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A Note on Fair Value and Illiquid Markets

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A Note on Fair Value and Illiquid Markets

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Abstract

We present in this paper a method to extract fair prices from observable prices in an illiquid market. The dynamics of fair prices have a general form encompassing random walks. In fact, only a part of a movement in price is assumed to reflect fundamental changes, the rest is considered to be friction. That part is optimally estimated by a *Kalman filter*. The model allows also to recover liquidity premia as a product of innovations times an illiquidity multiplier. Thus the higher the difference between observed and filtered prices (prices obtained under "normal" market dynamics), the higher liquidity premium will be. The model can be adapted to various kind of instruments and calibrated in different ways.

Keywords: Fair value - Illiquid Market - Kalman Filter - Mark to Model

JEL classification: .

1 Introduction

From summer of 2007, accumulating losses on US subprime mortgages triggered widespread disruption in the global financial system. Large losses were sustained on complex structured securities. Institutions reduced leverage resulting in forced selling and creating a disequilibrium between offer and demand and thus market illiquidity.

In this period of crisis, the notion of fair value has been widely discussed. As soon as the market becomes illiquid, certain stocks are no longer exchanged. Questions arise. For instance: is it possible to use a fair value in crisis period, is it linked to the transactions volumes? FAS 157 defines fair value as "the price that would be received to sell an asset or paid to transfer a liability in an orderly transaction between market participants at the measurement date". The definition of fair value often seems to be used in a way that assumes that supply and demand are in reasonable balance, in which case the fair value would be the amount at which an asset can be bought or sold in a current transaction (Generally Accepted Accounting

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Principles, (GAAP)). Is it possible to determine a "fair" value during a period of crisis?

Thus, the fair value is often associated with the transaction volume which is a way to measure the activity of the market, Li (2009), Lee and Swaminathan (2000) or LLorente, Michaely, Saar and Wang (2002). Although this assumption is popular, no precise work has been done to prove it. Here, we have not analysed this approach as we consider that the problem is more complex. Instead, we have considered an alternative issue.

The problem we highlight is that of fair price in illiquid market. In such a market, the mispricing, or the difference between the observed price and the fundamental/fair price, is major. But what is a fundamental price and how should it be modelled ?

One approach consists in applying the pricing in incomplete market framework by valuing an instrument as the expected utility of its discounted cash flows[1]. However, the latter are also uncertain so another problem is how to specify the various scenario probabilities, the form of the utility function and the calibration of model parameters.

A different, yet more empirical, approach consists in defining the price as the sum of the fair price and a residual component. As both are unobservable, one must identify one in order to deduce the other. Thus, Bao, Pan and Wang (2009) define the price of a corporate bond as the sum of a liquidity premium and a fair price, the latter being a random walk or a frictionless price. The amplitude of the illiquidity process is defined as the opposite of the covariance of the first difference of prices which is equal to that of illiquidity since fair prices are assumed random walks independent of illiquidity. The authors define the data generator process of the residual component as an AR(1) which denotes the transitory character of illiquidity risk. Cholette, Naes and Skjeltrop (2006) use a similar approach but with a different liquidy measure. One major drawback of such approach resides in the transitory character of the illiquidity. A short memory process on residuals does not take into account persistence in illiquidity risk nor sustainable mispricing like the one we have seen in the current crisis especially on distressed instruments. Besides, assuming random walk fair price and AR(1) residuals imply that observed prices are also AR(1).

On the other hand, can one question the existence of a reliable liquidity measure? By defining a liquid market as one in which every agent can buy and sell at any time a large quantity rapidly at low cost, Harris (1990) distinguishes four interrelated dimensions for liquidity:

- width that measures the cost per share of liquidity
- depth which is the number of shares that can be traded at a given price
- immediacy which captures how quickly a given number of shares can be traded at a given cost

[1] Interested readers may refer to Duffie (2001).

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- resiliency that indicates the ability to trade at minimal price impact

It is indeed very difficult to measure liquidity risk and to capture all of its aspects. Besides, modelling the fundamental price as a random walk is appealing but does not match stylized facts like fat tails, volatility clusters, etc.

The objective of a fair value measurement is to determine the price that would of selling the asset at the measurement date (an exit price) - such a measurement, by definition, requires consideration of current market conditions, including the relative liquidity of the market. It would not be appropriate to disregard observable prices, even if that market is relatively thinner as compared to previous market volumes. In fact only part of the price movement is due to illiquidity, the rest translates fundamental changes and should not be ignored.

