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## NCER Working Paper Series

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Working Paper #1

April 2006

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Don Harding\* and Adrian Pagan†

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Key Words: Business cycle; binary variable, Markov chain, probit model, yield curve

JEL Code C22, C53, E32, E37

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# 1 Introduction

Macroeconometric and financial econometric research often feature binary random variables. We will designate such a random variable as  $S_t$ , and assume that it takes the values of unity and zero. Such binary random variables arise in a number of ways, although they differ in their origin. Because of this it is useful to distinguish between binary random variables that are *primary* and those that are *secondary* or *constructed*. In the first set one would include most of those that arise in micro-econometrics. If a time series is involved there will generally be a panel of data on whether an individual makes a particular decision. In these cases the binary variable is often thought of as deriving from an underlying *continuous latent variable* (as in the Probit model). Also in this set would be cases where a continuous random variable - on which there are realizations - depends upon a latent binary random variable. The clearest example of the latter would be Markov Switching (MS) models - Hamilton (1989). In contrast to those cases, this paper is concerned with secondary binary random variables which are constructed from the realizations of a continuous random variable (or variables)  $y_t$ . This case does not seem to have been studied much, a notable exception being Kedem (1980). However, as we will try to illustrate, quite a few interesting econometric issues arise when such variables are used in empirical work.

Are there many examples of constructed binary time series? There seem to be quite a few, among which we can mention the following.

1. Cycles in economic activity. Here a series  $y_t$  is chosen to represent economic activity and a cycle in it involves expansions,  $S_t = 1$ , and contractions,  $S_t = 0$ . In the event that the series  $y_t$  represents the level of economic activity then it is the *business cycle* that is being isolated. If a permanent component is taken away from  $y_t$  we are investigating the *growth cycle*. In the case of the NBER's dating of the business cycle the variable used for  $y_t$  is the equivalent of the log of GDP - see *The NBER's Recession Dating Procedure at <http://www.nber.org/cycles/recessions.html>*.
2. Bull and bear markets. The underlying variable here will be some asset price e.g. the Dow-Jones or the S&P500 and similar sets of rules as in dating business cycles can be used to perform the segmentation of history into periods of bull and bear markets.

3. Financial crises. Here a unity indicates that a crisis is occurring while a zero indicates that this is not a crisis period -see Eichengreen et al (1985) and Kaminsky and Reinhart (1999) and Bordo et al (2001). The latter state (p 55) that “We construct the familiar index of exchange market pressure (calculated as a weighted average of exchange rate change, short-term interest rate change, and reserve change...). A crisis is said to occur when this index exceeds a critical threshold”.
4. IPO markets are often classified as hot ( $S_t = 1$ ) and cold ( $S_t = 0$ ) depending upon the volume of new offers - see Ibbotson et al. (1994).
5. Commodity and real estate markets are often classified as booms and slumps depending upon movements in the underlying prices.

One could continue on in this vein but as the examples above indicate, there are many situations in which binary random variables are constructed from some observed continuous random variable. The prevalence of them raises the issue of why this is such a popular strategy. We might put forward a number of reasons:

1. The  $S_t$  may be chosen to emphasize some feature in  $y_t$  that is not immediately obvious e.g. in U.S. business cycles expansions are not smooth but generally feature a period of very fast growth - see Sichel (1994) and Harding and Pagan (2002). This has also been observed in bull markets- see Pagan and Sussonov (2003).
2. Meaningful to decision makers. Because of the well documented phenomenon of loss aversion it is probably not surprising that decision makers are very sensitive to whether there has been a decline (turning point) in series such as GDP and the S&P. Reactions to such an event from the electorate or clients are often very strong and this has led to great interest in being able to predict these events and to examine their causes. This motivates why one might wish to determine the DGP of the  $S_t$  given a known DGP for  $y_t$ .
3. Often the  $S_t$  are objects of interest. An example would be if one wanted to ask whether cycles are synchronized across sectors or countries. Because business cycle dating agencies like the NBER utilize many series in determining the month that a turning point occurred, it is more

convenient to examine the coherence of the two cycles, as measured by their representative  $S_t$ , then to try to find correlations with the underlying series that they might have been derived from, since the latter may not even be known to an outside observer.

4. Sometimes there may be large short lived movements in  $\Delta y_t$  that can affect statistics based upon  $\Delta y_t$  but which have little effect upon the constructed  $S_t$  e.g. the stock market crash of October 1987 and the decline in output during the Great Depression. In this instance one might wish to obtain a more robust measure of some feature using the  $S_t$  rather than the  $\Delta y_t$ .
5. There are also many situations in which binary variables are used as inputs into a measure of fit. By far the most common examples are those assessing predictive success. Thus Pesaran and Timmerman (1992) have a sign test of predictive accuracy which has been used to compare output gap estimates by Camba-Mendez and Rodriguez-Palenzuela (2003).

In the next section we will discuss ways of constructing the  $S_t$  from the  $y_t$ . We will distinguish two classes of methods for doing this that are referred to as *turning point* and *termination* rules and give some illustrations of these in the contexts distinguished above. The nature of these rules turns out to be very important in the determination of the DGP of  $S_t$ , and the latter always needs to be carefully derived from that of  $y_t$ , since it is unlikely that the DGPs of  $y_t$  and  $S_t$  will be the same. In particular, it is rare for a constructed  $S_t$  to be *i.d.*, as is typically assumed in micro-econometrics. As should be expected the DGP of  $S_t$  will be determined by the interaction of the dating rule and the DGP for  $y_t$ . Section 3 provides some illustrations of this, using a combination of theoretical analysis and an examination of some of the actual  $S_t$  which are used in work connected with business cycles, stock market cycles and financial crises. A failure to make an allowance for the fact that  $S_t$  is not *i.d.* is therefore a potential problem with many existing studies using these variables. Section 4 then uses the principles established in section 3 to look at the econometric issues that arise when the  $S_t$  are used in regression models.

## 2 Constructing the States

The variable  $S_t$  is found by *segmenting* a period of time  $t = 1, \dots, T$  into a history of binary outcomes using information on some underlying continuous random variable. This segmentation requires a *rule* and, depending on what one is studying, this could be classified as either a *turning point* or a *termination* rule. A turning point rule performs the segmentation based upon the location of local maxima and minima in the series  $y_t$ . A termination rule is one which prescribes an event which would cause a change in the value of the state  $S_t$ . In turn termination rules could either be *non-parametric* or derive from a *parametric* model of  $y_t$ .

To give some examples suppose we consider the  $S_t$  that define a cycle. Perhaps the simplest definition is what might be termed the *calculus rule*. This says that a peak in a series occurs at time  $t$  if  $\Delta y_t > 0$  and  $\Delta y_{t+1} < 0$ . The reason for the name is the result in calculus that identifies a maximum with a change in sign of the first derivative from being positive to negative. A trough (or local minimum) can be found using the outcomes  $\Delta y_t < 0$  and  $\Delta y_{t+1} > 0$ . The states  $S_t$  are simply defined in this case as  $S_t = 1(\Delta y_t > 0)$ . This rule has been popular when  $y_t$  is yearly data, see Cashin and McDermott(2002 ) and Neftci (1984).

When data occurs at (say) the quarterly or monthly frequency one needs to recognize that common usage of a word like “recession” would identify it with a *sustained* decline in the *level* of economic activity i.e. something that lasts for several periods. If one applied the calculus rule there would be too many turning points since the growth rate might regularly switch sign between one period and the next. Visualizing a peak in a series leads one to the idea that a local peak in  $y_t$  occurs at time  $t$  if  $y_t$  exceeds values  $y_s$  for  $t-k < s < t$  and  $t+k > s > t$ , where  $k$  delineates some symmetric window in time around  $t$ . One can define a trough in a similar way. By making  $k$  large enough we also capture the idea that the level of activity has declined (or increased) in a sustained way. Of course we need to limit the window in time over which this test is applied when performing the test. It is this simple idea that is the basis of the NBER business cycle dating procedures summarized in the Bry and Boschan (1971) dating algorithm. In that program, designed for the analysis of monthly data,  $k = 5$ . However, because much analysis is conducted with quarterly data, the analogue would seem to be  $k = 2$ . We will refer to this latter rule as the BBQ rule.

