Strategic voting and nomination

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Abstract: Using computer simulations based on three separate data generating processes, I estimate the fraction of elections in which sincere voting will be a core equilibrium given each of eight single-winner voting rules. Additionally, I determine how often each voting rule is vulnerable to simple voting strategies such as 'burying' and 'compromising', and how often each voting rule gives an incentive for non-winning candidates to enter or leave races. I find that Hare is least vulnerable to strategic voting in general, whereas Borda, Coombs, approval, and range are most vulnerable. I find that plurality is most vulnerable to compromising and strategic exit (which can both reinforce two-party systems), and that Borda is most vulnerable to strategic entry. I support my key results with analytical proofs.

1. Introduction

For many who seek to improve the political process, alternative voting rules offer the possibility of transformative change; however, there is no consensus on which rule is best. When evaluating these systems, we must consider the extent to which they will encourage strategic behavior. I distinguish between two basic types of election strategy: The first is strategic voting, which means voters reporting preferences that differ from their sincere appraisal of the candidates. The second is strategic nomination, which means non-winning candidates attempting to change the result by entering or exiting races.

Since Gibbard (1973) and Satterthwaite (1975) demonstrated that all reasonable voting rules create incentives for strategic voting in at least some situations,[1] several authors have attempted to assess the degree to which different voting rules are susceptible to manipulation. There is no universally accepted way to measure this vulnerability, but one of the most common approaches has been to estimate the fraction of elections in which manipulation is logically possible, given some assumption about the distribution function that governs voters' preferences over candidates. Some papers are concerned with the probability that an individual voter will be able to change the result to his own benefit by voting insincerely,[2] while others are concerned with the probability that a

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[1] Specifically, if there are more than two candidates for a single office, and a non-dictatorial election method allows voters to rank the candidates in any order, then there must be some profile of voter preferences under which at least one voter can get a preferred result by voting insincerely. This well-known 'Gibbard-Satterthwaite theorem' relies in turn on the even more well-known 'Arrow theorem'—for this, see Arrow (1951, rev. ed. 1963).

[2] For example, Nitzan (1985), Kelley (1993), Smith (1999), and Aleskerov and Kurbanov (1999). Saari (1990) focuses on 'micro manipulations', i.e. strategic incursions by groups of arbitrarily small size.

coalition of voters will be able to change the result to all of its members' mutual benefit by voting insincerely,[3] and still others are concerned with both.[4] Here, I focus on coalitional manipulation.

In this paper, I extend the literature in at least five ways. First, I produce separate results for each of two distinct types of strategic voting – 'compromising' and 'burying' – which have different implications for political behavior, and I show the effect of limiting voters to a 'simple' strategy that combines these. Second, I extend the methodology of the strategic voting literature to the phenomenon of strategic nomination, thus permitting a more holistic understanding of the types of strategic behavior that each voting rule encourages. Third, whereas most papers that give numerical estimates of voting rules' vulnerability to coalitional manipulation are limited to a fixed number of candidates,[5] this paper presents algorithms that can generate estimates for any number of candidates. It is not practical to solve this problem using brute force, so I create a fundamentally distinct algorithm for each voting rule, based on the logical conditions that determine whether manipulation is possible. Fourth, whereas most papers in the literature have based their results on the assumption of a single data generating process,[6] I perform each of my strategic voting analyses three times: once with a spatial model, once with an impartial culture model, and once using survey responses from the American National Election Studies. With the latter, I bring some real preferences of citizens over politicians into a literature that has mostly used relatively stylized models of voter preferences. By performing the same analyses with multiple data generating processes, I'm able to make distinctions between artifacts of particular specifications, and more general patterns. Fifth, I introduce a number of original analytical results concerning burying, compromising, strategic voting given 'almost-symmetrical preferences', core equilibrium existence in voting, and strategic nomination.

I focus on eight relatively well-known single-winner voting rules that I consider to be broadly representative of single-winner rules in general: these are plurality, runoff, Hare, minimax, Borda, Coombs, range voting, and approval voting.

The remainder of this paper is organized as follows: In section two, I define the voting rules and the types of strategic behavior that the paper focuses on, and briefly discuss the strategic incentives

[3] For example, Chamberlin (1985), Lepelley and Mbih (1994), Kim and Roush (1996), and Tideman (2006).
[4] For example, Favardin, Lepelley, and Serais (2002), and Favardin and Lepelley (2006).
[5] Chamberlain (1985) considers only the four candidate case, while Lepelley and Mbih (1994), Favardin, Lepelley, and Serais (2002) and Favardin and Lepelley (2006) consider only the three candidate case.
[6] Nitzan (1985), Kim and Roush (1996), Smith (1999), Saari (1990) and Kelley (1993) all use an 'impartial culture' model, while Lepelley and Mbih (1994), Favardin, Lepelley, and Serais (2002), and Favaradin and Lepelley (2006) use an 'impartial anonymous culture' model. Tideman (2006) uses a data set consisting of 87 elections. Chamberlin (1985) provides an exception to this, as he presents results based on a spatial model, in addition to results based on an impartial culture model.

created by the plurality system, relative to those created by other single-winner systems. In section three, I describe the models and data that I use to generate elections. In sections four and five, I describe how the voting and nomination strategy simulations are constructed, and in sections six and seven, I present the results. In section eight, I present analytical results that complement the simulation results. In section nine, I conclude.

2. Preliminary definitions

Notation: Let C be the number of candidates, and V be the number of voters. Let c, x, and y serve as candidate indexes, and let v serve as a voter index. Let w denote the winning candidate. Let R_{vc} be the ranking that voter v gives to candidate c (such that lower-numbered rankings are better), and let U_{vc} be the utility that voter v gets if candidate c is elected. Let $x \succ y$ indicate that x is ranked ahead of y, or preferred to y, depending on context; likewise, let $x \sim y$ indicate that x is given the same ranking as y, or that a voter is indifferent between x and y. Let τ be a tiebreaking vector that gives a unique fractional score $\tau_c \in (0,1)$ to each candidate, and let E be a vector of candidate eliminations, such that E_c is initially set to zero for each candidate c.

2.1. Voting rule definitions

2.1.1. Plurality: Each voter votes for one candidate, and the candidate with the most votes wins. To facilitate comparison with other methods, plurality can also be thought of as a ranked ballot system that awards one point to the candidate listed at the top of each voter's rankings, and zero points to the rest. Plurality is used as the primary means of electing the national legislature or lower house of 47 countries, including the US, the UK, Canada, and India.[7]

The formal (ranked ballot) definition of plurality is as follows: $f_{vc} = 1\{R_{vc} = 1\}, \forall v, c$, $F_c = \sum_{v=1}^{V} f_{vc} + \tau_c, \forall c$, and $w = \text{argmax}(F)$. Here, f is a V by C matrix that keeps track of individual voters' first choice votes, and F is a length-C vector of the candidates' totals of first choice votes.

2.1.2. Two-round runoff: Each voter chooses one candidate, and the two candidates who receive the most votes compete in a runoff election. This system, or some variation on it, is used to elect the legislatures of 22 countries, including France, Vietnam, Mali, and the Central African Republic.[8]

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[7] Reynolds et al (2005)
[8] ibid

2.1.3. Hare:[9] (Also known as the alternative vote, or instant runoff voting.) Each voter ranks the candidates in order of preference. The candidate with the fewest first choice votes (ballots ranking them ahead of all other candidates in the race) is eliminated. The process repeats until one candidate remains. Hare is used for elections to the lower houses of Australia and Ireland, for mayoral elections in England, and for local elections in fifteen American cities.[10] As of this writing, a referendum is planned for May 5, 2011, to determine whether Hare will replace plurality as the system used to elect the British House of Commons.[11]

Formally, in each round $r = 1, \dots, C-1$, Hare performs the following calculations: $f_{vc} = 1\{[E_c = 0] \wedge [R_{vc} < R_{vx}, \forall x: (E_x = 0 \wedge x \neq c)]\}, \forall v, c.$ $F_c = \sum_{v=1}^{V} f_{vc} + \tau_c + E_c, \forall c.$ $z = \operatorname{argmin}(F)$. $E_z = \infty$. After round $C-1$, $w = \operatorname{argmin}(E)$.

2.1.4. Coombs:[12] This method is the same as Hare, except that instead of eliminating the candidate with the fewest first-choice votes in each round, it eliminates the candidate with the most last-choice votes in each round.

2.1.5. Minimax:[13] Before defining minimax, it is helpful to define a few related concepts. A **pairwise comparison** is an imaginary head-to-head contest between two candidates, in which each voter is assumed to vote for the candidate whom he gives a better ranking to. A **Condorcet winner** is a candidate who wins all of his pairwise comparisons. A **Condorcet method** (or a Condorcet-efficient voting rule), is any single-winner voting rule that always elects the Condorcet winner when one exists. A **majority rule cycle** is a situation in which each of the candidates suffers at least one pairwise defeat, so that there is no Condorcet winner.[14] Minimax is a Condorcet method that uses ranked ballots. Each candidate receives a score equal to the greatest number of voters who oppose him in any pairwise comparison, and the candidate who receives the lowest score is the winner.

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[9] This system is the application to the single-winner case of proportional representation by the single transferable vote, which is often named for Thomas Hare because he was highly influential in its development. See Hoag and Hallett (1926, 162-95).
[10] Center for Voting and Democracy, http://www.fairvote.org/where-instant-runoff-voting-has-been-adopted
[11] "Referendum on voting system goes ahead after Lords vote." *BBC News*, February 17, 2011.
[12] See Coombs (1964).
[13] Black (1958), page 175, develops the minimax method as a possible interpretation of Condorcet's intended proposal. Levin and Nalebuff (1995) label this method as the "Simpson-Kramer min-max rule"; presumably the reference is to Simpson (1969) and Kramer (1977). Nurmi (1999) refers to it as "Condorcet's successive reversal procedure", on page 18. Tideman (2006) refers to it as "maximin", on page 212.
[14] Condorcet (1785) describes the pairwise comparison method and the Condorcet winner. He also observes the possibility of a majority rule cycle emerging despite transitive voter preferences – this is known as the Condorcet paradox.

We can calculate minimax as follows: $p_{xyv} = 1\{R_{vx} < R_{vy}\}, \forall x, y, v$. $P_{xy} = \sum_{v=1}^{V} p_{xyv}, \forall x, y$. $\mathrm{M}_y = \max_{x=1}^{C} P_{xy} - \tau_y$. $w = \mathrm{argmin}(\mathrm{M})$. Here, P is the **pairwise matrix**, which keeps track of the pairwise comparisons; P_{xy} gives the number of voters who rank candidate x ahead of candidate y. A Condorcet winner is a candidate x such that $P_{xy} > P_{yx}, \forall y$. A majority rule cycle is a situation in which $\forall x, \exists y: P_{yx} > P_{xy}$.

2.1.6. Borda count:[15] Each voter ranks the candidates in order of preference. Each first-choice vote is counted as *C* points; each second-choice vote as *C*-1 points, and so on. The winner is the candidate with the most points. Equivalently, each candidate may receive one point for each candidate who is ranked above him on each ballot; the winner in this case is the candidate with the fewest points.

Using the latter definition, we can calculate the Borda winner using the pairwise matrix as follows: $\mathrm{B}_y = \sum_{x=1}^{C} P_{xy} - \tau_y$, and $w = \mathrm{argmin}(\mathrm{B})$.

2.1.7. Approval voting:[16] Each voter chooses whether or not to ‘approve’ each candidate; that is, each voter can give each candidate either one point or zero points. The winner is the candidate with the most points.

2.1.8. Range voting: Each voter may give each candidate any real number of points within a specified range (e.g. 0 to 1 or 0 to 100). The winner is the candidate with the most points.

2.2. Strategy definitions

In the case of ranking-based methods, **strategic voting** means providing a ranking of the candidates that differs from one’s true preference ordering, for example, my voting $\mathrm{A} \succ \mathrm{C} \succ \mathrm{B}$ when my sincere preference ordering is $\mathrm{A} \succ \mathrm{B} \succ \mathrm{C}$. In the case of plurality, it means voting for a candidate other than one’s sincere favorite, and in the case of approval voting or range voting, it means departing from one’s sincere cardinal ratings of the candidates.

Two subsidiary types of strategic voting that will provide important analytical distinctions are the ‘compromising’ and ‘burying’ strategies.[17] The **compromising strategy** entails voters improving the ranking or rating of a candidate, in order to cause that candidate to win. For example, a voter with sincere preferences $\mathrm{A} \succ \mathrm{B} \succ \mathrm{C} \succ \mathrm{D}$ could compromise in favor of B by voting $\mathrm{B} \succ \mathrm{A} \succ$

[15] This method was proposed by Jean-Charles de Borda in 1770; see Mclean and Iain (1995), page 81. Saari (2001) gives a contemporary argument in favor of it.
[16] See Brams and Fishburn (1978) and Brams and Fishburn (1983).
[17] This terminology was used by Blake Cretney, in the currently-defunct web site condorcet.org.

