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A note on competitive prices in multilateral assignment markets

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Abstract

A multilateral assignment market with buyers and a number of different types of firms can be modeled by a multi-sided assignment game. We prove that core allocations of the latter are in a one-to-one correspondence with competitive prices of the former, where the notion of competitive price extends that of Roth and Sotomayor (1990). This result generalizes to multi-sided assignment markets the characterization of competitive prices known for the two-sided case.

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1 Introduction

Consider a market in which there are buyers and two types of firms producing respectively two types of indivisible and perfectly complementary goods. Let $S = \{S_1, ..., S_{n_s}\}$ and $H = \{H_1, ..., H_{n_h}\}$ be the two sets of goods. For instance, S could be a set of software products and H a set of hardware products. Suppose there are n_s firms each one of them producing exactly one different unit of S at unitary costs $c_1^S, ..., c_{n_s}^S$ respectively and n_h firms each one of them producing exactly one different unit of H at unitary costs $c_1^H, ..., c_{n_h}^H$ respectively. Each buyer is only interested on buying at most one unit of each type of product, but has no utility on buying separately either a unit of S or a unit of H. If $B = \{B_1, ..., B_{n_b}\}$ is the set of n_b buyers, we denote by w_{ij}^k the willingness-to-pay of buyer B_k for the pair (S_i, H_j).

In this market, a transaction can only be carried out when a buyer acquires exactly a unit of S and a unit of H. Let p_i and q_j be the prices the buyer B_k pays for S_i and H_j, respectively. At such prices, her utility is given by $w_{ij}^k - p_i - q_j$, whereas the benefit of the firm producing S_i is $p_i - c_i^S$ and the benefit of the firm producing H_j is $q_j - c_j^H$. If we assume that the utility of the agents is monetary and transferable the total surplus generated by a transaction is $\left(w_{ij}^k - p_i - q_j\right) + \left(p_i - c_i^S\right) + \left(q_j - c_j^H\right) = w_{ij}^k - c_i^S - c_j^H$. If $w_{ij}^k - c_i^S - c_j^H < 0$ no transaction will be carried out because there are no prices that simultaneously give non-negative utility to the buyer and the two producers. Let $a_{ijk} = \max\left\{0, w_{ij}^k - c_i^S - c_j^H\right\}$ be the 'social surplus' generated when S_i, H_j and B_k are 'assigned', i.e: when buyer B_k buys S_i and H_j. If we are only interested on how the social surplus is divided among agents, the above market is completely determined by giving the sets of buyers and firms and the set of parameters a_{ijk}.

The model can be generalized to an arbitrary m-sided market, with $m-1$ different types of goods and can be studied in the framework of multi-sided assignment games, that were introduced by Quint (1991) as the generalization of two-sided assignment games (Shapley and Shubik, 1972).

In the present paper we show that competitive prices of any of such markets are in one-to-one correspondence with core allocations of an associated cooperative game, namely a multi-sided assignment game, where the notion of competitive price extends that of Roth and Sotomayor (1990). In Section 2 we present the necessary notation and background results concerning multi-sided assignment games, whereas in Section 3 the notion of competitive price is introduced and the main result of the paper is proved.

2 Multi-sided assignment games

An *m-sided assignment problem* (m-SAP) denoted by $(N^1, ..., N^m; A)$, is given by $m \geq 2$ different nonempty finite sets (or types) of agents $N^1, ..., N^m$ and a nonnegative m-dimensional matrix $A = (a_E)_{E \in \prod_{j=1}^m N^j}$. With some abuse of notation, let $N^k = \{1, 2, ..., n_k\}$ for all $k = 1, ..., m$. We shall refer to the i-th agent of type k as $i \in N^k$. We name any m-tuple of agents $E \in \prod_{k=1}^m N^k$ an *essential coalition*. Each entry $a_E \geq 0$ represents the profit associated to the essential coalition E. As an abuse of notation, and when no confusion is possible, we shall also use E to denote the set of agents that form the essential coalition.

