UNIVERSITÉ PARIS 1 – PANTHÉON - SORBONNE –

Documents de Travail du
Centre d'Economie de la Sorbonne

CES Working Papers

General equilibrium and fixed point theory :

A partial survey

Hichem BEN-EL-MECHAIEKH, Philippe BICH, Monique FLORENZANO

2009.51

Version révisée

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://centredeconomiesorbonne.univ-paris1.fr/bandeau-haut/documents-de-travail/
ISSN : 1955-611X

GENERAL EQUILIBRIUM AND FIXED POINT THEORY: A PARTIAL SURVEY

HICHEM BEN-EL-MECHAIEKH, PHILIPPE BICH, AND MONIQUE FLORENZANO

This paper is dedicated to Steve Smale, with our admiration

Abstract. Focusing mainly on equilibrium existence results, this paper emphasizes the role of fixed point theorems in the development of general equilibrium theory, as well for its standard definition as for some of its extensions.

Mathematics Subject Classification: Primary 47H10, 47H11, 47N10, 49J53, 54H25, 58C30

Keywords: fixed point, equilibrium, quasiequilibrium, abstract economy, Clarke's normal cone, financial equilibrium, Grassmanian manifold, degree theory

1. Introduction

General equilibrium is a unified framework for studying, in the Walras tradition, the general interdependence of economic activities: consumption, production, exchange. Arrow–Debreu's (1954) paper (2) gives at the same time the seminal definition of a so-called 'private ownership economy' and an equilibrium existence result proving consistency of the model.

The list of data

$$\mathcal{E} = \left(\mathbb{R}^L, (X_i, P_i, e_i)_{i\in I}, (Y_j)_{j\in J}, (\theta_{ij})_{\substack{i\in I\\ j\in J}}\right)$$

is the prototype description of an economy. L is a (finite) set of goods, so that $\mathbb{R}^L$ is the commodity space and the price space of the model. I is a (finite) set of consumers and $X_i \subset \mathbb{R}^L$, P_i (whose precise definition will be given later) and $e_i \in \mathbb{R}^L$ represent respectively the set of possible consumption plans, the preferences and the initial endowment of consumer $i \in I$. J is a (finite) set of producers, and $Y_j \subset \mathbb{R}^L$ is the set of possible production plans of firm $j \in J$. For each i and j, $\theta_{ij} \geq 0$ represents the share of consumer i in the profit of firm j, under the assumption that for each $j \in J$, $\sum_{i\in I} \theta_{ij} = 1$. These shares together with their initial endowment determine the budget constraint of each consumer. Equilibrium is basically defined as the solution of a simultaneous optimization problem for the consumers, producers and an hypothetic additional agent, the *Walrasian auctioneer*, who set prices so as to maximize the value of the excess demand. At equilibrium, the excess demand is equal to zero (markets clear) and, for equilibrium prices, the resulting allocation is optimal for each one of the economic agents.

Date: December 22, 2009.

Since then, the initial formalization has been progressively enriched in order to accommodate, one after the other, most of the different issues successively tackled by economic theory: intertemporal equilibrium, risk and uncertainty, financial markets, asymmetry of information, to quote only important issues among many others. Assumptions as the finite number of commodities, the finite number of agents, as well as some facilitating assumptions on the characteristics of the agents have been partially relaxed. Both in the initial framework as in various generalized settings, the research program of equilibrium theory concentrates on:

- Sufficient conditions for equilibrium existence, a constantly revisited issue with each generalization or extension of the model.
- From consumer's point of view, (Pareto) optimality properties of equilibrium.
- For the sake of comparative analysis, local uniqueness and continuity properties of equilibria with respect to the initial data of the economy.
- And obviously, computation of equilibria.

Fixed point theory is solicited at each step of this program. Fixed points of functions and correspondences for existence of equilibria, of Pareto optimal allocations and more generally of core allocations, degree theory for local uniqueness and more generally continuity properties of equilibria with respect to the initial data of the economy, fixed point algorithms for the computation of equilibria. In this (partial) survey, we will focus on equilibrium existence, first in the classical case (Section 3), then (Sections 4 and 5) in two settings of economic interest with different imperfections as discontinuities or nonconvexities. Fixed point results related with a social equilibrium point of view will be the thread of our exposition.

2. A social equilibrium point of view

Among the few technical approaches to an equilibrium existence proof, the least demanding strategy in terms of sufficient assumptions consists in associating with the economy under consideration an *abstract economy* or generalized game. Such a game is completely specified by

$$\Gamma = \big(N, (X_i, \alpha_i, P_i)_{i \in N}\big)$$

where N is a finite set of agents (players) and, for each $i \in N$, X_i is a choice set (or strategy set), $\alpha_i \colon \prod_{k \in N} X_k \to X_i$ is a correspondence , called *constraint correspondence* and $P_i \colon \prod_{k \in N} X_k \to X_i$ is a correspondence termed *preference correspondence*. Under the condition $x_i \notin P_i(x)$, $P_i(x)$ is interpreted as the set of elements of X_i strictly preferred by player i to x_i, when the choice of the other players is $(x_k)_{k \neq i}$. This interpretation encompasses the case when the preferences of each player i are represented by a utility function $u_i \colon \prod_{k \in N} X_k \to \mathbb{R}$. Using the notation $x_{-i} = (x_k)_{k \neq i}$, then $P_i(x) = \{x_i' \in X_i \colon u_i(x_{-i}, x_i') > u_i(x_{-i}, x_i)\}$. The abstract economy $\Gamma = \big(N, (X_i, \alpha_i, u_i)_{i \in N}\big)$ is, for game theorists, a *generalized game*. If moreover, for every $i \in N$, for every $x \in \prod_{k \in N} X_k$, $\alpha_i(x) = X_i$, then Γ is a standard game in normal form.

Set $X = \prod_{i \in N} X_i$. Then a vector $\overline{x} = (\overline{x}_i)_{i \in N} \in X$ is an *equilibrium* of Γ if for each $i \in N$, both conditions hold:

$$\overline{x}_i \in \alpha_i(\overline{x}) \tag{2.1}$$

$$P_i(\overline{x}) \cap \alpha_i(\overline{x}) = \emptyset\,. \tag{2.2}$$

It is a *β-quasiequilibrium* of Γ if the last condition is replaced by $P_i(\overline{x}) \cap \beta_i(\overline{x}) = \emptyset$, where the correspondences $\beta_i \colon \prod_{k \in N} X_k \to X_i$ satisfy for all $x \in \prod_{k \in N} X_k$,

$$\beta_i(x) \subset \alpha_i(x), \tag{2.3}$$

$$\text{if } \beta_i(x) \neq \emptyset, \text{ then } \operatorname{cl}\big(\beta_i(x)\big) = \operatorname{cl}\big(\alpha_i(x)\big). \tag{2.4}$$

If the correspondences β_i are assumed to be nonempty valued, then a β-quasiequilibrium is what is called an equilibrium in Borglin–Keiding (16) and Yannelis–Prabhakar (68).

For an abstract economy whose strategy sets are finite dimensional compact convex sets, and under mild appropriate convexity and continuity assumptions on the correspondences, the existence proof in (35) of a β-quasiequilibrium, and then of an equilibrium of Γ, strongly relies on the following celebrated social equilibrium existence result due to Gale and Mas-Colell (40).

