.5	114.9
5,000- 6,000	126,554	155,513	160,366	81.4	78.9
6,000- 7,000	79,152	99,102	101,389	79.9	78.1
7,000- 8,000	51,381	67,184	68,258	76.5	75.3
8,000- 9,000	35,117	47,740	48,266	73.6	72.8
9,000- 10,000	27,152	35,628	35,795	76.2	75.9
10,000- 11,000	20,414	26,793	26,832	76.2	76.1
11,000- 12,000	16,371	21,283	21,231	76.9	77.1
12,000- 13,000	13,202	16,999	16,873	77.7	78.2
13,000- 14,000	10,882	13,638	13,515	79.8	80.5
14,000- 15,000	9,123	11,328	11,165	80.5	81.7
15,000- 20,000	30,227	35,214	34,486	85.8	87.7
20,000- 25,000	16,350	17,654	17,097	92.6	95.6
25,000- 30,000	10,206	10,181	9,762	100.2	104.5
30,000- 40,000	11,887	10,886	10,336	109.2	115.0
40,000- 50,000	6,449	5,458	5.121	118.2	125.9
50,000- 60,000	3,720	3,147	2,928	118.2	127.0
60,000- 70,000	2,441	2,006	1,852	121.7	131.8
70,000- 80,000	1,691	1,359.5	1,246	124.4	135.7
80,000- 90,000	1,210	966.2	881.4	125.2	137.3
90,000- 100,000	934	721.0	653.7	129.5	142.9
100,000- 150,000	2,358	1,822.3	1,636.3	129.4	144.1
150,000- 200,000	866	712.7	629.8	121.5	137.5
200,000- 250,000	401	357.3	312.1	112.2	128.5
250,000- 300,000	247	206.0	178.3	119.9	138.5
300,000- 400,000	260	220.3	188.7	118.0	137.8
400,000- 500,000	122	110.5	93.55	110.4	130.4
500,000- 750,000	132	119.28	99.70	110.7	132.4
750,000-1,000,000	46	46.66	38.36	98.6	119.9
1,000,000-1,500,000	33	36.88	29.88	89.5	110.4
1,500,000-2,000,000	16	14.42	11.50	111.0	139.1
2,000,000-3,000,000	11	11.40	8.96	96.5	122.8
3,000,000-4,000,000	4	4.46	3.44	89.7	116.3
4,000,000-5,000,000	2	2.24	1.71	89.3	117.0
5,000,000 and over	1	4.86	3.60	20.6	27.8
Total (over $2,000)	2,908,176	2,769,408.00	2,908,176.00		

TABLE 28G

UNITED STATES INCOME-TAX RETURNS, 1919

	A	B	C		
Income class	U. S. income-tax returns	Least-squares straight line	Straight line giving correct total returns and income	Per cent A is of B	Per cent A is of C
$ 1,000–$ 2,000	(1,924,872)				
2,000– 3,000	1,569,741	1,984,285	1,673,688	79.1	93.8
3,000– 4,000	742,334	764,739	660,950	97.1	112.3
4,000– 5,000	438,154	379,330	333,645	115.5	131.3
5,000– 6,000	167,005	216,921	193,470	77.0	86.3
6,000– 7,000	109.674	137,278	123,953	79.9	88.5
7,000– 8,000	73,719	92,511	84,273	79.7	87.5
8,000– 9,000	50,486	65,403	60,066	77.2	84.1
9,000– 10,000	37,967	48,583	44,980	78.1	84.4
10,000– 11,000	28,499	36,386	33,887	78.3	84.1
11,000– 12,000	22,841	28,796	27,027	79.3	84.5
12,000– 13,000	18,423	22,921	21,600	80.4	85.3
13,000– 14,000	15,248	18,329	17,395	83.2	87.7
14,000– 15,000	12,841	15,181	14,459	84.6	88.8
15,000– 20,000	42,028	46,868	45,162	89.7	93.1
20,000– 25,000	22,605	23,249	22,797	97.2	99.2
25,000– 30,000	13,769	13,294	13,228	103.6	104.1
30,000– 40,000	15,410	14,084	14,219	109.4	108.4
40,000– 50,000	8,298	6,986	7,178	118.8	115.6
50,000– 60,000	5,213	3,994	4,162	130.5	125.3
60,000– 70,000	3,196	2,528	2,665	126.4	119.9
70,000– 80,000	2,237	1,704	1,813	131.3	123.4
80,000– 90,000	1,561	1,205	1,292	129.5	120.8
90,000– 100,000	1,113	894	968.3	124.5	114.9
100,000– 150,000	2,983	2,240	2,461.5	133.2	121.2
150,000– 200,000	1,092	863.2	971.6	126.5	112.4
200,000– 250,000	522	428.1	490.4	121.9	106.4
250,000– 300,000	250	245.0	284.4	102.0	87.9
300,000– 400,000	285	259.2	306.0	110.0	93.1
400,000– 500,000	140	128.6	154.4	108.9	90.7
500,000– 750,000	129	137.32	168.2	93.9	76.7
750,000–1,000,000	60	52.89	66.4	113.4	90.4
1,000,000–1,500,000	34	41.25	52.95	82.4	64.2
1,500,000–2,000,000	13	15.89	20.90	81.8	62.2
2,000,000–3,000,000	7	12.40	16.68	56.5	42.0
3,000,000 and over	11	12.15	17.27	90.5	63.7
Total (over $2,000)	3,407,888	3,929,905.00	3,407,888.00		

Why do the least-squares straight lines appear graphically such good fits to the cumulative distributions (for at least the later years) when a merely arithmetic analysis shows even this fit to the cumulative data to be so illusory? *Because the percentage range in the number of persons is so extremely wide.* The deviations of the cumulative data on a double log scale from the least-squares straight line are minute *when compared with the percentage changes in the data from the smallest to the largest incomes.* But this is not helpful. The fact that there are 100,000 times as many persons having incomes over $2,000 per annum as there are persons having incomes over $5,000,000 per annum, does not make a theoretical reading for a particular income interval of twenty or thirty per cent over or under the data reading an unimportant deviation. Charting data on a double log scale may thus become a fertile source of error unless accompanied by careful interpretation.[1] This fact has long been recognized by engineers and others who have had much experience with similar problems in curve fitting.

Another matter of some importance must be noted here. The deviations of the data from the straight lines might be much less than they are and yet constitute extremely bad fits. *The data points (even on a non-cumulative basis) do not flutter erratically from side to side of the fitted lines; they run smoothly, passing through the fitted line at small angles in the way that one curve cuts another.* Now, in curve fitting, such a condition always strongly suggests that the particular mathematical curve used is not in any sense the "law" of the data.

2. Are the slopes of the straight lines fitted to income data from different times and places similar in any significant degree?

[1] The dangers of fitting curves with such a combination as a cumulative distribution and a double log scale, without further analysis, is well illustrated by the results Professor Pareto obtained for Oldenburg. To the Oldenburg data he fitted the rather complicated equation $\log N = \log A - \alpha \log (x + a) - \beta x$ and obtained the following results. (The value Pareto gives for β, namely .0000631, does not check with his calculated figures given below. $\beta =$ .0000274 is evidently what he intended.)

Income in marks (over)	N	Logarithms of N: Observed	Logarithms of N: Calculated	Δ
300	54,309	4.7349	4.7349	
600	24,043	4.3810	4.4368	—.0558
900	16,660	4.2217	4.2304	—.0086
1,500	9,631	3.9837	3.9409	+.0428
3,000	3,502	3.5443	3.5008	+.0435
6,000	994	2.9974	2.9997	—.0023
9,000	445	2.6484	2.6671	—.0187
15,300	140	2.1461	2.1838	—.0377
30,000	25	1.3979	1.3364	+.0615

(From *Cours d'Economie Politique*, vol. II, p. 307.)

The above table may give the reader a vague idea that the fit is rather good. However, from the above table the following table may be directly derived:

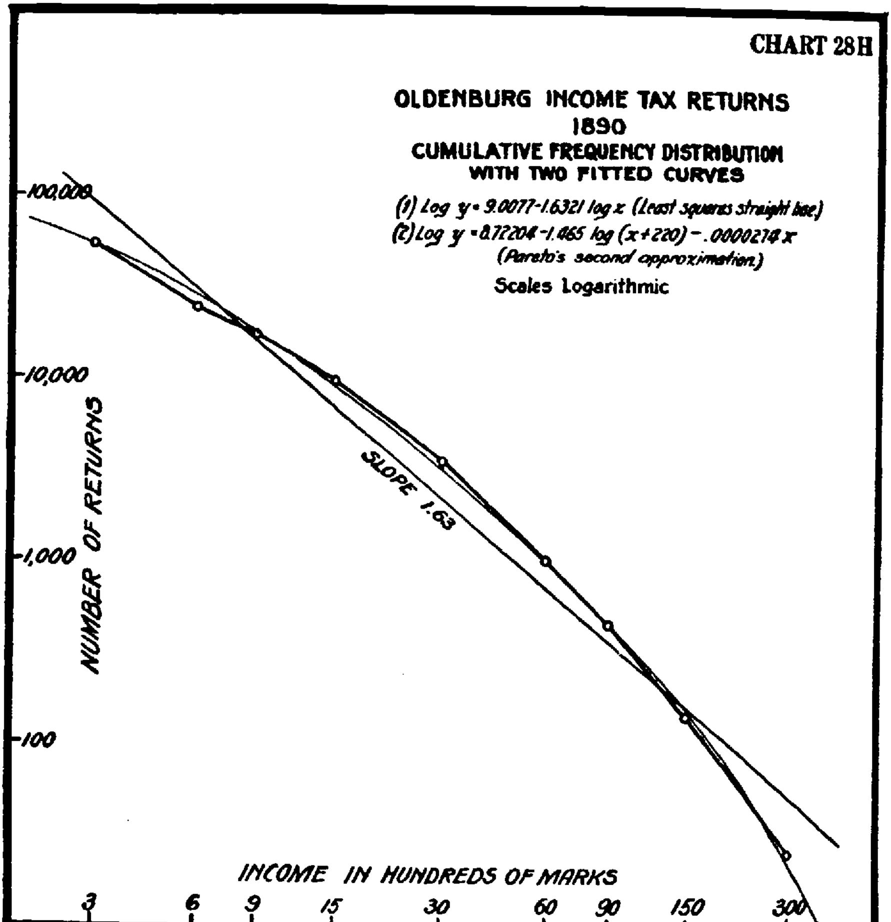
(Note concluded page 364.)

If income distributions charted on a double log scale not only cannot be approximately represented by straight lines, but also differ radically

(Note 1 page 363 concluded.)

Income in marks	Number of persons		Per cent actual are of computed
	Actual	Computed	
300– 600	30,266	26,969	112.2
600– 900	7,383	10,342	71.4
900– 1,500	7,029	8,270	85.0
1,500– 3,000	6,129	5,560	110.2
3,000– 6,000	2,508	2,169	115.6
6,000– 9,000	549	534	102.8
9,000–15,300	305	312	97.8
15,300–30,000	115	131	87.8
Over 30,000	25	22	113.6
Total	54,309	54,309	100.0

The fit no longer impresses one as quite so good. See Chart 28H below.



in shape, it is of course not of great importance whether the straight lines fitted to such data from different times and places have or have not approximately constant slopes. For example, a comparison of Chart 28C showing the cumulative distribution of United States income-tax returns for 1915 on a double log scale and Chart 28F showing similar data for 1918, makes it plain that, even were the slopes of the fitted straight lines for the two years identical, the data curves would still be so different as to make the similarity of slope of the fitted lines of almost no significance.[1]

In considering slopes, let us examine further both the data and the fitted lines for these two years 1915 and 1918. Tables 28I and 28J give some numerical illustrations of the differences between the distributions for the two years. Table 28I gives the number of returns in each income interval each year and the percentages that the 1918 figures are of the 1915 figures.