$C \succ D$, or, in plurality, by simply voting for B. The **burying strategy** entails voters worsening the ranking or rating of one candidate, in order to cause another candidate to win. For example, a voter with sincere preferences $A \succ B \succ C \succ D$ could bury C (in order to help A or B) by voting $A \succ B \succ D \succ C$.

When citizens cast their votes in a plurality election for candidates they consider to be the 'lesser of two evils', rather than for their sincere favorites, this is an example of the compromising strategy. For example, suppose that 49% of voters have the preference ordering $A \succ B \succ C$, 24% of voters prefer $B \succ A \succ C$, 24% of voters prefer $B \succ C \succ A$, and 3% of voters prefer $C \succ B \succ A$. (This example may be more intuitive if one imagines that candidate A is George W. Bush, candidate B is Al Gore, and candidate C is Ralph Nader.) If all voters vote for their sincere favorites, A will win with 49% of the vote, but if the $C \succ B \succ A$ voters compromise by voting for candidate B, B will win with 51% of the vote.

To see an example of the burying strategy, suppose that voters have the same preferences as above, but that the election method is Borda or minimax instead of plurality. The sincere winner given either rule will be candidate B, but if the $A \succ B \succ C$ voters all bury B by voting $A \succ C \succ B$, then A will win.

Strategic nomination means non-winning candidates entering or leaving a race in order to change the outcome to one they prefer; I describe these as **strategic entry** and **strategic exit**, respectively. The custom of strategic nomination can be seen in the party primaries that are a regular feature of American democracy. That is, if two or more candidates with similar views run in the same plurality election, then the voters who support those views will be divided among them, giving an advantage to other candidates with opposed views. Therefore, it is helpful for groups of fairly like-minded people to form some kind of association – that is, a political party – which fields only one candidate per election, and which provides some kind of process for deciding whom this one candidate should be – that is, a primary.

In this paper, I find that plurality has more frequent incentives for the compromising strategy, and for strategic exit, than any of the other voting rules that I analyze. Since strategic exit gives third party candidates a disincentive to run, and frequent use of the compromising strategy gives voters a disincentive to support third party candidates who do run, these phenomena together may provide much of the explanation for Duverger's Law,[18] which states that countries using the plurality voting rule will tend to have two dominant political parties. Therefore, switching to one of the alternative systems described here could decrease the extent to which a two party system prevails, and increase the competitiveness of elections.

However, I do not find that plurality is most vulnerable to strategic voting overall; instead, I find that the most vulnerable methods in my study are range voting, Coombs, approval voting, and Borda. Although these methods create less frequent incentives for compromising, they create frequent incentives for burying, whereas plurality is immune to burying (as are two round runoff and Hare).[19] Also, I find that Borda, not plurality, gives the most frequent incentives for strategic entry. Whereas the effects of compromising and strategic exit are relatively well-understood by virtue of the long history of the plurality system, adopting a voting rule that creates frequent burying or strategic entry incentives would bring us into relatively unknown territory.

3. Models and data

3.1. Spatial voting model: The spatial voting model used here distributes both voters and candidates randomly in S-dimensional issue space, according to a multivariate normal distribution without covariance. Voters are assumed to prefer candidates who are closer to them in this issue space. Formally, $L_{vs} \sim \mathcal{N}(0,1)$, $\forall v, s$. $\Lambda_{cs} \sim \mathcal{N}(0,1)$, $\forall c, s$. $U_{vc} = -\sqrt{\sum_{s=1}^{S}(L_{vs} - \Lambda_{cs})^2}$, $\forall v, c$. (The L and Λ matrices give the voter and candidate locations, respectively.)

3.2. Impartial culture model: The impartial culture model used here simply treats each voter's utility over each candidate as an independent draw from a uniform distribution, thus making each ranking equally probable, independent of other voters' rankings. Formally, $U_{vc} \sim \mathcal{U}(0,1), \forall v, c$.

3.3. ANES Time Series Study: I use the June 24, 2010 version of the Time Series Cumulative Data File, published by the American National Election Studies project. In its entirely, this data set includes approximately 50,000 survey respondents, going back to the year 1948, but they don't begin to ask the questions I'm using until 1968, which leaves us with just under 37,000 observations, from the years 1968 to 2008, or approximately 21,000 if we only include presidential election years. I follow Tideman and Plassmann (2011) in using the 'political figure thermometer'

[18] See Duverger (1964).

[19] I demonstrate in propositions 1-3 below that plurality, runoff, and Hare are not vulnerable to burying. Woodall (1997) demonstrates that Condorcet-efficient methods can't satisfy his 'later-no-help' and 'later-no-harm' criteria; a similar proof can be used to show that they must be vulnerable to the burying strategy in some situations.

questions, which ask respondents to rate particular politicians on a scale from 0 to 100. The list of politicians varies from year to year; current presidents and vice-presidents are always included, as are Democratic and Republican presidential and vice-presidential candidates, during presidential election years. In addition to this, there are various other figures who are included in the survey even when they don't hold any of these positions, for example Ted Kennedy (from 1970 to 1988), Ronald Reagan (from 1968 to 1990, and again in 2004), Hillary Clinton, Ross Perot, and so on. Since the survey doesn't determine any actual electoral outcome, there is no obvious incentive for the respondents to report insincere ratings; thus it is not too much of a stretch to treat the thermometer ratings as the voters' sincere cardinal ratings of the candidates, and to use them to derive sincere ordinal preferences.

In some presidential election years, respondents are rating as many as 12 politicians; in others, as few as 7. For a given number of candidates C, I generate $\binom{A_t}{C}$ imaginary elections in each presidential election year, where A_t is the number of politicians rated by survey respondents in year t. (Thus, I explore all possible C-candidate subsets of the rated politicians.) In each of these simulated elections, I treat each of the survey respondents as one voter; although the data set includes some weighting variables, I don't make use of them here. To get the score for each year t, I find the fraction of these $\binom{A_t}{C}$ elections that are vulnerable to strategic manipulation. I then take the average over these yearly scores to get the overall score for the given value of C.

4. Strategic voting simulation design

4.1. How often is sincere voting a core equilibrium? (analysis V1)

My primary approach to strategic voting is to ask how often sincere voting is a core equilibrium. That is, I begin with sincere votes, and ask whether there is a group of voters who can change the winner to one whom they all prefer over the sincere winner, by changing their votes. If this is true for a given voting method, then the method is vulnerable to strategic manipulation in that example; otherwise, sincere voting is a core equilibrium.

As it turns out, it is difficult to test for core equilibria in strategic voting using brute force. That is, for a ranking-based method, there are $C!$ possible rankings of C candidates, and thus $C!^V$ ways in which V voters can rank them. As for approval voting, there are 2^{CV} possible voting profiles, or $(2^C - 1)^V$ if we consider approving all and approving none as being equivalent, and as for plurality, there are C^V functionally unique voting profiles. Thus, even with a fast computer, it can be a daunting task to search over every one of these ranking profiles to determine whether any of them give an advantage to all of the voters whose votes differ from their sincere preferences. Therefore, I've written separate programs to determine whether each of the eight voting rules is vulnerable to manipulation. To give a sense of how these operate, I describe then briefly below.[20]

In these descriptions, let $\psi_v = 1\{U_{vq} > U_{vw}\}$ indicate whether voter v prefers candidate q – the potential winner by strategy – to the sincere winner w, and let $\Psi = \sum_{v=1}^{V} \psi_v$ be the number of potential strategists. Also, let a tilde mark indicate a version of an existing variable that is altered by omitting these potential strategists; for example, $\widetilde{P_{xy}} = \sum_{v=1}^{V}\left(1\{\psi_v = 0\} \cdot p_{xyv}\right), \forall x, y$. Let $\Omega_q = 1$ indicate that manipulation on behalf of candidate q is feasible, and let $\Omega_q = 0$ indicate that it is not.

4.1.1. Plurality: First, I calculate the sincere winner w using the first choice votes vector F, and I find the pairwise matrix P, as described in subsections 2.1.1 and 2.1.5. Then, I loop through possible strategic challengers $q \neq w$ to determine whether q would win if all the voters who prefer q to w voted for q; this is the necessary and sufficient condition for strategic incursion on behalf of q to be possible. Formally, $\Omega_{\mathrm{q}} = 1 \leftrightarrow P_{qw} + \tau_q > F_w$.

4.1.2. Approval voting: In my simulations, I suppose that voters' sincere inclination is to approve candidates who give them greater than average utility. (In the spatial voting model, this means that they approve candidates who are closer to them than average.) Alternative assumptions are possible, but this seems as straightforward as any. $\alpha_{vc} = 1\left\{U_{vc} \geq \frac{1}{C}\sum_{x=1}^{C} U_{vx}\right\}$ indicates whether voter v approves of candidate c, and $\mathrm{A}_c = \sum_{\mathrm{v}=1}^{\mathrm{V}} \alpha_{vc} + \tau_c$ gives the number of voters who approve of c, plus a fractional tiebreaker. Strategic incursion is possible on behalf of a challenger candidate $q \neq w$ if and only if q wins when all of the voters who prefer q to w vote to approve q and no one else.

Formally, for each $q \neq w$, I make the following calculations: $\psi_v = 1\{U_{vq} > U_{vw}\}, \forall v$. $\Psi = \sum_{v=1}^{V} \psi_v$. $\widetilde{\alpha_{vc}} = 1\{\psi_v = 0\} \cdot \alpha_{vc}, \forall v, c$. $\mathrm{A}_c = \sum_{v=1}^{V} \widetilde{\alpha_{vc}} + \tau_c, \forall c$. $\mathrm{A}'_c = \widetilde{\mathrm{A}_c}, \forall c \neq q$. $\mathrm{A}'_q = \widetilde{\mathrm{A}_q} + \Psi$. $\Omega_q = 1 \leftrightarrow q = \mathrm{argmax}(\mathrm{A}')$.

4.1.3. Range voting: I convert voter utilities into sincere ratings on the [0,1] interval, and sum them to find the sincere winner. Formally, $\xi_{vc} = \frac{U_{vc} - \min_{x=1}^{C} U_{vx}}{\max_{x=1}^{C} U_{vx} - \min_{x=1}^{C} U_{vx}}, \forall v, c,\ \Xi_c = \sum_{v=1}^{V} \xi_{vc}, \forall c,\ \acute{\Xi}_c = \Xi_c + \tau_c \cdot 1\{\Xi_c = \max(\Xi)\}$, and $w = \mathrm{argmax}(\acute{\Xi})$. The program to detect manipulability is similar to the approval voting program.

[20] More detailed descriptions, along with the codes themselves, are available from the author on request.

4.1.4. Two round runoff: The sincere winner is determined by the following calculations: $f_{vc} = 1\{U_{vc} = \max_{x=1}^{C} U_{vx}\},\ \forall v, c.\ \ F_c = \sum_{v=1}^{V} f_{vc} + \tau_c, \forall c.\ \ x = \text{argmax}(F).\ \ y = \text{argmax}(F_{-x}).\ \ P_{xy} + \tau_x > P_{yx} + \tau_y \leftrightarrow w = x. P_{yx} + \tau_y > P_{xy} + \tau_x \leftrightarrow w = y.$

To cause $q \neq w$ to win in the runoff system, strategists must cause the runoff to be between q and some other candidate d, whom q can beat. Therefore, within the loop over $q \neq w$, the program loops over $d \neq q$, and determines whether (1) those who prefer q to d or q to w (or both) constitute a majority, enabling q to win the runoff, and (2) the strategists can cause q and d to be the top two finishers in the first round. Strategic incursion is possible if and only if both of these conditions are true.

Formally, given that $\Delta = \sum_{v=1}^{V} 1\{(U_{vq} > U_{vd}) \vee (U_{vq} > U_{vw})\}$, the first condition is true if and only if $\Delta + \tau_q > V - \Delta + \tau_w$. Given that $\widetilde{f_{vc}} = 1\{\psi_v = 0\} \cdot f_{vc}, \forall v, c,\ \ \widetilde{F_c} = \sum_{v=1}^{V} \widetilde{f_{vc}} + \tau_c,\ \ \forall c,$ $\ddot{F_c} = 1\{c \neq q \wedge c \neq d\} \cdot \widetilde{F_c}, \forall c,$ and $\sigma = \max(\ddot{F})$, the second condition is true if and only if $\Psi \geq \max\{0, \lceil \sigma - \widetilde{F_q} \rceil\} + \max\{0, \lceil \sigma - \widetilde{F_d} \rceil\}$.