The calculus rule can also be formulated as a termination rule by express-

ing it as

$$S_t = 1(\Delta y_t > 0) \quad (1)$$

where  $I(A)$  is the indicator function having value one if the event  $A$  is true. There is no dependence here on  $S_{t-1}$ . A termination rule that does have such dependence is the “two quarters rule” that often appears in the financial press and which can be summarized as

$$\begin{aligned} S_t &= 1 \text{ if } (\Delta y_{t+1} > 0, \Delta y_{t+2} > 0 | S_{t-1} = 0). \\ S_t &= 0 \text{ if } 1(\Delta y_{t+1} < 0, \Delta y_{t+2} < 0 | S_{t-1} = 1) \\ S_t &= S_{t-1} \text{ otherwise} \end{aligned} \quad (2)$$

All of the rules above are non-parametric in the sense that they simply look for patterns in the data without making any assumptions about the DGP of  $y_t$ . Parametric (model-based) termination rules proceed by working with a parametric model of  $\Delta y_t$ . Perhaps the best known of these arises by assuming that  $\Delta y_t$  is a function of a latent binary variable  $\xi_t$  that follows a Markov chain and to then construct a series of binary states using the *MS rule*  $\zeta_t = 1[\Pr(\xi_t = 1 | F_t) - .5]$ , where  $F_t$  is a set containing either the past history of the observed random variable  $\Delta y_t$  or perhaps the complete sample of observations - see Hamilton (1989). Of course one could use other parametric models of  $\Delta y_t$  to produce  $\zeta_t$  e.g. a SETAR model. In all these cases a classification into binary outcomes is produced which is based on whether movements in some function of the  $\Delta y_t$  (and its lags) exceeds a threshold, and the magnitude of the movements involves the parameters of the model.

Each type of rule generates binary random variables but they will not be the same. For this reason we will use the symbol  $S_t$  to designate those that come from either a turning point or non-parametric termination rule and reserve  $\zeta_t$  for those that come from a parametric termination rule. Applied to the same data series  $y_t$  the states  $\zeta_t$  and the states  $S_t$  are conceptually distinct but, in practice, they are often quite close. Thus the  $\zeta_t$  states estimated in Hamilton (1989) with his MS-based termination rule were close to the  $S_t$  coming from using the NBER type rules. Harding and Pagan (2003a) looked at the way in which they differed by using some approximations for getting

the  $\zeta_t$  from the history of  $\Delta y_t$ . From that analysis it was clear that the MS rule used a broader information set than the NBER-type turning point rule (in the sense that the latter uses  $\{\Delta y_{t\pm j}\}_{j\leq 2}$  whereas the former uses  $\{\Delta y_{t\pm j}\}_{j\leq T}$ , with downweighting as  $j$  rises). Notice that the latent states in the MS model  $\xi_t$  are *not the same* as the  $\zeta_t$  and so  $\Pr(\xi_t = 1) \neq \Pr(\zeta_t = 1)$  – failure to recognize this is a common error in many studies that use parametric dating rules.<sup>1</sup> We will focus upon  $S_t$  type measures in this paper but everything said about these holds for the  $\zeta_t$  type measures. The states  $S_t$  are then how one summarizes information on the cycle in  $y_t$  and it is possible to use these to investigate questions such as synchronization of business cycles across countries, regions and sectors.

Pagan and Sussonov (2003) provide a set of rules for locating turning points in the equivalent of the S&P500, while Lunde and Timmermann(2004) use a non-parametric termination rule. There have been quite a few adaptations of this approach e.g. to study booms and slumps in commodity markets - Cashin, McDermott and Scott (2002). There is also a literature which uses parametric termination rules e.g. the MS model in Maheu and McCurdy(2000).

Non-parametric termination rules to construct the indicators of financial crises most often involve a consideration of the size of the movements in a combination of a number of series. Thus Eichengreen et al. (1995) define a crisis as occurring whether a weighted average of changes in exchange rates, reserves and interest rates exceeds some threshold value. Parametric termination rules have also been applied, mainly based on an MS model e.g. Abiad (2003).

Another important feature of constructed states is that extra censoring rules concerning the minimum or maximum time that can be spent in a particular state are often applied. Thus in the case of the business cycle dating by the NBER, recessions and expansions must be five months long and a complete cycle must last for 15 months. In quarterly terms these are best interpreted as requiring two quarters as minimum phase lengths and 5 quarters for a complete cycle. We will illustrate how these impact upon the DGP for  $S_t$  later using the quarterly versions.

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<sup>1</sup>Indeed, it is often asserted that the duration of time spent in the state  $\zeta_t$  (or  $S_t$ ) can be determined from the transition probabilities associated with  $\xi_t$ . It is easy to see that this is incorrect since the former will depend on all the parameters of the MS process, including the mean values of  $\Delta y_t$  in each of the regimes, whereas the transition probabilities for  $\xi_t$  do not depend on the mean values.



### 3 The DGP of The Binary States

As mentioned in the introduction the DGP of constructed binary states is not one that the investigator is free to prescribe. It is determined by the interaction of the DGP of the variable they are constructed from and the type of rule used for mapping the observable variable into the binary states. As we will note the DGP is generally a high-order Markov process. Nevertheless, it is useful for understanding the DGP to think of approximating the higher order process with a first order one. This is not an unfamiliar process. A high order autoregression in a continuous random variable can always be approximated by a first order AR and, if one wished (say) to measure the degree of persistence in the process, this approximation often gives a very good indication of that quantity. We follow this approximation strategy in the next sub-section.

#### 3.1 Serial Correlation in the States

If the states evolve as a first-order Markov process then Hamilton (1994 p684) shows that the following identity holds:

$$S_t = p_{01} + (1 - p_{01} - p_{10})S_{t-1} + \eta_t, \quad (3)$$

where  $\eta_t$  is discrete and conditionally heteroskedastic since it depends upon  $S_{t-1}$  and

$$p_{jk} = \Pr(S_{t+1} = k | S_t = j) \quad (4)$$

The determinants of  $p_{jk}$  will depend upon the nature of the DGP for  $y_t$  and the type of rule employed to construct  $S_t$ . To illustrate this we suppose that

$$\Delta y_t = \mu + \sigma e_t \quad (5)$$

where  $e_t$  is *i.i.d*(0,  $\sigma^2$ ). Now if the calculus rule is employed i.e.  $S_t = 1(\Delta y_t > 0)$ ,

$$\begin{aligned} p_{10} &= \Pr(S_{t+1} = 0 | S_t = 1) \\ &= \Pr(\Delta y_{t+1} < 0 | \Delta y_t > 0) \\ &= \Pr(\Delta y_{t+1} < 0) = \psi \end{aligned}$$

due to independence of  $\Delta y_t$ . In the same way  $p_{01} = 1 - \psi$  and, from (3),

$$S_t = 1 - \psi + (0 \times S_{t-1}) + \eta_t, \quad (6)$$

showing that there is no serial correlation in the states  $S_t$ .