We propose hereafter a different approach to recover fair prices. Instead of modelling all the characteristics of complex instruments and then computing a price after calibrating certain parameters of the model on liquid markets, we focus on the only observable data: market prices that we try to split into its two components: fair price and a residual or liquidity premium. Instead of specifying the dynamics of liquidity risk, we specify those of the fair price itself. The advantage of our approach lies in the fact that it does not require any additional source of information other than market prices. It takes into account the market sentiment as well as liquidity persistence. As such, the focus, as well as the contribution of our paper, is mainly empirical.

In the next section, we introduce our model and set out the arguments in support. Section Three is devoted to the description of the estimation algorithm. In Section Four, after describing the data set, we show how our approach permits the recovery of the liquidity premia in periods of crisis. We compare our approach with classical ones. Section Five concludes.

2 The Model

Let S_t denote the observed market price of a given financial instrument at time t. Following the previous discussion, we introduce a simple model decomposing S_t as the sum of a hidden fair price Z_t and a residual process u_t according to the following measurement equation:

$$S_t = Z_t + u_t \tag{1}$$

We assume that the fair price dynamics follow the transition equation:

$$Z_t = \phi.Z_{t-1} + (1-\phi).N + v_t \tag{2}$$

where $\phi \in [0,1]$ is a scalar, N the par value or the nominal of the instrument and v_t a noise. The transition equation (2) states that the fair price is a noisy weighted average of the previous fair price and the long run price. The lower ϕ is, the closer Z_t will be to N.

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The rationale behind transition equation (2) is easier to understand in the case of a plain vanilla bond of nominal N, coupon rate c and a yield to maturity R. We give now an intuitive approach. The fair price can thus be written as the sum of its discounted cash flows:

$$Z_t = N.\left[\sum_{s=t+1}^{T} \frac{c}{(1+R)^{s-t}} + \frac{1}{(1+R)^{T-t}}\right]$$

and satisfies the following recursive equation:

$$Z_t = (1+R).Z_{t-1} - c.N$$

In the following, we use this working assumption:

(H_0): R is close to c: $R - c \sim 0$

This assumption means that the fair yield to maturity is close to its coupon rate and that the instrument is trading at its par value. It is a reasonable assumption in terms of market activity. Let $\psi = (1+R) \geq 1$, then:

$$Z_t = \psi.Z_{t-1} + (1-\psi).N \tag{3}$$

Let $0 \leq \phi \leq 1$ then equation (3) becomes :

$$\begin{aligned} Z_t &= \phi.Z_{t-1} + (1-\phi).N + \underbrace{(\psi-\phi).(Z_{t-1}-N)}_{\to 0 \ under \ H0} \\ &= \phi.Z_{t-1} + (1-\phi).N + v_t \end{aligned}$$

The process described in equation (2) encompasses many specifications:

- Random Price for $\phi = 0$: prices evolve around a constant N, previous prices do not impact today's prices: $Z_t = N + v_t$
- Random Walk for $\phi = 1$: prices follow a random walk, any forecasting exercice is useless: $Z_t = Z_{t-1} + v_t$
- Mean Reverting as in our model for $0 < \phi < 1$. In fact, equation (2) can also be written as follows: $Z_t = N + \phi.(Z_{t-1} - N) + v_t$. We can see the mean reversion character of fair prices around N.

3 Implementing the model

3.1 *Kalman filter*

Transition equation (2) and measurement equation (1) define a state space model whose parameters should be estimated given a sample of observable prices $S_{1:T} = (S_1, \ldots S_T)$. We use the *Kalman filter*[2] to optimally compute the state variable Z_t corresponding to the fair price. For that we assume:

[2]Interested readers may refer to Harvey A. (1989) or Hamilton J.D. (1994) chap. 13.

- $(H1)$: $(u_t, v_t)'$ is a white noise identically and normally distributed:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} \rightsquigarrow \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_v^2 \end{pmatrix}\right) \tag{4}$$

- $(H2)$: Z_0 the initial state is independant of noises (u_t) and (v_t) and has a variance $\sigma^2_{Z,0}$

These assumptions are usually required to implement the *Kalman filter* and allow the obtention of robust estimates. They imply the independance of noises u_t and v_t of observations $S_{1:t-1} = (S_1, S_2, \ldots, S_{t-1})$ and the optimality of the *Kalman filter*. In case of failure of the Gaussian assumption, *Kalman* estimates remain optimal among the only linear ones.