4.1.5. Hare: The Hare program is somewhat similar to the two round runoff program, but more complex. To determine whether those who prefer a given candidate q can change their votes so that q is elected, I examine each of the $(C-1)!$ elimination orders that result in q's victory, and determine whether the $q \succ w$ voters can cause any of them to occur. To determine whether an elimination order is feasible, I examine each of the rounds from $r = 1, \ldots, C-1$, continuing as long as the strategists can cause the elimination of the candidate who is supposed to be eliminated in round r, according to the given elimination order. In determining this, I need to keep track of votes that strategists must commit to particular candidates in order to ensure a given elimination, and bind them to these votes until the candidates are eliminated.

4.1.6. Coombs: The structure of this program is similar to that of the Hare program, although it is somewhat less complex, because it doesn't need to keep track of strategists' commitments. That is, rather than adding first choice votes to candidates whom they want to survive, strategists are adding last choice votes to candidate whom they want to be eliminated; as long as this elimination is successful (which is necessary to the strategy in any case), there are no restrictions on whom the voter can name as his last choice in the next round.

4.1.7. Minimax: To determine whether minimax is vulnerable to strategic manipulation on behalf of some candidate q, I begin by finding the nonstrategic pairwise matrix $\tilde{P}$, the corresponding minimax scores $\widetilde{\mathrm{M}}$, and the value T, which is defined as q's entry in $\widetilde{\mathrm{M}}$. Formally, $\widetilde{p_{xyv}} = 1\{\psi_v = 0\} \cdot 1\{U_{vx} > U_{vy}\}, \forall x, y, v$, $\widetilde{P_{xy}} = \sum_{v=1}^{V} \widetilde{p_{xyv}}, \forall x, y$, $\widetilde{\mathrm{M}_y} = \max_{x=1}^{C} \widetilde{P_{xy}} - \tau_y, \forall y$, and $\mathrm{T} = \widetilde{\mathrm{M}_q}$.

Because strategic voters can do nothing to reduce T, they must arrange for all of the other candidates to have higher (worse) scores in order to elect q. This means that each of the other candidates needs to have a certain number of votes against him in at least one pairwise contest. As long as this is the case, it doesn't matter what happens in the other pairwise contests, so there are only $C - 1$ 'beats' that we need to focus on.

I proceed by giving separate consideration to each of several possible 'defeat profiles', which, for each candidate other than q, names another candidate who will give him a pairwise beat stronger than T. An exhaustive list of these is given by the Γ array; $x = \Gamma_{\pi y q}$ tells us the candidate x who is supposed to beat candidate y, given profile π, when the strategist candidate is q. Γ will sometimes list q as the candidate doing the beating, but it will not require any candidate to beat q (because this never helps q to win), so $\Gamma_{\pi y q} = 0$ when $y = q$. For example, when $C = 3$, $\Gamma = \begin{bmatrix} 0 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 3 & 1 & 0 \\ 3 & 3 & 0 \end{bmatrix}$, where π is the row dimension, y is the column dimension, and q is the matrix dimension.

Given a strategist candidate q, and given a defeat profile π, I create a 'need' matrix η, such that η_{xy} tells us how many votes the strategists need to add to x's side of the x vs. y pairwise contest, once the non-strategists' votes have already been taken into account. Formally, if $\Gamma_{\pi y q} = x$, then $\eta_{xy} = \max\{0, \lceil \mathrm{T} - \widetilde{P_{xy}} + \tau_y \rceil\}$; otherwise, $\eta_{xy} = 0$.

If voters weren't required to submit transitive rankings (e.g. if someone could cast a vote characterized by $A \succ B$, $B \succ C$, and $C \succ A$), then the number of strategists needed would simply be the largest value in η. However, I do assume that voters must submit transitive rankings, and so I need to do a few more calculations. In short, to complete a 'loop' that is formed in η with K beats, whose entries in η sum to Σ, the number of strategists needed is given by the greatest of these entries, or by $\frac{\Sigma}{\mathrm{K}-1}$, whichever is larger. (For example, $\eta_{1,2} = 5$, $\eta_{2,3} = 6$, and $\eta_{3,1} = 7$ forms a loop with three beats, and the number of strategists needed to ensure the defeat of all three candidates in the loop is $\frac{5+6+7}{3-1} = 9$.)

Θ_c, which is the number of strategists needed to ensure that c can be given a defeat of the necessary magnitude, is determined by this formula if c is in such a loop, and is otherwise simply c's nonzero entry in the η matrix. If the number of strategists is greater than or equal to the maximum of the Θ vector, for any defeat profile π, then the result is manipulable by supporters of q; otherwise, it is not.

4.1.8. Borda: ψ, Ψ, $\tilde{p}$, and $\tilde{P}$ are calculated as above. Then, $\widetilde{\mathrm{B}_y} = \sum_{x=1}^{C} \widetilde{P_{xy}} - \tau_y, \forall y$ gives the Borda scores from non-strategic voters, and $\Phi = \widetilde{B_q}$ gives the minimum Borda score of q, which strategists can't reduce. The strategists' goal is to form their own 'strategic pairwise matrix' $\hat{P}$, such that q is the winner according to the combined pairwise matrix $P' = \hat{P} + \tilde{P}$, which requires that $\sum_{x=1}^{C} \widehat{P_{xy}} + \sum_{x=1}^{C} \widetilde{P_{xy}} - \tau_y > \Phi, \forall y \neq q$.

In short, the method of searching for a successful $\hat{P}$ is as follows. $\hat{P}$ begins as a matrix of zeros, and then is updated so that $\widehat{P_{qy}} = \Psi, \forall q \neq y$. (As $\hat{P}$ is updated, $P' = \hat{P} + \tilde{P}$ is updated accordingly.) If there are any 'covered' candidates $c \neq q$ such that $\sum_{x=1}^{C} P'_{xc} - \tau_c > \Phi$, the strategists 'lift' them, i.e. rank them between q and the remaining candidates. (Thus, $\widehat{P_{cy}} = \Psi$, for all candidates $y \neq q$ who are not yet lifted.) If this causes other candidates to be covered in turn, then they are lifted as well, though they are still ranked behind candidates who were lifted earlier.

If the iteration of this process leads to every candidate being covered, then $\Omega_q = 1$. Otherwise, strategic voters are committed, one at a time, to ranking the remaining uncovered candidates as tied for last choice. (I assume that if voters give equal rankings to two or more candidates, then their votes are cast as the average of all strict rankings that can be formed by resolving expressed indifferences – for example, an $\mathrm{A} \succ \mathrm{B} \sim \mathrm{C}$ vote is treated as one half of a $\mathrm{A} \succ \mathrm{B} \succ \mathrm{C}$ vote, and one half of a $\mathrm{A} \succ \mathrm{C} \succ \mathrm{B}$ vote.) If and when this process causes additional candidates to be covered, then they are lifted as well, by the strategists who haven't yet committed to ranking them as tied for last. This process continues until all candidates other than q are covered, in which case $\Omega_q = 1$, or until the supply of strategists is exhausted, in which case $\Omega_q = 0$.

4.2. How often can simple strategies succeed? (analysis V2)

4.2.1. Compromising and burying together: We have seen that some strategies are highly complex, and require both precise knowledge of other voters' preferences and precise coordination to be successful. Thus, as a complement to the primary analysis, it might be interesting to know how often each method is vulnerable to simpler voting strategies.

This analysis works as follows: For each method, I begin by finding the sincere winner, w. Then, for all other candidates $q \neq w$, I check to see whether q would win if the $q \succ w$ voters were to simultaneously bury w and compromise in favor of q. That is, I suppose that the $q \succ w$ voters give the best possible ranking or rating to q, and the worst possible ranking or rating to w. Certainly there may be other ideas about what a 'simple' strategy might entail, but this is one of the more obvious ones, it has the advantage of being applicable to all of the voting methods we're examining, and as we'll see below, it can succeed in most of the cases in which strategy is possible.

4.2.2. Compromising: For each voting rule, in each example, I first find the sincere winner, w. Then, for all other candidates $q \neq w$, I check to see whether q would win if the $q \succ w$ voters were to change their votes to give q the best possible ranking or rating.

4.2.3. Burying: For each voting rule, in each example, I first find the sincere winner, w. Then, for all other candidates $q \neq w$, I check to see whether q would win if the $q \succ w$ voters were to change their votes to give w the worst possible ranking or rating.

5. Strategic nomination simulation design

In order to provide a relative measure of how frequently different voting methods will have incentives for strategic nomination, I start with the assumption that there are CI candidates who are in the race by default, and CO candidates who are out of the race by default, but who would be prepared to enter it. (Thus, there are $C = CI + CO$ candidates overall.)

The V by C matrix of voter utilities over candidates is generated as before, using the spatial model. In addition to this, I generate a C by C matrix Υ, such that $\Upsilon_{xy} = -\sqrt{\sum_{s=1}^{S}\left(\Lambda_{xs} - \Lambda_{ys}\right)^2}$ (the additive inverse of the Euclidean distance) gives the utility that candidate x experiences if candidate y wins (and vice versa). This definition of Υ implies that all candidates prefer their own election to the election of any other candidate. I focus on the spatial model because it gives us the most natural means of calculating candidates' preferences over other candidates.

5.1. Strategic nomination incentive for individual candidates (analysis N1): Starting from the default set of 'in' candidates and 'out' candidates, I ask whether any individual candidate can get a result that he prefers by either leaving the race or by entering it. If so, I record this as an example of a strategic nomination incentive, except in the case in which a candidate enters the race and wins. I record incentives to quit the race separately from incentives to enter the race.

5.2. Strategic nomination incentive for groups of candidates (analysis N2): Starting again from the default ballot, I ask whether any *groups* of candidates could conspire to simultaneously change their status (either all from in to out, or all from out to in) so that the result changes to one that they all prefer. Again, I don't record it as strategic nomination when one of the status-changing candidates enters the race and wins it. Because 'groups' of one candidate are allowed in N2, N1 vulnerability implies N2 vulnerability.

6. Strategic voting results

6.1. General voting strategy analysis

Tables 1-3 and figures 1-3 give the results of the general voting strategy analysis using the spatial model, the impartial culture model, and the ANES data set. Each data point indicates the share of trials in which a group of voters can change the result to all of its members' mutual benefit by voting insincerely, using a given voting rule, and a given specification. I use 10,000 trials for each (non-ANES) data point, which causes the margin of error to be .0098 or less, with 95% confidence.[21]

Using the spatial model, the ANES data, and the impartial culture model with relatively few voters, there is a clear stratification between Hare and runoff, which are vulnerable to manipulation with low frequency, minimax and plurality, which are vulnerable to manipulation with moderate frequency, and approval, Borda, range, and Coombs, which are vulnerable to manipulation with high frequency. Within these groups, Hare is almost always better than runoff, and minimax is almost always better than plurality.

In the impartial culture model, when the number of voters is large, most of the eight methods are vulnerable approximately 100% of the time, but Hare and runoff are not. The manipulability of runoff is not close to 100% when $C = 3$, but it is close to 100% when $C \geq 4$. The manipulability of Hare is not close to 100% for any $C = 3, 4, 5, 6$. Propositions 10-12 below provide some intuition for these results. Changing the number of voters has much less of an impact in the spatial model, as shown in table 4.

These results are broadly consistent with the existing literature on coalitional manipulation. For example, out of plurality, Borda, Hare, and Coombs, Chamberlin (1985) finds that Borda is most manipulable, and Hare is least manipulable, in both the impartial culture model and in a spatial

[21] A margin of error of ±.0098 is the upper bound, which applies when the true probability is exactly one half. I further reduce the random error in the difference between the scores that the various voting methods receive by using the same set of randomly generated elections for each method.

model. Chamberlin also finds that all of the methods other than Hare are manipulable in the impartial culture model in 100% of trials, given $V = 1000$. Lepelley and Mbih (1994) use an impartial anonymous culture (IAC) model,[22] and find the following ordering of methods from least to most manipulable: Hare, plurality, Coombs. Also using an IAC model, Favardin and Lepelley (2006) find the ordering Hare, runoff, minimax, plurality, Coombs, Borda (in what they call case 3, which is closest to the analysis here). Tideman (2006) uses a data set consisting of 87 elections, and finds the ordering Hare, minimax, runoff, plurality, Borda, range, approval.