Now, what happens if one relaxes the assumption that  $y_t$  follows a random walk with drift? Using the calculus rule, combined with  $\Delta y_t$  being a mean-zero stationary Gaussian process, Kedem(1980, p34) sets out the relation between the autocorrelations of the  $\Delta y_t$  and  $S(t)$  processes. Letting  $\rho_{\Delta y}(k) = \text{corr}(\Delta y_t, \Delta y_{t-k})$ , and  $\rho_S(k) = \text{corr}(S_t, S_{t-k})$ , he determines that

$$\rho_S(k) = \frac{2}{\pi} \arcsin(\rho_{\Delta y}(k)). \quad (7)$$

Thus, given an estimate of  $\rho_{\Delta y}(k)$ , we can immediately find an estimate of

$\rho_S(k)$  and *vice versa*, making it clear that an AR process for  $\Delta y_t$  will result in a much more complex DGP for  $S_t$  than an  $AR(1)$ .

So the autocovariances in  $S_t$  depend upon whether there is serial correlation in  $\Delta y_t$ . But, even if there is no serial correlation in  $\Delta y_t$ , the dating rule itself can induce it into  $S_t$ . The analysis will be done assuming that  $y_t$  follows (5) and the “two quarters rule” for dating phase shifts. The complications in working with this rule come from the fact that the conditioning event  $S_{t-1} = 1$  will place some restrictions upon the signs of past sample paths for  $\{\Delta y_t\}$  that are associated with an expansion terminating sequence defining the move from  $S_{t-1} = 1$  to  $S_t = 0$ . For example the sequence

$$\{\Delta y_{t+1}, \Delta y_t, \Delta y_{t-1}, \Delta y_{t-2}, \dots\} = \{-, -, -, +, \dots\} \quad (8)$$

would be incompatible with  $S_{t-1} = 1$ , since the negative growth at  $t - 1$  would match with the negative growth at  $t$ , and so the expansion would have been terminated at  $t - 1$ . The appendix shows that

$$p_{10} = \frac{\psi^2}{(1 + \psi)}, p_{01} = \frac{(1 - \psi)^2}{2 - \psi} \quad (9)$$

$$p_{11} = \frac{1 + \psi - \psi^2}{(1 + \psi)}, p_{00} = \frac{1 + \psi - \psi^2}{2 - \psi}. \quad (10)$$

Hence, using (3), we will have

$$S_t = \frac{(1 - \psi)^2}{2 - \psi} + [1 - \frac{(1 - \psi)^2}{2 - \psi} - \frac{\psi^2}{(1 + \psi)}]S_{t-1} + \eta_t \quad (11)$$

To get some feel for the magnitude of the coefficients in this relation assume that  $\Delta y_t$  is  $N(\mu, \sigma^2)$ , so that  $\psi = \Phi(-\frac{\mu}{\sigma})$ , where  $\Phi(u)$  is the cumulative

standard normal distribution function. Using sample estimates of  $\mu$  and  $\sigma^2$  for US GDP over the period 1959/1-1997/2 (the same sample as used in the application by Estrella and Mishkin (1998)) gives  $\psi = .21$ . Inserting this into (11) produces

$$S_t = .35 + .62S_{t-1} + \eta_t, \quad (12)$$

showing that there is substantial serial correlation in the states. Fitting an AR(1) to the quarterly “NBER business cycle states” (found from their web page) over the same period yields

$$S_t = .29 + .67S_{t-1} + \eta_t, \quad (13)$$

which shows that the predictions about the nature of the business cycle states identified by the NBER, and those using the “two quarters rule” are quite good.

These results also continue to hold for the  $S_t$  found from the monthly S&P500 using the turning point definitions in Pagan and Sussonov (2003). The regression over 1854/6-1997/12 gives

$$S_t = .07 + .89S_{t-1} + \eta_t \quad (14)$$

(8.1)    (80.5)

So it is very likely that there will be serial correlation in the states  $S_t$ . This is important since it means that secondary (constructed) states cannot be treated as if they were primary states. In particular, it will not be correct to assume that they are realizations from an *i.i.d.* process, as is done in the micro-econometrics literature and in the derivation of many tests based on these states. Examples of the latter in the time series literature would be the market timing test of Pesaran and Timmermann (1992) and its close relative, Pearson’s test of independence in a contingency table (see Artis et al (1997)). t ratios underlying these tests are effectively constructed under the *i.i.d.* assumption. In Pesaran and Timmermann’s context this may be a valid assumption, since the  $y_t$  are forecast errors and the  $S_t = 1(y_t > 0)$ . But others have applied it to  $y_t$  that are possibly serially correlated e.g. in Camba-Mendez and Rodriguez-Palenzuela (2003) the  $y_t$  are the revision errors in output gaps and there is no reason to think that these would be serially uncorrelated. In all instances an adjustment needs to be made for the serial correlation in the  $S_t$ . As seen in Harding and Pagan(2006), the requisite adjustment to t-statistics of the Artis et al (1997) test of synchronization of cycles can be very large indeed.

### 3.2 Effects of Censoring Rules on the DGP of the States

Rules that involve restrictions on the duration of time spent in a phase, such as the minimum two-quarter restriction for recessions and expansions used by the “two quarters” and BBQ rules, also place strong restrictions on the nature of the DGP for  $S_t$ . To examine this in more detail, let  $\Delta y_t - \mu$  be a mean-zero covariance stationary process. Then, under most dating rules, the  $S_t$  are generated as nonlinear functions of  $\Delta y_t$  and  $\Delta y_{t+1}$ , and thus  $S_t$  is covariance stationary.

Now let us look at the case where the calculus rule is adopted but some censoring of binary variables is employed to impose phase-length restrictions. We have already mentioned that in these circumstances, and without phase restrictions, Kedem has shown that one can represent  $S_t$  as a  $K$ 'th order Markov process, where  $K$  may need to be infinite. A first order Markov process can always be written as

$$S_t = \mu_0 + \phi_1 S_{t-1} + \varepsilon_t \quad (15)$$

while a second order one has the form

$$S_t = \mu_0 + \phi_1 S_{t-1} + \phi_2 S_{t-2} + \psi_1 S_{t-1} S_{t-2} + \varepsilon_t. \quad (16)$$

Higher order processes consist of the linear form plus all the interaction terms. Notice that terms like  $S_{t-1}^k$  are just  $S_{t-1}$  so that we only need to consider interaction terms using  $k = 1$ .

Using Kedem's result, the binary process  $S_t$  can be represented as

$$S_t = \mu_0 + \phi_1 S_{t-1} + \phi_2 S_{t-2} + f(S_{t-1}, S_{t-2}, \tilde{S}_{t-3}) + \varepsilon_t, \quad (17)$$

where  $\tilde{S}_t = \{S_{t-i}\}_{i=0}^{\infty}$  and the notation  $f(\cdot)$  means the sum of all interaction terms of  $S_{t-1}$  and  $S_{t-2}$  with  $\tilde{S}_{t-3}$  e.g.  $S_{t-1} S_{t-2} \tilde{S}_{t-3}$ , as well as elementary and interaction terms formed from the elements of  $\tilde{S}_{t-3}$  alone e.g.  $S_{t-3}, S_{t-3} S_{t-4}$ . Then

$$E(S_t | \tilde{S}_{t-1}) = \mu_0 + \phi_1 S_{t-1} + \phi_2 S_{t-2} + f(S_{t-1}, S_{t-2}, \tilde{S}_{t-3}) \quad (18)$$