TABLE 28I

COMPARISON OF UNITED STATES INCOME-TAX RETURNS FOR 1915 AND 1918

Income class	Number of returns		Ratio of 1918 to 1915
	1915	1918	
$ 4,000 [a]–$ 5,000	58,949	322,241	5.4664
5,000– 10,000	120,402	319,356	2.6524
10,000– 15,000	34,102	69,992	2.0524
15,000– 20,000	16,475	30,227	1.8347
20,000– 25,000	9,707	16,350	1.6844
25,000– 30,000	6,196	10,206	1.6472
30,000– 40,000	7,005	11,887	1.6969
40,000– 50,000	4,100	6,449	1.5729
50,000– 100,000	6,847	9,996	1.4599
100,000– 150,000	1,793	2,358	1.3151
150,000– 200,000	724	866	1.1961
200,000– 250,000	386	401	1.0389
250,000– 300,000	216	247	1.1435
300,000– 400,000	254	260	1.0236
400,000– 500,000	122	122	1.0000
500,000-1,000,000	209	178	.8517
1,000,000 and over	120	67	.5583

[a] The $3,000-$4,000 class is not included, as in 1915 married persons in that class were exempted while in 1918 they were not.

The change as we pass from the $4,000–$5,000 interval, where the 1918 figures are nearly five-and-a-half times the 1915 figures, to the intervals above $500,000, where the 1918 figures are actually less than the 1915 figures, illustrates the great and fundamental difference between the slopes of the two distributions. However, such a comparison of unadjusted

[1] Compare also the deviations from the fitted lines as given in Tables 28C and 28F.

money intervals, while it throws into relief the differences in slope of the two distributions, is by no means as enlightening for purposes of exhibiting their other essential dissimilarities as a comparison of the two sets of data after they have been adjusted for changes in average (per capita) income and changes in population. Table 28J gives some comparisons between the data for the two years and between the fitted lines for the two years on such an adjusted basis. Two intervals, one in the relatively low income range and the other in the high income range, are used to illustrate the essentially different character of the distributions for the two years.

TABLE 28J

COMPARISONS OF UNITED STATES INCOME-TAX RETURNS FOR THE YEARS 1915 AND 1918 ADJUSTED FOR CHANGES IN AVERAGE (PER CAPITA) INCOME AND CHANGES IN POPULATION

ACTUAL INCOME-TAX DATA

Income intervals	Number of returns (1)	(2)	Fraction of population (3)	(4)	Ratio of Column (4) to Column (3)
	1915	1918	1915	1918	
Between 12 and 13 times average income	21.190	31.197	.00021099	.00029945	1.4193
Between 1.200 and 1.300 times average income	43.85	20.37	.0000004366	.0000001955	.4478
Over 12 times average income	248.600	271.452	.00247536	.00260561	1.0526
	Amount in dollars		Per cent of total income		
	1915	1918	1915	1918	
Over 12 times average income	$4.283.010.735	$5.312.832.516	11.9%	8.7%	.7311

LEAST-SQUARES STRAIGHT LINES

Income intervals	Number of returns (1)	(2)	Fraction of population (3)	(4)	Ratio of Column (4) to Column (3)
	1915	1918	1915	1918	
Between 12 and 13 times average income	32.886	41.730	.00032745	.00040056	1.2233
Between 1.200 and 1,300 times average income	47.63	17.10	.0000004743	.0000001641	.3460

STRAIGHT LINES FITTED TO GIVE THE SAME TOTAL NUMBER OF RETURNS AND THE SAME TOTAL INCOME AS THE INCOME-TAX DATA

Income intervals	Number of returns (1)	(2)	Fraction of population (3)	(4)	Ratio of Column (4) to Column (3)
	1915	1918	1915	1918	
Between 12 and 13 times average income	24.510	42.460	.00024405	.00040756	1.6700
Between 1.200 and 1.300 times average income	54.73	14.15	.0000005450	.0000001358	.2492

NOTES TO TABLE 28J

"Average Income" Intervals

	1915	1918
Average income	$ 358	$ 586
12 times average income	4.296	7.032
13 " " "	4.654	7.618
1.200 " " "	429.600	703.200
1.300 " " "	465.400	761.800

Equations of Fitted Straight Lines on a Cumulative Double Log Basis

	Least-squares lines	Lines giving correct total number of returns and total income
1914	$y = 11.153322 - 1.559256\,x$	$y = 10.557242 - 1.420936\,x$
1915	$y = 10.643299 - 1.419579\,x$	$y = 10.202382 - 1.325598\,x$
1916	$y = 10.839435 - 1.424638\,x$	$y = 10.212702 - 1.298088\,x$
1917	$y = 11.410606 - 1.539996\,x$	$y = 11.170980 - 1.486817\,x$
1918	$y = 12.033697 - 1.693823\,x$	$y = 12.202452 - 1.738497\,x$
1919	$y = 12.320963 - 1.734802\,x$	$y = 12.036155 - 1.667258\,x$

Table 28J needs little discussion. In the section treating actual income-tax data we notice that while the adjusted number of returns in the lower income interval [1] *increased* 41.93 per cent from 1915 to 1918, the adjusted number of returns in the upper income interval [2] *decreased* 55.22 per cent. Moreover, while the adjusted total *number* of returns above the "12-times-average-income" point increased 5.26 per cent, the adjusted *amount of income* reported in these returns decreased 26.89 per cent.

Such figures suggest a rather radical change in the distribution of income during this short three-year period. Similar conclusions may be drawn from the figures for the two pairs of fitted lines, though we must of course remember that these lines describe only very inadequately the actual data. The lines so fitted as to give each year the same total number of returns and total amount of income as the data for that year yield sensational results. While the adjusted number of returns in the lower income-interval *increased* 67 per cent, the adjusted number of returns in the upper income-interval *decreased* 75.08 per cent.

Finally, it has been suggested that changes in the characteristics of the tax-income-distribution in the United States from 1915 to 1918 may be accounted for as the results of the increase in the surtax rates with 1917. We do not believe any large part of these changes can be so accounted for. Notwithstanding the fact that the country entered the European war during the interval, the difference between the 1915 distribution and the 1918 distribution in the United States, extreme as it is, cannot be said to be unreasonably or unbelievably great. Even the changes in the slope of the least-squares line are not phenomenal. Pareto's Prussian figures contain fluctuations in slope from —1.60 to —1.89 while the slope of the least-squares straight line fitted to his Basle data is only —1.25. The

[1] Between 12 and 13 times the average income (per capita) each year.

[2] Between 1.200 and 1,300 times the average income (per capita) each year.

slopes of the least-squares straight lines fitted to the American data are —1.42 for 1915 and —1.69 for 1918.

> 3. If the upper income ranges (or "tails") of income distributions were, when charted on a double log scale, closely similar in shape, would that fact justify the assumption that the lower income ranges were likewise closely similar?

Before attempting to answer the above question, let us summarize the case we have just made against believing the "tails" significantly similar. We can then discuss how much importance such similarity would have did it exist.

We have found upon examination that the approximation to straight lines of the tails of income distributions plotted on double log scales is specious; that the slopes of the fitted straight lines differ sufficiently to produce extreme variations in the relative number of income recipients in the upper as compared with the lower income ranges of the tails; that the upper and lower income ranges of the actual data for different times or places tell a similar story of extreme variation; and that the irregularities in shape of the tails of the actual data, entirely aside from any question of approximating or not approximating straight lines of constant slope, vary greatly from year to year and from country to country, ranging all the way from the irregularities of such distributions as the Oldenburg data, through the American data for 1914, 1915 and 1916 to such an entirely different set of irregularities as those seen in the American data for 1918[1].

At this stage of the discussion the reader may ask whether a general appearance of approximating straight lines on a double log scale, poor as the actual fit may be found to be under analysis, has not some meaning, some significance. The answer to this question must be that, if we were not dealing with a frequency distribution but with a correlation table showing a relationship between *two variables*, an approximation of the regression lines to linearity when charted on a double log scale might easily be the clue to a first approximation to a rational law; but that, on the other hand, approximate linearity in the *tail of a frequency distribution* charted on a double log scale signifies relatively little because it is such a common characteristic of frequency distributions of many and varied types.

The straight line on a double log scale or, in other words, the equation $y = bx^m$, when used to express a relationship between two variables, is, to quote a well-known text on engineering mathematics, "one of the most useful classes of curves in engineering." [2] In deciding what type of equation to use in fitting curves by the method of least squares to data con-

[1] Compare Charts 28H, 28B, 28C, 28D and 28F.
[2] P. Steinmetz, *Engineering Mathematics*, p. 216.

cerning two variables the texts usually mention $y = bx^m$ as "a quite common case." [1] A recent author writes, "simple curves which approximate a large number of empirical data are the parabolic and hyperbolic curves. The equation of such a curve is $y = ax^b$ [$y = bx^m$], parabolic for b positive and hyperbolic for b negative." [2] A widely used text on elementary mathematics speaks of the equation $y = bx^m$ as one of "the three fundamental functions" in practical mathematics.[3] The market for "logarithmic paper" shows what a large number of two-variable relationships may be approximated by this equation. Moreover this equation is often a close first approximation to a rational law. Witness "Boyle's Law." Indeed, sufficient use has not been made of this curve in economic discussions of two-variable problems.

The primary reason why approximation to linearity on a double log scale has no such significance in the case of the *tail of a frequency distribution* as it often has in the case of a two-variable problem is because of the very fact that we are considering the *tail* of the distribution, in other words, a mere fraction of the data. While frequency distributions which can be described throughout their length by a curve of the type $y = bx^m$ are extremely rare, a large percentage of all frequency distributions have *tails* approximating straight lines on a double log scale.[4] It is astonishing how many homogeneous frequency distributions of all kinds may be described with a fair degree of adequacy by means of hyperbolas [5] fitted to the data on a double log scale. Along with this characteristic goes, of course, the possibility of fitting to the tails of such distributions straight lines approximately parallel to the asymptotes of the fitted hyperbola. However we have by no means adequately described an hyperbola when we have stated the fact that one of its asymptotes is (of course) a straight line and that its slope is such and such. Had we even similar information concerning the other asymptote also, we should know little about the hyperbola or the frequency distribution which it would describe on a double log scale. The hyperbola might coincide with its asymptotes and hence have an *angle* at the mode or it might have a very much rounded "top." Such a variation in the shape of the top of the hyperbola [6] would generally correspond to a very great variation in the scatter or "inequality" of the distribution as well as many other characteristics.

[1] D. P. Bartlett, *Method of Least Squares*, p. 33.

[2] J. Lipka, *Graphical and Mechanical Computation*, p. 128.

[3] C. S. Slichter, *Elementary Mathematical Analysis*, preface.

[4] A very large percentage of the remainder have tails approximating straight lines on a natural x log y basis.

[5] N. B. Not a *straight line on the double log scale*, which is a so-called hyperbola on the natural scale, but a true conic section *hyperbola on the double log scale*.

Charts 28K and 28L (Earnings per Hour of 318,946 Male Employees in 1919) illustrate how excellent a fit may often be obtained by means of an hyperbola even though fitted only by selected points. A comparison of the least-squares parabola and the selected-points hyperbola on Chart 28K illustrates also the straight-tail effect.

[6] Compare Karl Pearson's concept of "kurtosis."