Table 1: Analysis V1, spatial model

V	S	C	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
99	1	3	.549	.509	.186	.171	.153	.282	.594	.181
99	2	3	.497	.461	.333	.065	.187	.229	.503	.069
99	4	3	.472	.449	.397	.031	.198	.212	.469	.032
99	8	3	.448	.428	.420	.017	.202	.207	.442	.017
99	16	3	.442	.423	.429	.012	.192	.193	.438	.012
99	1	4	.817	.904	.432	.397	.388	.553	.877	.398
99	2	4	.726	.759	.624	.171	.357	.482	.776	.210
99	4	4	.667	.702	.662	.070	.329	.425	.716	.109
99	8	4	.648	.671	.673	.036	.303	.391	.682	.069
99	16	4	.624	.650	.668	.026	.286	.362	.656	.054
99	1	5	.910	.983	.635	.585	.584	.727	.959	.573
99	2	5	.840	.906	.801	.297	.489	.680	.907	.380
99	4	5	.783	.834	.807	.118	.409	.595	.843	.219
99	8	5	.753	.792	.807	.061	.377	.540	.799	.153
99	16	5	.732	.769	.806	.045	.356	.503	.773	.123
99	1	6	.961	.998	.784	.733	.742	.849	.989	.708
99	2	6	.909	.968	.917	.441	.597	.825	.962	.553
99	4	6	.851	.909	.907	.185	.486	.736	.917	.358
99	8	6	.815	.858	.894	.081	.424	.647	.865	.245
99	16	6	.814	.845	.890	.054	.400	.603	.846	.193

[22] Despite the similar name, this is not equivalent to the impartial culture model. Rather, the IAC model supposes that every possible division of the voters among the $C!$ possible preference orderings is equally likely.

Table 2: Analysis V1, impartial culture

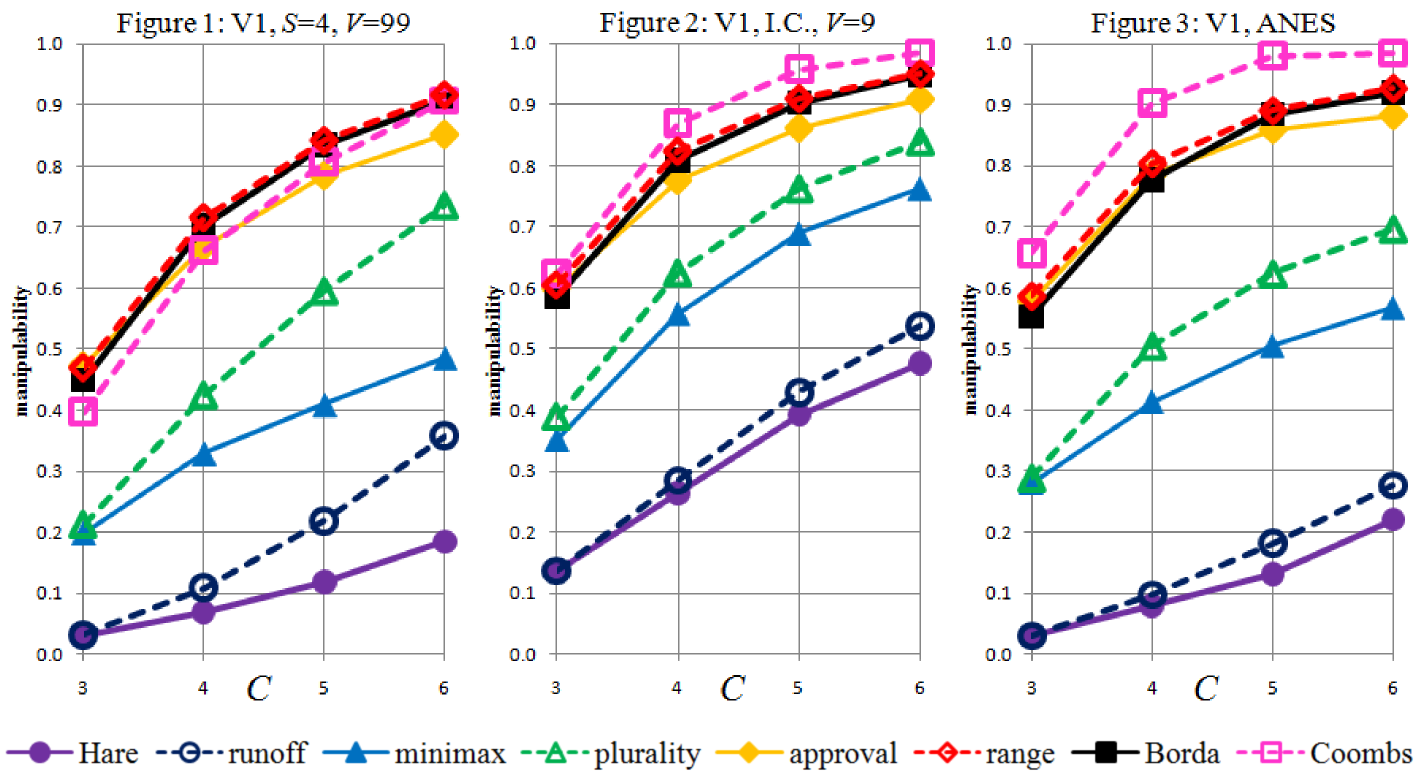
V	*C*	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
9	3	.599	.586	.623	.136	.352	.389	.606	.136
29	3	.837	.836	.918	.147	.676	.694	.843	.147
99	3	.986	.990	.998	.160	.951	.951	.990	.160
999	3	1.000	1.000	1.000	.166	1.000	1.000	1.000	.166
9	4	.776	.809	.869	.264	.559	.624	.825	.284
29	4	.948	.975	.998	.292	.848	.922	.976	.433
99	4	.999	1.000	1.000	.330	.987	.999	1.000	.667
999	4	1.000	1.000	1.000	.341	1.000	1.000	1.000	.999
9	5	.862	.903	.957	.392	.690	.763	.910	.429
29	5	.978	.995	1.000	.427	.909	.976	.994	.655
99	5	1.000	1.000	1.000	.470	.995	1.000	1.000	.924
999	5	1.000	1.000	1.000	.489	1.000	1.000	1.000	1.000
9	6	.909	.949	.986	.476	.764	.840	.951	.538
29	6	.989	.998	1.000	.541	.945	.992	.998	.803
99	6	1.000	1.000	1.000	.585	.998	1.000	1.000	.984
999	6	1.000	1.000	1.000	.607	1.000	1.000	1.000	1.000

Table 3: Analysis V1, ANES

C	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
3	.578	.553	.657	.031	.280	.289	.587	.030
4	.786	.778	.903	.079	.413	.505	.805	.097
5	.859	.885	.980	.132	.506	.623	.891	.182
6	.882	.921	.985	.221	.568	.696	.928	.276

Table 4: Analysis V1, spatial model, impact of *V*

S	*C*	*V*	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
4	3	9	.431	.369	.278	.052	.135	.160	.417	.054
4	3	39	.446	.408	.335	.041	.163	.187	.439	.042
4	3	159	.452	.431	.371	.039	.186	.203	.450	.041
4	3	639	.454	.433	.393	.030	.196	.209	.450	.031
4	3	2559	.468	.447	.405	.029	.197	.208	.467	.030
4	4	9	.465	.443	.402	.028	.201	.214	.461	.028
4	4	39	.464	.442	.410	.027	.204	.216	.457	.027
4	4	159	.476	.451	.412	.028	.213	.226	.472	.029
4	4	639	.472	.452	.413	.029	.209	.220	.470	.029
4	4	2559	.634	.632	.522	.118	.264	.322	.652	.131



6.2. Simple strategies results

6.2.1. Compromising and burying together: Tables 5-7 and figures 4-6 show the fraction of cases in which groups strategic voters can elect a mutually preferred candidate by simultaneously burying the sincere winner and compromising in favor of their preferred candidate. Table 8 compares the frequency of strategic opportunities using this simple strategy to the overall frequency of strategic opportunity as determined in the general voting strategy analysis. The last column of the table shows that taking all of the voting rules together, approximately 94% of the strategically vulnerable cases in the spatial model, 94% of the vulnerable cases in the impartial model, and 97% of the vulnerable ANES elections are also vulnerable to this simple strategy. This tells us that the simple combination of compromising and burying tends to be quite effective, and it tells us that most examples of strategic vulnerability do not require voters to orchestrate very complex manipulation schemes to be successful. Thus, looking at figures 4-6, we see that they very closely resemble figures 1-3.

Table 5: Analysis V2, compromising and burying, spatial model

V	*S*	*C*	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
99	1	3	.549	.395	.186	.140	.153	.282	.594	.140
99	2	3	.496	.422	.333	.047	.187	.229	.503	.047
99	4	3	.471	.435	.397	.020	.198	.212	.469	.020
99	8	3	.447	.422	.420	.011	.202	.207	.442	.011
99	16	3	.442	.421	.429	.007	.192	.193	.438	.007
99	1	4	.811	.595	.388	.328	.296	.553	.871	.296
99	2	4	.717	.659	.601	.114	.331	.482	.773	.129
99	4	4	.664	.672	.655	.039	.322	.425	.715	.050
99	8	4	.648	.661	.671	.021	.300	.391	.681	.024
99	16	4	.623	.647	.667	.015	.284	.362	.656	.018
99	1	5	.901	.722	.549	.485	.379	.727	.956	.405
99	2	5	.829	.797	.753	.198	.422	.680	.904	.231
99	4	5	.777	.799	.790	.068	.388	.595	.842	.093
99	8	5	.751	.781	.800	.032	.369	.540	.798	.044
99	16	5	.731	.764	.802	.024	.352	.503	.773	.033
99	1	6	.957	.868	.684	.623	.452	.850	.987	.513
99	2	6	.896	.882	.856	.296	.506	.825	.962	.337
99	4	6	.838	.876	.874	.095	.445	.721	.910	.140
99	8	6	.813	.856	.884	.045	.410	.647	.868	.065
99	16	6	.811	.845	.883	.026	.392	.605	.849	.041
99	4	10	.953	.983	.980	.210	.588	.959	.991	.338
99	4	20	.996	1.000	.999	.504	.712	1.000	1.000	.679
99	4	30	1.000	1.000	1.000	.715	.759	1.000	1.000	.822