The implementation of *Kalman filter* is based on two steps: forecasting and updating. We briefly recall the algorithm we have used.

Forecasting Taking expectation on transition equation conditional on information available up to $t-1$, denoted $I_{t-1} = \sigma(S_1, \cdots, S_{t-1})$, we compute the conditional mean and variance of the fair price:

$$\begin{aligned} Z_{t|t-1} &= \mathbb{E}[Z_t \mid I_{t-1}] = \phi.Z_{t-1|t-1} + (1-\phi).N & (5) \\ \sigma^2_{Z,t|t-1} &= \mathbb{V}[Z_t \mid I_{t-1}] = \phi^2.\sigma^2_{Z,t-1|t-1} + \sigma_v^2 & (6) \end{aligned}$$

where $\bar{Z}$ is the sample mean. Given these values, one can also compute the conditional expectation and the variance of the price:

$$\begin{aligned} S_{t|t-1} &= \mathbb{E}[S_t \mid I_{t-1}] = Z_{t|t-1} & (7) \\ \sigma^2_{S,t|t-1} &= \mathbb{V}[S_t \mid I_{t-1}] = \sigma^2_{Z,t|t-1} + \sigma_u^2 & (8) \end{aligned}$$

Updating After observing S_t, we update the values of Z and σ^2_Z:

$$\begin{aligned} Z_{t|t} &= \mathbb{E}[S_t \mid I_{t-1}] = Z_{t|t-1} + \mathcal{K}_t.\eta_t & (9) \\ \sigma^2_{Z,t|t} &= \mathbb{V}[S_t \mid I_{t-1}] = \sigma^2_{Z,t|t-1} - \mathcal{K}_t^2.\sigma^2_{S,t|t-1} & (10) \end{aligned}$$

with, $Z_{t|t}$ being the estimate of the fair price given the new observed prices and $\sigma^2_{Z,t|t}$ its variance, $\eta_t = S_t - S_{t|t-1}$ the innovation and $\mathcal{K}_t = \frac{\sigma^2_{Z,t|t-1}}{\sigma^2_{S,t|t-1}} = \frac{\sigma^2_{Z,t|t-1}}{\sigma^2_{Z,t|t-1}+\sigma_u^2}$ the *Kalman gain*. Note that since $\sigma_u^2 > 0$ we have $\mathcal{K}_t < 1$. One can see that $Z_{t|t}$ can be written as the weighted average of $S_{t|t-1}$ and S_t:

$$\begin{aligned} Z_{t|t} &= \mathcal{K}_t.S_t + (1-\mathcal{K}_t).S_{t|t-1} \\ &= \frac{\sigma^2_{Z,t|t-1}}{\sigma^2_{Z,t|t-1}+\sigma_u^2}.S_t + \frac{\sigma_u^2}{\sigma^2_{Z,t|t-1}+\sigma_u^2}.S_{t|t-1} \end{aligned}$$

The higher the σ_u^2, the lower $\mathcal{K}_t$ and the lesser is the importance of observed prices in the fair prices.

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3.2 Statistical Inference

Given the model parameters $\Theta = \{S_0, \sigma_0^2, \phi, \sigma_v^2, \sigma_u^2\}$, the likelihood from the observations set $S_{1:T}$ is given by:

$$\mathcal{L}(S_1, \ldots, S_T \mid \Theta) = \prod_{t=1}^{T} l(S_t \mid \Theta) = \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi.\sigma_{S,t|t-1}^2}} e^{-\frac{1}{2}\frac{\eta_t^2}{\sigma_{S,t|t-1}^2}}$$

and the log likelihood:

$$\mathcal{LL} = \log \mathcal{L}(S_1, \ldots, S_T \mid \Theta) \quad = \quad -\frac{T}{2}\ln 2\pi - \frac{1}{2}\sum_{t=1}^{T} \ln \sigma_{S,t|t-1}^2 - \frac{1}{2}\sum_{t=1}^{T} \frac{\eta_t^2}{\sigma_{S,t|t-1}^2} \qquad (11)$$

Thus, the maximum likelihood estimator $\hat{\Theta}$ is obtained by maximizing the log likelihood given in equation (11). An estimate of its covariance is obtained by using the inverse of the second derivative of the log likelihood function evaluated in $\hat{\Theta}$:

$$Cov(\hat{\Theta}) = \left[-\frac{\partial^2 \mathcal{LL}}{\partial\Theta\partial\Theta'} \mid \Theta = \hat{\Theta} \right]^{-1}$$

The properties of these estimates are discussed in Harvey (1989).