A *matching* $\mu = \{E^1, ..., E^t\}$ among $N^1, ..., N^m$ is a set of essential coalitions such that $|\mu| = t = \min_{1 \leq j \leq m} |N^j|$ and any agent belongs at most to one essential coalition

$E^1, ..., E^t$. We say that an agent is *unassigned* by μ if she does not belong to E^k for any $1 \leq k \leq t$. We denote by $\mathcal{M}(N^1, ..., N^m)$ the set of all matchings among $N^1, ..., N^m$. Given a m-SAP $(N^1, ..., N^m; A)$, a matching μ^* is optimal if $\sum_{E\in\mu^*} a_E \geq \sum_{E\in\mu} a_E$, for any $\mu \in \mathcal{M}(N^1, ..., N^m)$. We denote by $\mathcal{M}^*_A(N^1, ..., N^m)$ the set of all optimal matchings of $(N^1, ..., N^m; A)$. Since $n_1, ..., n_m$ are finite, at least one optimal matching always exists and thus $\mathcal{M}^*_A(N^1, ..., N^m)$ is always nonempty.

Following Shapley and Shubik (1972) and Quint (1991), for each multi-sided assignment problem $(N^1, ..., N^m; A)$ the *m-sided assignment game* (m-SAG) is the cooperative game[1] (N, ω_A) with set of players $N = \cup_{j=1}^m N^j$ composed of all agents of all types and characteristic function

$$\omega_A(S) = \max_{\mu\in\mathcal{M}(N^1\cap S, ..., N^m\cap S)} \left\{ \sum_{E\in\mu} a_E \right\},$$

for any $S \subseteq N$, where the summation over the empty set is zero. If $m = 2$, the previous setting reduces to the classic Shapley-Shubik assignment game.

The core of a game is the set of allocations that cannot be improved upon by any coalition on its own[2]. It is easy to check that the core $C(\omega_A)$ of a given m-SAG (N, ω_A) coincides with the set of nonnegative vectors $x = ((x_i)_{i\in N^j})_{j=1}^m$ that satisfy $\sum_{i\in E} x_i \geq a_E$ for any $E \in \prod_{k=1}^m N^k$, where the inequality must be tight if E belongs to some optimal matching, and $x_i = 0$ for any agent $i \in N$ that is unassigned under some optimal matching.

3 Competitive prices of multilateral markets

An arbitrary assignment market like the one described in the Introduction shall be denoted by $AM(c^1, ..., c^{m-1}; w)$, where $w \in M_{n_m\times k}(\mathbb{R}_+)$, $k = n_1 \cdot ... \cdot n_{m-1}$ is the matrix of willingness-to-pay of buyers and $c^j \in \mathbb{R}^{n_j}$ is the vector of unitary costs of firms of type j, for all $1 \leq j \leq m-1$.

To any m-sided market $AM(c_1, ..., c_{m-1}; w)$ we associate a m-SAP $(N^1, ..., N^m; A)$, where $N^1, ..., N^{m-1}$ are the sets of firms of different sectors, N^m is the set of buyers and $a_E = \max\left\{0, w^{i_m}_{i_1 ... i_{m-1}} - \sum_{j=1}^{m-1} c_{ji_j}\right\}$, for all $E = (i_1, ..., i_m) \in \prod_{j=1}^m N^j$.

On the one hand, for each $x = ((x_i)_{i\in N^j})_{j=1}^m$, we define a unique vector of prices (one price for each firm) $p^x = ((p^x_i)_{i\in N^j})_{j=1}^{m-1}$ by $p^x_{i_j} = x_{i_j} + c^j_{i_j}$, for all $i_j \in N^j$ and $1 \leq j \leq m-1$. We say that $\{p^x : x \in C(\omega_A)\}$ is the set of *core prices* of $AM(c_1, ..., c_{m-1}; w)$.

On the other hand, we follow Roth and Sotomayor (1990) and define the *demand set* of the i_m-th buyer at prices $p = ((p_i)_{i\in N^j})_{j=1}^{m-1}$ by[3]

$$D_{i_m}(p) = \arg\max_{(i_1,...,i_{m-1})\in\prod_{j=1}^{m-1} N^j} \left\{ \begin{array}{c} w^{i_m}_{i_1...i_{m-1}} - \sum_{i_j\in N^j, 1\leq j\leq m-1} p_{i_j} : \\ w^{i_m}_{i_1...i_{m-1}} - \sum_{i_j\in N^j, 1\leq j\leq m-1} p_{i_j} \geq 0 \end{array} \right\}.$$

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[1] A cooperative game is a pair (N, v), where N is the set of players and v, the *characteristic function*, associates a numerical value $v(S) \in \mathbb{R}$ to any coalition $S \subseteq N$, being $v(\varnothing) = 0$.
[2] Formally, given (N, v), the core is the set $C(v) := \{x \in \mathbb{R}^n : x(N) = v(N) \text{ and } x(S) \geq v(S)$ for all $S \subset N\}$, where $x(S) := \sum_{i\in S} x_i$.
[3] Notice that $D_i(p)$ may be empty.