Table 6: Analysis V2, compromising and burying, impartial culture model

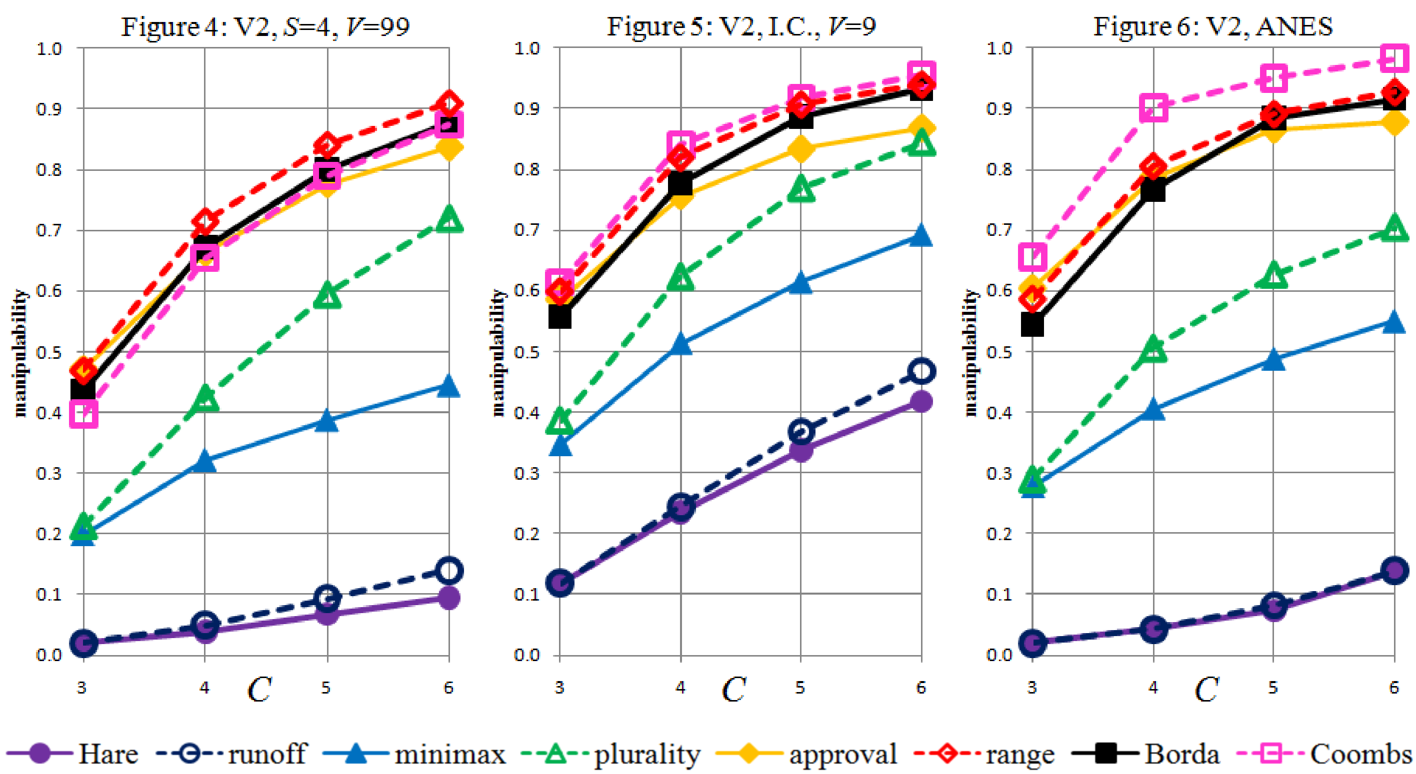
C	*V*	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
3	9	.585	.557	.614	.118	.347	.387	.599	.118
3	29	.842	.825	.923	.125	.681	.695	.850	.125
3	99	.986	.988	.998	.124	.955	.955	.989	.124
3	999	1.000	1.000	1.000	.123	1.000	1.000	1.000	.123
4	9	.756	.778	.840	.234	.513	.623	.820	.245
4	29	.946	.974	.997	.231	.828	.917	.975	.265
4	99	.999	1.000	1.000	.238	.986	.999	1.000	.266
4	999	1.000	1.000	1.000	.231	1.000	1.000	1.000	.265
5	9	.833	.886	.918	.338	.614	.769	.907	.368
5	29	.977	.993	.999	.332	.889	.977	.994	.388
5	99	1.000	1.000	1.000	.316	.994	1.000	1.000	.376
5	999	1.000	1.000	1.000	.326	1.000	1.000	1.000	.390
6	9	.868	.933	.955	.418	.692	.844	.941	.468
6	29	.987	.998	1.000	.405	.917	.993	.998	.481
6	99	1.000	1.000	1.000	.408	.998	1.000	1.000	.482
6	999	1.000	1.000	1.000	.403	1.000	1.000	1.000	.491

Table 7: Analysis V2, compromising and burying, ANES

C	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
3	.605	.545	.656	.020	.277	.291	.587	.020
4	.787	.766	.901	.044	.406	.506	.806	.043
5	.866	.884	.950	.073	.488	.627	.892	.082
6	.878	.916	.982	.138	.550	.704	.928	.139

Table 8: Simple strategic opportunities, as share of all opportunities

	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff	total
spatial	99%	92%	97%	64%	92%	100%	100%	52%	94%
IC	99%	99%	99%	77%	98%	100%	100%	63%	94%
ANES	100%	99%	99%	60%	98%	100%	100%	52%	97%
average	100%	97%	98%	67%	96%	100%	100%	55%	95%



6.2.2. Compromising strategy results: Tables 9-11 and figures 7-9 show the voting rules' vulnerability to the compromising strategy, given various specifications. As shown in proposition 4, Coombs is immune to the compromising strategy. Minimax is next-least vulnerable to the compromising strategy, as it is immune to compromising when there is a sincere Condorcet winner. (This is demonstrated in proposition 5.) Plurality is the most vulnerable to compromising in all specifications (propositions 7 and 8 show that it is dominated by both Hare and runoff in this respect), and approval, range, and Borda are consistently more vulnerable than Hare and runoff.

Table 9: Compromising strategy, spatial model

V	*S*	*C*	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
99	4	3	.118	.118	.000	.018	.003	.202	.118	.018
99	4	4	.148	.219	.000	.044	.006	.422	.212	.051
99	4	5	.152	.308	.000	.066	.008	.594	.298	.093
99	4	6	.145	.360	.000	.092	.009	.724	.365	.136

Table 10: Compromising strategy, impartial culture model

V	*C*	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
29	3	.451	.493	.000	.118	.084	.700	.481	.118
29	4	.567	.728	.000	.230	.170	.925	.714	.262
29	5	.627	.832	.000	.335	.245	.981	.807	.398
29	6	.657	.880	.000	.407	.304	.992	.869	.480

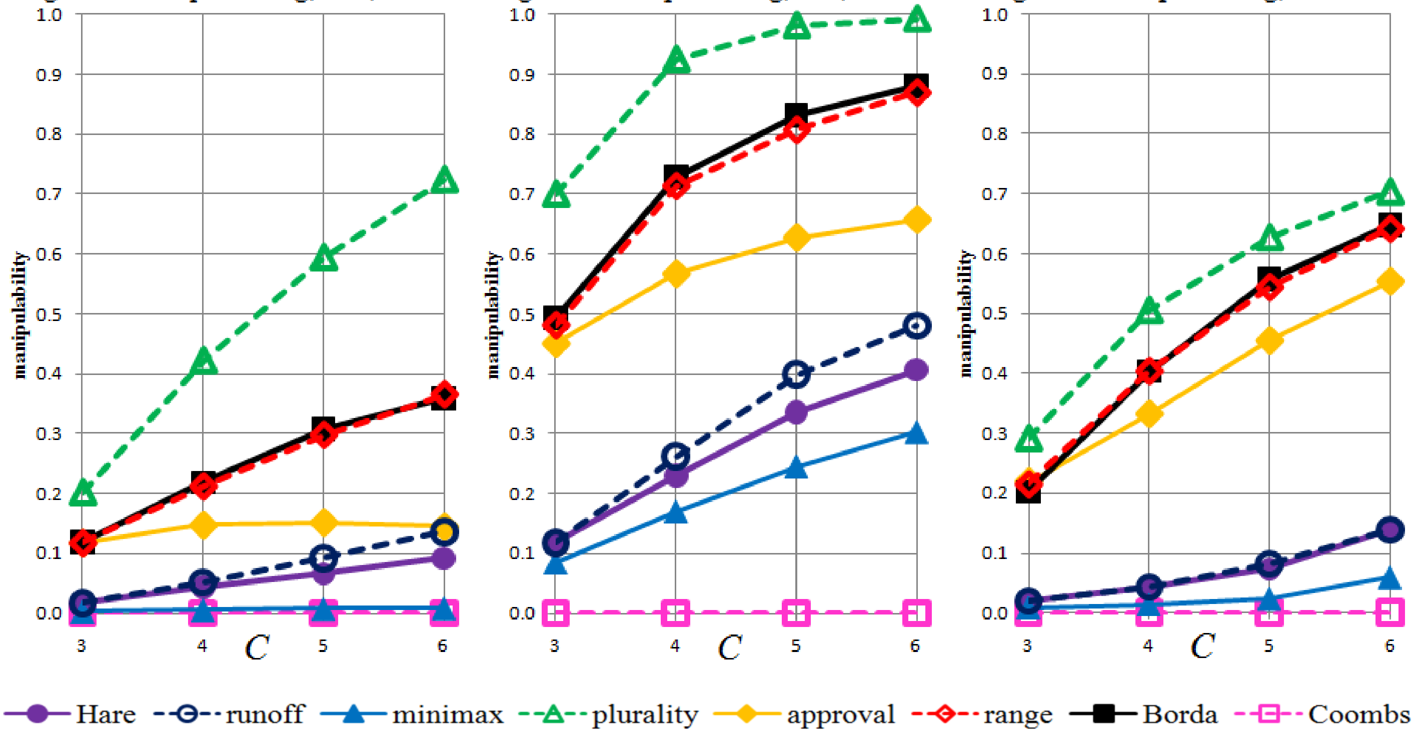
Table 11: Compromising strategy, ANES

C	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
3	.219	.202	.000	.020	.008	.291	.216	.020
4	.332	.403	.000	.044	.013	.506	.405	.043
5	.455	.557	.000	.073	.023	.627	.545	.082
6	.554	.648	.000	.138	.059	.704	.642	.139

Figure 7: Compromising, S=4, V=99

Figure 8: Compromising, I.C., V=29

Figure 9: Compromising, ANES



6.2.3. Burying strategy results: Tables 12-14 and figures 10-12 show the voting rules' vulnerability to the burying strategy, given various specifications. As demonstrated in propositions 1-3, plurality, runoff, and Hare are immune to the burying strategy. Coombs, range voting, and approval voting are consistently the most vulnerable to burying, while Borda and minimax form an intermediate category.

Table 12: Burying strategy, spatial model

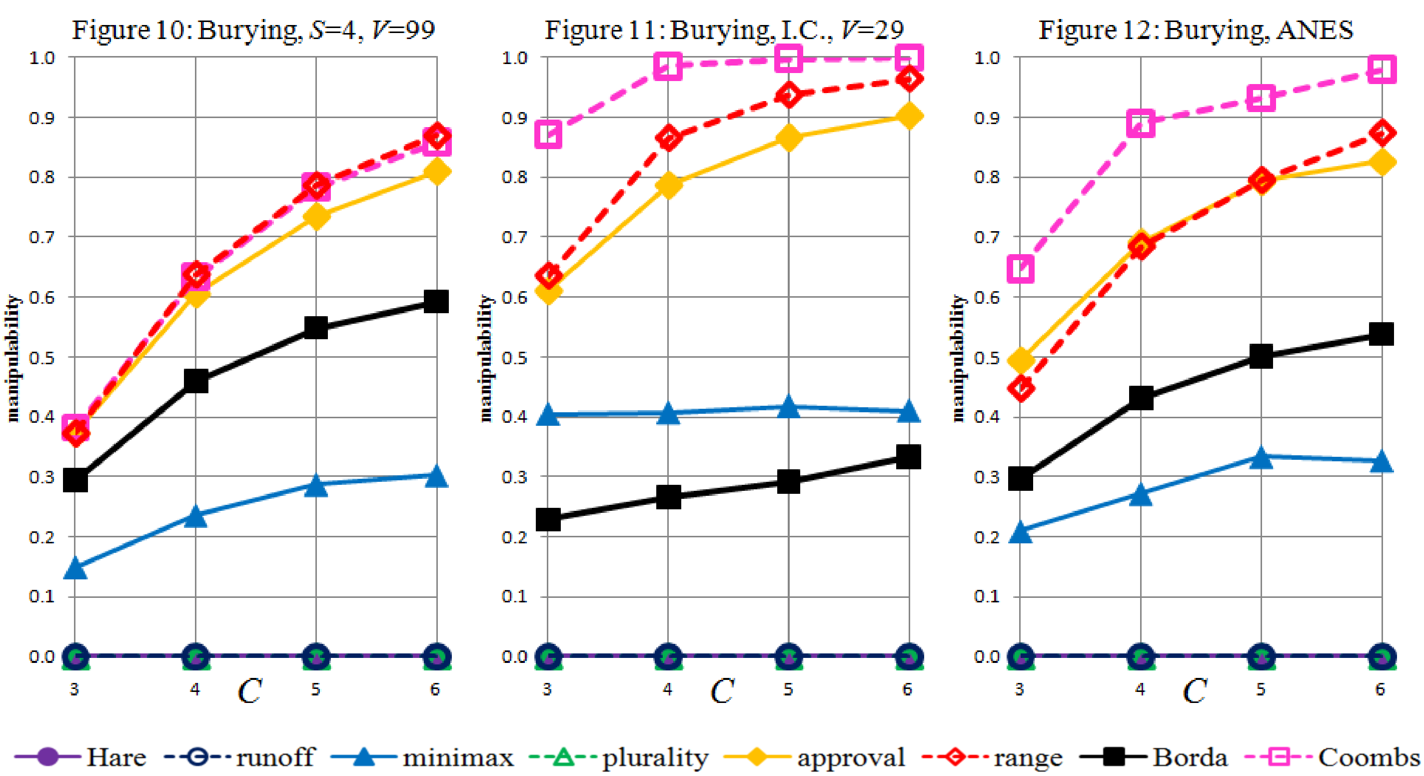
V	S	C	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
99	4	3	.374	.294	.382	.000	.148	.000	.374	.000
99	4	4	.605	.459	.634	.000	.235	.000	.639	.000
99	4	5	.735	.549	.784	.000	.288	.000	.787	.000
99	4	6	.810	.593	.859	.000	.303	.000	.869	.000

Table 13: Burying strategy, impartial culture model

V	C	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
29	3	.610	.229	.871	.000	.404	.000	.635	.000
29	4	.786	.267	.986	.000	.407	.000	.865	.000
29	5	.865	.293	.998	.000	.418	.000	.937	.000
29	6	.902	.333	.999	.000	.410	.000	.964	.000

Table 14: Burying strategy, ANES

C	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
3	.495	.297	.647	.000	.209	.000	.448	.000
4	.692	.431	.890	.000	.272	.000	.684	.000
5	.794	.501	.932	.000	.334	.000	.795	.000
6	.827	.538	.979	.000	.328	.000	.873	.000



7. Strategic nomination results

Tables 15-18 and figures 13-16 show the voting rules' vulnerability to strategic exit and entry, first by single candidates, and then by coordinated groups of candidates. I find that plurality is most frequently vulnerable to strategic exit (proposition 20 gives some intuition for this result), although with large numbers of candidates in the race, runoff and Hare are vulnerable with similar frequency. I find that Borda is most vulnerable to strategic entry, and that Coombs is second-most vulnerable. (Propositions 21 and 22 give some intuition for the vulnerability of Borda and Coombs to strategic entry.) Minimax is vulnerable to both exit and entry with very low frequency, because its vulnerability depends on the existence of a majority rule cycle, as demonstrated in propositions 18 and 19.