Lemma 2.1 (Gale–Mas-Colell)**.** *Given $X = \prod_{i=1}^m X_i$ where each X_i is a nonempty compact convex subset of some finite dimensional Euclidean vector space, for each i let $\varphi_i \colon X \to X_i$ be a convex (possibly empty) valued correspondence. Assume that for every i, φ_i is lower semicontinuous. Then there exists $\overline{x} \in X$ such that for each $i = 1, \ldots, m$,*

$$\textit{either } \varphi_i(\overline{x}) = \emptyset \ \textit{ or } \overline{x}_i \in \varphi_i(\overline{x}).$$

The proof of Lemma 2.1 uses a finite dimensional version of Michael's selection theorem (56) and an easy construction allowing one to invoke Kakutani's theorem (53). An infinite dimensional version of Lemma 2.1 is easy to obtain if the correspondences φ_i have open lower sections in X. Using the same construction as in the finite dimensional case, the proof invokes a Kakutani type theorem in topological vector spaces (18; 33). For the applications, the interest of the finite dimensional version of Lemma 2.1 is in only assuming the lower semicontinuity of the correspondences φ_i.

An interesting generalization of Lemma 2.1 is proved by Gourdel (44) using an extension theorem due to Cellina (22):

Lemma 2.2 (Gourdel)**.** *Given $X = \prod_{i=1}^{m+n} X_i$ where each X_i is a nonempty compact convex subset of some finite dimensional Euclidean vector space, let for each i: $\varphi_i \colon X \to X_i$ be a convex (possibly empty) valued correspondence. Assume that for every $i = 1, \ldots, m$, φ_i is lower semicontinuous, and that for every $i = m+1, \ldots, m+n$, φ_i is upper semicontinuous with compact values. Then there exists $\overline{x} \in X$ such that for each $i = 1, \ldots, m+n$,*

$$\textit{either } \varphi_i(\overline{x}) = \emptyset \ \textit{ or } \overline{x}_i \in \varphi_i(\overline{x}).$$

Lemma 2.2 has useful applications that we will see in Section 4. An immediate corollary of Lemma 2.2 is the following variant of the celebrated Gale–Nikaido–Debreu lemma (29; 39; 57), stated and proved in (35; 36) directly from Brouwer's theorem using the concept of continuous partition of unity.

Lemma 2.3 (Gale–Nikaido–Debreu)**.** *Let P be a convex cone with vertex 0 in $\mathbb{R}^L$, B the closed convex ball with center 0 and radius 1, S the corresponding sphere, and ζ an upper semicontinuous and nonempty compact, convex valued correspondence from $B \cap P$ into $\mathbb{R}^L$. If ζ satisfies the following boundary condition*

$$\forall p \in S \cap P, \ \forall z \in \zeta(p), \ p \cdot z \leq 0 \tag{2.5}$$

then there exists $\overline{p} \in B \cap P$ such that $\zeta(\overline{p}) \cap P^0 \neq \emptyset$, where $P^0 = \{z \in \mathbb{R}^L \mid z \cdot p \leq 0, \ \forall p \in P\}$ is the polar cone of P.

To see that Lemma 2.3 is a corollary of Lemma 2.2, if Z is a convex compact subset of $\mathbb{R}^L$ containing $\zeta(B \cap P)$, it suffices to apply Lemma 2.2 to the correspondences $\zeta\colon B \cap P \to Z$ and $\varphi\colon Z \times (B \cap P) \to B \cap P$ defined by $\varphi(p, z) = \{q \in B \cap P\colon q \cdot z > p \cdot z\}$. The conclusion of Lemma 2.3 easily follows from condition (2.5).

For a general equilibrium, $P = \mathbb{R}^L$ and $P = \mathbb{R}^L_+$ are the most interesting cases of Lemma 2.3.

Corollary 2.4. *Let, in $\mathbb{R}^L$, B be the closed unit-ball, S the unit-sphere, and ζ an upper semicontinuous and nonempty compact, convex valued correspondence from B into $\mathbb{R}^L$. If ζ satisfies the boundary condition*

$$\forall p \in S,\ \forall z \in \zeta(p),\ p \cdot z \leq 0 \tag{2.6}$$

then there exists $\overline{p} \in B$ such that $0 \in \zeta(\overline{p})$.

Corollary 2.5. *Let, in $\mathbb{R}^L$, B be the unit-ball, S the unit-sphere, and ζ an upper semicontinuous and nonempty compact, convex valued correspondence from $B \cap \mathbb{R}^L_+$ into $\mathbb{R}^L$. If ζ satisfies the boundary condition*

$$\forall p \in S \cap \mathbb{R}^L_+,\ \forall z \in \zeta(p),\ p \cdot z \leq 0 \tag{2.7}$$

then there exists $\overline{p} \in B \cap \mathbb{R}^L_+$ such that $\zeta(\overline{p}) \cap (-\mathbb{R}^L_+) \neq \emptyset$.

In the next section, the correspondence ζ will be interpreted as an excess demand correspondence and the condition (2.5) will be called *Walras law*. Corollary 2.4 will lead to the existence of an (exact) *equilibrium*. Corollary 2.5 will be used for proving the existence of a *free-disposal equilibrium* with positive prices.

From a mathematical point of view, Condition (2.5) in Lemma 2.3 is nothing but the boundary condition in the Ky Fan coincidence theorem (Theorem 5 in (34)) whose Lemma 2.3 is a simple consequence. It was first noted by Uzawa (62) that Brouwer and Kakutani theorems can be proved using the classical statement of the Gale–Debreu–Nikaido lemma. Here, Kakutani's theorem is a particular case of Lemma 2.2 and may also be easily deduced from Lemma 2.3, establishing that the fixed point lemmas most often used in equilibrium existence proofs form a circle of equivalent results.

3. Quasiequilibrium and equilibrium of a classical private ownership economy

Recall that an *equilibrium* of the private ownership economy $\mathcal{E}$ is a pair of an allocation $\big((\overline{x}_i)_{i\in I}, (\overline{y}_j)_{j\in J}\big) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$ and of a non-zero price $\overline{p} \in \mathbb{R}^L \setminus \{0\}$ such that

(1) for every $j \in J$, for every $y_j \in Y_j$, $\overline{p} \cdot y_j \leq \overline{p} \cdot \overline{y}_j$,
(2) for every $i \in I$, $\overline{x}_i$ is optimal for preferences P_i in the budget set

$$B_i(\overline{p}, (\overline{y}_j)_{j\in J}) := \Big\{x_i \in X_i\colon \overline{p} \cdot x_i \leq \overline{p} \cdot e_i + \sum_{j\in J} \theta_{ij} \overline{p} \cdot \overline{y}_j\Big\},$$

(3) $\sum_{i\in I} \overline{x}_i = \sum_{i\in I} e_i + \sum_{j\in J} \overline{y}_j$.

Condition (1) states that each producer maximizes his profit taking the equilibrium prices $\overline{p}$ as given. Condition (2) states that each consumer optimizes his preferences in his budget set taking as given equilibrium prices, his equilibrium revenue and, according to the definition of consumer i's preferences by the correspondence $P_i\colon \prod_{k\in I} X_k \to X_i$, the equilibrium consumption of the other consumers. Condition (3) states the exact feasibility (without disposal) of the equilibrium allocation. It is because the equilibrium definition requires the exact feasibility of the equilibrium allocation that the resulting equilibrium price need not be positive.