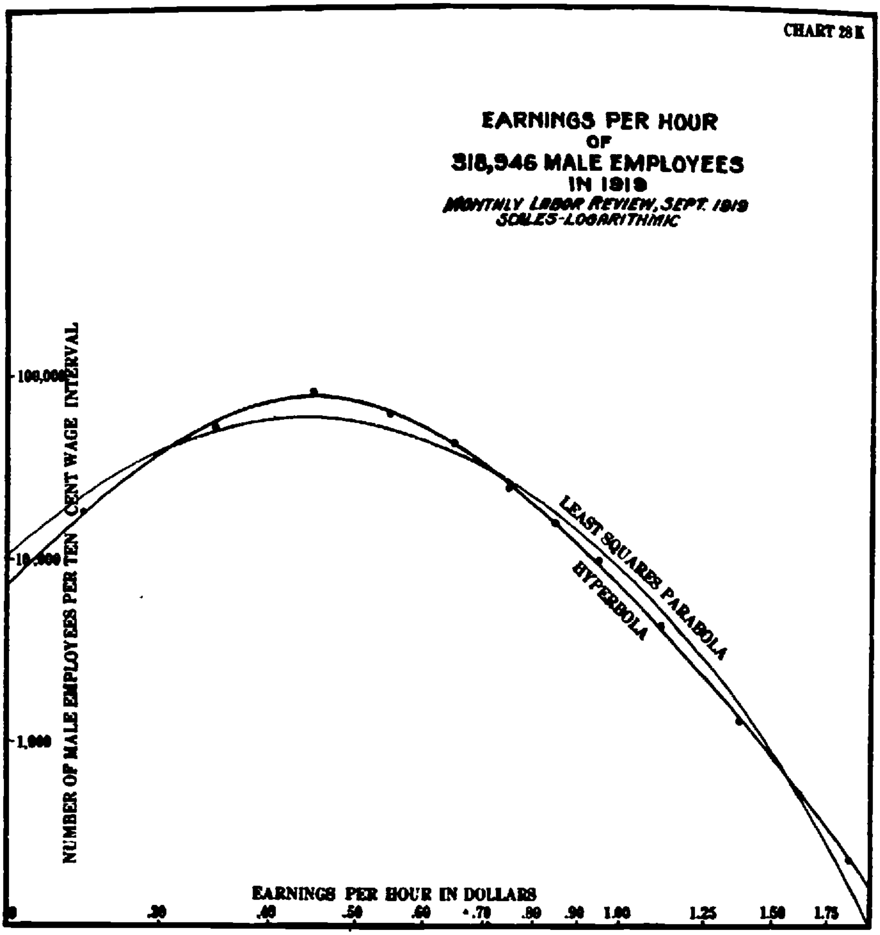
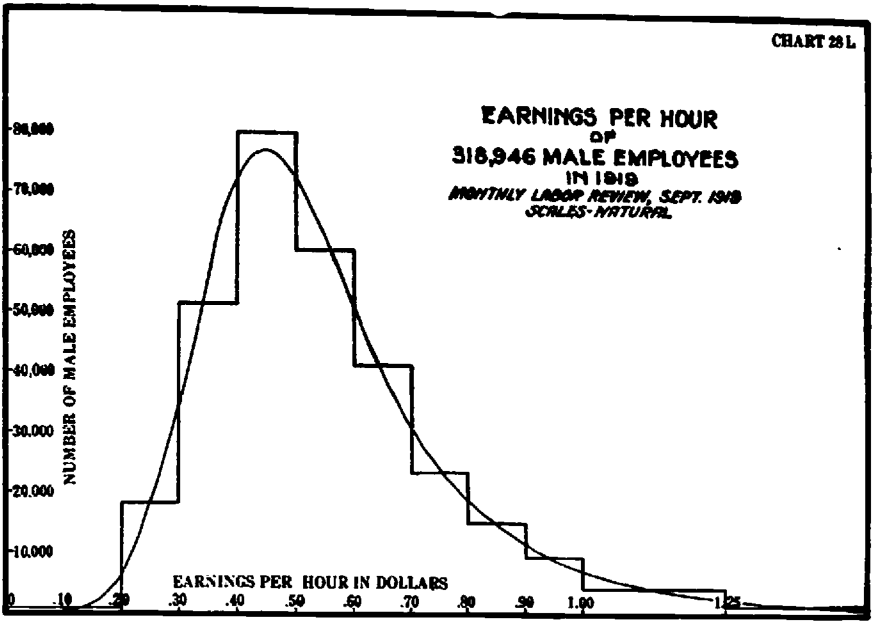
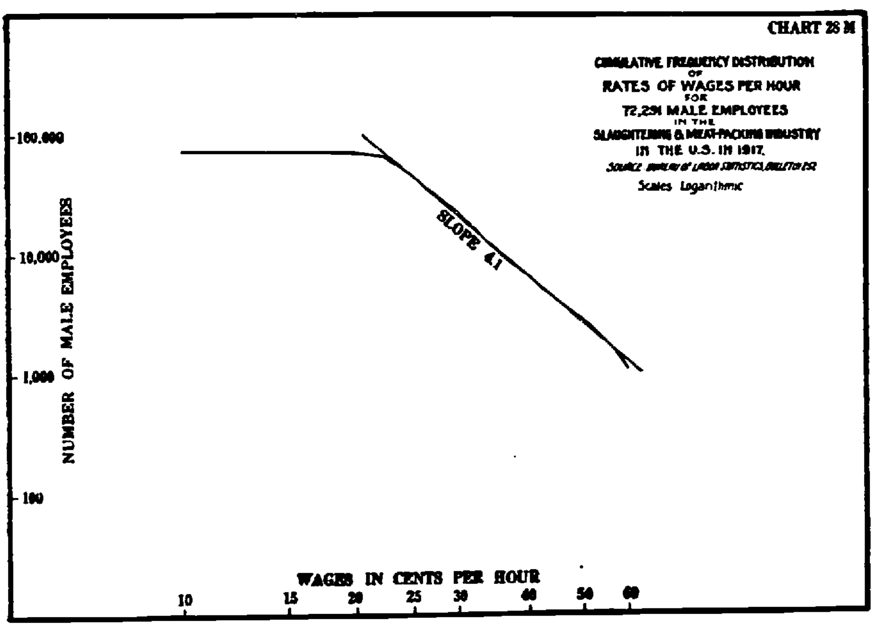
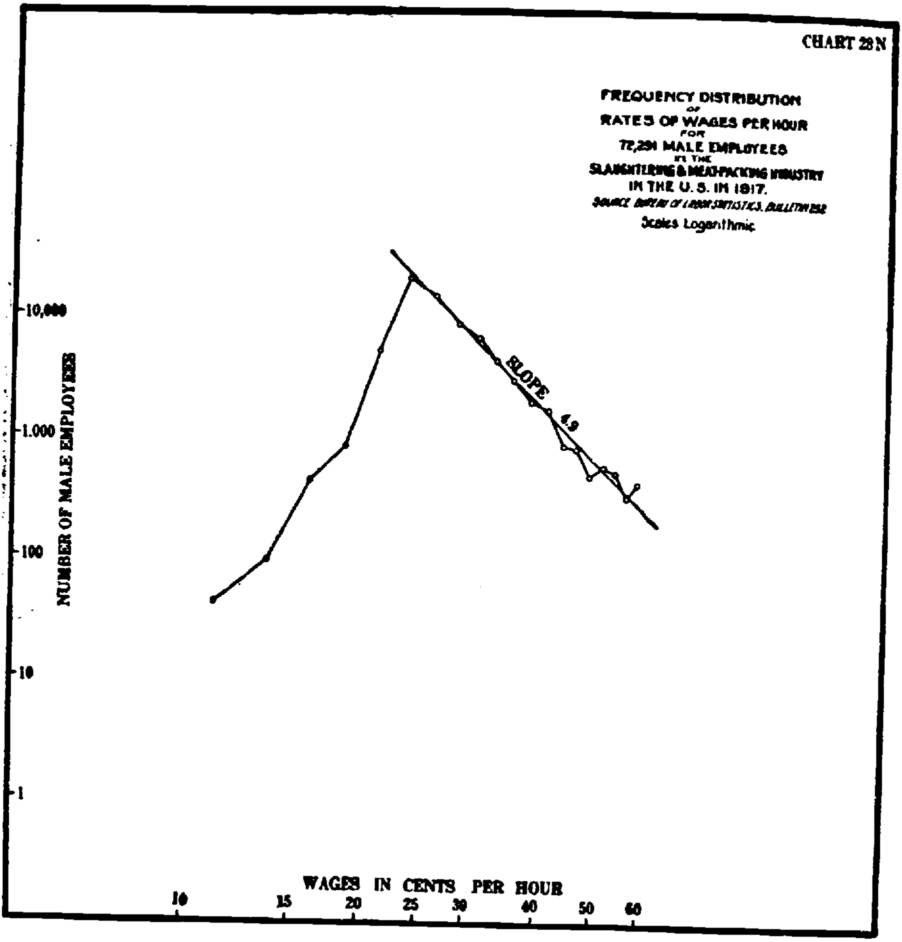


CHART 28 K
EARNINGS PER HOUR
OF
318,946 MALE EMPLOYEES
IN 1919
MONTHLY LABOR REVIEW, SEPT. 1919
SCALES-LOGARITHMIC
NUMBER OF MALE EMPLOYEES PER TEN CENT WAGE INTERVAL
100,000
10,000
1,000
LEAST SQUARES PARABOLA
HYPERBOLA
EARNINGS PER HOUR IN DOLLARS
.30
.40
.50
.60
.70
.80
.90
1.00
1.25
1.50
1.75



Rough similarity in the *tails* of two distributions on a double log scale by no means proves even rough similarity in the remainder of the distributions. Charts 28M, 28N, 28O and 28P illustrate both cumulatively





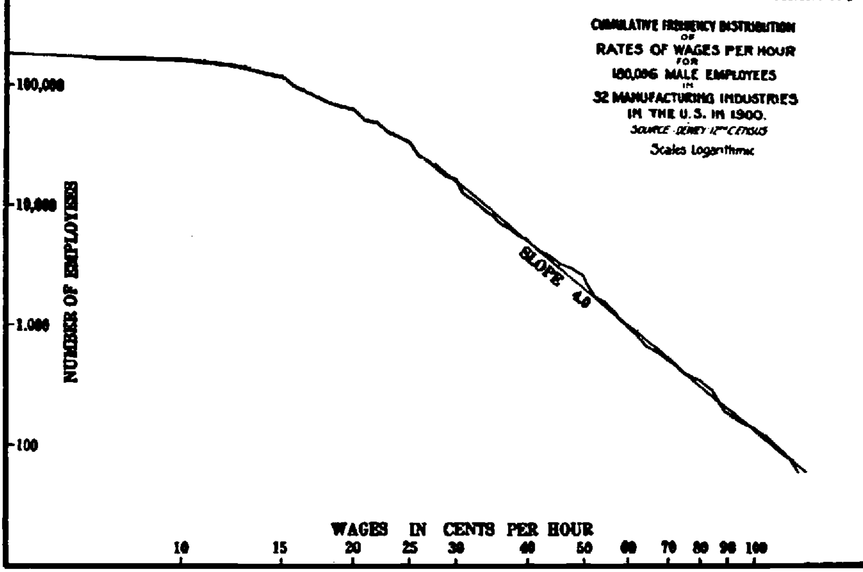
and non-cumulatively on a double log scale two wages distributions whose extreme tails appear roughly to approximate straight lines of about equal slope.[1] Charts 28M and 28N are from data concerning wages per hour of 72,291 male employees in the slaughtering and meat-packing industry in 1917;[2] Charts 28O and 28P are from data concerning wages per hour of 180,096 male employees in 32 manufacturing industries in the United States in 1900.[3] A mere glance at the two non-cumulative distributions will bring home the fact that while they show considerable similarity in the upper income range tails, they are quite dissimilar in the remainder

[1] The illustration shows only "rough similarity" in the extreme tails. However, there seems no good reason for believing that even great similarity in the tails proves similarity in the rest of the distribution. It certainly cannot do so in the case of essentially heterogeneous distributions, such as income distributions.

[2] Bureau of Labor Statistics; *Bulletin* No. 252.