Table 15: Strategic exit, single candidates

S	*V*	*CO*	*CI*	Borda	Coombs	Hare	minimax	plurality	runoff
4	99	0	3	.006	.001	.015	.001	.091	.015
4	99	0	5	.013	.002	.060	.004	.251	.093
4	99	0	7	.017	.005	.104	.006	.356	.175
4	99	0	9	.021	.007	.151	.008	.434	.267
4	99	0	11	.018	.011	.193	.010	.490	.344
4	99	0	13	.022	.013	.245	.012	.526	.402
4	99	0	15	.026	.016	.298	.013	.546	.448
4	99	0	19	.027	.021	.389	.015	.588	.532
4	99	0	23	.026	.023	.468	.017	.605	.572
4	99	0	27	.022	.031	.533	.018	.627	.597
4	99	0	31	.027	.035	.587	.018	.641	.624
4	99	0	35	.026	.040	.640	.022	.654	.649

Table 16: Strategic entry, single candidates

S	*V*	*CI*	*CO*	Borda	Coombs	Hare	minimax	plurality	runoff
4	99	2	1	.015	.004	.000	.001	.001	.000
4	99	2	2	.029	.006	.000	.001	.004	.000
4	99	2	3	.038	.010	.001	.002	.003	.001
4	99	2	5	.059	.014	.001	.002	.008	.001
4	99	2	7	.065	.018	.002	.002	.009	.002
4	99	2	9	.076	.025	.002	.003	.009	.002
4	99	2	11	.094	.031	.002	.005	.013	.002
4	99	2	13	.103	.036	.002	.005	.016	.002
4	99	2	15	.101	.037	.002	.005	.018	.002
4	99	2	19	.113	.043	.004	.006	.020	.004
4	99	2	23	.118	.050	.004	.008	.020	.004
4	99	2	27	.131	.058	.006	.009	.027	.006
4	99	2	31	.135	.065	.005	.009	.029	.005
4	99	2	35	.139	.071	.004	.010	.030	.004

Figure 13: N1, exit, V=99, S=4

Figure 14: N1, entry, V=99, S=4, CI=2

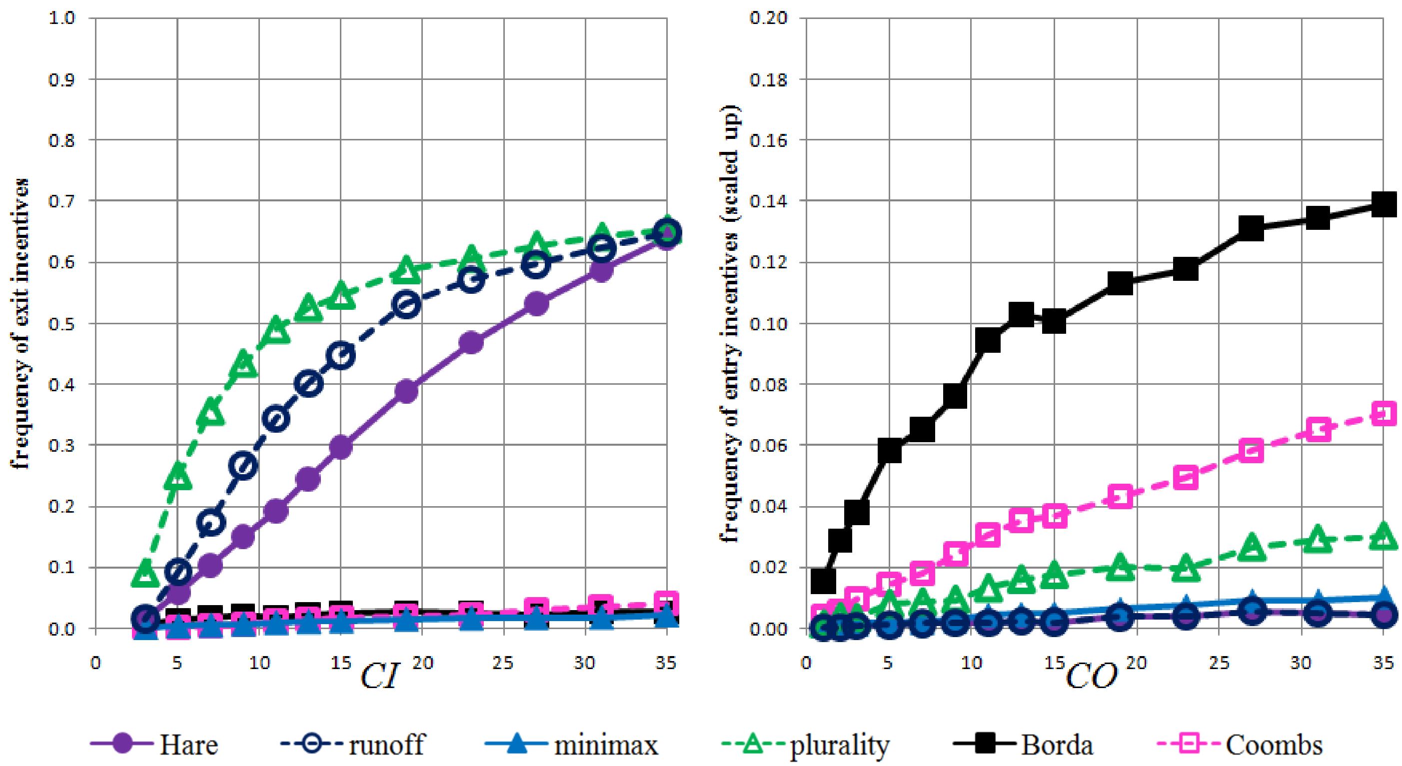


Table 17: Candidate groups, strategic exit

V	S	CI	CO	Borda	Coombs	Hare	minimax	plurality	runoff
29	4	3	0	.010	.002	.023	.004	.098	.023
29	4	5	0	.033	.009	.084	.011	.331	.119
29	4	7	0	.045	.017	.156	.018	.545	.255
29	4	9	0	.053	.023	.228	.024	.684	.380
29	4	11	0	.063	.030	.305	.031	.790	.506

Table 18: Candidate groups, strategic entry

V	S	CI	CO	Borda	Coombs	Hare	minimax	plurality	runoff
29	4	2	1	.042	.013	.002	.004	.010	.003
29	4	2	3	.077	.022	.004	.006	.021	.005
29	4	2	5	.087	.030	.004	.007	.028	.006
29	4	2	7	.108	.038	.007	.011	.035	.009
29	4	2	9	.124	.042	.007	.014	.043	.013

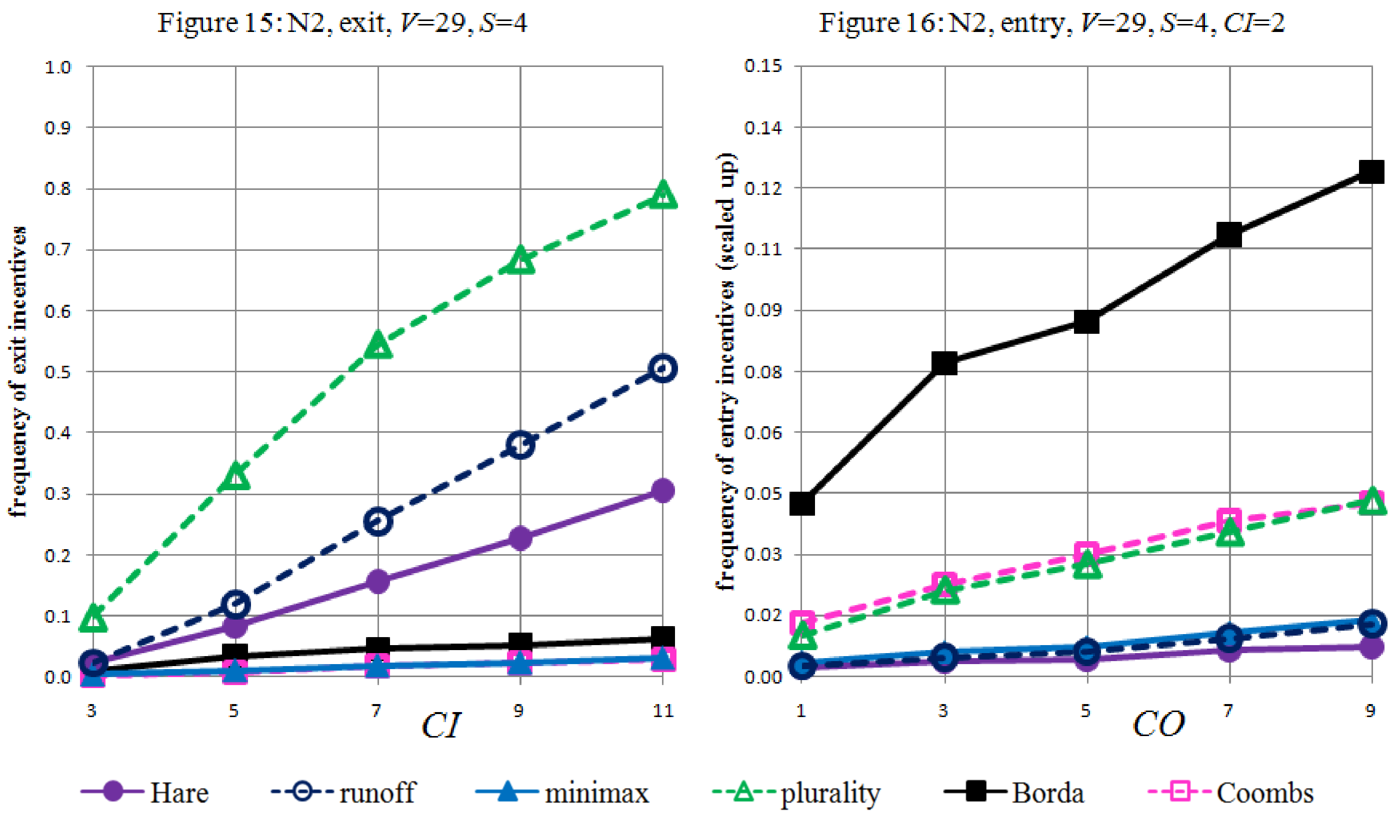


Figure 15: N2, exit, V=29, S=4

Figure 16: N2, entry, V=29, S=4, CI=2

8. Analytical results

Notation: Let an overscore denote that a variable is defined with respect to voters' sincere preferences; for example, the sincere ranking matrix is $\bar{R}$, the sincere pairwise matrix is $\bar{P}$, and so on.

Burying strategy

Proposition 1: Plurality is immune to the burying strategy.

Proof: If voter v prefers candidate q to the sincere winner w, then v's sincere ranking will not place w first, which means that w will gain zero points from v's sincere ballot. If v were to attempt to use a burying strategy against w, this would entail giving w a worse ranking, $R'_{vw} > \overline{R_{vw}}$, but because both of these rankings give zero points to w, the change can't affect the outcome of the election. ■

Proposition 2: Runoff is immune to the burying strategy.

Proof: If voter v prefers q to w, then v's sincere first-round vote will not be for w. Therefore, v can't affect whether or not w makes it to the second round by burying w. If q and w are both in the second round, then v's sincere second-round vote will not be for w. Therefore, v can't help q to get elected by burying w. ■

Proposition 3: Hare is immune to the burying strategy.