4 Case Study: Results and Analysis

4.1 Calibration

In order to calibrate our model, we run the *Kalman filter* on the observed prices during a period of quiet market then we use the estimated parameters to filter fair prices $Z_{t|t}$ during distressed periods[3]. Only a part of price movements is assumed to reflect fundamental changes, the rest is thus considered as friction. That part is optimally estimated by the *Kalman filter*.

In what follows we present an application for filtering fair prices for a Soft Bullet *Residential Mortgage Backed Security* tranche: *Storm 2006-1 C*. Our sample of daily prices ranging from March 2006 to August 2009 is represented in Figure (1a). *Storm 2006-1 C* prices dropped in the aftermath of August 2007 as we can see in Figure (1c).

We estimate our model given by equations (1)-(2) on the subperiod prior to the crises that is before August 2007. Estimated parameters are given in Table (1), with their standard deviation in brackets.

[3]One could estimate the model parameters on a similar liquid market and then use the estimated parameters to filter the fair prices of the instrument. However, similar liquid markets may not exist.

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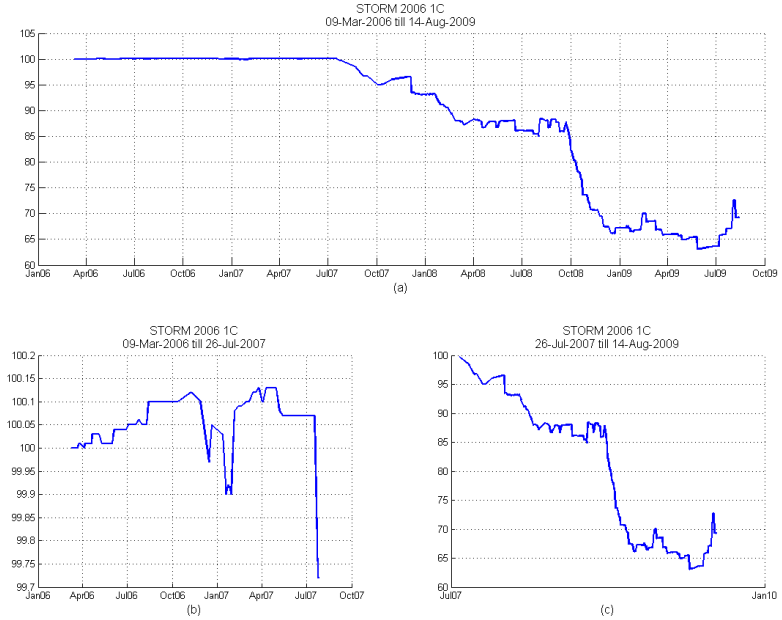

Figure 1: Observed prices

S_0	σ_0^2	ϕ	σ_v^2	σ_u^2
100.0005	0.00290	0.8488	0.0032	0.0062
(0.0453)	(0.0016)	(0.8688)	(0.0016)	(0.0017)

Table 1: Estimated parameters and the corresponding standard deviation between brackets

4.2 Liquidity premium

We filter observed prices using parameters given in Table (1), and their corresponding fair prices are shown for each subperiod in figure (2). The full period is given in (2 a); the subperiod January 2006-July 2007 in (2 b) and subperiod July 2007 - August 2009 in (2 c). We provide also in figure (3) the bimodal distribution of innovations η_t, after August 2007. These two modes capture the market dislocations in both August 2007 and October 2008.