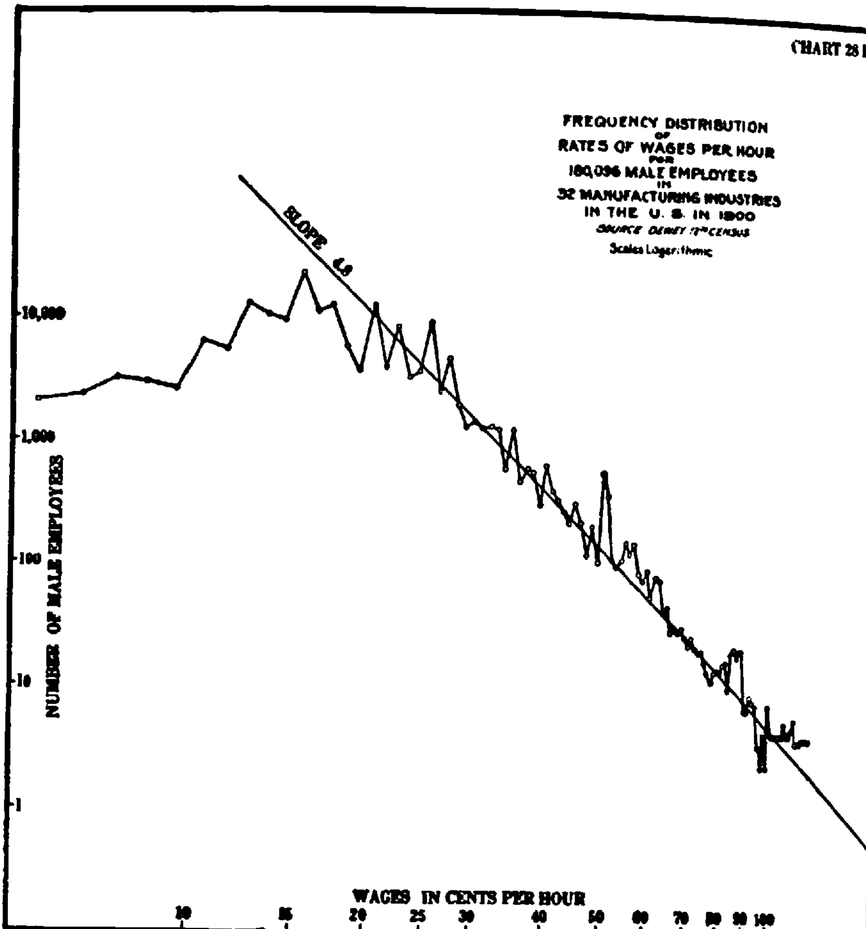
[3] Twelfth Census of the United States (1900), *Special Report on Employees and Wages*, Davis R. Dewey.

CHART 280



of the curves. Moreover, in spite of this similarity of tails, the slaughtering and meat-packing distribution has a coefficient of variation of 30.5 while the manufacturing distribution has a coefficient of 47.7. In other words, the relative scatter or "inequality of distribution" is more than one-and-a-half times as great in the manufacturing data as it is in the slaughtering and meat-packing data. Furthermore, no discussion and explanation of greater essential heterogeneity in the one distribution than in the other will offset the fact that the tails are similar but the distributions are different. There seems indeed to be almost no correlation between the slope of the upper-range tail and the degree of scatter in wages distributions. Some distributions showing extremely great scatter have very steep tails, some have not.[1] The frequency curve for the distribution of income in Australia in 1915 is radically different from either the curve for the United States in 1910 constructed by Mr. W. I. King or the curve for the United States in 1918 constructed by the National Bureau of Economic Research.

[1] The tails of wage distributions have in general much greater slopes than those of the upper (i. e., income-tax) range of income distributions. This is an outstanding difference between the two distributions. Pareto's conclusions with respect to the convex appearance of the curve for wages are consistent with curves showing *number of dollars* per income-tax interval traceable to wages but not with actual wage distributions showing *number of recipients* per wage interval. Distributions based upon income from effort and distributions based upon income from such sources (mostly profits and income from property) as yield the higher incomes seem to have tails the one as roughly straight as the other. Indeed many wage distributions have tails more closely approximating straight lines than do income-tax data.



Yet all three curves have tails on a double log scale quite as similar as is common with income-tax returns.[1]

From this discussion we may draw the corollary that it is futile to attempt to measure changes in the inequality of distribution of income throughout its range by any function of the mere tail of the income frequency distribution. It seems unnecessary therefore to discuss Pareto's suggestions on this subject.

4. Is it probable that the distribution of income is similar enough from year to year in the same country to make the formulation of any useful general "law" possible?

[1] As will be seen in Chapter 29, there seems reason for believing that the extreme difference between the distribution of incomes obtained by the Australian Census and the estimate made by the National Bureau of Economic Research is due largely to difference in definition of *income* and *income recipient*. However, this does not alter the fact that we have here again two distributions with tails as similar as is usual with income-tax distributions and lower ranges about as different as it is possible to imagine.

Before answering this question we must decide what we should mean by the word *similar*. If income distributions for two years in the same country were such that each distribution included the same individuals and each individual's *income* was twice as large in the second year as it had been in the first year, it would seem reasonable to speak of the distributions as strictly similar. If in a third year (because of a doubling of population due to some hypothetical immigration) the *number* of persons receiving each specified income size was exactly twice what it was in the second year, it would still seem reasonable to speak of the distributions as strictly similar. Tested by any statistical criterion of dispersion which takes account of relative size (such as the coefficient of variation), the dispersion is precisely the same in each of the three years. Moreover the three distributions mentioned above [1] must necessarily have identically the same shape on a double log scale, and furthermore any two distributions which have identically the same shape on a double log scale [2] must necessarily have the same relative dispersion as measured by such indices as the coefficient of variation, interquartile range divided by median, etc. Approximation to identity of shape on a double log scale seems then a useful concept of "similarity." It is the concept implicit in Pareto's work.[3]

Now we have already found considerable evidence that income distributions are not, to a significant degree, similar in shape on a double log scale. The income-tax tails of income distributions for different times and places neither approximate straight lines of constant slope nor approximate one another; they are of distinctly different shapes. Moreover, such tails do not show in respect of their numbers of income recipients and

[1] Or. any distributions whose equations may be reduced to one another by substituting k_1x for x and k_2y for y.

[2] The curve may be thought of as consisting of two parts, which before reduction to logarithms, would be (1) the positive income section and (2) the negative income section with positive signs.

[3] While approximate identity of shape on a natural scale, a natural x and log y scale, or any other similar criterion would constitute a "law." no such approximate identity of shape on such scales has yet been discovered and it seems difficult to advance any very cogent *a priori* reasons for expecting it.

In this connection we must remember that had we the exact figures for the entire frequency curves of the distribution of income in the United States from year to year, if moreover we could imagine definitions of *income* and *income recipient* which would be philosophically satisfactory and statistically usable—and if further we managed year by year to describe our data curves adequately by generalized mathematical frequency curves of more or less complicated variety we should not *necessarily* have arrived at any particularly valuable results. Any series of data may be described to any specified degree of approximation by a power series of the type $y = A + Bx + Cx^2 + Dx^3 + \ldots\ldots$ but such fit is purely empirical and absolutely meaningless except as an illustration of MacLaurin's theorem in the differential calculus. We might be able to describe each year's data rather well by one of Karl Pearson's generalized frequency curves, but if the essential characteristics of the curve—skewness, kurtosis, etc., changed radically from year to year, description of the data by such a curve might well give no clue whatever as to any "law." Not only might the years be different but the fits might be empirical. Professor Edgeworth has well said that "a close fit of a curve to given statistics is not, *per se* and apart from *a priori* reasons, a proof that the curve in question is *the* form proper to the matter in hand. The curve may be adapted to the phenomena merely as the empirically justified system of cycles and epicycles to the planetary movements, not like the ellipse, in favor of which there is the Newtonian demonstration, as well as the Keplerian observations." *Journal of the Royal Statistical Society*, vol. 59, p. 533.

total amounts of income any uniformity of relation to the total number of income recipients and total amount of income in the country, even after adjustments have been made for variations in population and average income.[1] Considerations such as these, reënforce the conclusion which we arrived at from an examination of wage distributions, namely, that there is little necessary relation between the shape of the tail and the shape of the body of a frequency distribution, and have led us to suspect that, even if the tails of income distributions were practically identical in shape, it would be extremely dangerous to conclude therefore that the lower income ranges of the curves were in any way similar.

A most important matter remains to be discussed. What right have we to assume that the heterogeneity necessarily inherent in all income distribution data is not such as inevitably to preclude not only uniformity of shape of the frequency curve from year to year and country to country but also the very possibility of rational mathematical description of any kind unless based upon *parts* rather than the *whole?* What evidence have we as to the extent and nature of heterogeneity in income distribution data?

In the first place we must remember that lower range incomes are predominantly from wages and salaries, while upper range incomes are predominantly from rent, interest, dividends and profits.[2] While 74.67 per cent of the total income reported in the United States in the \$1,000–\$2,000 income interval in 1918 was traceable to *wages* and *salaries*, only 33.10 per cent of the income in the \$10,000–\$20,000 interval was from those sources, and only 15.92 per cent of the income in the \$100,000–\$150,000 interval and 3.27 per cent of the income in the over-\$500,000 intervals. On the other hand, while only 1.93 per cent of the total income reported in the \$1,000–\$2,000 interval in 1918 was traceable to *dividends*, 23.73 per cent was so traceable in the \$10,000–\$20,000 interval, 43.18 per cent in the \$100,000–\$150,000 interval, and 59.44 per cent in the over-\$500,000 intervals.[3] The difference in constitution of the income at the upper and

[1] Estimated per cent of total income received by highest 5% of income receivers in United States:

1913	.33
1914	.32
1915	.32
1916	.34
1917	.29
1918	.26
1919	.24

National Bureau of Economic Research. *Income in the United States*, vol. 1, p. 116.

[2] Compare Professor A. L. Bowley's paper on "The British Super-Tax and the Distribution of Income," *Quarterly Journal of Economics*, February, 1914.

[3] *Statistics of Income 1918*, pp. 10 and 44.

While the reporting of dividends was almost certainly less complete in the lower than in the upper income classes, the difference could not be sufficient to invalidate the general conclusion. Lower range incomes are predominantly wage and salary incomes; upper range incomes are not.

lower ends of the distribution is sufficient to justify the statement that most of the individuals going to make up the lower income range of the frequency curve are wage earners, while the individuals going to make up the upper income range are capitalists and entrepreneurs.[1] What do we know about the shapes of these component distributions? Is the fundamental difference in their relative positions on the income scale their only dissimilarity?

In any particular year the upper income tail of the frequency distribution of income among *capitalists and entrepreneurs* seems not greatly different from the extreme upper income tail of the frequency distribution of income among all classes. This is what we might expect. Not only is the percentage of the total income in the extreme upper income ranges reported as coming from wages and salaries small but much of this so-called wages and salaries income must be merely technical. For example, it is often highly "convenient" to pay "salary" rather than dividends. Furthermore, in so far as the tail of the curve of distribution of income among capitalists and entrepreneurs is not identical with the tail of the general curve, it will show a *smaller* rather than a larger slope, because the percentage of the number of persons in each income interval who are capitalists and entrepreneurs increases as we pass from lower to higher incomes.[2] Now the slopes of the straight lines fitted to the extreme tails of non-cumulative income distributions on a double log scale fluctuate within a range of about 2.4 to 3.0.

The upper range tails of *wages* distributions tell an entirely different story. Aside from surface irregularities often quite evidently traceable to concentration on certain round numbers, the majority of wages distributions have tails which, on a double log scale, are roughly linear.[3] However the *slopes* of straight lines fitted to these tails are much greater than the slopes of corresponding straight lines fitted to income distribution tails.[4] While the slopes of income distribution tails range from about 2.4

[1] Many individuals in the middle income ranges must necessarily be difficult to classify. This does not mean that the concept of heterogeneity is inapplicable. There are countries in which the population is a mixture of Spanish, American Indian, and Negro blood. Now such a population must, for many statistical purposes, be considered extremely heterogeneous even though the percentage of the population which is of *any* pure blood be quite negligible.