Proof: If voter v prefers q to w, then q will be ahead of w in v's sincere rankings, $\overline{R_{vq}} < \overline{R_{vw}}$. v may give w a still-worse ranking, $R'_{vw} > \overline{R_{vw}}$, but since no ranks behind R_{vq} will be counted unless q is eliminated, this can't improve q's chances of winning. ■

Note: Minimax, Borda, approval, range, runoff, and Coombs are vulnerable to the burying strategy.

Compromising strategy

Proposition 4: Coombs is immune to the compromising strategy.

Proof: If voter v prefers q to w, then q will be ahead of w in v's sincere rankings, $\overline{R_{vq}} < \overline{R_{vw}}$. v may give q a still-better ranking, $R'_{vq} > \overline{R_{vq}}$, but this can't affect the outcome of the race until after w is eliminated. Therefore, the strategy can't have an effect until after w has been eliminated, and w will not be eliminated until the strategy has an effect. Therefore, the strategy can't work. (This is logically similar to proposition 3 in reverse.) ■

Note: The 'anti-plurality' system, which elects the candidate with the fewest last choice votes, is another method that is immune to compromising. Plurality, runoff, Hare, minimax, Borda, approval, and range are all vulnerable to compromising.

Proposition 5: If there is a sincere Condorcet winner, Minimax is not vulnerable to the compromising strategy.

Proof: If voter v prefers q to w, then q will be ahead of w in v's sincere rankings, which means that the q, w entry in v's sincere individual pairwise matrix will be 1. Formally, $U_{vq} > U_{vw} \leftrightarrow \overline{R_{vq}} < \overline{R_{vw}} \leftrightarrow \overline{p_{qvw}} = 1$. If voter v gives q a still-better ranking, this will not change P_{qw}, because v's q, w ordering will be unchanged; nor will it change any other P_{cw}, because w isn't moving in q's ranking relative to any other candidate. Formally, $p'_{cwv} = 1 \leftrightarrow \overline{p_{cwv}} = 1, \forall c$, which implies that $P'_{cw} = \overline{P_{cw}}, \forall c.$

Because w is the sincere Condorcet winner, $\overline{P_{wc}} > \overline{P_{cw}}, \forall c \neq w$. Combining this with the above, $P'_{wc} > P'_{cw}, \forall c \neq w$, which implies that w is still the Condorcet winner and therefore the minimax winner. ■

Proposition 6: Plurality, runoff, Hare, and minimax are vulnerable to the compromising strategy when there is a sincere majority rule cycle.

Proof: If there is a sincere majority rule cycle, then for any given sincere winner w, there will be some alternative candidate q such that a majority prefers q to the sincere winner w. Formally,

$\exists q: \overline{P_{qw}} > \overline{P_{wq}}$. If all $q \succ w$ voters rank q as their first choice, then q will be the winner in plurality, runoff, Hare, and minimax. ■

Discussion: From propositions 5 and 6, we see that, in the absence of pairwise ties (which are unlikely in large elections), Plurality, runoff, and Hare are vulnerable to compromising whenever minimax is vulnerable to compromising, but minimax is not necessarily vulnerable to compromising when plurality, Hare, or runoff is vulnerable to compromising.

Proposition 7: If plurality isn't vulnerable to compromising, then runoff isn't vulnerable to compromising.

Proof: If plurality isn't vulnerable to compromising, then the number of voters whose sincere first choice is the plurality winner must be greater than the number of voters who prefer any alternative candidate q to w. Formally, $\overline{F_w} > \overline{P_{xw}} \wedge \overline{F_w} > \overline{F_x}, \forall x \neq w$. Since $F_w > F_x, \forall x \neq w$, w must have the most votes, and must advance to the second round of a runoff election, given sincerity. Since $\overline{P_{wx}} \geq \overline{F_w}$ by definition of P and F, $\overline{F_w} > \overline{P_{xw}}, \forall x$ implies that $\overline{P_{wx}} > \overline{P_{xw}}, \forall x$; that is, w must be a Condorcet winner. Therefore, w wins the second round (and the election) given sincere voting.

If voters compromise in favor of a candidate $q \neq w$, since $\overline{F_w} > \overline{P_{qw}}$, w will still have the most votes, which means that w will still advance into the second round. If q advances into the second round as well, the second round still amounts to a pairwise comparison between w and q; because w is a Condorcet winner, w will win the runoff. ■

Proposition 8: If plurality isn't vulnerable to compromising, then Hare isn't vulnerable to compromising.

Proof: If plurality isn't vulnerable to compromising, then the number of voters whose sincere first choice is the plurality winner must be greater than the number of voters who prefer any alternative candidate q to w. Formally, $\exists w | \overline{F_w} > \overline{P_{xw}}, \forall x \neq w$. Given sincere voting, in any given round of counting, any vote counting for any candidate $x \neq w$ must come from a voter who prefers x to w. Therefore, because $\overline{F_w} > \overline{P_{xw}}$, and because votes listing w as the top choice will be counted for w in every round as long as w is not eliminated, the number of votes counting for x in any given round must be less than the number of votes counting for w. Therefore, w can't be eliminated in any round, so w will be the sincere winner in Hare.

Given a compromising strategy on behalf of q, the logic above will still hold. That is, in any round in which w has not been eliminated, any vote held by any candidate $x \neq w$ must come from a voter who prefers $x \succ w$; if voters are compromising in favor of x, then only $x \succ v$ voters will be strategists, and otherwise, the x, w ordering will be reported sincerely. Therefore, w will still have the most votes in every round of counting, and w will still be the winner in Hare. ■

Discussion: With regard to resistance to the compromising strategy, we see from propositions 7 and 8 that runoff and Hare dominate plurality, and we see from propositions 5 and 6 that all three of these are all but entirely dominated by minimax.

General voting strategy

Proposition 9: If the sincere Condorcet winner w is also the sincere first choice of more than one third of the voters, then both Hare and runoff will elect w and be non-manipulable.

Runoff: Because w is the first choice of more than one third of the voters, no group of strategists will be able to cause his elimination in the first round, because it's impossible for two other candidates to have more than one third of the votes. Because w is the Condorcet winner, no candidate will be able to defeat him in the runoff.

Hare: As in runoff, a candidate with more than $\frac{V}{3}$ first choice votes will not be eliminated before the last round, because each of the prior rounds will include three or more candidates, and because none of the voters whose sincere first choice is w will have an incentive to defect on behalf of an alternative candidate q. Because the last round is equivalent to a pairwise comparison, and because w is the Condorcet winner, w will win. ■

Note: The property described in proposition 9 is not shared by plurality, minimax, Borda, Coombs, approval, or range.

Proposition 10: Given the Hare system, with candidates $x_1, \dots, x_C$, if there are N voters with each possible preference ordering, plus one additional voter with an $x_1 \succ \cdots \succ x_N$ preference ordering (a set of 'almost symmetrical' preferences), x_1 will be the winner, and the result will not be manipulable.

Proof: In the first round of counting, $\overline{F_1} = N(C-1)! + 1$, and $\overline{F_c} = N(C-1)!$, $\forall c \neq 1$. That is, the votes are divided evenly, except for the one extra vote that gives x_1 his advantage. Strategists can't cause x_1 to be eliminated in the first round, because x_1 is the first choice of more than $\frac{V}{C}$ voters, which means that it's not possible for strategists to arrange for all of the other candidates to have more votes.

Similar logic holds in later rounds of counting. That is, when D candidates remain, no candidate can have the fewest first choice votes if he has at least $\frac{V}{D}$ votes; because x_1 will have $\frac{V-1}{D}+1=\frac{V+D-1}{D}$ sincere votes, he can't be eliminated. Because x_1 is their first choice among non-eliminated candidates, none of these voters will be interested in participating in a strategy on behalf of any non-eliminated candidate q. Therefore, strategic incursion against x_1 can't succeed. ■

Discussion: The purpose of propositions 10 through 12 is to shed some light on a dynamic observed in the impartial culture simulation results: Given $C \geq 4$, Hare is the only one of my eight voting methods that doesn't converge towards 100% manipulability as the number of voters gets large. Given $C=3$, this property is shared by the runoff system. When V is large in an impartial culture model, each preference order appears in approximately equal proportion; therefore, the winner's margin of victory tends to be very small relative to the number of voters who prefer an alternative candidate to the sincere winner. These features are captured in the 'almost symmetrical preferences' scenario that provides the basis for these propositions.

Proposition 11: Given $C \geq 3$ candidates $x_1, \dots, x_C$, if there are N voters with each possible preference ordering, plus one additional voter with an $x_1 \succ \cdots \succ x_N$ preference ordering, the result will be manipulable in plurality, minimax, Borda, and Coombs, given a sufficiently large value of N.

Plurality: With $\frac{V-1}{C}+1$ first choice votes (where $V=NC!+1$), to every other candidate's $\frac{V-1}{C}$ first choice votes, x_1 is the sincere winner. For any $q \neq 1$, the number of potential $x_q \succ x_1$ strategists is given by $\overline{P_{q1}}=\frac{V-1}{2}$, which is greater than $\overline{F}_1=\frac{V+C-1}{C}$, for $V>\frac{3C-2}{C-2}$. Therefore, if these strategists vote for x_q, x_q will win.

Minimax: For $i<j$, $\overline{P_{ij}}=\frac{1}{2}NC!+1$, and for $i>j$, $\overline{P_{ij}}=\frac{1}{2}NC!$. Therefore, for $j=1$, $\max_{i=1}^{C} P_{ij}=\frac{1}{2}NC!$, and for all other j, $\max_{i=1}^{C} P_{ij}=\frac{1}{2}NC!+1$. Therefore, x_1 is the sincere winner.

Suppose, however, that all of the voters who prefer $x_2 \succ x_1$ vote $x_2 \succ x_3 \succ \cdots \succ x_C \succ x_1$. Then, we still have $P'_{1,2}=\frac{1}{2}NC!+1$, but now, $P'_{c1}=\frac{2}{3}NC!, \forall c \geq 3$, and $P'_{cd}=\frac{2}{3}NC!+1, \forall c,d: c<d \wedge c \geq 2$. Therefore, there will be a majority rule cycle, but the defeat against x_2 will be the weakest, so x_2 will win.

Borda: From the sincere pairwise matrix $\bar{P}$ described above, we calculate that the sincere Borda score for each candidate $j = 1, \ldots, C$ is $\bar{\mathrm{B}}_j = \frac{1}{2}NC!\,(C-1) + j - 1$, and the sincere winner is x_1. Suppose, however, that all of the voters who prefer $x_2 \succ x_1$ vote $x_2 \succ x_3 \succ \cdots \succ x_C \succ x_1$. The resulting pairwise matrix will be $P' = \begin{bmatrix} 0 & \frac{1}{2}NC!+1 & \frac{1}{3}NC!+1 & \ldots & \frac{1}{3}NC+1 \\ \frac{1}{2}NC! & 0 & \frac{2}{3}NC!+1 & \ldots & \frac{2}{3}NC!+1 \\ \frac{2}{3}NC! & \frac{1}{3}NC! & 0 & \ldots & \frac{2}{3}NC!+1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{2}{3}NC! & \frac{1}{3}NC! & \frac{1}{3}NC! & \ldots & 0 \end{bmatrix}$, and the resulting Borda scores will be $\mathrm{B}'_1 = NC!\left[\frac{1}{2} + \frac{2}{3}(C-2)\right]$, $\mathrm{B}'_2 = NC!\left[\frac{1}{2} + \frac{1}{3}(C-2)\right] + 1$, and $\mathrm{B}'_j = NC!\left[\frac{1}{3} + \frac{2}{3}(j-2) + \frac{1}{3}(C-j)\right] + j - 1, \forall j \neq 1{,}2$. Therefore, the new winner will be x_2.

Coombs: Given sincere voting, each candidate will have the same number of last choice votes in each round of counting, except for the one symmetry-breaking vote, which will cause the elimination order to be $x_N, x_{N-1}, \ldots, x_2$, so that x_1 is the winner. However, if the voters who prefer $x_2 \succ x_1$ change their votes so that x_1 is ranked last, but the rankings are otherwise the same, x_1 will be eliminated in the first round (with $\frac{1}{2}NC!$ last choice votes, which is greater than x_N's total of $\frac{1}{C}NC! + 1$ last choice votes), and then the remaining eliminations will occur in their original order, leaving x_2 as the winner. ■

Proposition 12: Given the runoff system, with candidates $x_1, \ldots, x_C$, if there are N voters with each possible preference ordering, plus one additional voter with an $x_1 \succ \cdots \succ x_N$ preference ordering, x_1 will be the winner. If $C \leq 3$, the result will not be manipulable, but if $C \geq 4$, the result will be manipulable.