The difference, we observe on figure (2), between fair prices (red line) and market prices (blue line) accounts for liquidity risk in the second subperiod. It is proportional to innovation η_t:

$$Z_{t|t} - S_t = -(1 - \mathcal{K}_t).\eta_t$$

The coefficient $(1 - \mathcal{K}_t) = \frac{\sigma_u^2}{\sigma_{Z,t|t-1}^2 + \sigma_u^2}$ appears as an illiquidity/friction multiplier. The higher σ_u^2 is, the higher the illiquidity would be. We define the liquidity premium at time t as the absolute value of the difference between observed prices and prices:

$$\lambda_t = |-(1 - \mathcal{K}_t).\eta_t|$$

We plot λ_t in figure (4). We notice the increase in liquity premium in the aftermath of the Lehman Brothers default in September 2008 and its relative decrease after June 2009. At

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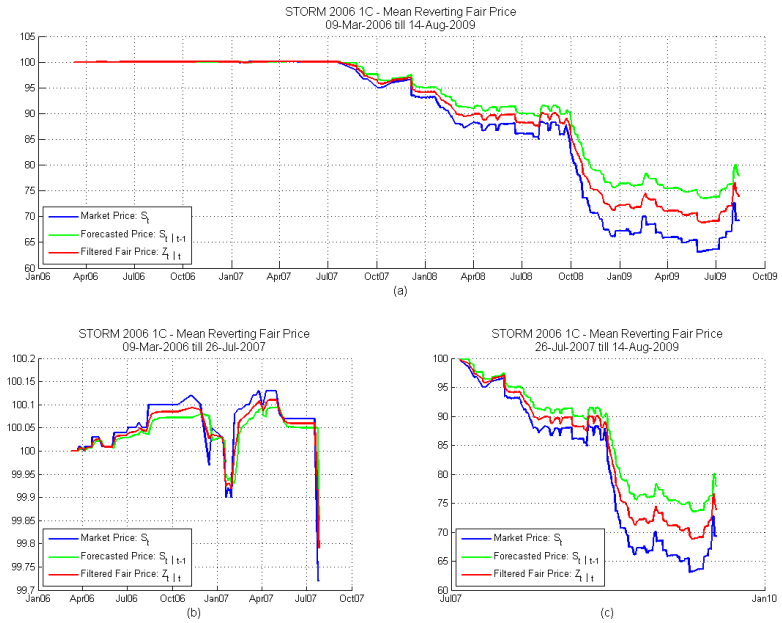

Figure 2: Observed and fair prices

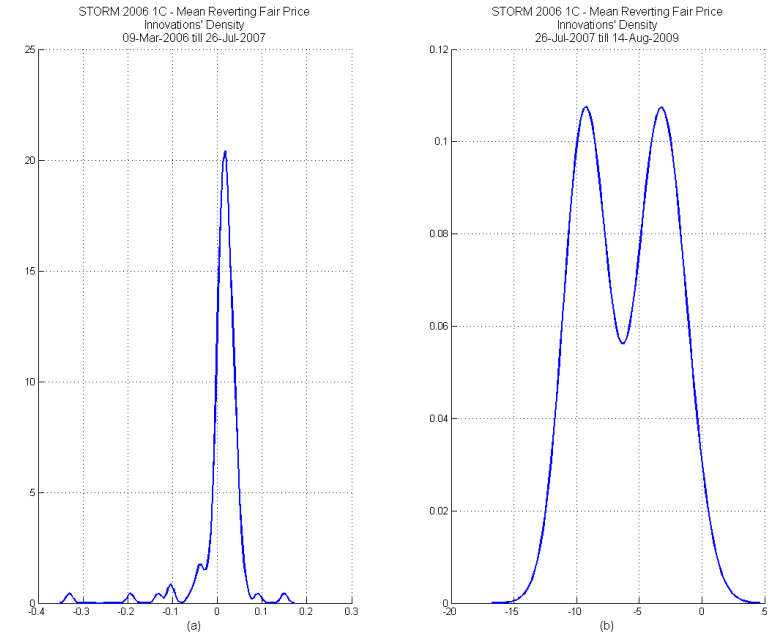

Figure 3: Innovations' density

this stage we have recovered both the fair price and the liquidity premium.

In order to evaluate the ability of our model to recover the liquidity premium, we also consider other dynamics for the fair price.