[2] In 1917, the only year in which returns are classified according to "principal source of income" (wages and salaries, income from business, income from investment) the difference in slope, in the income range $100,000 to $2,000,000, between the distribution for *all returns* and the distribution for those returns which did not report wages and salaries as their principal source of income was less than .05. The slope in this range of the line fitted to all returns was about 2.64; the business and investment line was about 2.59 and the wages line about 3.21. In 1916, the only year in which returns are classified according to occupations, the distribution of income among *capitalists* shows a slope of only 2.08 while *public service employees* (*civil*) show a slope of 2.70 and *skilled and unskilled laborers* a slope of 2.74.

[3] Attention has already been drawn to the fact that this is a characteristic of many frequency distributions of various kinds.

[4] A further difference between the upper range income distribution among capitalists and entrepreneurs and the upper range of the distribution among all persons seems to be, from the 1916 occupation distributions, that the distribution among all persons shows less of a roll, i. e., is straighter.

to 3.0, the slopes of wages distributions tails commonly range between 4.0 and 6.0. They seldom run below about 4.5; they sometimes run as high as 10.0 and 11.0.

A distribution of wages per hour for 26,183 male employees in iron and steel mills in the United States in 1900 [1] shows a tail with a slope of about 3.35. However, the total of which this is a part, the distribution of wages per hour among 180,096 male employees in 32 manufacturing industries in 1900, shows a tail-slope of about 4.8. The estimated distribution of weekly earnings of 5,470,321 wage earners in the United States in 1905 [2] shows a tail-slope of about 5.0. The distribution of earnings per hour among 318,946 male employees in 29 different industries in the United States in 1919 [3] shows a tail-slope of about 5.86. The distribution of wages per month among 1,939,399 railroad employees in the United States in 1917 [4] shows a tail-slope of about 6.25. The distribution of wages per hour among 43,343 male employees in the foundries and metal working industry of the United States in 1900 [5] shows a tail-slope of about 7.8. The distribution of earnings in a week among 9,633 male employees in the woodworking industry—agricultural implements—in the United States in 1900[6] shows a tail-slope of over 11.0. At the other extreme was the case of the wages-per-hour distribution among 26,183 male employees in American iron and steel mills in 1900 with a slope of 3.35. Both 11.0 and 3.35 are exceptional, but the available data make it clear that wages distributions of either earnings or rates have tail-slopes which are always much greater than the maximum tail-slope of income distributions.

The illustrations in the preceding paragraph are illustrations of the tail-slopes of *wages* distributions among wage earners. However all the evidence points to frequency distributions of *income* among wage earners having tail-slopes only very slightly less steep than the tail-slopes of wages distributions. We have almost no usable data concerning the relation between individual wage distributions and income distributions for the same individuals, but we have a few samples showing the relation between family earnings distributions and family income distributions.[7] Moreover, we can without great risk base certain extremely general conclusions

[1] Twelfth Census of the United States (1900), *Special Report on Employees and Wages*, Davis R. Dewey.

[2] *1905 Census of Manufactures*, Part IV, p. 647.

[3] *Monthly Labor Review*, Sept., 1919.

[4] *Report of the Railroad Wage Commission to the Director General of Railroads*, 1919, p. 96.

[5] Twelfth Census of the United States (1900), *Special Report on Employees and Wages*, Davis R. Dewey.

[6] Twelfth Census of the United States (1900), *Special Report on Employees and Wages*, Davis R. Dewey.

[7] The reader must not confuse the percentage of the income not derived from wages going to *wage-earners* in any particular income class with the percentage of the income not derived from wages going to *all income recipients* in any particular income class. Some of these last recipients are not wage earners at all, they receive no wages. Information concerning the second of these relations *but not the first* is given in the income tax reports.

concerning individual wage-earners' income distributions on these family data. The upper tails of the family-wage distributions *are* the tails of the wage distributions for the individuals who are the heads of the families. This is apparent from an analysis of the samples. Now income from rents and investments belongs almost totally to heads of families. Such income is however so small in amount that it cannot alter appreciably the slope of the tail.[1] While income from other sources than rents and investments (lodgers, garden and poultry, gifts and miscellaneous) may not be so confidently placed to the credit of the head of the family, this item changes its percentage relation to the total income so slowly as to be negligible in its effect upon the tail-slope of the distribution.[2] Notwithstanding the danger of reasoning too assuredly about individuals from these picked family distributions, we seem justified in believing that the tail-slopes of income distributions among individual wage earners are not very different from the tail-slopes of wage distributions among the same individuals.[3]

The upper tail-slopes of income distributions among typical wage earners

[1] For example. in the report on the incomes of 12,096 white families published in the *Monthly Labor Review* for December, 1919, we find the income from rents and investments less than one per cent of the total family income for each of the income intervals.

Income group	Percentage income from rents and investments is of total income
Under $900	.079
$ 900-$1.200	.176
1.200- 1.500	.410
1.500- 1.800	.551
1.800- 2.100	.606
2,100- 2,500	.998
2.500 and over	.778

[2] As a somewhat extreme example. the Bureau of Labor investigation mentioned in the preceding note shows the following relations between total family earnings and total family income (including income from rents and investments, lodgers, garden and poultry, gifts and miscellaneous).

Income group	Percentage that total earnings are of total income
Under $900	96.2
$ 900-$1.200	96.5
1.200- 1.500	96.3
1,500- 1,800	96.0
1,800- 2.100	96.3
2.100- 2.500	95.1
2,500 and over	96.2

[3] Further corroboratory evidence. of some slight importance. that the tail-slopes of wage
distributions among wage earners are not very different from the tail-slopes of income dis-
tributions among wage earners is yielded by the fact that the tail-slopes of income distribu-
tions among families (which are virtually identical with the tail-slopes of both income and
wage distributions among the heads of these families) have roughly the same range as the
tail-slopes of wage distributions among individuals. The British investigation into the in-
comes of 7.616 workingmen's families in the United States in 1909 shows a tail-slope of about
3.5. (Report of the British Board of Trade on *Cost of Living in American Towns*. 1911. [Cd.
5609], p. XLIV.) The Bureau of Labor's investigation into the income of 12,096 white fam-
ilies in 1919 shows a tail-slope of about 4.0. Mr. Arthur T. Emery's extremely careful in-
vestigation into the incomes of 2.000 Chicago households in 1918 shows a tail-slope of
about 4.4. At the other extreme we find that the Bureau of Labor's investigation into the
income of 11,156 families in 1903 (*Eighteenth Annual Report of the Commissioner of Labor*.
1903, p. 558) shows a tail-slope of about 10.0. and that Mr. R. C. Chapin's investigation into
the income of 391 workingmen's families in New York City (*Standard of Living Among Work-
ingmen's Families in New York City*, p. 44) also shows a slope of about 10.0. The tails of
these last two cases are very irregular so that the slope itself is not determinable with much
precision.

may then be assumed to have much greater slopes than the upper tail-slopes of income distributions among capitalists and entrepreneurs. It does not seem possible to make any very definite statement concerning the body and lower tail of the capitalist and entrepreneurial distribution—even in so far as that term is a significant one.[1] All the evidence suggests that the mode of what we have termed the capitalist-entrepreneurial distribution is consistently higher than the wage-earners' mode.[2] Its lower income tail undoubtedly reaches out into the negative income range, which the tail of the wage-earners' distribution may, both *a priori* and from evidence, be assumed not to do. It seems a not irrational conclusion then to speak of the capitalist-entrepreneurial distribution as having a lesser tail-slope than the wage-earners' distribution on the *lower* income side as well as on the upper income side,[3] and as a corollary almost certainly a much greater dispersion both actual and relative than the wage-earners' distribution.

Though the above generalizations concerning differences between the wage-earners' income distribution and the capitalist-entrepreneurial income distribution seem sound, they tell but a fraction of the story. Aside from the difficulty of classifying all income recipients in one or the other of these two classes, we are faced with the further fact that investigation suggests that our two component distributions are themselves exceedingly heterogeneous.[4] We have already noted that wage distributions for different occupations and times are extremely dissimilar in shape and we suspect that the same applies to capitalist-entrepreneurial distributions. For example, what little data we possess suggest that the distribution of income among farmers has little in common with other entrepreneurial distributions.

Moreover, the component distributions, into which it would seem necessary to break up the complete income distribution before any rational description would be possible, not only have different shapes and different positions on the income scale (i. e., different modes, arithmetic averages, etc.), but *the relative position with respect to one another on the income scale* of these different component distributions changes from year to year.[5]

[1] In the total income curve there is a broad twilight zone where individuals are often both wage or salary earners and capitalists or even entrepreneurs.

[2] In the 1916 occupation distributions the only occupations showing more returns for the \$4,000–\$5,000 interval than the \$3,000–\$4,000 (that is the only occupations showing any suggestion of a mode) are of a capitalistic or entrepreneurial description—bankers; stock-brokers; insurance brokers; other brokers; hotel proprietors and restaurateurs; manufacturers; merchants; storekeepers; jobbers; commission merchants, etc.; mine owners and mine operators; saloon keepers; sportsmen and turfmen.

[3] Of course the very word *slope* is an ambiguous term to use concerning the tail of a curve which enters the second quadrant.

[4] Evidence suggesting definite heterogeneity in the "wage and salary" figures of the income-tax returns is presented in Chapter 30.

[5] This fact is one of the simpler pieces of evidence against the existence of a "law." Of course, even though the income distribution were made up of heterogeneous material, if the

Table 28Q [1] is interesting as showing the changes in the relative positions of the arithmetic averages of different wage distributions in 1909, 1913 and 1918.

TABLE 28Q

CHANGES IN THE RELATIVE POSITIONS OF THE AVERAGE ANNUAL EARNINGS OF EMPLOYEES ENGAGED IN VARIOUS INDUSTRIES

Industry	1909	1913	1918
All Industries	100.0	100.0	100.0
Agriculture	48.2	45.4	54.7
Production of Minerals	95.7	104.4	119.0
Manufacturing:			
Factories	91.2	97.5	103.5
Hand Trades	111.7	103.5	110.8
All Transportation	104.9	105.4	119.3
Railway, Express, Pullman, Switching and Terminal Cos	104.0	108.2	129.3
Street Railway, Electric Light and Power, Telegraph and Telephone Cos	99.5	93.8	81.4
Transportation by Water	123.5	114.1	147.5
Banking	123.0	128.6	135.5
Government	118.1	113.8	83.0
Unclassified Industries	114.4	107.7	97.8

The data are so inadequate that the construction of a similar table for capitalist-entrepreneurial distributions is not feasible. However, there are comparatively good figures for total income of farmers and total number of farmers year by year.[2] The average incomes of farmers, year by year, were the following percentages of the estimated average incomes of all persons gainfully employed in the country.

	Percentages
1910	75.19
1911	69.13
1912	72.41
1913	74.88
1914	76.33
1915	80.45
1916	82.85
1917	104.51
1918	109.68
1919	103.95
1920	63.88

This is a wide range.

Exactly what effects have such internal movements of the component distributions upon the total income frequency distribution curve? This is a difficult question to answer as we have not sufficient data to break

component parts remained constant in shape *and in their relative positions with respect to one another on the income scale*, these relations would of themselves constitute a "law."

[1] Based upon *Income in the United States*, Vol. I, pp. 102 and 103.

[2] See *Income in the United States*, Vol. I, p. 112.

down the total, composite, curve into its component parts with any degree of confidence.[1] However, the movements of wages in recent years would appear to give us a clue to the sort of phenomena we might expect to find if we had complete and adequate data.