Notice that, in view of Condition (3), Condition (2) can be rephrased as

$$\text{for every } i \in I,\ \overline{p} \cdot \overline{x}_i = \overline{p} \cdot e_i + \textstyle\sum_{j \in J} \theta_{ij} \overline{p} \cdot \overline{y}_j \text{ and } [x_i \in P_i(\overline{x}) \Longrightarrow \overline{p} \cdot x_i > \overline{p} \cdot \overline{x}_i].$$

The *quasiequilibrium* definition keeps the profit maximization and feasibility conditions (1) and (3) of the previous definition and replaces preference optimization in the budget set by the requirement that each consumer binds his budget constraint and could not be strictly better off spending strictly less:

(2') for every $i \in I$, $\overline{p} \cdot \overline{x}_i = \overline{p} \cdot e_i + \sum_{j \in J} \theta_{ij} \overline{p} \cdot \overline{y}_j$ and $[x_i \in P_i(\overline{x}) \Longrightarrow \overline{p} \cdot x_i \geq \overline{p} \cdot \overline{x}_i]$.

When for each $i \in I$, consumer i's preferences do not depend on the consumption of the other agents, in particular when correspondences $P_i \colon X_i \to X_i$ are derived from utility functions $u_i \colon X_i \to \mathbb{R}$ and represented by $P_i(x_i) = \{x'_i \in X_i \colon u_i(x'_i) > u_i(x_i)\}$, let us define for every $p \in \mathbb{R}^L$ the functions and the (possibly empty valued) correspondences:

$$\pi_j(p) = \sup\{p \cdot y_j \colon y_j \in Y_j\} \text{ and } w_i(p) = p \cdot e_i + \textstyle\sum_{j \in J} \theta_{ij} \pi_j(p),$$
$$\psi_j(p) = \{y_j \in Y_j \colon p \cdot y_j = \pi_j(p)\},$$
$$\gamma_i(p) = \{x_i \in X_i \colon p \cdot x_i \leq w_i(p)\},$$
$$\delta_i(p) = \{x_i \in X_i \colon p \cdot x_i < w_i(p)\},$$
$$\xi_i(p) = \{x_i \in \gamma_i(p) \colon \gamma_i(p) \cap P_i(x_i) = \emptyset\},$$
$$\chi_i(p) = \{x_i \in \gamma_i(p) \colon \delta_i(p) \cap P_i(x_i) = \emptyset\},$$
$$\zeta(p) = \textstyle\sum_{i \in I} \xi_i(p) - \sum_{j \in J} \psi_j(p) - \sum_{i \in I} e_i,$$
$$\chi(p) = \textstyle\sum_{i \in I} \chi_i(p) - \sum_{j \in J} \psi_j(p) - \sum_{i \in I} e_i.$$

Each ψ_j can be thought of as the supply correspondence of producer j; each ξ_i can be seen as the demand correspondence of consumer i, each χ_i as his quasi-demand. Finally, ζ defines the *excess demand correspondence* in the economy and χ defines the *excess quasi-demand correspondence.* An equilibrium price of $\mathcal{E}$ is easily seen to be a zero of the excess demand correspondence: $0 \in \zeta(\overline{p})$; a quasiequilibrium price is defined as a zero of the excess quasi-demand correspondence: $0 \in \chi(\overline{p})$.

Since the seventies, following Gale and Mas-Colell (40), equilibrium existence for a finite dimensional classical economy is commonly proved using explicitly or implicitly equilibrium existence for the associated abstract economy (see (6; 16; 35; 40; 45; 59; 60; 61)) whose agents are the consumers, the producers and the Walrasian auctioneer. In this approach, using a quasiequilibrium existence result for compact abstract economies related with lemma 2.1 and standard techniques and tricks as explained in (7; 38) and summarized below, the existence of a quasiequilibrium can be obtained under minimal assumptions. Namely,

Consumption and production sets are convex and closed, initial endowments and profit shares allow for the *autarky* of each consumer ($e_i \in X_i - \sum_{j \in J} \theta_{ij} Y_j$), consumers' preference correspondences $P_i \colon \prod_{h \in I} X_h \to X_i$ are lower semi-continuous and, at any component of a feasible consumption allocation, have convex values and satisfy *local no satiation* and *irreflexivity* ($x_i \in \mathrm{cl}\, P_i(x) \setminus P_i(x)$); the set of feasible allocations is compact. The quasiequilibrium is an equilibrium if, in addition, preference correspondences have open values at any component of a feasible consumption allocation and if the economy, as a whole, satisfies *survival* and *irreduciblity* assumptions.

Actually, the idea of deducing equilibrium existence for an economy from the existence of an equilibrium in an abstract economy traces back to Debreu (28) and Arrow–Debreu (2). However, when applied in Arrow-Debreu (2) to an equilibrium existence proof of the abstract economy associated with the economy, the generalization in Debreu (28) of the Nash theorem is based on the convexity of values of correspondences which are nothing other than consumers' demand correspondences. It is well known that such a convexity property is strongly related with the Arrow-Debreu assumption of transitive and complete consumers' preferences on their consumption set. In this case, that is, when each consumer is assumed to have a complete preference preorder on his consumption set, quasiequilibrium existence can be based on Lemma 2.3 and its corollaries applied to the excess quasi-demand correspondence χ, easily proved to have convex values under the classical convexity assumption of individual complete preference preorders and to satisfy the Walras law. At the cost of a slight strengthening of continuity and convexity of preference, quasiequilibrium existence as well as equilibrium existence are obtained under basically the same other assumptions as when using the Gale and Mas-Colell lemma 2.1.

Whether the fixed point lemma used is Lemma 2.1 or Lemma 2.3, the main steps of the quasiequilibrium existence proof are the same. Taking advantage of the compactness of the set of all feasible allocations, the economy $\mathcal{E}$ is first replaced by a compact economy whose consumption and production sets are the intersection of the original ones with a closed ball of $\mathbb{R}^L$ whose interior contains any component of a feasible allocation. Prices are restricted to the closed unit-ball of $\mathbb{R}^L$. In order to get a non zero equilibrium price, budget constraint are suitably modified. Finally, a limiting process allows one to deduce quasiequilibrium existence in the original economy from its existence in the compact economy.

We conclude this section by pointing out that the Gale and Mas-Colell paper's fundamental contribution, and the main interest of a simultaneous optimization approach, is the existence of equilibrium without any convexity assumption on the values of consumers' demand correspondences, together with the possibility of considering dependent preferences (which may depend on the consumption of other consumers and even on prices and on the production allocation). Though requiring, with the existence of a complete preference preorder on their consumption set, stronger assumptions on the rationality of consumers, the excess demand approach is for economists the preferred approach to equilibrium existence. In addition, the existence of utility functions to describe consumers' preferences is required as soon as one wants to obtain some continuity properties of equilibrium.

Depending on the considered problem dealt with, various approaches are used when studying non classical economies. In the next sections, we will concentrate on two extensions of the classical economic model for which an adaptation of the fixed point methods is the key tool for obtaining equilibrium existence.

4. Equilibrium existence in economies with nonconvex technologies

A number of extensions of the classical model aim at improving its economic relevance by incorporating new types of commodities and agents (such as financial instruments and institutions) or by considering more complex but realistic assumptions both on the consumers and producers data. For instance, the recognition that some goods are indivisible, that there are anti-complementarities between commodities (e.g. non aversion to risk for agents facing uncertainty) translates into the nonconvexity of consumers' consumption sets and / or preferences.

On the production side, convexity of production sets implies, for instance, for producers who produce only one good, the convexity of the input requirement sets $V_j(y_j)$ (consisting of all input bundles from which it is possible to produce at least y_j units of output) and includes the case of so-called *constant returns to scale* (scaling output up by a constant can be achieved by scaling inputs equally, i.e. $u_j \in V_j(z_j) \Longrightarrow tu_j \in V_j(tz_j)$, for every $t \geq 0$, $(-u_j, z_j) \in Y_j$). This replication hypothesis is however not true in situations where, for instance, outputs increase by more than the scale of inputs (for example, an upside increase of input allowing for increased efficiency in modes of production and higher increase of output). A technology exhibiting such a pattern, known as *increasing returns to scale,* has a nonconvex production set.