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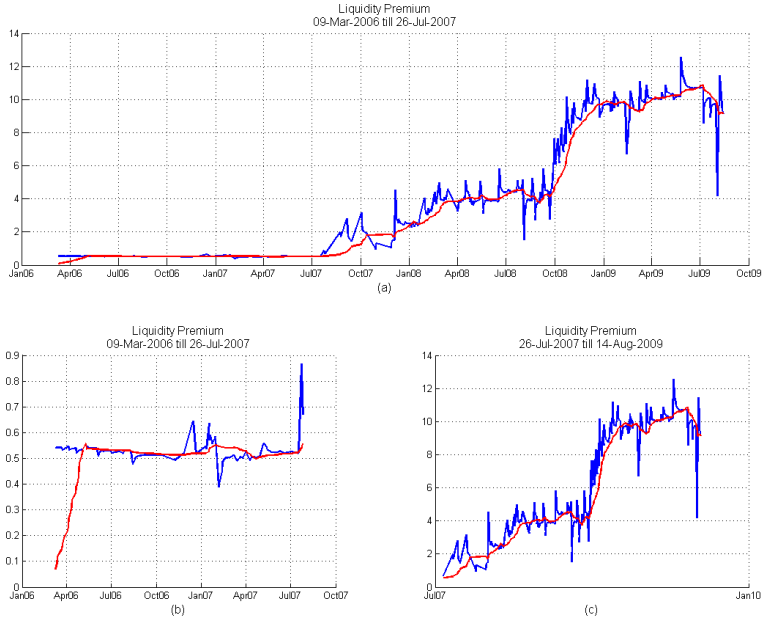

Figure 4: Daily liquidity premia in blue (22 days moving average in red)

4.3 The Random Walk model

In Figures (5) and (6) we show results obtained when specifying a random walk dynamic on fair prices as we would have done if markets were complete ($\phi = 1$ in equation (2)). We notice that under such dynamics, filtered prices are very close to observed ones. Illiquidity risk is not captured and any price variation is directly repercuted on fair prices. Random walk dynamics are not suitable for illiquid markets.

4.4 The Random Price model

In figures (7) and (8) we show results obtained when specifying a random dynamic on fair prices ($\phi = 0$ in equation (2)). We notice that under such dynamics, filtered prices diverge significantly from observed prices as forecasted prices $S_{t|t-1}$ will be equal to a constant N. Illiquidity risk is overestimated and filtered prices are less sensitive to price variation than in the case of Random Walk. Our Mean Reverting specification offers a good trade off between Random Walk and Random Prices. In the mean reversion case, only a part of price variation is considered as an illiquidity risk, the rest is interpreted as a fundamental change and thus affects fair prices.

5 Conclusion

This paper deals with pricing aspects in illiquid markets from a Mark to Model approach. In such markets observed prices do not reflect fundamental prices. We present an approach to filter fair prices from those observed and to recover liquidity premia.

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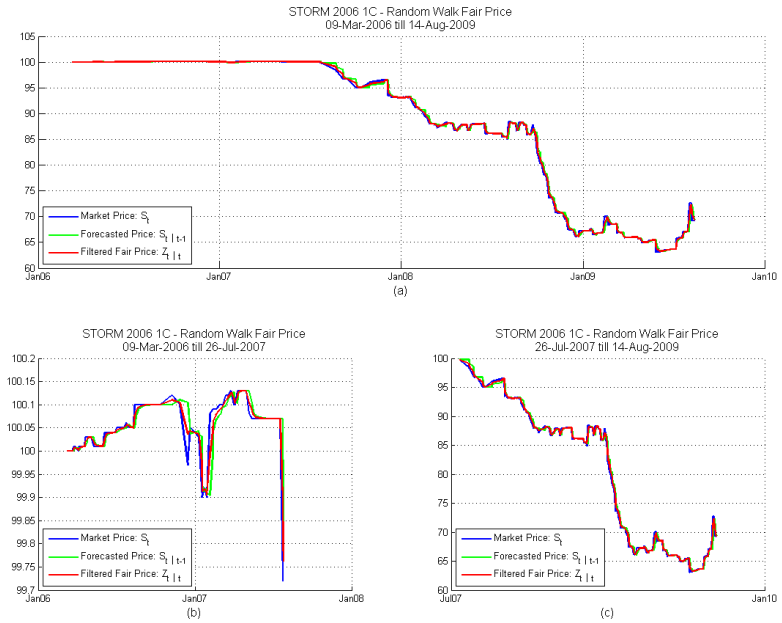