The slopes of the upper income tails of wages distributions are great, 4 to 5 or more.[2] Now the wage curve moved up strongly from 1917 to 1918 if we may judge by averages. The average wage of all wage earners in the United States [3] increased 15.6 per cent [4] from 1917 to 1918. During the same period the average income of farmers increased 19.1 per cent [5] and the average income of persons other than wage earners and farmers remained nearly constant. Total amounts of income by sources in millions of dollars were:

	1917	1918	Percentage 1918 was of 1917
Total Wages [a]	$27,795	$32,575	117.20
Total Farmers' Income	8,800	10,500	119.32
All other Income	17,265	17,291	100.15
Total Income	$53,860	$60,366	112.08

[a] Includes pensions, etc., and includes soldiers, sailors, and marines.

Stockholders in corporations saw income from that source actually decline from 1917 to 1918.[6] What happened to American income-tax returns during this time?

[1] The processes by which the income distribution curve published in *Income in the United States*, Vol. I, pp. 132–135 was arrived at were such that to use that material here would practically amount to circular reasoning. The conclusions arrived at here were used in building up that curve.

[2] The slope of the tail of the wage and salary curve in the 1917 income tax returns is only about 3.21 (compare, note 2, p. 377). However we must remember that the individuals there classified are largely of an entirely different type of "wage-earner" from those in the lower groups. In this upper group occur the salaried entrepreneurs, professional men, etc., and those whose "salaries" are really profits or dividends. The evidence points to a rather distinct and significant heterogeneity along this division in the wage and salary distribution. See Chapter 30.

[3] Excluding soldiers, sailors, and marines, and professional classes but including officials and "salaried entrepreneurs."

[4] From $945 per annum in 1917 to $1,092 per annum in 1918.

[5] From $1,370 per annum in 1917 to $1,632 per annum in 1918.

[6] CORPORATION DIVIDENDS, SURPLUS AND EARNINGS

(In millions of dollars)

	Dividends	Surplus	Net earnings
1917	3,995	3,963	7,958
1918	2,568	1,945	4,513

See page 324.

TOTAL AMOUNT OF NET INCOME RETURNED BY SOURCES (RETURNS REPORTING OVER $2,000 PER ANNUM NET INCOME) [a]

(Millions of dollars)

Income class	Wages and salaries		All other sources [b]	
	1917	1918	1917	1918
Over $2,000	$3,648	$6,493	$7,543	$7,198
2,000- 4,000	1,553	3,687	1,799	2,036
4,000- 5,000	301	703	528	736
5,000-10,000	661	849	1,167	1,296
Over 10,000	1,133	1,254	4,049	3,130

[a] Wages income from returns reporting between $1,000 and $2.000 per annum is not available for 1917.

[b] "Other sources" are total *net* income minus wages and salaries, i. e.. total *general deductions* have been assumed as deductible from other sources (gross). All things considered, this seems proper here though it may easily be criticised. In connection with changes in the relation between *net* and *gross* income from 1917 to 1918 see Chapter 30. pp. 401 and 402.

While reported income from all other sources than wages and salaries declined 4.6 per cent,[1] reported income from wages and salaries increased 78.0 per cent.[2] Moreover, the great increases in wages and salaries were in the lowest intervals. The wage curve with its steep tail-slope was moving over into the income tax ranges.[3] The effect upon the total curve is very pronounced, as may be seen from Table 28R.

TABLE 28R

AMERICAN INCOME TAX RETURNS IN 1917 AND 1918

Total Number of Returns
(In thousands)

	1917	1918	Percentage 1918 was of 1917
$2,000-$4,000	1,214	2,107	173.56
4,000- 5,000	186	322	173.12
5,000-10,000	271	319	117.71
Over 10,000	162	160	98.77

On a double log scale we see the curve changing its shape radically. While the 1917 curve is comparatively smooth and regular, the 1918 curve develops a distinct "bulge" in the lower ranges.[4]

The preceding discussion has been concerned with equal dollar-income

[1] Had "other sources" been taken gross instead of net, that item would have shown an increase of 5.3 per cent instead of a decrease of 4.6 per cent.

[2] The actual spread is still greater than the figures show. Income from professions, which in 1917 was classed under *wages*. in 1918 and 1919 was classed under *business*.

[3] This seems to be a fact though it is not the whole story. The "intensive drive" of 1919 may easily account for some of the increase. See Chapter 30 for a discussion of the probable extent of this influence.

[4] See *Income in the United States*, Vol. I, Charts 28 and 30.

intervals. However, $2,000 income in 1918 was relatively less than $2,000 income in 1917. The average (per capita) income of the country was $523 in 1917 and $586 in 1918.[1] The adjustment is theoretically crude, but $2,241 [2] in 1918 might be considered as in one sense equivalent to $2,000 in 1917. The results of comparisons of the two years upon this basis are given in Table 28S.[3]

TABLE 28S

INCOME RETURNED—BY SOURCES

(Millions of dollars)

Income class	Wages and salaries	Total *net* income	Total *net* income minus wages and salaries	Total *gross* income	Total *gross* income minus wages and salaries
1917					
$2,000–$4,000....	$1,553	$3,352	$1,799	$3,713	$2,161
4,000– 5,000....	301	829	528	895	594
5,000–10,000....	661	1,828	1,167	1,951	1,290
Over 10,000....	1,133	5,182	4,049	5,518	4,384
1918					
$2,241–$4,482....	$3,236	$5,359	$2,123	$5,766	$2,530
4,482– 5,602....	498	1,111	613	1,247	749
5,602–11,205....	773	1,960	1,187	2,315	1,542
Over 11,205.....	1,153	4,129	2,976	4,842	3,689
(Multiplied by $\frac{523}{586}$, that is reduced to "1917 dollars")					
$2,241–$4,482....	$2,888	$4,783	$1,895	$5,146	$2,258
4.482– 5,602....	445	992	547	1,113	668
5,602–11,205....	690	1,749	1,059	2.066	1,376
Over 11,205....	1,029	3,685	2,656	4,321	3,292
(Percentages of Total Income of Country)					
1917					
$2,000–$4,000....	2.88	6.22	3.34	6.89	4 01
4,000– 5,000....	.56	1.54	.98	1.66	1.10
5,000–10,000....	1.23	3.39	2.16	3.62	2.39
Over 10,00....	2.10	9.61	7.51	10.24	8.14
1918					
$2,241–$4,482....	5.30	8.78	3.48	9.45	4.15
4,482– 5,602....	.82	1.82	1.00	2.05	1.23
5,602–11,205....	1.27	3.21	1.94	3.80	2.53
Over 11,205....	1.89	6.77	4.88	7.94	6.05

[1] *Income in the United States*, Vol. I. p. 76.

[2] $\$2,000 \times \frac{586}{523}$.

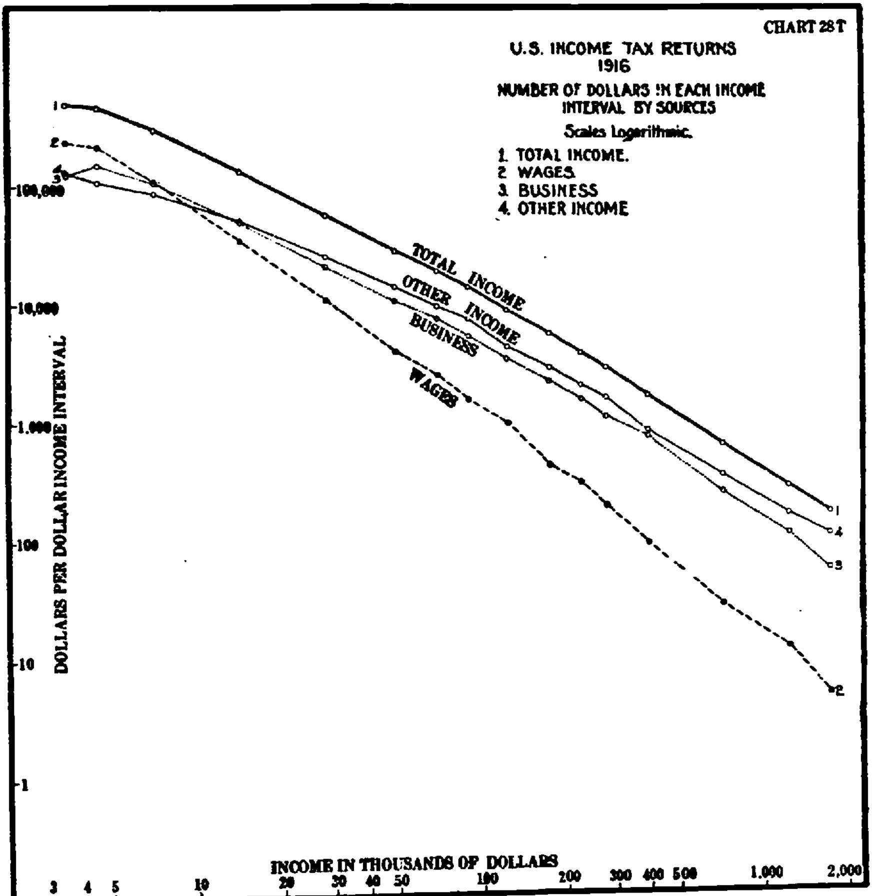
[3] The figures for the amounts of income in the irregular 1918 income intervals of that table ($2,241–$4,482, etc.) were calculated by straight line interpolation on a double log scale applied to the even thousand dollar intervals of the income-tax returns. Though the total income curve does not approximate linearity it may be assumed linear within the small range of one income tax interval without serious error.

(Table 288 concluded.)

NUMBER OF RETURNS
(Thousands)

Income class	1917	Income class	1918	Percentage 1918 was of 1917
\$2,000–\$4,000	1,214	\$2,241–\$4,482	1,758	144.81
4,000– 5,000	186	4,482– 5,602	220	118.28
5,000–10,000	271	5,602–11,205	260	95.94
Over 10,000	162	Over 11,205	136	83.95

It is from this table once again apparent that the wage distribution moved independently up on the income scale and that the effect of this movement was confined to the lowest income intervals. Charts 28T, 28U, 28V, 28W, 28X, 28Y, 28Z, and 28AA which show the number of dollars income per dollar-income interval, by sources, are enlightening as illustrating in still