Now the restriction that a phase must last two quarters implies that  $\Pr(S_t = 1 | S_{t-1} = 0, S_{t-2} = 1, \tilde{S}_{t-3}) = 0$ . Since the probability equals  $E(S_t | S_{t-1} = 0, S_{t-2} = 1, \tilde{S}_{t-3})$  we see that

$$\mu_0 + \phi_2 + f(0, 1, \tilde{S}_{t-3}) = 0 \quad (19)$$

and this can only occur for arbitrary  $\tilde{S}_{t-3}$  if  $f(0, 1, \tilde{S}_{t-3}) = 0$ . Hence  $S_t$  must be second order Markov with  $\mu_0 = -\phi_2$ . Since it is also the case that  $\Pr(S_t = 0 | S_{t-1} = 1, S_{t-2} = 0, \tilde{S}_{t-3}) = 0$  we find that  $\mu_0 + \phi_1 = 1$ . Thus the presence of censoring has influenced both the order of the Markov Chain that is the DGP of the states and has also imposed restrictions upon the coefficients of the linear (in parameters) representation that the chain can be given. To check this prediction out we fitted a second order Markov Chain to the business cycle states found by applying the BBQ rule ( with minimum phase restriction of two periods imposed) to US quarterly GDP over 1947/1-2002/2. The resulting parameter estimates clearly satisfy the predicted relations between the coefficients (all coefficients were highly significant):

$$S_t = .45 + .55S_{t-1} - .45S_{t-2} + .40S_{t-1}S_{t-2} + \eta_t. \quad (20)$$

If we had dropped the interactive term then the regression would be

$$S_t = .38 + .78S_{t-1} - .22S_{t-2} + \eta_t, \quad (21)$$

with heteroskedastic robust t ratios for the AR parameters of 13.2 and 3.8 respectively.

The same type of effects can be seen in other series, even when there is no explicit censoring, but rather is due to the fact that a minimum duration to phases might occur within the sample. For example fitting a second order Markov process to the data for financial crises in the United Kingdom over 1883 to 1998 from Bordo et al (2001) we get

$$S_t = .07 + .05S_{t-1} + .05S_{t-2} + .49S_{t-1}S_{t-2} \quad (22)$$

The  $S_{t-1}$  and  $S_{t-2}$  terms are not significant but the interaction term is, and after dropping the insignificant terms we get

$$S_t = .08 + .59S_{t-1}S_{t-2}, \quad (23)$$

with the t ratio on the interaction term being 3.54. Thus the type of serial correlation in the states can be quite complex.

### 3.3 Testing the Order of a Markov Chain

If we assume that  $S_t$  is generated by a  $k'$ th order Markov Chain it can be given the linear form

$$S_t = X_t^k \beta_k + \eta_t \quad (24)$$

where the  $X_t^k$  are generated from a recursion as follows

$$\begin{aligned} X_t^0 &= 1 \\ X_t^j &= \begin{bmatrix} X_t^{j-1} & X_t^{j-1} S_{t-j} \end{bmatrix}. \end{aligned}$$

Thus we would have

$$\begin{aligned} X_t^1 &= \begin{bmatrix} 1 & S_{t-1} \end{bmatrix} \\ X_t^2 &= \begin{bmatrix} 1 & S_{t-1} & S_{t-2} & S_{t-1}S_{t-2} \end{bmatrix} \\ X_t^3 &= \begin{bmatrix} 1 & S_{t-1} & S_{t-2} & S_{t-1}S_{t-2} & S_{t-3} & S_{t-1}S_{t-3} & S_{t-2}S_{t-3} & S_{t-1}S_{t-2}S_{t-3} \end{bmatrix} \end{aligned}$$

Now the number of parameters in a Markov chain grows as  $2^k$ . Consequently, high order processes would be hard to estimate unless there are large numbers of observations. It is also the case that, with relatively small sample sizes, the matrix  $X_t^k$  can be singular, since the summation of some of the terms may be identical. Thus, in testing the order of the business cycle states using the NBER states in Estrella and Mishkin's application one cannot estimate a third order Markov chain as  $S_{t-1}S_{t-2}S_{t-3}$  and  $S_{t-1}S_{t-3}$  are perfectly correlated and  $S_{t-1}S_{t-2}$  can be perfectly predicted by  $S_{t-1}S_{t-2}, 1, S_{t-2}$  and  $S_{t-2}S_{t-3}$ .

Now in the event that a minimum phase is imposed upon the states the Markov chain becomes a restricted one in the sense that the parameters in the linear (in variables) relation obey linear restrictions. We have already seen an example of that in (20). The restrictions matrix has the form,

$$R_k \beta = r_k \tag{25}$$

and can be built up via recursion as follows. Starting with the restrictions on a second order system of

$$R_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{26}$$

those on higher order processes are generated via the following recursion:

$$R_j = \begin{bmatrix} R_{j-1} & 0 \\ R_{j-1} & R_{j-1} \end{bmatrix}, \quad r_j = \begin{bmatrix} r_{j-1} \\ r_{j-1} \end{bmatrix} \tag{27}$$

Since there are  $2^k$  parameters in the unrestricted  $k^{th}$  order Markov chain there will be  $2^{k-1}$  parameters in the restricted version and thus the latter

may be possible to estimate when the former is not. By multiplying  $R_j$  and  $r_j$  by the matrix  $T_j$  defined by

$$T_j = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}, \quad (28)$$

we obtain a simpler expression for the restrictions of the form:

$$R_j^* \beta_j = r_j^*, \quad (29)$$

where

$$R_j^* = \begin{bmatrix} R_{j-1} & 0 \\ 0 & R_{j-1} \end{bmatrix}, r_j^* = \begin{bmatrix} r_{j-1} \\ 0 \end{bmatrix}. \quad (30)$$

When estimating the model with restrictions imposed it is useful to re-write the model as

$$\beta_j = C_j \alpha_j + d_j, \quad (31)$$

where  $R_j C_j = 0$  and  $R_j d_j = r_j$ . Then  $C_j$  can be obtained recursively as,

$$C_j = \begin{bmatrix} C_{j-1} & 0 \\ 0 & C_{j-1} \end{bmatrix}. \quad (32)$$

Since  $T_j R_j d_j = T_j r_j$  it follows that

$$\begin{bmatrix} R_{j-1} & 0 \\ 0 & R_{j-1} \end{bmatrix} \begin{bmatrix} d_{j-1} \\ d^* \end{bmatrix} = \begin{bmatrix} r_{j-1} \\ 0 \end{bmatrix} \quad (33)$$

This must imply that  $d^* = 0$  and so  $d_j$  can be generated recursively as

$$d_j = \begin{bmatrix} d_{j-1} \\ 0 \end{bmatrix}. \quad (34)$$

Thus,  $C_2$ ,  $d_2$  and the recursion relations (32) and (34) are all that are required to construct the restriction matrices.

For later use we wish to determine the order of the Markov chain in the business cycle states used by Estrella and Mishkin. We initially started with a fourth order Markov chain as the general model, but found that one could not estimate this model, even with the phase length restrictions imposed. Thus the maximum order of the chain needed to be reduced to three, in which case it has the form

$$\begin{aligned}
S_t = & \beta_1 + \beta_2 S_{t-1} + \beta_3 S_{t-2} + \beta_4 S_{t-1} S_{t-2} + \beta_5 S_{t-3} \\
& + \beta_6 S_{t-1} S_{t-3} + \beta_7 S_{t-2} S_{t-3} + \beta_8 S_{t-1} S_{t-2} S_{t-3}.
\end{aligned} \tag{35}$$

This model was estimated after imposing the two quarter minimum phase restrictions i.e.

$$\begin{aligned}
\beta_1 + \beta_2 &= 1 \\
\beta_1 + \beta_3 &= 0 \\
\beta_1 + \beta_2 + \beta_5 + \beta_6 &= 1 \\
\beta_1 + \beta_3 + \beta_5 + \beta_7 &= 0.
\end{aligned}$$

Imposing these restrictions on the third order Markov process the residual sum of squares was 9.228. We then estimated a restricted second order process and, as it had virtually the same sum of squares, it was easy to accept a second order restricted process. Proceeding to the next order in the sequence it was easy to reject the hypothesis of a first order process since the sum of squares from a regression of  $S_t$  on a constant and  $S_{t-1}$  was 9.99. One question that might arise however is whether the dependence in the process for  $S_t$  might be somewhere between a second and third order Markov chain. We therefore asked if the second order process might be augmented by the variables  $S_{t-3}$  and  $S_{t-2}S_{t-3}$ . The heteroskedasticity adjusted Wald test for this is 6.26, with a  $p$  value of .044. Given the fact that this is only an asymptotic test it seems reasonable to conclude that a second order process is a reasonable approximation for this data set.