Figure 5: Observed and random walk fair prices

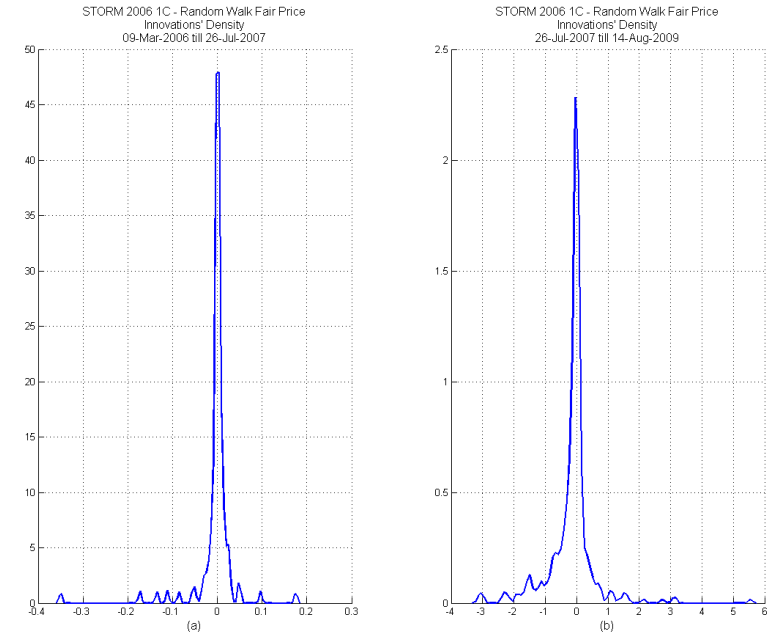

Figure 6: Innovations' density under the assumption of random walk fair prices

The main advantage of our approach is its simplicity. It applies directly to prices without requiring any specific modelling for future cash flows. It allows the recovery of fair prices and gives an estimate for illiquidity risk as the product of innovation times an illiquidity multiplier.

The part of price changes that should account for market friction depends not only on fair price dynamics (random walk vs mean reverting) but also on the model parameters. Besides

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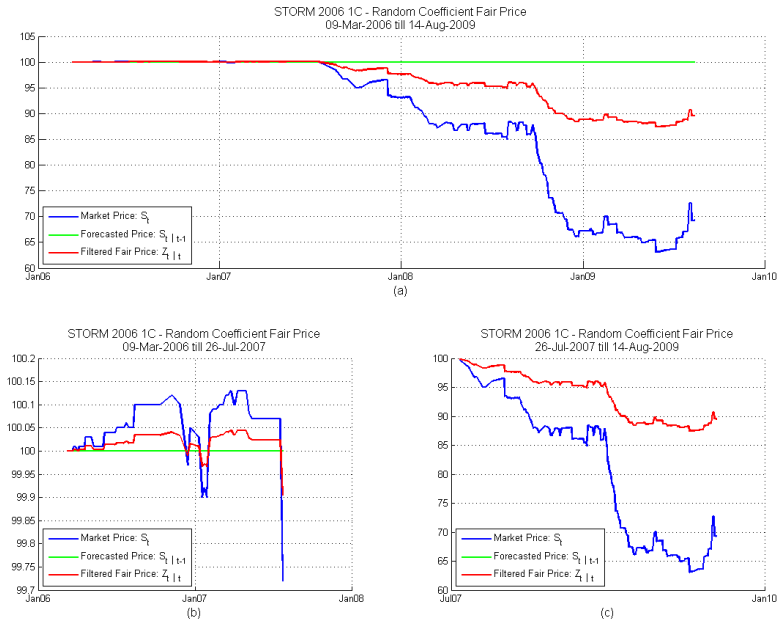