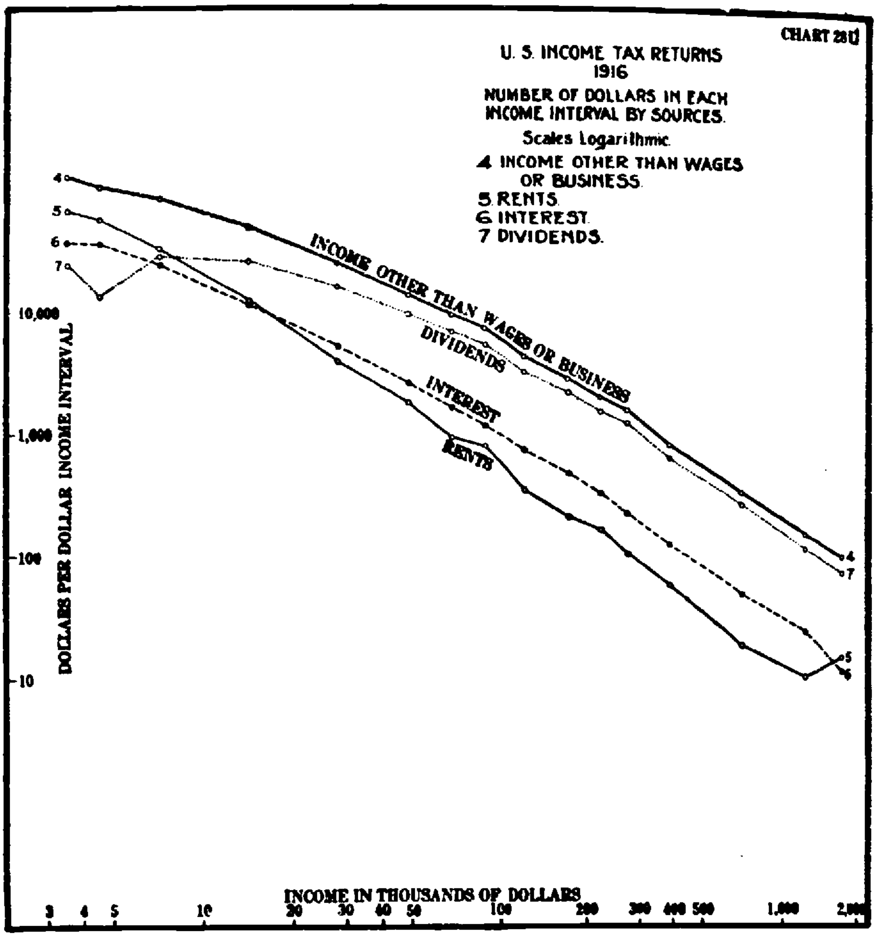


CHART 28U
U. S. INCOME TAX RETURNS
1916
NUMBER OF DOLLARS IN EACH
INCOME INTERVAL BY SOURCES.
Scales Logarithmic.
4 INCOME OTHER THAN WAGES
OR BUSINESS.
5 RENTS.
6 INTEREST.
7 DIVIDENDS.
INCOME OTHER THAN WAGES OR BUSINESS
DIVIDENDS
INTEREST
RENTS
DOLLARS PER DOLLAR INCOME INTERVAL
10,000
1,000
100
10
INCOME IN THOUSANDS OF DOLLARS
3
4
5
10
20
30
40
50
100
200
300
400
500
1,000
2,000

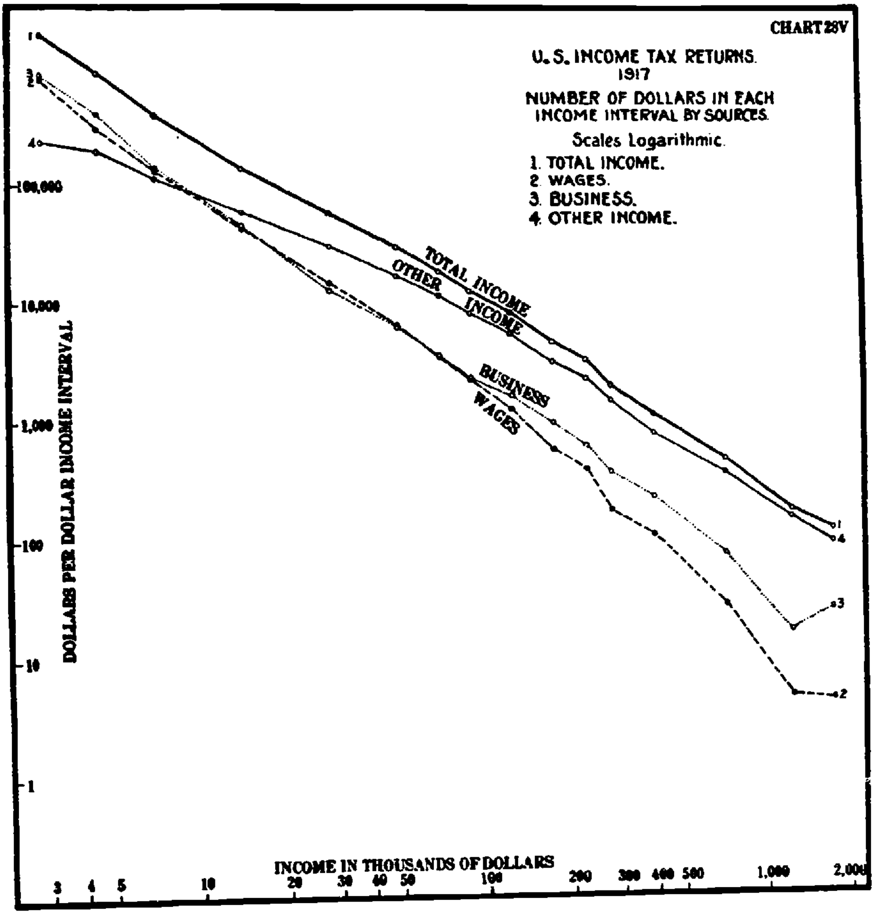


CHART 28V
U. S. INCOME TAX RETURNS.
1917
NUMBER OF DOLLARS IN EACH INCOME INTERVAL BY SOURCES.
Scales Logarithmic.
1. TOTAL INCOME.
2. WAGES.
3. BUSINESS.
4. OTHER INCOME.
TOTAL INCOME
OTHER INCOME
BUSINESS
WAGES
DOLLARS PER DOLLAR INCOME INTERVAL
100,000
10,000
1,000
100
10
1
INCOME IN THOUSANDS OF DOLLARS
3
4
5
10
20
30
40
50
100
200
300
400
500
1,000
2,000

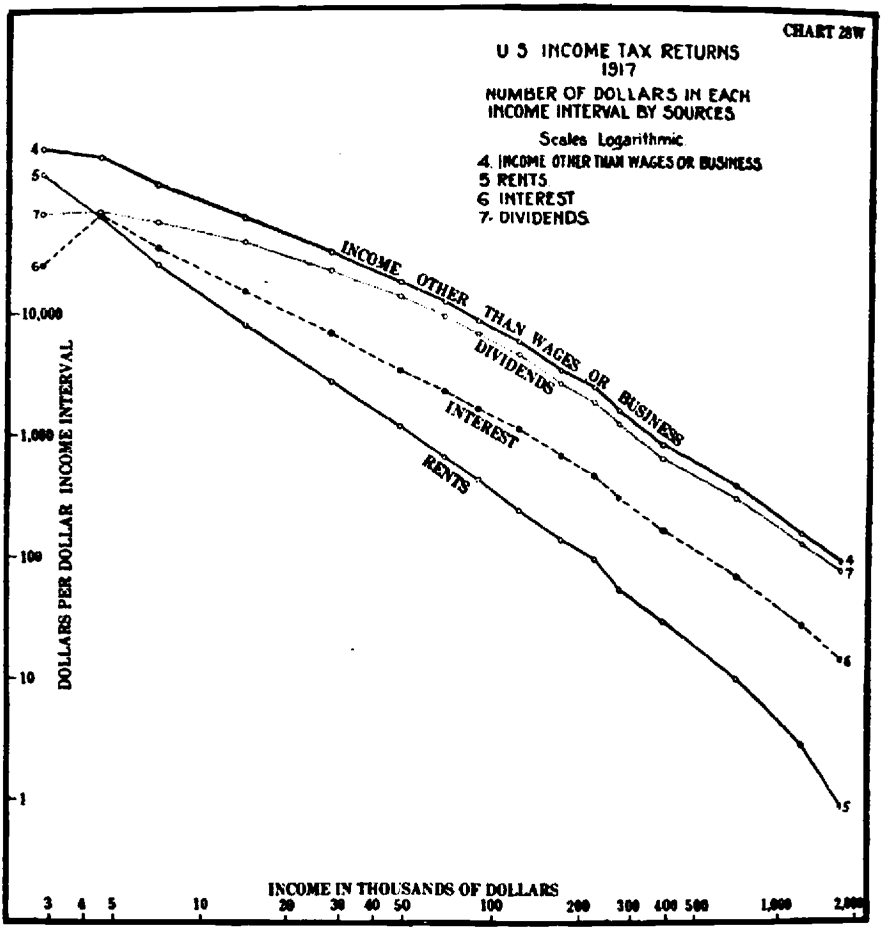


CHART 28W
U S INCOME TAX RETURNS
1917
NUMBER OF DOLLARS IN EACH
INCOME INTERVAL BY SOURCES
Scales Logarithmic
4. INCOME OTHER THAN WAGES OR BUSINESS
5 RENTS
6 INTEREST
7. DIVIDENDS
INCOME OTHER THAN WAGES OR BUSINESS
DIVIDENDS
INTEREST
RENTS
DOLLARS PER DOLLAR INCOME INTERVAL
10,000
1,000
100
10
1
INCOME IN THOUSANDS OF DOLLARS
3
4
5
10
20
30
40
50
100
200
300
400
500
1,000
2,000

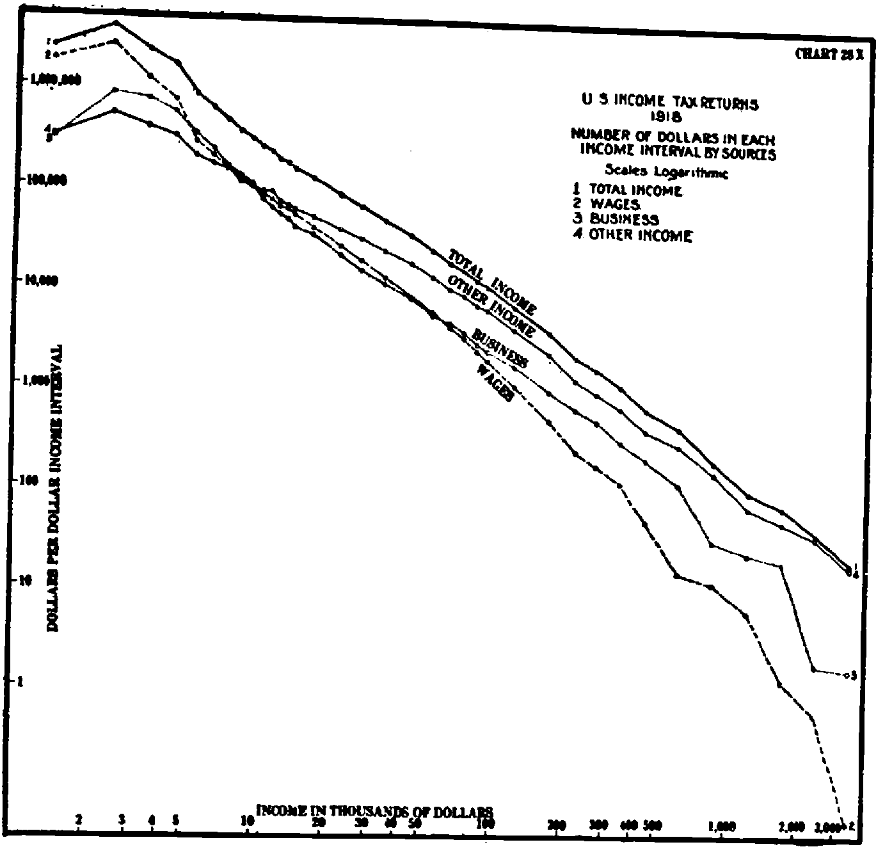


CHART 28 X
U S INCOME TAX RETURNS
1918
NUMBER OF DOLLARS IN EACH INCOME INTERVAL BY SOURCES
Scales Logarithmic
1 TOTAL INCOME
2 WAGES
3 BUSINESS
4 OTHER INCOME
TOTAL INCOME
OTHER INCOME
BUSINESS
WAGES
DOLLARS PER DOLLAR INCOME INTERVAL
1,000,000
100,000
10,000
1,000
100
10
1
INCOME IN THOUSANDS OF DOLLARS
2
3
4
5
10
20
30
40
50
100
200
300
400
500
1,000
2,000
3,000