## 4 Using the States as Regressors and Regressands

So far we have demonstrated how the DGP of the constructed states needs to be treated quite carefully. In particular, because they are constructed variables, they cannot be treated in the same way as they would be in microeconomic work, where information is directly available (say) on whether a person is unemployed or not. The states have also often been used in regressions, either as regressors or as the dependent variable, and we therefore need to look at the implications of the results established in the previous sections for these uses.



## 4.1 Constructed Binary Variables as Regressors

There is an emerging tendency to utilize  $S_t$  as regressors. In these applications they are included in a regression such as

$$y_t = a + bx_t + cS_t + dx_tS_t + e_t, \quad (36)$$

where the effect of  $x_t$  upon  $y_t$  may change according to the value of  $S_t$ . Thus one might have either  $y_t$  as output and  $x_t$  an interest rate or  $y_t$  might be inflation and  $x_t$  an output gap. Such possibilities are often mentioned. In particular there have been tests for the asymmetric effects of monetary policy e.g. Cover (1992) but in the past these tests have been done through a definition like  $S_t = 1(w_t > 0)$ , where  $w_t$  might be  $e_t$  or  $\Delta y_t$ . Clearly such tests do not effectively address whether the impact of monetary policy is different in different phases of the cycle, since the resulting  $S_t$  do not delineate the business cycle phases.

To illustrate the complications that are caused by using  $S_t$  as a regressor we assume it has been generated with the BBQ rule. Then we know that

$$S_t = 1(\Delta_2 y_t > 0, \Delta y_t > 0, \Delta_2 y_{t+2} < 0, \Delta y_{t+1} < 0). \quad (37)$$

It is therefore clear that we cannot use  $S_t$  as a regressor, since it is a function of  $e_{t+2}$ ,  $e_{t+1}$  and  $e_t$ . It would however be possible to use  $S_{t-3}$  as an instrument for  $S_t$ .

Another type of equation in which these regressors have appeared is in VARs e.g. as in Dueker (2005). Thus a typical equation in the VAR might involve output growth and be of the form

$$\Delta y_t = a + b\Delta y_{t-1} + cS_{t-1} + e_t \quad (38)$$

Using  $S_t$  found from the quarterly turning points provided on the NBER web page and  $\Delta y_t$  as US GDP growth, we find that the estimate of  $c$  is .008 from a regression on (38), with a t ratio of 3.6 . But, if we recognize that  $S_{t-1}$  will be correlated with the error term and use  $S_{t-3}$  as an instrument, the corresponding values are -.012 and -1.0. So one would be misled by the regression if one did not take account of the nature of  $S_t$ . Indeed, if  $S_t$  is to measure a turning point at time  $t$ , it is inevitable that information at points in time past  $t$  will be used in its construction.<sup>2</sup> Difficulties arise if the

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<sup>2</sup>This is one argument why we might want to work with an estimate of the  $\zeta_t$  associated with parametric dating rules, as there we can control the conditioning information.

$S_t$  are just provided with little in the way of a precise account of how they are constructed but, as the example above shows, one needs to treat these variables with great caution.

## 4.2 Constructed Binary Variables as Regressands

Often  $S_t$  are the variables to be explained or predicted and it is desired to test if the regressors have an influence upon  $S_t$ . A simple case of this is testing for synchronization of cycles where  $x_t$  is another cyclical indicator. It is clear that, if one runs a regression of  $S_t$  upon  $x_t$ , and tests the null hypothesis that the coefficient of  $x_t$  is zero in such a regression, then one must take account of the fact that, under the null hypothesis,  $S_t$  has extensive serial correlation and heteroskedasticity and test statistics must be made robust to those features. The necessary adjustments were shown to be large in Harding and Pagan (2006).

Now motivated by the micro-econometric literature it is often felt desirable to test if  $\Pr(S_t = 1)$  is a function of some determinants  $x_t$ . In one approach it is first postulated that  $\Pr(S_t = 1) = F(-x_t'\beta)$ , where  $F(\cdot)$  is a c.d.f. and then a likelihood is established under the assumption that  $S_t$  are *i.i.d.* An alternative is to assume that there is some latent variable process  $y_t^*$  that is a linear function of a single index  $x_t'\beta$  with the format

$$y_t^* = x_t'\beta + u_t^*, \quad (39)$$

where  $u_t^*$  are *i.i.d.* Then a density for  $u_t^*$  is prescribed and  $S_t = 1(y_t^* > 0)$  is taken to be the rule for generating the  $S_t$ . The  $F(\cdot)$  of the first approach is then just the distribution function corresponding to the density function for  $u_t^*$ . Examples of these methodologies are Estrella and Mishkin (1998), Birchenall et al (1999) and Chen et al (2000).

We can see that there are two problems with these approaches. One is that the  $S_t$  are not *i.i.d.* and so one cannot write a likelihood as

$$\prod_{t=1, S_t=1}^T F(-x_t'\beta) \prod_{t=1, S_t=0}^T (1 - F(-x_t'\beta)). \quad (40)$$

Nevertheless, it is this form that has been used in the literature to date. Chen et al and Birchenall et al. are explicit about the fact that  $S_t$  is a constructed variable in time series but then ignore the method of construction when

specifying and estimating the models that seek to explain  $S_t$ . If the  $S_t$  were first order Markov the likelihood could be formed as the product of transition probabilities distinguishing the various state shifts that can occur. In theory one might do this for higher order processes, but it would become increasingly complex. Moreover, as we have emphasized before, it seems likely that, in many cases, we would not know the exact order of the Markov process. An alternative is to work with the latent variable model. If the dating rule is known it may be possible to find the likelihood using computer simulation methods. We have worked through a simple case in the Appendix where we found that the transition probabilities at time  $t$  would depend on the complete past history of  $x_t$ . But, in general it will be very difficult to derive a likelihood for  $S_t$  conditional upon  $x_t$ , simply because we often know little precisely about the DGP for  $y_t$  and the precise dating rules that are used.

Some feasible method of accounting for the nature of  $S_t$  in determining relations with  $x_t$  is needed, even if it is approximate and not fully efficient. We will assume that the states are  $S_t$  and the question to be answered is whether transition probabilities vary with some regressor  $x_t$ . Thus we have

$$\Pr(S_t|\tilde{S}_{t-1}, x_t) = h(\tilde{S}_{t-1}, x_t), \quad (41)$$

where  $\tilde{S}_t$  is some conditioning set whose nature depends upon the order of the Markov chain.