The presence of increasing returns in production sectors such as railways or electricity production is recognized by economists since the first part of the past century. To guarantee an optimal provision, such goods, called 'public utilities', should be priced at their marginal cost (hence at a price satisfying the first order necessary conditions for optimality); the possible deficit in their optimal production explains the State intervention in non-profit maximizing sectors. This problem is at the origin of a long standing puzzle of economic theory: how to define a (decentralized) equilibrium, and a fortiori prove its existence, in case of increasing returns to scale in production. It is on this problem that we will focus our attention in this section, restricting ourselves to the case where only production sets may be nonconvex. Equilibrium with financial markets will be studied in the next section.

The marginal cost pricing rule and the correlative necessity of financing possible deficits motivate the following model of a nonconvex production economy and its equilibrium concept:

$$\mathcal{E} = \Big(\mathbb{R}^L, (X_i, P_i, e_i, r_i)_{i\in I}, (Y_j, \Phi_j)_{j\in J}\Big).$$

- As in the convex case, L is the finite set of goods. Each member i of a finite set of consumers I has a consumption set $X_i \subset \mathbb{R}^L$, an initial endowment $e_i \in \mathbb{R}^L$, a preference correspondence $P_i\colon \prod_{h\in I} X_h \to X_i$ assigning to each $x \in \prod_{h\in I} X_h$ the set of consumption vectors preferred to x_i given the consumption choices $(x_h)_{h\neq i}$ of the other consumers.
- Each firm of a finite set of producers J is characterized by a production set $Y_j \subset \mathbb{R}^L$ (whose boundary is denoted ∂Y_j) and by a *pricing rule* $\Phi_j\colon \mathbb{R}^L \setminus \{0\} \times \prod_{k\in J} \partial Y_k \to \mathbb{R}^L$ which establishes the jth firm's set of admissible prices as a function of market conditions.
- Finally, for each $i \in I$, the function $r_i\colon \mathbb{R}^L \setminus \{0\} \times \prod_{j\in J} \partial Y_j \to \mathbb{R}$, continuous and homogeneous of degree 1 with respect to its first variable, defines the wealth of the ith consumer for prices p and production plans $(y_j)_{j\in J} \in \prod_{j\in J} \partial Y_j$, under the assumption that $\sum_{i\in I} r_i(p, (y_j)_{j\in J}) = p \cdot (\sum_{i\in I} e_i + \sum_{j\in J} y_j)$. This abstract wealth structure clearly encompasses the case of the private ownership economy studied in Section 3 with profit maximizing producers where for each $i \in I$, $r_i(p, (y_j)_{j\in J}) = p \cdot e_i + \sum_{j\in J} \theta_{ij} p \cdot y_j$.

Here, an *equilibrium* for the economy $\mathcal{E}$ is a pair of an allocation $\big((\overline{x}_i)_{\in I}, (\overline{y}_j)_{j\in J}\big) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$ and of a non-zero price vector $\overline{p} \in \mathbb{R}^L \setminus \{0\}$ such that

(1) for every $j \in J$, $\overline{y}_j \in \partial Y_j$ and $\overline{p} \in \Phi_j\big(\overline{p}, (\overline{y}_k)_{k\in J}\big)$,
(2) for every $i \in I$, $\overline{x}_i$ is optimal for preferences P_i in the budget set

$$B_i(\overline{p}, (\overline{y}_j)_{j\in J}) := \{x_i \in X_i\colon \overline{p} \cdot x_i \leq r_i\big(\overline{p}, (\overline{y}_j)_{j\in J})\},$$

(3) $\sum_{i\in I} \overline{x}_i = \sum_{i\in I} e_i + \sum_{j\in J} \overline{y}_j$.

Condition (1) states that the equilibrium production plan of j belongs to the boundary of Y_j and that, given the prevailing market conditions, $\overline{p}$ is for j an admissible price. Condition (2) states that, taking as given the equilibrium prices and the equilibrium consumption plans of the other consumers, $\overline{x}_i$ is an optimal choice for consumer i in the budget set defined by his equilibrium revenue. Condition (3) states the exact feasibility (without disposal) of the equilibrium allocation.

Let $S = \{p \in \mathbb{R}^L_+ \colon \sum_{k\in L} p_k = 1\}$ be the unit-simplex of $\mathbb{R}^L$. In view of the free-disposal assumption which will be made on production sets, the price set can be assumed to be restricted to S. Let

$$A(\mathcal{E}) = \Big\{\big((x_i)_{i\in I}, (y_j)_{j\in J}\big) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j \colon \sum_{i\in I} x_i \le \sum_{i\in I} e_i + \sum_{j\in J} y_j\Big\}$$

be the set of free-disposal feasible allocations, and for every $e' \ge e = \sum_{i\in I} e_i$, let

$$A(e') = \Big\{(y_j)_{j\in J} \in \prod_{j\in J} Y_j \colon \sum_{j\in J} y_j + e' \in \sum_{i\in I} X_i + \mathbb{R}^L_+\Big\}$$

be the set of feasible productions when the total endowment is e'.

Denote also

$$\mathrm{PE} = \Big\{\big(p, (y_j)_{j\in J}\big) \in S \times \prod_{j\in J} \partial Y_j \colon p \in \bigcap_{j\in J} \Phi_j\big(p, (y_j)_{j\in J}\big)\Big\}$$

$$\mathrm{APE} = \Big\{\big(p, (y_j)_{j\in J}\big) \in \mathrm{PE}\colon \sum_{j\in J} y_j \in -\sum_{i\in I} e_i + \sum_{i\in I} X_i + \mathbb{R}^L_+\Big\}$$

the set of production equilibria and the set of free-disposal feasible production equilibria respectively.

The first equilibrium existence results (3; 4; 19; 55) have been obtained assuming that nonconvex production sets have a smooth boundary. Without this hypothesis, and under basically the same assumptions on the consumption side as in section 3 (see the framed text), the following set of assumptions, listed by Gourdel (44), covers a significant number of existing results asserting that , under appropriate boundedness and survival assumptions, an equilibrium exists when firms follow 'regular' pricing rules that guarantee 'bounded losses' and when the wealth structure is 'compatible' with firm's behavior:

*Assumption (**P**):* Y_j is nonempty and closed for all j, and $Y_j - \mathbb{R}^L_+ \subset Y_j$.

*Assumption (**B**):* For every i, the feasible set $\widehat{X}_i$ (projection of $A(\mathcal{E})$ on X_i) is bounded and for every $e' \ge e$, the set $A(e')$ is compact.

*Assumption (**PR**):* For every $j \in J$, the pricing rule correspondence $\Phi_j \colon S \times \prod_{k\in J} \partial Y_k \longrightarrow \mathbb{R}^L$ is upper semicontinuous with nonempty convex compact values.

*Assumption (**BL**):* (bounded losses) There exists $\alpha \in \mathbb{R}$ such that for every $\big(p, (y_k)_{k\in J}\big) \in S \times \prod_{k\in J} \partial Y_k$, for every $j \in J$, and for every $q \in \Phi_j(p, (y_k))$, $q \cdot y_j \ge \alpha$.