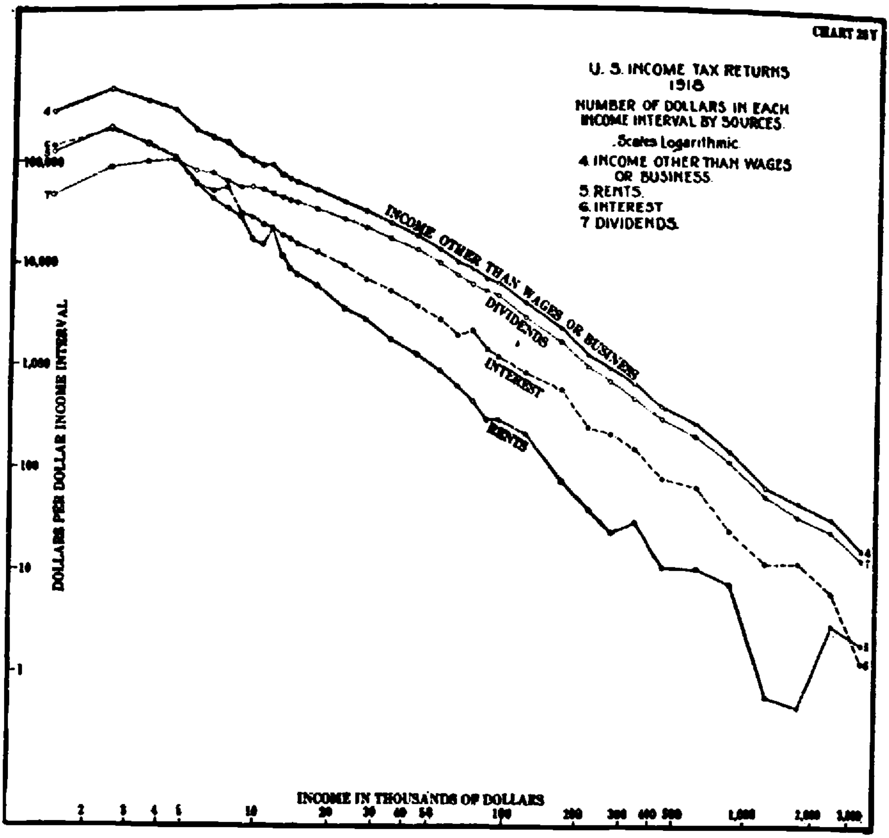


CHART 28Y
U. S. INCOME TAX RETURNS
1918
NUMBER OF DOLLARS IN EACH INCOME INTERVAL BY SOURCES.
Scales Logarithmic
4 INCOME OTHER THAN WAGES OR BUSINESS
5 RENTS
6 INTEREST
7 DIVIDENDS
INCOME OTHER THAN WAGES OR BUSINESS
DIVIDENDS
INTEREST
RENTS
DOLLARS PER DOLLAR INCOME INTERVAL
100,000
10,000
1,000
100
10
1
INCOME IN THOUSANDS OF DOLLARS
2
3
4
5
10
20
30
40
50
100
200
300
400
500
1,000
2,000
3,000

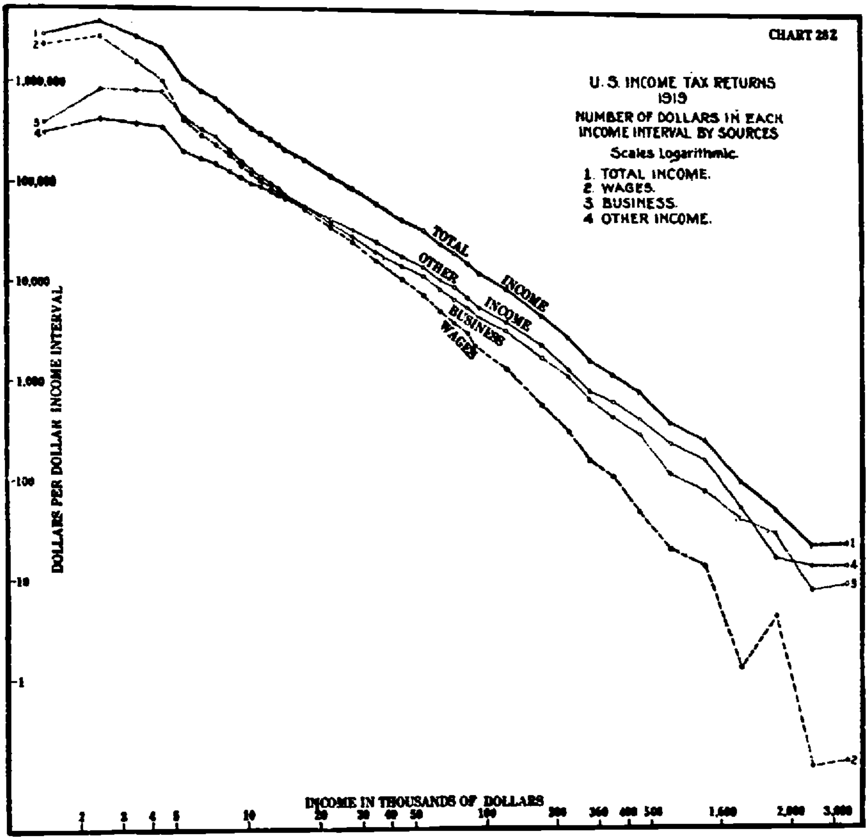


CHART 282
U. S. INCOME TAX RETURNS
1919
NUMBER OF DOLLARS IN EACH
INCOME INTERVAL BY SOURCES
Scales Logarithmic.
1. TOTAL INCOME.
2. WAGES.
3. BUSINESS.
4. OTHER INCOME.
TOTAL INCOME
OTHER INCOME
BUSINESS
WAGES
DOLLARS PER DOLLAR INCOME INTERVAL
1,000,000
100,000
10,000
1,000
100
10
1
INCOME IN THOUSANDS OF DOLLARS
2
3
4
5
10
20
30
40
50
100
200
300
400
500
1,000
2,000
3,000

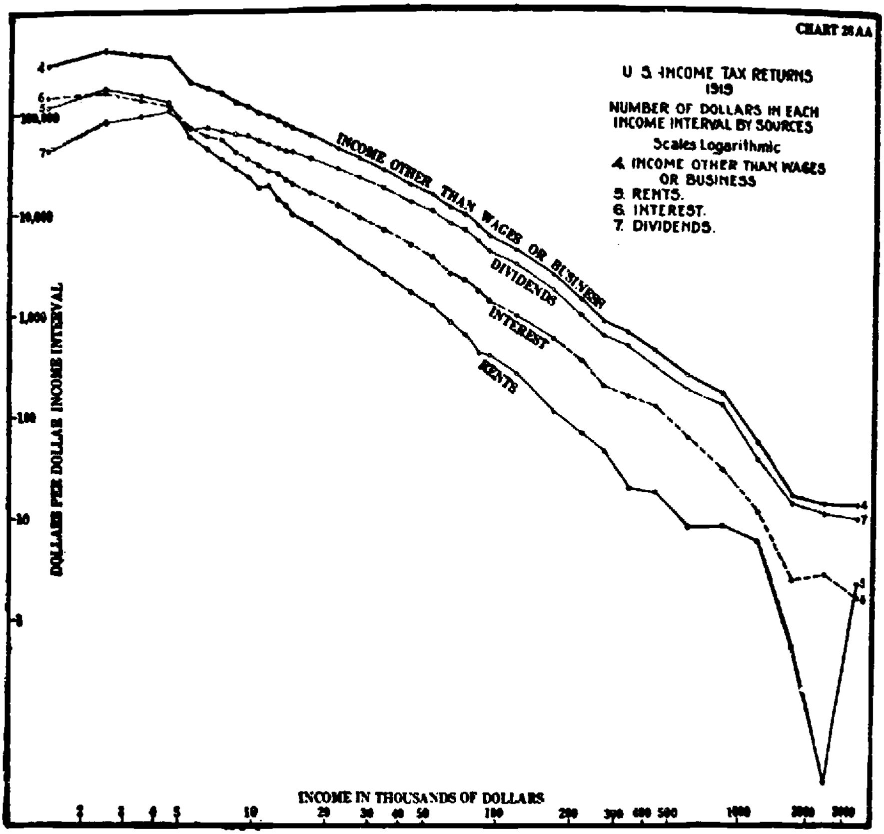


CHART 28 AA
U. S. INCOME TAX RETURNS
1919
NUMBER OF DOLLARS IN EACH
INCOME INTERVAL BY SOURCES
Scales Logarithmic
4. INCOME OTHER THAN WAGES OR BUSINESS
5. RENTS.
6. INTEREST.
7. DIVIDENDS.
INCOME OTHER THAN WAGES OR BUSINESS
DIVIDENDS
INTEREST
RENTS
DOLLARS PER DOLLAR INCOME INTERVAL
100,000
10,000
1,000
100
10
INCOME IN THOUSANDS OF DOLLARS
10
20
30
40
50
100
200
300
400
500
1000
3000

greater detail the changes in the constitution of the returns from year to year.

Such material and the appearance of the "bulge" on the income-tax curve in the lowest income ranges [1] in the years 1918 and 1919 when wages and salaries were high and average (per capita) incomes also high [2] strongly suggest that the income curve, in so far as it shows any similarity from year to year, changes its general appearance and turns up (on a double log scale) as it approaches those ranges where wages and salaries are of predominant influence.[3] The great *slopes* of wage distributions are on this hypothesis not inconsistent with the smaller *slope* of the general income curve in its higher (income-tax) ranges.[4]

Conclusions:

(1) Pareto's Law is quite inadequate as a mathematical generalization, for the following reasons:
   - (a) The tails of the distributions on a double log scale are not, in a significant degree, linear;
   - (b) They could be much more nearly linear than they are without that condition being especially significant, as so many distributions of various kinds have tails roughly approaching linearity;
   - (c) The straight lines fitted to the tails do not show even approximately constant slopes from year to year or between country and country;
   - (d) The tails are not only not straight lines of constant slope but are not of the same shape from year to year or between country and country.

(2) It seems unlikely that any useful mathematical law describing the entire distribution can ever be formulated, because:
   - (a) Changes in the shape of the income curve from year to year seem traceable in considerable measure to the evident heterogeneity of the data;
   - (b) Because of such heterogeneity it seems useless to attempt to

[1] See Chapter 30 for further discussion of this "bulge" in connection with an examination of how far it may be the result of irregularity in reporting.

[2] Average (per capita) incomes being high means that a definite money income (such as $2,000) takes us relatively further down the income curve than if average incomes were low.

[3] It is difficult to say just where the "bulge" might have appeared in the 1917 distribution if as great efforts had been made to obtain correct returns in that year as were made under the "intensive drive" for 1918 returns. The *wages* line on the 1917 number of dollars income per dollar-income interval chart (Chart 28V) shows signs of turning up somewhere between $4,000 and $5,000 and the *business* line somewhere in the $5,000–$10,000 interval. However neither movement is large nor can their positions be accurately determined on account of the size of the reporting intervals. See also Chapter 30, p. 412.

[4] The "bulge" on the income from wages and salaries curve itself, as seen in the income-tax returns for 1918 and 1919 (see Charts 28X and 28Z), seems the result of heterogeneity in these wage and salary data themselves. This hypothesis is considered in Chapter 30.

describe the whole distribution by any mathematical curve designed to describe homogeneous distributions (as any *simple* mathematical expression must almost necessarily be designed to do);

(c) Furthermore, the existing data are not adequate to break up the income curve into its constituent elements;

(d) If the data were complete and adequate we might still remain in our present position of knowing next to nothing of the nature of any "laws" describing the elements.[1]

(3) Pareto's conclusion that economic welfare can be increased only through increased production is based upon erroneous premises. The income curve is not constant in shape. The internal movements of its elements strongly suggest the possibility of important changes in distribution. The radically different mortality curves for Roman Egypt and modern England,[2] and the decrease in infant mortality in the last fifty years illustrate well what may happen to heterogeneous distributions.

The next four chapters review the data from which any income frequency distribution for the United States must be constructed.

[1] Though all the evidence points to hope of further progress lying in the analysis of the parts rather than in any direct attack upon the unbroken heterogeneous whole.

[2] See *Biometrika*, Vol. I, pp. 261–264.