We say that a price vector p is *quasi-competitive* if $p_{i_j} \geq c^j_{i_j}$, for all $i_j \in N^j$ and $1 \leq j \leq m-1$, and there is a matching $\mu \in \mathcal{M}(N^1, ..., N^m)$ such that, if $D_{i_m}(p) \neq \varnothing$ for some $i_m \in N^m$, there is $(i_1, ..., i_{m-1}, i_m) \in \mu$ satisfying $(i_1, ..., i_{m-1}) \in D_i(p)$. The matching μ is said to be *compatible* with p. The pair (p, μ) is a competitive equilibrium (and the price p is called competitive) if p is quasi-competitive, μ is compatible with p and $p_k = c^j_k$ for all $k \in N^j$, $1 \leq j \leq m-1$, such that k is either unassigned or assigned to a buyer with an empty demand set under μ. We next prove that competitive prices are core prices, and vice versa.

Theorem 1 *Let $AM(c_1, ..., c_{m-1}; w)$ be an arbitrary m-sided assignment market. Then, the set of core prices coincides with the set of competitive prices.*

Proof. Let $(N^1, ..., N^m; A)$ be the m-SAP associated to $AM(c_1, ..., c_{m-1}; w)$. By adding dummy agents, we can assume without loss of generality that there is the same number of buyers and firms of each sector $n_1 = ... = n_m = n$, which implies that no agent is unmatched under any matching. First we prove that if $p = ((p_i)_{i \in N^j})_{j=1}^{m-1} \in \mathbb{R}^{n(m-1)}$ is a competitive price then it is a core price, that is, $p = p^x$ for some $x \in C(\omega_A)$. Indeed, let μ be a compatible matching with p. Then define $x^p := ((x^p_i)_{i \in N^j})_{j=1}^{m} \in \mathbb{R}^{nm}$ by $x^p_{i_j} = p_{i_j} - c^j_i$, for all $i_j \in N^j$ and $1 \leq j \leq m-1$, and $x^p_{i_m} = w^{i_m}_{i_1 \ldots i_{m-1}} - \sum_{i_j \in N^j, 1 \leq j \leq m-1} p_{i_j}$ if $D_{i_m}(p) \neq \varnothing$, where $(i_1, ..., i_{m-1}, i_m) \in \mu$, or $x^p_{i_m} = 0$ otherwise, for all $i_m \in N^m$. Since p is competitive, $x^p \geq 0$. Furthermore, we have that, for any arbitrary $E' = (k_1, ..., k_m) \in \prod_{j=1}^m N^j$,

$$\sum_{j=1}^{m-1} x^p_{k_j} + x^p_{k_m} \geq \sum_{j=1}^{m-1} \left(p_{k_j} - c^j_{k_j}\right) + w^{k_m}_{k_1 \ldots k_{m-1}} - \sum_{k_j \in N^j, 1 \leq j \leq m-1} p_{k_j} = w^{k_m}_{k_1 \ldots k_{m-1}} - \sum_{j=1}^{m-1} c^j_{k_j},$$

which, by nonnegativeness of x^p, implies that

$$\sum_{i_j \in N^j, 1 \leq j \leq m-1} x^p_{k_j} + x^p_{k_m} \geq a_{E'} = \max\left\{0, w^{k_m}_{k_1 \ldots k_{m-1}} - \sum_{j=1}^{m-1} c^j_{i_{kj}}\right\},$$