*Assumption (**SA**):* (survival) for every $\big(p, (y_j)_{j\in J}\big) \in \mathrm{PE}$, for every $e' \gg e$, $(\sum_{j\in J} y_j + e' \in \sum_{i\in I} X_i + \mathbb{R}^L_+$ implies $p \cdot (\sum_{j\in J} y_j + e') > \inf p \cdot \sum_{i\in I} X_i$

*Assumption (**R**):* for every $(p,(y_j)_{j\in J})\in APE$

$$[p\cdot(\sum_{j\in J}y_j+e)>\inf p\cdot\sum_{i\in I}X_i]\Longrightarrow r_i(p,(y_j)_{j\in J}))>\inf p\cdot X_i\ \forall i\in I.$$

Among the above assumptions whose similar statements may differ according to the authors, the most questionable is the monotonicity assumption in (**P**): $\forall j\in J$, $Y_j-\mathbb{R}^L_+\subset Y_j$ (anything less than a feasible for j production plan is also feasible for him). Termed as *free-disposal*, it implies that unwanted goods can be disposed off at no cost; clearly an unwarranted assumption in many situations (nuclear wastes' disposal, for instance, is very costly).

Under Assumption (**P**), ∂Y_j, the boundary of the production set Y_j, coincides with the set of the (weakly) efficient production plans and (normalized) equilibrium prices can be researched in S. Moreover, under the free-disposal assumption, the set ∂Y_j can be made homeomorphic to an Euclidean space of dimension $(|L|-1)$, and hence, as written by Villar (64), "the nonconvexity can be handled in the convex mirror's image".

Assumption (**B**) is obviously stronger than the compactness of the set of feasible allocations assumed in the convex case. It is satisfied whenever consumption sets are bounded below and $\mathbf{A}(\sum_{j\in J}Y_j)\cap-\mathbf{A}(\sum_{j\in J}Y_j)=\{0\}$ where $\mathbf{A}(\sum_{j\in J}Y_j)$ is the asymptotic cone of the aggregate production set (see Hurwicz and Reiter (49))

Assumption (**PR**) covers the case where firms follow the *marginal cost pricing rule* (MPR)

$$\Phi_j(p,(y_k)_{k\in J}))=N_{Y_j}(y_j)\cap S$$

where $N_{Y_j}(y_j)$ is the normal cone of nonsmooth analysis (see Clarke (24; 25)) introduced by Cornet (26) as the proper way of defining marginal pricing in the general case. When Y_j is convex, $N_{Y_j}(y_j)$ is the normal cone of convex analysis and $p\in\Phi_j(y_j)$ is simply profit maximization as in the definition of equilibrium in Section 3 above. When Y_j has smooth boundary, $N_{Y_j}(y_j)$ is simply the outer normal (see e.g. Beato (3; 4), Mantel (55), Brown and Heal (19)).

Assumption (**SA**) strengthens the global survival of the economy assumed in the convex case. It expresses the fact that if at a production equilibrium, production is feasible with a larger initial endowment $e'\geq e$, then the total wealth may be distributed among consumers so as to keep each one of them above his subsistence level. Kamiya (52) points out that it is not possible to restrict (SA) to feasible production equilibria only (see also counterexample in Bonnisseau and Cornet (13)).

Assumption (**R**) corresponds to the irreducibility condition assumed in Section 3.

The equilibrium existence proof given in (44) is rather intricate but its principle is simple. After compactification of the economy, the existence of a quasiequilibrium is obtained using the hybrid lsc/usc fixed point on components result (Lemma 2.2) obviously tailored for this situation. The most interesting feature of this simultaneous optimization approach is that it requires, aside from consumers, producers and the usual Walrasian auctioneer, the intervention of an additional agent, a government regulator who, roughly speaking, chooses new firm's production plans in function of the difference between each pricing rule and the proposed price. Such a regulation still exists but is less visible in the equilibrium process related with excess demand approaches. Going from quasiequilibrium existence to equilibrium existence is standard.

Most of existing equilibrium existence results in nonconvex economies require free-disposal in production, see e.g. (3; 4; 5; 11; 12; 13; 14; 15; 19; 20; 26; 27; 31; 52; 55; 63; 64; 65; 66). Free-disposal for nonconvex economies is relaxed in Jouini (51) to weak free elimination, a weaker version of free-disposal, while Hamano (46) establishes the existence of marginal cost pricing equilibria without free-disposal as well as with general pricing rules for technologies under increasing returns to scale with nonempty interior and strictly star-shaped production sets.

5. Fixed point and Incomplete markets theory

In the Arrow–Debreu model studied in Section 3, all commodities are traded at once, no matter when they are consumed, or under what state of nature. We now turn to exchange models where consumers face a multiplicity of budget constraints at different times and under different states of nature, and hold assets in order to transfer wealth between budget constraints. The purpose of this section is to explain some developments connected with fixed point theory for equilibrium existence in these models.

Actually, as explained in (37; 41; 54), such an integration of finance, time and uncertainty may be easy. In the model defined below, under some specific assumptions, like:

- completeness of the markets (1), which means that there are enough independent assets to offset the uncertainty in the model
- short-sales constraints on the possible portfolios (see 5.2 below)
- specific classes of assets, for example nominal assets (21; 67) or numeraire assets (23; 42)

the existence of an equilibrium can be proved using the same fixed-point methods as in Section 3. We will focus in this section on the equilibrium existence problem under the more realistic assumption of incompleteness of the markets, without short-sales constraints and for general classes of assets. Then, the fixed-point results of Section 2 cannot be used anymore to yield the existence of an equilibrium and have to be extended. A mathematical reason is the following: the principal new object capturing the incompleteness of the markets and the generality of assets is the return matrix $V(p)$, a matrix specifying, for each price vector p of the economy, the returns of all the assets in each possible state of nature. The drops in rank of $V(p)$ at some price vectors p (called 'bad prices') create some discontinuities of the *market span*, by definition $\operatorname{span} V(p)$ (intuitively, the space of financial opportunities); this may create discontinuities of the budget correspondence of the consumers, and consequently of their best-reply correspondence. A possible solution to this problem is to notice that the set of bad prices should be exceptional (for sufficiently well-behaved mapping $V(p)$), and, using an extended fixed point argument, to prove the generic existence of equilibrium.

5.1. The General Equilibrium with Incomplete markets model (GEI).

Let us consider two time periods, today and tomorrow, an a priori uncertainty about which of a finite set S of states of nature, $s = 1, \ldots, S$, will occur tomorrow, and a positive finite number K of physical goods, $k = 1, \ldots, K$, available today and in each state of nature tomorrow. For convenience, $s = 0$ denotes the state of nature (known with certainty) today.

A GEI model is a two-period stochastic exchange economy described by the list

$$\mathcal{E} = \big(\mathbb{R}^L, \mathbb{R}^J, (X_i, Z_i, P_i, e_i)_{i\in I}, V\big).$$

Here, I is the finite set of agents. The set $X_i \subset R^L$ is the consumption set of the i-th consumer. where L is the number of possible contingents goods ($L = K(1+S)$). The correspondence $P_i \colon \prod_{h \in I} X_h \to X_i$ is the preference correspondence of agent i, defined as in Section 3; the vector $e_i \in X_i$ represents his endowment in each physical good at each possible state of nature. There are J assets and the set $Z_i \subset \mathbb{R}^J$, the portfolio set of agent i, is the set of all possible (for i) portfolios, a portfolio specifying the amount of each asset. Last, $V \colon \mathcal{P} \to \mathcal{M}(S \times J)$ is a continuous mapping from the price set $\mathcal{P} \subset \mathbb{R}^L$ (to be precised later) to the set of $(S \times J)$-matrices denoted $\mathcal{M}(S \times J)$. Asset $j \in \{1, ..., J\}$, represented for each $p \in \mathcal{P}$ by the $j-$th column of $V(p)$, is a (random) variable specifying the return of this asset for each state of nature $s \in \{1, ..., S\}$, such return continuously depending on the price $p \in \mathcal{P}$.