Figure 7: Observed and random fair prices

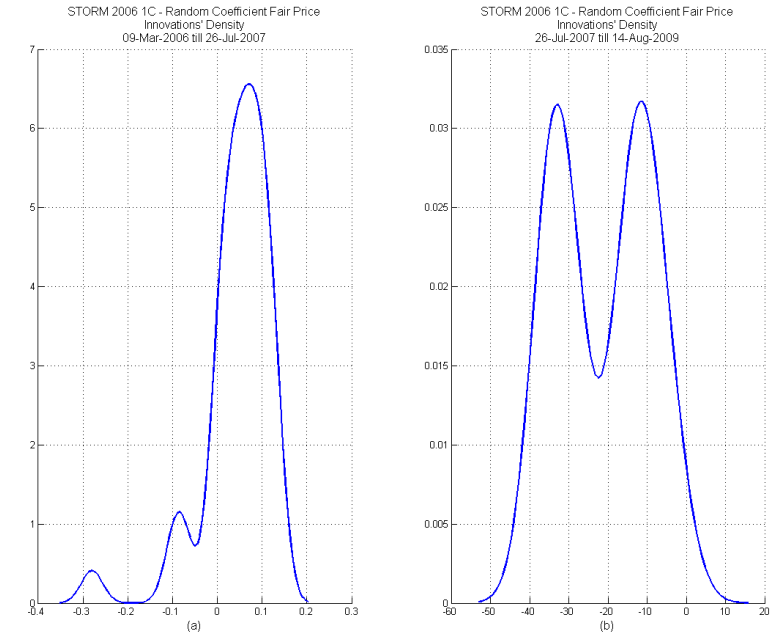

Figure 8: Innovations' density under the assumption of random fair prices

ϕ that should range between 0 and 1 and given initial parameters (S_0, σ_0^2), one may argue that σ_v^2 and σ_u^2 should be bounded in order to have an economic sense. For instance, one should expect σ_v^2 to be lower than σ_u^2 since uncertainty on observations should be greater than that on fair prices. σ_u^2 is a key parameter as it enters directly the *Kalman gain*. Taking variances on both sides of equation (1) in the stationary case we have: $\sigma_S^2 = \sigma_Z^2 + \sigma_u^2$. σ_S^2 can be estimated directly from observed prices. We can see that σ_u^2 may not exceed σ_S^2. As for

its lower boundary, we have from equation (2) that: $\sigma_Z^2 = \frac{\sigma_v^2}{1-\phi^2} \leq \frac{\sigma_u^2}{1-\phi^2}$. Therefore, we have: $\sigma_u^2 + \frac{\sigma_u^2}{1-\phi^2} \geq \sigma_S^2$ and thus: $\sigma_u^2 \geq \sigma_S^2 . \frac{1-\phi^2}{2-\phi^2}$. For example for $\phi = 0.95$ we have: $\sigma_u^2 \geq \sigma_S^2 \times 0.09$.

Another way to introduce boundaries on σ_u^2 consists in assuming a *Constant Absolute Risk Aversion* utility function of parameter γ and given a required risk premium P for investing in a risky asset then we have: $P = \gamma.\sigma_S^2 = \gamma.(\sigma_Z^2 + \sigma_u^2) \Leftrightarrow \frac{P}{\gamma} = (\sigma_Z^2 + \sigma_u^2)$.

One possible extension of our work would be to check whether our dynamic illiquidity measure is coherent with the four liquidity dimensions as highlighted by Harris (1990). It would also be interesting to check the consistency of our approach for different asset classes by intoducing other price drivers (DV01, volatility, momentum ...) and by refining the dynamics in the transition equation.

References

[1] Bao J., Pan J., Wang J. (2009) Liquidity of Corporate Bonds, available at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1106852.

[2] Chollete L., Næs R., Skjeltorp J. (2006) Pricing Implications of Shared Variance in Liquidity Measures, Discussion paper, Institutt for Foretaks0konomi, Department of Finance and Management Science, Norway.

[3] Duffie D. (2001) Dynamic Asset Pricing Theory (Third Edition), Princeton University Press.

[4] Financial Acounting Standard Boards (FAS) (2009) Pre-codification standards, Statement 157, Norwalk, Connecticut, USA.

[5] Hamilton J.D. (1994) Time Series Analysis, Princeton University Press.

[6] Harris, L. (1990) Statistical Properties of the Roll Serial Covariance Bid/Ask Spread Estimator, The Journal of Finance, 45, 579 - 590.

[7] Harvey A. (1989) Forecasting Structural Time Series Models and the Kalman Filter, Cambridge University Press.

[8] Lee C.M.C., Swaminathan B. (2000) Price momemtum and trading volume, The Journal of Finance, 55, 2017 - 2069.

[9] Li M-Y L. (2009) Value or volume strategy, Finance Research Letters, 6, (4), December 2009, 210 - 218.

[10] Llorente G., Michaely R., Saar G., Wang J. (2002) Dynamic volume-return relation of individual stocks, The Review of Financial Studies, 15, 1005 - 1047.

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