Case 1, $C = 3$: Because x_1 is the sincere Condorcet winner, and because x_1 has more than $\frac{V}{3}$ votes, x_1 is the sincere winner, and the result is non-manipulable, by proposition 9 above.

Case 2, $C \geq 4$: As in the $C = 3$ case, x_1 is the sincere winner. x_1 has $\frac{1}{C}NC! + 1$ first choice votes, and there are $\frac{1}{2}NC!$ voters who prefer $x_2 \succ x_1$. Aside from these $x_2 \succ x_1$ voters, x_3 has $\frac{1}{2C}NC!$ first choice votes. The $x_2 \succ x_1$ voters can prevent x_1 from reaching the runoff by distributing their votes between x_2 and x_3 so that x_1 has fewer votes than each of them. This requires $\left[\frac{1}{2C}NC! + 1\right] +$

$\left\lceil \frac{1}{C} NC! + 1 \right\rceil = \frac{3}{2C} NC! + 2$ votes, which is less than the number of $x_2 \succ x_1$ voters, so the strategy is feasible. ■

Core equilibria in voting

Proposition 13: In plurality, runoff, Hare, minimax, Coombs, approval, or range, a coordinated majority of voters can cause the election of any candidate they wish, regardless of how the remaining voters vote.

Proof: Suppose that the majority consists of $M > V/2$ voters, and the remaining minority consists of $N = V - M$ voters. If all M voters rank a given candidate x in first place, then x will surely be the winner in plurality, runoff, Hare, and minimax (as well as any other Condorcet method). In approval, if all M voters in the majority approve only candidate x, then x will have at least M points, and all other candidates will have at most N points. In range, given an interval [0,1] (without loss of generality), if all M voters in the majority give 1 point to candidate x, and 0 points to all other candidates, then x will have at least M points, and all other candidates will have at most N points. In Coombs, if all voters in the majority agree on any ordering of the candidates that puts x in first place, then the reverse of this order will necessarily be the elimination order, and x will win. ■

Discussion: Propositions 13-15 primarily serve as the basis for propositions 16 and 17.

Proposition 14: In the Borda count, a coordinated majority can cause the election of any candidate they wish if they know how the remaining voters are voting.

Proof: Suppose that for every voter in the minority, there is a voter in the majority who casts the exact opposite ranking. Given a minority of N voters, the resulting pairwise matrix is characterized by $P_{xy} = N, \forall x, y\!: x \neq y$, giving all candidates a Borda score of $(C-1)N$. Then, if the remaining voters in the majority all submit a ranking that lists x in first place, x will still have a Borda score of $(C-1)N$, but the remaining candidates will all have greater Borda scores. ■

Proposition 15: In the Borda count, a coordinated majority of M voters can vote in a way that causes the election of any candidate they wish, regardless of how the remaining N voters vote, if and only if $M > \left(2 - \frac{2}{C}\right) N$.

Proof: Suppose that the majority wishes to elect candidate x_1. From the ballots of the minority voters, x_1 can have a Borda score of up to $(C-1)N$ points. (This maximum results from all minority voters ranking x_1 last.) Since it is possible for the minority ballots to add zero points to the

Borda score of any given x_n, the majority voters must themselves add at least $(C-1)N$ points to the scores of all candidates other than x_1, in order to guarantee the election of x_1. If each of the M majority voters cast their votes as $x_1 \succ x_2 \sim \ldots \sim x_C$, then they will collectively add $\left[1+\frac{1}{2}(C-2)\right]M = \frac{1}{2}CM$ points to every other candidate's score. Therefore, x_1 will certainly have the lowest score only if $\frac{1}{2}CM > (C-1)N$, or equivalently, only if $M > \left(2-\frac{2}{C}\right)N$, or $M > \frac{2C-2}{3C-2}V$. Note that if the majority comprises over two thirds of the electorate (that is, if $M > 2N$), it can ensure the election of its chosen candidate given any number of other candidates. ■

Proposition 16: If there is a majority rule cycle in sincere preferences, there is no core equilibrium in voting, under plurality, runoff, Hare, minimax, Coombs, approval, range, or Borda.

Proof: If there is a majority rule cycle, then for any given winner w, there will be some alternative candidate q such that a majority prefers q to w. Formally, $\forall w, \exists q | \overline{P_{qw}} > \overline{P_{wq}}$. By propositions 13 and 14, this majority can change the winner from w to q, holding the other votes constant, which shows that the initial profile is not a core equilibrium. Since this applies to all possible profiles, there is no core. ■

Proposition 17: If there exists a candidate w who is a Condorcet winner in sincere preferences, then there is a core in plurality, runoff, Hare, minimax, Coombs, approval, and range, and all profiles in the core elect w. However, there may not be a core in Borda.

Proof: Given plurality, runoff, Hare, minimax, Coombs, approval, or range, suppose that all voters use the strategies described in proposition 13 to support the sincere Condorcet winner, w. By definition of a sincere Condorcet winner, any given faction of voters who prefer some candidate q to w will comprise only a minority of the electorate. Therefore, by proposition 13, w will win, regardless of how members of this minority alter their ballots. Therefore, the initial profile is in the core.

Given Borda, however, the above logic doesn't hold. For example, consider the case in which 55 voters have preferences $\mathrm{A} \succ \mathrm{B} \succ \mathrm{C}$, and 45 voters have preferences $\mathrm{C} \succ \mathrm{B} \succ \mathrm{A}$. A is a sincere Condorcet winner, but by proposition 15, there is no way the majority can vote to unconditionally elect A, and there is no way the minority can vote to unconditionally prevent the election of A. Thus, there is no core. ■

Strategic nomination

Proposition 18: Minimax is not vulnerable to strategic exit if there is a Condorcet winner among the candidates initially on the ballot.

Proof: If w is a Condorcet winner, then w will also be a minimax winner. If candidate q exits the race, the pairwise contests between the remaining candidates will not be changed. Therefore, w will still be the Condorcet winner, and the minimax winner. ■

Proposition 19: Minimax is not vulnerable to strategy entry, unless the final ballot (after entry) lacks a Condorcet winner.

Proof: If the final ballot has a Condorcet winner, then this candidate must be the minimax winner. By definition of strategic entry, none of the newly-entered candidates may be the winner. Therefore, the winning candidate is a candidate w who was on the old ballot, and who has pairwise defeats against every other candidate on the new ballot. Because all candidates on the new ballot are also on the old ballot, w has pairwise defeats against all candidates on the old ballot. Therefore, w is the winner given the old ballot as well as the new ballot. ■

Proposition 20: Given the plurality rule, if the electorate consists of $\boldsymbol{A}$ voters whose preferences are $\boldsymbol{x_1 \sim \ldots \sim x_N \succ y}$, and $\boldsymbol{B}$ voters whose preferences are $\boldsymbol{y \succ x_1 \sim \ldots \sim x_N}$, and votes are sincere, $\boldsymbol{y}$ will win if and only if $\boldsymbol{N > \frac{A}{B}}$.

Proof: Assume that the plurality rule asks voters to rank the candidates in order of preference, and then chooses the candidate with the most first choice rankings. (Or, in the case of a tie, that it is broken lexicographically, such that x_1 is most favored by the tiebreaker, and y is least favored.) Assume that if voters give equal rankings to two or more candidates, then their votes are cast as the average of all strict rankings that can be formed by resolving expressed indifferences.

If votes are sincere, then each x_n will receive $\frac{A}{N}$ votes, and y will receive B votes. Therefore, y wins if and only if $B > \frac{A}{N}$, or equivalently, $N > \frac{A}{B}$. Therefore, in this type of situation, having more x candidates is disadvantageous for the $x \succ y$ group. ■

Proposition 21: Given the Borda rule, if the electorate consists of $\boldsymbol{A}$ voters whose preferences are $\boldsymbol{x_1 \succ \cdots \succ x_N \succ y}$, and $\boldsymbol{B}$ voters whose preferences are $\boldsymbol{y \succ x_1 \succ \cdots \succ x_N}$, and votes are sincere, $\boldsymbol{y}$ will win if and only if $\boldsymbol{N < \frac{B}{A}}$.

Proof: Make the same assumptions as above. y has N candidates ranked above him on A ballots, so his Borda score is $\mathrm{B}_y = NA$. x_1 has one candidate (y) ranked above him on B ballots, so his Borda score is $B_{x_1} = B$. Therefore, y wins if and only if $B > NA$, or equivalently, $N < \frac{B}{A}$. Therefore, in this type of situation, having more x candidates is advantageous for the $x \succ y$ group. ■

Proposition 22: Given the Coombs rule, if the electorate consists of A voters whose preferences are $x_1 \sim \ldots \sim x_N \succ y$, and B voters whose preferences are $y \succ x_1 \sim \ldots \sim x_N$, and votes are sincere, y will win if and only if $N < \frac{B}{A}$.

Proof: Make the same assumptions as above. For any value of N, A voters will rank y in last place, and, given the treatment of equally-ranked candidates described above, $\frac{B}{N}$ voters will effectively rank x_n in last place, where x_n is any candidate in the x group. Therefore, y will avoid first-round elimination if and only if $A < \frac{B}{N}$, or equivalently, if $N < \frac{B}{A}$. If y is not eliminated in the first round, then y will not be eliminated in any subsequent round, because y's last choice vote total will remain at A, while the last choice vote total for each x_n will increase to $\frac{B}{N-1}$, $\frac{B}{N-2}$, and so on, until all of the x_n's have been eliminated. Therefore, in this type of situation, having more x candidates is advantageous for the $x \succ y$ group. ■

Note: Given Hare or minimax, and the scenarios described in propositions 20-22, y wins if and only if $B > A$.

Discussion: Propositions 20 through 22 explore strategic nomination using simple two-group scenarios. They suggest that plurality should be particularly vulnerable to strategic entry, and that Borda and Coombs should be particularly vulnerable to strategic exit. This is consistent with the simulation results.

9. Conclusion

9.1. Summary of simulation results

Table 19 below summarizes the results presented in sections 6 and 7, by qualitatively characterizing the relative vulnerability of each method to each type of strategic manipulation.

Table 19: Overall summary

	Hare	runoff	minimax	plurality	approval	range	Coombs	Borda
general voting	very low	low	moderate	moderate	high	high	high	high
compromising	low	low	very low	highest	high	high	none	high
burying	none	none	moderate	none	high	high	high	moderate
exit	moderate	moderate	very low	highest	minimal	minimal	very low	very low
entry	very low	very low	very low	low	minimal	minimal	moderate	highest

9.2. General discussion

With regard to strategic voting, there is a clear stratification between frequently-manipulable methods such as range, Coombs, Borda, and approval, moderately-manipulable methods such as plurality and minimax, and infrequently-manipulable methods such as Hare and runoff: this pattern emerges in nearly all specifications, regardless of the data generating process that is used. Plurality is clearly most vulnerable to compromising and strategic exit, while Coombs, range, and approval are most vulnerable to burying, and Borda is most vulnerable to strategic entry. If there is no strategic voting, then we would expect approval, range, and minimax to have very infrequent strategic nomination incentives, though this might not hold if candidates take the possibility of strategic voting into account during the nomination stage. Thus, an analysis that combines strategic nomination and strategic voting into a two-stage game would be an interesting topic for further study.

Among the eight methods that are covered here, Hare has the advantage of being the least frequently vulnerable to strategic voting, but minimax has an advantage in resistance to strategic nomination, particularly strategic exit.[23]

Aside from counting the raw frequency with which strategic manipulation is possible, there are many other interesting questions to be explored, such as the likelihood that manipulation will actually occur, and the effect on social welfare if manipulation is successful. These questions

[23] This result leads one to wonder whether it might be possible to construct Condorcet-Hare hybrid methods that possess both of these advantages. Green-Armytage (2011) identifies four methods that fit this description.

require us to make more assumptions to generate results, but they are nonetheless worth asking, and they have already formed the basis for much interesting research in this area.

A broad lesson from this paper is that all voting rules are vulnerable to strategic manipulation in some non-insignificant fraction of elections. Looking at the bottom of table 5, we see that even Hare is vulnerable to strategic voting in a majority of cases if the number of candidates is sufficiently large. Proportional representation may provide a partial solution to this predicament. For example, Tullock (1967) describes a system in which anyone who wishes to can serve as a representative, and in which each representative's voting weight is determined by the number of people who vote for them, whether this number is one, or several million. Since this system would allow all voters to have their first choice of representative, there is arguably no incentive for strategic voting over candidates, though once elected, representatives may still engage in strategic voting over issues, using whatever parliamentary rules they have established. As for the question of which parliamentary rules are least susceptible to manipulation, I leave this for future study.

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