Before defining equilibrium of the model, and in order to simplify the presentation, let us introduce the following notation: for every price $p = (p(0), p(1), ..., p(S)) \in R^{K(1+S)}$, where $p(s) \in R^K$ for every $s \in \{0, 1, ..., S\}$, and for every consumption bundle $x = (x(0), ..., x(S)) \in R^{K(1+S)}$, where $x(s) \in R^K$ for every $s \in \{0, 1, ..., S\}$, we denote by $p \square x$ the vector in R^S defined by $p \square x = (p(1) \cdot x(1), p(2) \cdot x(2), ..., p(S) \cdot x(S))$: it is the vector of contingent values of x, given the contingent price $p \in \mathcal{P}$, for every state of nature $s \in \{1, ..., S\}$.

If we assume for simplicity the strict monotonicity of consumers' preferences in each state of nature, after some standard transformations using the necessary no-arbitrage property of equilibrium prices, the budget set of each agent i can be written for every $p \in \mathcal{P}$.

$$B_i(p) = \{(x_i, z_i) \in X_i \times Z_i \colon \; p \cdot (x_i - e_i) = 0, \; p \square (x_i - e_i) = V(p) z_i\}$$

Notice that the first equality is a standard budget constraint in the complete markets model; the second equality means roughly that, at each state of nature $s = 1, \ldots, S$, each consumption bundle of agent i can be financed by agent i's endowment e_i and by its portfolio return $V(p)z_i$. In addition, the price set $\mathcal{P}$ can be taken equal to the unit-simplex $S_+^{L-1} = \{p \in \mathbb{R}_+^L \colon \sum_{\ell=1}^L p_\ell = 1\}$ of $\mathbb{R}^L$. In the following, we define $S_{++}^{L-1} = \{p \in S_+^{L-1} \colon \forall \ell \in L, \; p_\ell > 0\}$.

A *financial equilibrium* for the economy $\mathcal{E}$ is defined as an element $\big((\overline{x}_i, \overline{z}_i)_{i \in I}, \overline{p}\big) \in \prod_{i \in I} (X_i \times Z_i) \times \mathcal{P}$ such that, if we let $\overline{x} = (\overline{x}_i)_{i \in I}$, one has:

(1) for every $i \in I$, $(\overline{x}_i, \overline{z}_i) \in B_i(\overline{p})$ and $(P_i(\overline{x}) \times Z_i) \cap B_i(\overline{p}) = \emptyset$,
(2) $\sum_{i \in I} \overline{x}_i = \sum_{i \in I} e_i$;
(3) $\sum_{i \in I} \overline{z}_i = 0$.

As usual, condition (1) expresses that, if consumer i takes as given the equilibrium consumption plans of the other agents and the equilibrium contingent prices, $(\overline{x}_i, \overline{z}_i)$ is an optimal choice for each consumer $i \in I$ in his budget set. Conditions (2) and (3) express market clearing on the markets for contingent goods and assets, assuming that agents have no initial endowment in assets.

5.2. **Existence of a financial equilibrium for bounded below portfolio sets.** Radner (58) was the first to formulate the archetype of a very general multi-period stochastic economy with multiple commodities, multiple budget constraints and really incomplete markets; he proved the existence of an equilibrium under the condition that the economy is bounded. In the much simpler GEI model, Radner's boundedness assumption amounts to assume that, for some negative constant C and for every $i \in I$, $Z_i = \{z \in \mathbb{R}^J \colon z_j \geq C \; \forall j \in J\}$

In view of the assumed continuity of V, the important property for existence of equilibrium is that the previously defined budget sets are closed correspondences. Under standard assumptions on consumption sets, preferences and endowments, similar to those of Section 3 and if portfolio sets are closed and bounded below, the same techniques can be used again. In particular, since each feasible portfolio set $\widehat{Z}_i = \{z_i \in Z_i \colon z_i + \sum_{h\neq i} z_h = 0 \text{ for some } (z_h) \in \prod_{h\neq i} Z_h\}$ is compact, one can replace the original economy by a compact one, intersecting original consumption and portfolio sets with balls of radius large enough to contain in their interior all feasible consumption and portfolio sets. Existence of a quasiequilibrium in the compact economy can be obtained using Lemmas 2.1 or 2.3 and then extended to the original economy. Under the survival assumption $e_i \in \operatorname{int} X_i$ for each $i \in I$, the quasi-equilibrium is an equilibrium.

5.3. **The Grassmanian approach to the definition of pseudo-equilibrium.** Suppose now that the Z_i are not bounded below, or even to simplify that $Z_i = \mathbb{R}^J$ for every $i \in I$. Clearly, the set of feasible portfolios need not be compact and the previous approach may not be used.

Despite this, one could try to adapt the previous approach, just using x and p as variables. Indeed, Condition (1) of the definition of a financial equilibrium may be written:

(1) for every $i \in I$, $\overline{x}_i \in B_i(\overline{p})$ and $P_i(\overline{x}) \cap B_i(\overline{p}) = \emptyset$

for budget correspondences $B_i(\overline{p})$ defined by

$$B_i(p) = \{x_i \in X_i,\ \overline{p}.(x_i - e_i) = 0,\ \overline{p}\Box(x_i - \overline{e}_i) \in \operatorname{span} V(\overline{p})\}.$$

The main problem is now that, for general structures of assets, the so-defined budget correspondences may not be upper semicontinuous because for some p the rank of $V(p)$ may drop. Thus, because of such discontinuities, any fixed point lemma of Section 2 cannot be used for proving the equilibrium existence. Moreover, in this case, equilibrium may fail to exist, as Hart's counterexample in (47) has shown.

As explained in (41; 54), a new idea is to relax the previous budget constraint, defining

$$B_i(p, E) := \{x_i \in X_i \colon\ p \cdot (x_i - e_i) = 0,\ p\Box(x_i - e_i) \in E\}$$

where E is some J-dimensional subspace of $\mathbb{R}^S$ containing (and not equal to) $\operatorname{span} V(p)$. This relaxation leads to the notion of pseudo-equilibrium.

In the following, let $G^J(\mathbb{R}^S)$ denotes the set of all J-subspaces of $\mathbb{R}^S$. As well-known, $G^J(\mathbb{R}^S)$ can be given a system of local neighborhoods with respect to which it becomes a smooth compact manifold without boundary of dimension $J(S-J)$ called *Grassmanian manifold*.

A *pseudo-equilibrium* of the economy $\mathcal{E}$ is an element $(\overline{x}, \overline{p}, \overline{E}) \in \prod_{i\in I} X_i \times S_+^{L-1} \times G^J(\mathbb{R}^S)$ such that

(1) for every $i \in I$, $\overline{x}_i \in B_i(\overline{p}, \overline{E})$ and $P_i(\overline{x}) \cap B_i(\overline{p}, \overline{E}) = \emptyset$,
(2) $\sum_{i\in I} \overline{x}_i = \sum_{i\in I} e_i$;
(3) $\operatorname{span} V(\overline{p}) \subset \overline{E}$.