Now we know that, for any finite order Markov chain, if  $g(x_t) = 0$  then we can write

$$\Pr(S_t|\tilde{S}_{t-1}) = E(S_t|\tilde{S}_{t-1}) = \tilde{S}'_{t-1}\delta. \quad (42)$$

Thus, for the second order case,

$$\tilde{S}'_{t-1}\delta = S_{t-1}\delta_1 + S_{t-2}\delta_2 + S_{t-1}S_{t-2}\delta_3. \quad (43)$$

This suggests that we consider a separable version of  $h(\cdot)$ , allowing the transition probability to be written as

$$\Pr(S_t|\tilde{S}_{t-1}, x_t) = \tilde{S}'_{t-1}\delta + g(x_t). \quad (44)$$

This has the regression format

$$S_t = \tilde{S}'_{t-1}\delta + g(x_t) + u_t, \quad (45)$$

where  $E(u_t|\tilde{S}_t, x_t) = 0$ . We then wish to estimate  $g(x_t)$ . To do this we can use the semi-parametric method of Robinson (1988). This proceeds by taking

the expectation of  $S_t$  given  $x_t$ ,

$$E(S_t|x_t) = E(\tilde{S}_{t-1}|x_t)'\delta + g(x_t) \quad (46)$$

and then forming

$$S_t - E(S_t|x_t) = [\tilde{S}_{t-1} - E(\tilde{S}_{t-1}|x_t)]'\delta + u_t. \quad (47)$$

The conditional expectations in (47) can be estimated quite accurately by non-parametric methods as there is only a scalar as the conditioning element. Once these have been found the regression of  $S_t - E(S_t|x_t)$  against  $[\tilde{S}_{t-1} - E(\tilde{S}_{t-1}|x_t)]$  provides an estimate of  $\delta$ . With this estimate  $g(x_t)$  can be extracted by non-parametrically computing  $E[(S_t - \tilde{S}_{t-1}'\hat{\delta})|x_t]$ .

Notice the importance of the separability assumption. If this is incorrect then we would need to estimate  $h(\cdot)$  by non-parametric methods, and then the number of conditioning elements will depend on the dimension of  $\tilde{S}_t$ , which can be quite high. With a low order Markov process this may be a feasible estimation strategy, but not if the order is above second. The motivation for the separability assumption is that it produces a model which nests those in the literature by setting  $\delta = 0$  and making  $g(\cdot)$  the cumulative normal (when a Probit model is adopted). Thus what we are doing here would be simplest possible generalization that retains the same structure as the Probit models in the literature but allows for dependence in the  $S_t$ . We could obviously compare the estimated  $g(\cdot)$  to the cumulative normal.

As an example we consider the influence of the yield spread upon the probability of moving to a recession. Estrella and Mishkin(1998) did this via a Probit model. As the  $S_t$  they used has previously been shown to be second order Markov we therefore fit the model

$$S_t = \delta_1 S_{t-1} + \delta_2 S_{t-1} + \delta_3 S_{t-1} S_{t-2} + g(x_t), \quad (48)$$

where  $x_t$  is the yield spread lagged two quarters. The intercept in this relation is absorbed into  $g(x_t)$  since we do not know the function  $g(\cdot)$ .

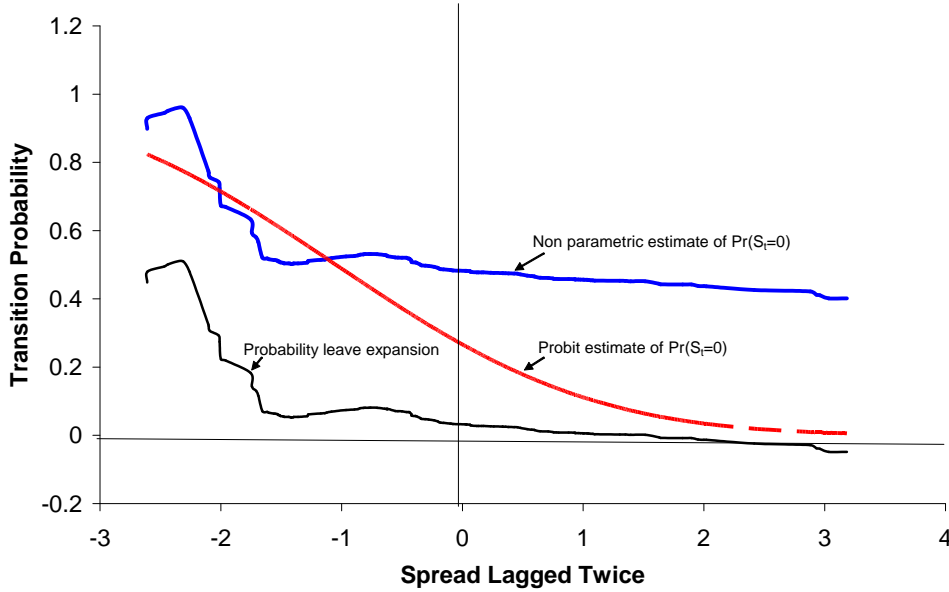
Fitting this model using Robinson's method produces

$$S_t = .5S_{t-1} - .32S_{t-1} + .27S_{t-1}S_{t-2} + g(x_t) \quad (49)$$

Figure 1 plots  $1 - \hat{g}(x_t)$  and the Probit estimate of  $\Pr(S_t = 0)$  against  $x_t$ . It is clear that there is a substantial difference between the Probit estimates and the non-parametric estimates of  $g(\cdot)$  when the yield spread becomes positive,

although for negative spreads of less than  $-1\%$  the correspondence is quite good. It might appear that there is an implication from the non-parametric estimates there is a high probability of being in a recession. This is not correct since the transition probability depends also upon the values of  $S_{t-1}$  and  $S_{t-2}$ . Thus, if  $E(1-S_t)$  is computed from the non-parametric relation it is .14, which agrees with the fraction of the sample spent in recessions. Another perspective is to be had by calculating  $P(S_t = 0|S_{t-1} = 1, S_{t-2} = 1, x_t)$  i.e. the probability of leaving an expansion as the yield spread varies. This is done by subtracting  $.5 - .32 + .27 = .45$  from the estimate  $1 - \hat{g}(x_t)$ . This is also plotted in Figure 1 and it shows that the probability of leaving an expansion is very high ( $> .4$ ) when the yield spread gets smaller than  $-2.2\%$ .

Figure 1: Probit and non parametric estimates of  $Pr(S_t = 0)$  and probability of leaving expansion



## 5 Conclusion

We have made the argument that constructed states  $S_t$  require careful treatment if they are to be used in econometric work since they are very different

in their nature to the binary states often modelled in micro-econometrics. One has to allow for the fact that they are essentially Markov Chains when engaging in a broad range of estimation and inference methods. But, to date, the nature of the  $S_t$  has mostly been ignored, with the potential for quite misleading estimates and inferences. We have suggested some methods to deal with this fact.

## 6 Appendix

The determination of these transition probabilities becomes much more complex with the “two quarters rule” as the conditioning event  $S_{t-1} = 1$  will place some restrictions upon the past sample paths for  $\{\Delta y_t\}$  that are associated with an *ETS*. For example the sequence

$$\{\Delta y_{t+1}, \Delta y_t, \Delta y_{t-1}, \Delta y_{t-2}, \dots\} = \{-, -, -, +, \dots\} \quad (50)$$

would be incompatible with  $S_{t-1} = 1$  since the negative growth at  $t-1$  would match with the negative growth at  $t$  and so the expansion would have been terminated at  $t-1$ . It is clear that the sample paths  $\{\Delta y_{t-1}, \Delta y_{t-2}, \dots\}$  that are compatible with  $S_{t-1} = 1$  and  $\{\Delta y_{t+1} < 0, \Delta y_t < 0\}$  must have the form  $\{+, \dots\}$  and in such paths we must encounter a  $\{+, +\}$  before we encounter a  $\{-, -\}$ . If this did not happen e.g. we had for  $\{\Delta y_{t-1}, \Delta y_{t-2}, \dots\}$  the path  $\{+, -, +, -, -, \dots\}$ , then the recession would have begun at  $t-5$  and would still be running when we reach  $t-5$ .

Now let us consider an enumeration of the paths that are consistent with  $S_{t-1} = 1$ . This is done in the matrix below where the first column represents time and subsequent columns represent paths along which we are assured that  $S_{t-1} = 1$ . The notation used is as follows:

- “+” indicates  $\Delta y_t > 0$ ;
- “-” indicates  $\Delta y_t < 0$ ;
- “\*” before a “-” indicates that any pattern for the observations can occur along the path up to and including that point;
- “\*” following a “+” indicates that any pattern for the observations can occur along the path from that point forward.