where the inequality is tight if $E' \in \mu$ and $D_i(p) \neq \varnothing$. Hence, $x^p(N) \geq \sum_{E' \in \mu'} a_{E'}$ for any $\mu' \in \mathcal{M}(N^1, ..., N^m)$ and $x^p(N) \geq \omega_A(N)$, where, as usual, $x(S) := \sum_{i \in S} x_i$. Lastly, given $E = (i_1, ..., i_m) \in \mu$ with $D_{i_m}(p) = \varnothing$, we have that $x^p_{i_m} = 0$ by construction of x^p, and $x^p_{i_j} = p_{i_j} - c^j_{i_j} = 0$, for all $i_j \in N^j$, $1 \leq j \leq m-1$, since p is competitive, and thus $\sum_{i_j \in N^j, 1 \leq j \leq m} x^p_{i_j} = 0$. Hence,

$$x^p(N) = \sum_{\substack{E=(i_1,...,i_m) \in \mu \\ D_{i_m}(p) \neq \varnothing}} \left(x^p_{i_m} + \sum_{i_j \in N^j, 1 \leq j \leq m-1} x^p_{i_j} \right) = \sum_{\substack{E=(i_1,...,i_m) \in \mu \\ D_{i_m}(p) \neq \varnothing}} a_E \leq \omega_A(N).$$

In conclusion, $x^p \in C(\omega_A)$ and p is a core price, since we trivially have $p^{x^p} = p$.

Second, we prove that if $p^x = ((p^x_i)_{i \in N^j})_{j=1}^{m-1} \in \mathbb{R}^{n(m-1)}$ is a core price, i.e. $p^x_{i_j} = c^j_{i_j} + x_{i_j}$ for all $i_j \in N^j$, $1 \leq j \leq m-1$, where $x \in C(\omega_A)$, then p^x is a competitive price. Let μ be an optimal matching. Without loss of generality suppose that there is $s \in \{1, ..., n\}$ such that, for all $E \in \mu$, either $a_E > 0$ if $1 \leq i_m \leq s$ or $a_E = 0$ otherwise, where $E = (i_1, ..., i_m)$. Since $x \in C(\omega_A)$, we have that $p^x_{i_j} - c_{i_j} \geq 0$, for all

$i_j \in N^j$, $1 \leq j \leq m-1$. Next we prove that μ is a compatible matching with p^x. Indeed, for all $E' = (k_1, ..., k_{m-1}, k_m) \in N^1 \times ... \times N^{m-1} \times \{1, ..., s\}$,

$$\begin{aligned}
& w^{k_m}_{k_1 \ldots k_{m-1}} - \sum_{k_j \in N^j, 1 \leq j \leq m-1} p^x_{k_j} \\
&= w^{k_m}_{k_1 \ldots k_{m-1}} - \sum_{k_j \in N^j, 1 \leq j \leq m-1} x_{k_j} - \sum_{j=1}^{m-1} c^j_{k_j} - x_{k_m} + x_{k_m} \\
&\leq w^{k_m}_{k_1 \ldots k_{m-1}} - a_{E'} - \sum_{j=1}^{m-1} c^j_{k_j} + x_{k_m} \\
&= w^{k_m}_{k_1 \ldots k_{m-1}} - \sum_{j=1}^{m-1} c^j_{k_j} - \max \left\{ 0, w^{k_m}_{k_1 \ldots k_{m-1}} - \sum_{j=1}^{m-1} c^j_{k_j} \right\} + x_{k_m} \leq x_{k_m},
\end{aligned}$$

where the first inequality holds since $x \in C(\omega_A)$ and it is tight if $E' \in \mu$, and the last inequality is tight if $w^{k_m}_{k_1 \ldots k_{m-1}} - \sum_{j=1}^{m-1} c^j_{k_j} \geq 0$. Since $1 \leq k_m \leq s$, we have $a_E > 0$, where $E = (i_1, ..., i_{m-1}, k_m) \in \mu$, which implies $w^{k_m}_{i_1 \ldots i_{m-1}} - \sum_{i_j \in N^j, 1 \leq j \leq m-1} p^x_{i_j} = x_{k_m}$ and, therefore, $(i_1, ..., i_{m-1}) \in D_{k_m}(p)$. Finally, if $s+1 \leq k_m \leq n$, we have $\sum_{i_j \in N^j, 1 \leq j \leq m-1} x_{i_j} + x_{k_m} = a_E = 0$, where $E = (i_1, ..., i_{m-1}, k_m) \in \mu$. As a consequence, $p^x_{i_j} = c^j_{i_j}$ for all $i_j \in N^j$, $1 \leq j \leq m-1$ and therefore μ is a compatible matching with p^x. ■

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