It is easily seen that a pseudo-equilibrium $(\overline{x}, \overline{p}, \overline{E})$ is a financial equilibrium if $\operatorname{span} V(\overline{p}) = \overline{E}$.

5.4. **Fixed-point-like theorems.** The following theorems allow for pseudo-equilibrium existence proofs based, after compactification of the GEI model, on excess demand and/or simultaneous optimization approaches.

The first one is the natural extension of the Debreu excess-demand approach in (30) to incomplete markets. It is proved in (48), (32), (43) and (50). On the consumption side, the assumptions are the usual ones for a differentiable approach. Preferences of the agents, represented by C^∞ utility functions on $\mathbb{R}^L_{++}$, are differentiably strictly monotone, differentiably strictly convex and such that every indifference surface is closed in $\mathbb{R}^L$. Theorem 5.1 states directly the existence of pseudo-equilibrium.

Theorem 5.1 (Duffie–Geanakoplos–Hirsch–Husseiny–Lasry–Magill–Mas-Colell–Shafer)**.** *Let* $V : S^{L-1}_{++} \to \mathcal{M}(S \times J)$ *be a continuous mapping, where* $\mathcal{M}(S \times J)$ *denotes the set of* $S \times J$*-matrices. Let* $z\colon S^{L-1}_{++} \times G^J(\mathbb{R}^S) \to \mathbb{R}^L$ *be a continuous function such that* $p \cdot z(p, E) = 0$ *for every* $(p, E) \in S^{L-1}_{++} \times G^J(\mathbb{R}^S)$*, and such that for every sequence* $(p^n, E^n) \in S^{L-1}_{++} \times G^J(\mathbb{R}^S)$ *converging to* $(p, E) \in \partial S^{L-1}_{++} \times G^J(\mathbb{R}^S)$*, then* $\lim_{n\to\infty} z_\ell(p^n, E^n) = +\infty$ *for every* ℓ *such that* $p_\ell = 0$*.*

Then there exists $(\overline{p}, \overline{E}) \in S^{L-1}_{++} \times G^J(\mathbb{R}^S)$ *such that* $z(\overline{p}, \overline{E}) = 0$ *and Span*$V(\overline{p}) \subset \overline{E}$*.*

The (first) proof of Duffie and Shafer (32) uses modulo 2 degree theory. Geanakoplos and Shafer (43) first prove that the set of solutions of the equation span $V(p) \subset E$ is a manifold, then they prove that there exists a solution of $z(p, E) = 0$ on this manifold, using the homotopy invariance of topological degree; Husseiny et al. (50) use characteristic classes of vector bundles, and Hirsch, Magill and Mas-Colell (48) use intersection theory of vector bundles. A byproduct is the following fixed-point-like theorem on subspaces that, as remarked by all, admits as corollaries the Brouwer and Borsuk–Ulam theorems.

Theorem 5.2. *For every* $j = 1, \dots, J$*, let* $\theta^j\colon G^J(\mathbb{R}^S) \to \mathbb{R}^S$ *be a continuous function. Then, there exists* $\overline{E} \in G^J(\mathbb{R}^S)$ *such that for every* $j = 1, \dots, J$*,* $\theta^j(\overline{E}) \in \overline{E}$*.*

The next theorem allows for proving the existence of pseudo-equilibrium using a simultaneous optimization approach

Theorem 5.3 (Bich–Cornet)**.** *For* $J \leq S$*, let* $G^J(\mathbb{R}^S)$ *be the Grassmanian manifold of all* J*-dimensional subspaces of* $\mathbb{R}^S$*, For each* $i = 1, \dots, n$*, let* C_i *be a nonempty, convex and compact subset of some Euclidean vector space. Defined on* $M := \prod_{i=1}^n C_i \times G^J(\mathbb{R}^S)$*, let us consider, for every* $i = 1, \dots n$*, lower semicontinuous and convex valued correspondences* $\varphi_i\colon M \to C_i$ *and, for each* $j = 1, \dots, J$*, continuous functions* $\theta^j\colon M \to \mathbb{R}^S$*. Then there exists* $\overline{m} = \big(\overline{x}_1, \dots, \overline{x}_n, \overline{E}\big) \in M$ *such that*

$$\forall i = 1, \dots, n, \ \text{either } \varphi_i(\overline{m}) = \emptyset \ \text{ or } \overline{x}_i \in \varphi_i(\overline{m})$$
$$\forall j = 1, \dots, J, \ \theta^j(\overline{m}) \in \overline{E}.$$

Lemma 2.1 is obviously a particular case of this result.

Theorem 5.3 is proved starting from a fixed-point-like theorem equivalent to Theorem 5.1. An hybrid version of Theorem 5.3 for convex valued correspondences which are either lsc or usc with compact values can be found in (10); it establishes that, as in Section 2, the different fixed-point-like theorems of this section form a circle of equivalent results.

A more recent approach for proving the existence of a pseudo-equilibrium (8) rests on the following discontinuous generalization of Brouwer's theorem. One considers a continuous mapping $V\colon B \to \mathcal{M}(S \times J)$ (where B is the closed unit ball of a Euclidean space); one says that a mapping $f : B \to B$ is V-continuous if for every sequence (x_n) of B converging to $x \in B$ and such that the sequence span$V(x_n)$ is a convergent sequence of $G^J(\mathbb{R}^S)$, then $f(x_n)$ converges.

Theorem 5.4. *For an open and dense subset of mappings V (for the topology of uniform convergence), every V-continuous mapping $f : B \to B$ admits an approximate fixed point x, which means that there exists a sequence (x_n) of $\frac{1}{n}$-fixed point converging to x such that $SpanV(x_n)$ is a sequence of $G^J(\mathbb{R}^S)$ which converges.*

It is easy to obtain from this theorem the existence of a pseudo-equilibrium: indeed, notice that the excess demand $z\big(p, \operatorname{span} V(p)\big)$ of a GEI economy is easily seen to be V-continuous, where V is the asset structure of the GEI model. Besides, one can classically formulate the equilibrium existence problem $z\big(p, \operatorname{span} V(p)\big) = 0$ as a fixed-point existence problem for which one can apply Theorem 5.4; this provides a sequence $(p_n, \operatorname{span} V(p_n)) \in S^{L-1}_{++} \times G^J(\mathbb{R}^S)$ converging (up to an extraction) to $(\overline{p}, \overline{E}) \in S^{L-1}_{++} \times G^J(\mathbb{R}^S)$ where $\lim_{n\to\infty} z\big(p_n, \operatorname{span} V(p_n)\big) = 0$; $(\overline{p}, \overline{E})$ is easily seen to be a pseudo-equilibrium.

5.5. **Pseudo-equilibrium and generic existence of a financial equilibrium.** Having the existence of a pseudo-equilibrium, one can obtain the *generic* existence of an equilibrium by making a perturbation argument which rests on Sard's theorem. Here generic means several things: Duffie and Shafer (32) have proved that for generic endowments and real asset structures (which means for an open and dense subset of initials endowment and real asset structures), then a pseudo-equilibrium is an equilibrium, which yields the existence of an equilibrium. Bottazzi (17) has improved the previous result, proving that for an explicit class of continuous asset structure (which strictly contains an open and dense class of real asset structures) and for generic endowments, then a pseudo-equilibrium is an equilibrium. Recall that, in view of Hart's counterexample, there is no hope of having a general non generic existence result.

References

[1] K.J. Arrow, *Le rôle des valeurs boursires pour la répartition la meilleure des risques.* Cahiers du Séminaire d'conométrie, Paris (1953), translated as *The role of securities in the optimal allocation of risk-bearing*, Review of Economic Studies **31** (1964), 91–96.