Thus looking at the second column the “+, +” at  $t$  and  $t - 1$  assures us that  $S_{t-1} = 1$  along all paths that exhibit this pattern at  $t$  and  $t - 1$ . Similarly, the “-” at  $t$  and the “+, +” at  $t - 1$  and  $t - 2$  assures us that all paths with this pattern are consistent with  $S_{t-1} = 1$ . Similar logic can be applied to all the subsequent paths.

$$\begin{bmatrix} t+1 & * & * & * & * & * & * & \cdots \\ t & + & - & + & - & + & - & \cdots \\ t-1 & + & + & - & + & - & + & \cdots \\ t-2 & * & + & + & - & + & - & \cdots \\ t-3 & & * & + & + & - & + & \cdots \\ t-4 & & & * & + & + & - & \cdots \\ t-5 & & & & * & + & + & \cdots \\ t-6 & & & & & * & + & \cdots \\ \vdots & & & & & & * & \ddots \end{bmatrix} \quad (51)$$

To understand the derivation of these paths suppose we start with the four possible outcomes for  $(\Delta y_t, \Delta y_{t-1})$ , namely  $\{+, +\}, \{-, +\}, \{+, -\}$  and  $\{-, -\}$ . The last would give  $S_{t-1} = 0$  and the first  $S_{t-1} = 1$ ; thus the first becomes the second column of the table. The other two outcomes do not enable us to decide what the state for  $S_{t-1}$  is and so we proceed to observation  $t - 2$  and consider what happens to each of them as we add on a - or a +. Thus  $\{-, +, +\}$  will give  $S_{t-1} = 1$  and that becomes the third column. But  $\{-, +, -\}$  produces no resolution and one needs to proceed to  $t - 3$ . Augmenting  $\{+, -\}$  with a + also fails to resolve the indeterminacy while adding on a - result in  $S_{t-1} = 0$ . Consequently that path has to be continued on to  $t - 3$  as well. The process continues in this way and all columns of the matrix will eventually be enumerated by such a strategy.

To formalize the discussion it is helpful to separate the set of paths that are consistent with  $S_{t-1} = 1$  into two subsets. Let  $E_t$  be the set of paths such that  $\{\Delta y_t > 0 \text{ and } S_{t-1} = 1\}$  and  $F_t$  be the set of paths such that  $\{\Delta y_t < 0 \text{ and } S_{t-1} = 1\}$ . If we introduce the notation that

- $[+-]_t^j$  represents the fragment of the path along which there are  $j$  repetitions of the pattern in the  $[+-]$ . with the leading term in the pattern being located at time  $t$ ,

- $[++]_t$  represents the fragment of path where the pattern "++" occurs with the first "+" being at  $t$  and the second at  $t-1$
- $[-]_t$  represents the case where  $\Delta y_t < 0$ ,

the sets  $E_t$  and  $F_t$  can be enumerated as

$$E_t = \left\{ [++]_t; [-]_t [++]_{t-2}; [-]^2_+ [++]_{t-4}; \dots; [-]^j_+ [++]_{t-2j}; \dots \right\} \quad (52)$$

$$F_t = \left\{ \begin{array}{l} [-]_t [++]_{t-1}; [-]_t [-]_{t-1} [++]_{t-3}; \\ [-]_t [-]^2_{t-1} [++]_{t-5}; \dots; [-]_t [-]^j_{t-1} [++]_{t-2j-1}; \dots \end{array} \right\}. \quad (53)$$

Thus, using the notation that  $\Pr(E_t)$  represents the probability that the path is drawn from the set  $E_t$ , and recognizing that the paths are mutually exclusive, we have<sup>3</sup>

$$\Pr(E_t) = \sum_{j=0}^{\infty} \Pr\left([-]^j_+ [++]_{t-2j}\right) \quad (54)$$

and

$$\Pr(F_t) = \sum_{j=0}^{\infty} \Pr\left([-]_t [-]^j_{t-1} [++]_{t-2j-1}\right). \quad (55)$$

By definition

$$\Pr(S_{t-1} = 1) = \Pr(E_t) + \Pr(F_t). \quad (56)$$

Interest also centres on the joint event  $\Pr\{S_t = 0, S_{t-1} = 1\}$ ; this will involve the set  $G_{t+1}$  defined as

$$G_{t+1} = \left\{ \begin{array}{l} [--]_{t+1} [++]_{t-1}; [--]_{t+1} [-]_{t-1} [++]_{t-3}; [--]_{t+1} [-]^2_{t-1} [++]_{t-5}; \dots \\ \dots; [--]_{t+1} [-]^j_{t-1} [++]_{t-2j-1}; \dots \end{array} \right\} \quad (57)$$

---

<sup>3</sup>To simplify notation we have omitted the conditioning on  $\mathfrak{S}_{t+1}$  in equations (54), (55), (56) and (58).



Then, since  $S_t$  is a stationary process,

$$p_{10} = \frac{\Pr(S_t = 0, S_{t-1} = 1)}{\Pr(S_{t-1} = 1)} = \frac{\Pr(G_{t+1})}{\Pr(E_t) + \Pr(F_t)}. \quad (58)$$

If  $\Pr(S_t = 1, S_{t-1} = 0)$  is constant<sup>4</sup> then  $\Pr(S_t = 1, S_{t-1} = 0) = \Pr(S_t = 0, S_{t-1} = 1)$  (as the number of peaks and troughs must be the same). Using this in conjunction with  $\Pr(S_t = 0) = 1 - \Pr(S_t = 1)$  we can directly derive  $p_{01}$  from the same information as used to construct  $p_{10}$ .<sup>5</sup>

Considering the limits of  $E_t$  etc we get

$$\begin{aligned} \Pr(E) &= \sum_{j=0}^{\infty} (1 - \psi)^2 [\psi(1 - \psi)]^j \\ &= \frac{(1 - \psi)^2}{1 - \psi(1 - \psi)} \end{aligned} \quad (59)$$

$$\begin{aligned} \Pr(F) &= \sum_{j=0}^{\infty} \psi (1 - \psi)^2 [\psi (1 - \psi)]^j \\ &= \frac{\psi(1 - \psi)^2}{1 - \psi(1 - \psi)} \end{aligned} \quad (60)$$

$$\begin{aligned} \Pr(G) &= \sum_{j=0}^{\infty} \psi^2 (1 - \psi)^2 [\psi (1 - \psi)]^j \\ &= \frac{\psi^2 (1 - \psi)^2}{1 - \psi(1 - \psi)} \end{aligned} \quad (61)$$

and so

$$p_{10} = \frac{\psi^2}{(1 + \psi)} \quad (62)$$

$$p_{11} = \frac{1 + \psi - \psi^2}{(1 + \psi)} \quad (63)$$

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<sup>4</sup>Essentially this requires  $\Delta y_t$  to be a random walk with time invariant drift and variance.

<sup>5</sup>If  $\Pr(S_t = 1, S_{t-1} = 0)$  is time varying (as would be the case where  $\mu_t$  depends on some exogenous variable) then one also needs to enumerate the various paths where  $S_{t-1} = 0$ .

$$p_{01} = \frac{(1 - \psi)^2}{2 - \psi} \quad (64)$$

$$p_{00} = \frac{1 + \psi - \psi^2}{2 - \psi} \quad (65)$$

Now in some of the literature we deal with it is assumed that the process for  $\Delta y_t$  depends linearly upon some other variable  $x_t$  in the following way:

$$\Delta y_t = a + bx_t + u_t \quad (66)$$

where the  $x_t$  are taken to be strictly exogenous (and so can be conditioned upon) and  $u_t$  is *n.i.d.*(0, 1). If  $\psi_t = \Phi(-a - bx_t)$ , applying the enumeration method results in

$$\begin{aligned} \Pr(E_{t+1}|\mathfrak{S}_{t+1}) &= \sum_{j=0}^{\infty} \Pr\left([+-]_{t+1}^j [++]_{t+1-2j}\right) \\ &= (1 - \psi_{t+1})(1 - \psi_t) \\ &\quad + \sum_{j=1}^{\infty} \left\{ \left[ \prod_{i=0}^{j-1} (1 - \psi_{t+1-i}) \psi_{t-i} \right] (1 - \psi_{t-2j+1})(1 - \psi_{t-2j}) \right\} \end{aligned} \quad (67)$$

and

$$\begin{aligned} \Pr(F_{t+1}|\mathfrak{S}_{t+1}) &= \sum_{j=0}^{\infty} \Pr\left([-]_{t+1} [+-]_t^j [++]_{t-2j}\right) \\ &= \psi_{t+1}(1 - \psi_t)(1 - \psi_{t-1}) + \\ &\quad \psi_{t+1} \sum_{j=1}^{\infty} \left\{ \left[ \prod_{i=0}^{j-1} (1 - \psi_{t-i}) \psi_{t-i-1} \right] (1 - \psi_{t-2j})(1 - \psi_{t-2j-1}) \right\}. \end{aligned} \quad (68)$$

Letting  $P_t = \Pr(S_t = 1|\mathfrak{S}_{t+1})$  under the two quarters rule gives

$$P_t = \Pr(E_{t+1}|\mathfrak{S}_{t+1}) + \Pr(F_{t+1}|\mathfrak{S}_{t+1}) \quad (69)$$

It is clear from this expression that the use of the two quarters dating rule means that  $P_t$  is a function not only of  $x_t$  but also of  $x_{t+1}$  and the entire past history of  $x_t$ . Moreover it does not have a single index form i.e. does not

depend upon  $\alpha + x_t\beta$  alone. Only if the dating rule had been the “calculus” one would  $\Pr(S_t = 1|\mathfrak{S}_{t+1}) = (1 - \psi_t)$  be a function of  $x_t$  only. Clearly the lesson of this analysis is that one cannot just assume that  $\Pr(S_t = 1)$  is a function of a contemporaneous variable only; it is necessary that one know how the  $S_t$  were generated in order to be able to write down the correct likelihood.

## 7 References

Abiad, A. (2003), “Early Warning Systems: A Survey and a Regime-Switching Approach”, *IMF Working Paper #32*.

Artis M.J. and W. Zhang W (1997), “International Business Cycles and the ERM”. *International Journal of Finance and Economics*, 2,1, January.

Artis M.J. and W. Zhang, (1999), “Further Evidence on International Business Cycle and the ERM: is there a European Business Cycle?”, *Oxford Economic Papers*, 51, 120-32.

Artis, M.J., Z.G., Kontolemis and D.R. Osborn (1997), “Business Cycles for G7 and European Countries”. *Journal of Business*, 70, 249-79.

Birchenall, C.R., H. Jessen, D.R. Osborne and P. Simpson (1999), “Predicting U.S. Business-Cycle Regimes”, *Journal of Business and Economic Statistics*, 17, 313-23.

Bordo, M, B. Eichengreen, D. Klingbiel and M. S. Martinez-Peria (2001), “Financial Crises”, *Economic Policy*, 54-82.

Bry, G., Boschan, C., (1971), *Cyclical Analysis of Time Series: Selected Procedures and Computer Programs*, New York, NBER.

Camba-Mendez, G. and D. Rodriguez-Palenzuela (2003), “Assessment Criteria for Output Gap Estimates”, *Economic Modelling*, 20, 529-62.

Cashin, P.,C.J. McDermott and A. Scott (2002), “Booms and Slumps in World Commodity Prices”, *Journal of Development Economics*, 69, 277-96.

Cashin, P. and C.J. McDermott (2002), "Riding on the Sheep's Back: Examining Australia's Dependence on Wool Exports", *Economic Record*, 78, 249-263.

Chin, D., Geweke, J., and P., Miller, (2000), “Predicting Turning Points”, *Federal Reserve Bank of Minneapolis Research Department Staff Report No. 267*.

Diebold, F.X. and G. D. Rudebusch (1990), “A Non-Parametric Investigation of Duration Dependence in the American Business Cycle”, *Journal of*

*Political Economy*, 98, 596-616.

Dueker, M. (2005), "Dynamic Forecasts of Qualitative Variables: A Qual VAR Model of U.S. Recessions", *Journal of Business and Economic Statistics*, 23, 96-104.

Durland, J.M. and T.H. McCurdy, T.H., (1994), "Duration-Dependent Transitions in a Markov Model of US GNP Growth". *Journal of Business and Economic Statistics*, 12, 279-88.

Eichengreen, B., A.K. Rose and C. Wyplosz (1995), "Exchange Rate Mayhem: The Antecedents and Aftermath of Speculative Attacks", *Economic Policy*, 21, 251-312.

Estrella, A. and F.S. Mishkin (1998), "Predicting US Recessions: Financial Variables as Leading Indicators", *Review of Economics and Statistics*, LXXX, 28-61.

Hamilton, J.D., (1989), "A New Approach to the Economic Analysis of Non-Stationary Times Series and the Business Cycle", *Econometrica*, 57, 357-84.

Harding D., and A.R. Pagan, (2002), "Dissecting the Cycle: A Methodological Investigation", *Journal of Monetary Economics*, 49(2), 365-81.

Harding, D. and A. R. Pagan (2003) "A Comparison of Two Business Cycle Dating Methods" *Journal of Economic Dynamics and Control*, 27, 1681-90.

Harding D., and A.R. Pagan, (2006), "Synchronisation of Cycles", *Journal of Econometrics* (In Press)

Ibbotson, R.G., J.L. Sindelar and J.R. Ritter (1994), "The Market's Problems with the Pricing of Initial Paper", *Journal of Applied Corporate Finance*, 7, 66-74.

Kaminsky, G.I. and C.M. Reinhart (1999), "The Twin Crises: The Causes of Banking and Balance-of- Payments Problems", *American Economic Review*, 89, 473-500.

Kedem, B., (1980), *Binary Time Series*, Marcel Dekker, New York.

Lunde, A. and A.Timmermann (2004), "Duration Dependence in Stock Prices: An Analysis of Bull and Bear Markets", *Journal of Economic and Business Statistics*, 22, 253-73.

Neftci, S.N. (1984), "Are Economic Times Series Asymmetric over the Business Cycle", *Journal of Political Economy*, 92, 307-28.

Newey, W.K. (1984), "A Method of Moments Interpretation of Sequential Estimators", *Economics Letters*, 14, 201-06.

Newey, W.K. and K. West (1994), “Automatic Lag Selection in Covariance Matrix Estimation”, *The Review of Economic Studies*, Vol 61, Issue 4, October.

Ohn, J., L. Taylor and A.R. Pagan (2004), “Testing for Duration Dependence in Economic Cycles”, *The Econometrics Journal*, 7, 528-49.

Pagan, A.R. and K. Sossounov (2003), “A Simple Framework for Analysing Bull and Bear Markets”, *Journal of Applied Econometrics*, 18, 23-46.

Pesaran, M.H. and A. Timmermann (1992), “A Simple Nonparametric Test of Predictive Performance”, *Journal of Business and Economic Statistics*, 10(4), 461-65.

Robinson, P. (1988), “Root-N-Consistent Semiparametric Regression”, *Econometrica*, 56, 931-54.

Sichel, D. E. (1994). “Inventories and the Three Phases of the Business Cycle”, *Journal of Business and Economic Statistics*, 12, 269-77.