[2] K.J. Arrow and G. Debreu, *Existence of an equilibrium for a competitive economy.* Econometrica **22** (1954), 265–290.

[3] P. Beato, *Marginal cost pricing and increasing returns.* PhD Dissertation, University of Minnesota, 1979.

[4] P. Beato, *The existence of marginal cost pricing equilibria with increasing returns.* Quart. J. Econom. **389** (1982), 669–688.

[5] P. Beato and A. Mas-Colell, *On marginal cost pricing with given tax-subsidy rules.* J. Econom. Theory **37**, 356–365.

[6] T.C. Bergstrom, *The existence of maximal elements and equilibria in the absence of transitivity.* Technical report, University of Michigan, 1975.

[7] T.C. Bergstrom, *How to discard "free disposability" - At no cost.* J. Math. Econom. **3** (1976), 131–134.

[8] P. Bich, *On the existence of approximated equilibria in discontinuous economies.* J. Math. Econom. **42** (2006), 452–470.

[9] P. Bich and B. Cornet, *Existence of financial equilibria: space of transfers of fixed dimension.* Working Paper, Paris 1 University (1998).

[10] P. Bich and B. Cornet, *Fixed-point-like theorems on subspaces.* Fixed Point Theory and Applications, Volume 2004 (2004), Issue 3, 159-171.
[11] J-M. Bonnisseau, *On two existence results of equilibria in economies with increasing returns.* J. Math. Econom. **17** (1988), 193–207.
[12] J-M. Bonnisseau, *Existence of equilibria in the presence of increasing returns: A synthesis.* J. Math. Econom. **21** (1992), 441–452.
[13] J-M. Bonnisseau and B. Cornet, *Existence of equilibria when firms follow bounded losses pricing rules.* J. Math. Econom. **17** (1988), 119–147.
[14] J-M. Bonnisseau and B. Cornet, *Existence of marginal cost pricing equilibria in economies with several nonconvex firms.* Econometrica **58** (1990), 661–682.
[15] J-M. Bonnisseau and B. Cornet, *Existence of marginal cost pricing equilibria: The nonsmooth case.* Internat. Econom. Rev. **31** (1990), 685–708.
[16] A. Borglin and H. Keiding, *Existence of equilibrium actions and of equilibrium: A note on the 'new' existence theorems.* J. Math. Econom. **3** (1976), 313–316.
[17] J.M. Bottazi, *Incomplete markets: transverse financial structures* Econom. Theory **20** (2002), 67–82
[18] F. Browder, *The fixed point theory of multivalued mappings in topological vector spaces.* Math. Ann. **177** (1968), 283–301.
[19] D.J. Brown and G. Heal, *Existence, local uniqueness and optimality of a marginal cost pricing equilibrium with increasing returns.* Social Science Working Paper No. 415, California Institute of Technology, Pasadena, CA, 1982.
[20] D.J. Brown, G. Heal, A. Kahn and R. Vohra, *On a general existence theorem for marginal cost pricing equilibria.* J. Econom. Theory **38** (1986), 371–379.
[21] D. Cass, *Competitive equilibrium with incomplete financial markets.* CARESS Working Paper 84-09, University of Pennsylvania (1984), published in J. Math. Econom. **42** (2006), 384–405.
[22] A. Cellina, *A theorem on the approximation of compact multivalued mappings.* Atti. Mat. Naz. Lincei **8** (1969), 149–153.
[23] S. Chae, *Existence of competitive equilibrium with incomplete markets.* J. Econom. Theory **44**, 179–188.
[24] F. Clarke, *Generalized gradients and applications.* Trans. AMS**207**, 247–262.
[25] F. Clarke, *Optimization and Nonsmooth Analysis.* New york, John Wiley, 1983.
[26] B. Cornet, *Existence of equilibria in economies with increasing returns*, Berkeley Working Paper (IP 311, 1982), published in *Contributions to Economics and Operations Research, The XXth Anniversary of the CORE* (B. Cornet and H. Tulkens eds.) The MIT Press, Cambridge Ms, 1990.
[27] B. Cornet, *General equilibrium theory and increasing returns.* J. Math. Econom. **17** (1988), 117–118.
[28] G. Debreu, *A social equilibrium existence theorem.* Proc.Natl. Acad. Sci. USA **38** (1952), 886–893.
[29] G. Debreu, *Market equilibrium.* Proc.Natl. Acad. Sci. USA **52** (1956), 876–878.
[30] G. Debreu, *Economies with a finite set of equilibria* Econometrica **38** (1970), 387–392.
[31] E. Dierker, R. Guesnerie and W. Neuefeind, *General equilibrium when some firms follow special pricing rules.* Econometrica **53** (1985), 1369–1393.
[32] D. Duffie and W. Shafer, *Equilibrium in incomplete markets: I A basic model of generic existence.* J. Math. Econom. **14** (1985), 285-300.
[33] K. Fan, *A generalization of Tichonoff' fixed point theorem.* Math. Ann. **142** (1961), 305–310.

[34] K. Fan, *A minimax inequality and applications.* In Inequalities III (O. Shisha ed.) Academic Press, New york and London, 1972, 103–113.
[35] M. Florenzano, *L'équilibre économique géneral transitif et intransitif: problèmes d'existence* Ed. CNRS, Paris, 1981.
[36] M. Florenzano, *The Gale–Nikaido–Debreu lemma and the existence of transitive equilibrium with or without the free-disposal assumption.* J. Math. Econom. **9** (1982), 113–134.
[37] M. Florenzano, *General equilibrium of financial markets: An introduction*, Cahiers de la MSE 1999.76, 1999.
[38] M. Florenzano, *General Equilibrium Analysis - Existence and Optimality Properties of Equilibria* Kluwer, Boston/Dordrecht/London, 2003.
[39] D. Gale, *The law of supply and demand.* Math. Scand. **3** (1955), 155–169.
[40] D. Gale and A. Mas-Colell, *An equilibrium existence theorem for a general model without ordered preferences.* J. Math. Econom. **2** (1975), 9–15 (Erratum ibidem **6**, 297–298 (1979))
[41] J. Geanakoplos, *An introduction to general equilibrium with incomplete asset markets.* J. Math. Econom. **19** (1990), 1–38.
[42] J. Geanakoplos and H. Polemarchakis, *Existence, regularity, and constraint suboptimality of competitive allocations when markets are incomplete.* In W. Haller, R. Starr and D. Starret, eds, *"Equilibrium Analysis", Essays in Honor of Kenneth Arrow* Vol. III, Chap. 3, Cambridge University Press, Cambridge (1986)
[43] J. Geanakoplos and W. Shafer, *Solving systems of simultaneous equations in economics.* J. Math. Econom. **19** (1990), 69–93.
[44] Gourdel, P., *Existence of intransitive equilibria in nonconvex economies.* Set-valued Analysis **3** (1995), 307–337.
[45] J. Greenberg, *Quasi-equilibrium in abstract economies without ordered preferences.* J. Math. Econom. **4** (1977), 163–165.
[46] T. Hamano, *On the existence of equilibria without convexity or free disposal on production technologies*, J. Math. Econom. **23** (1994), 565–583.
[47] O. Hart, *On the optimality of equilibrium when the market structure is incomplete.* J. of Econom. Theory **11** (1975), 418–443.
[48] M. Hirsch, M. Magill and A. Mas-Colell, *A geometric approach to a class of equilibrium existence theorems.* J. Math. Econom. **19** (1990), 95–106.
[49] L. Hurwicz and S. Reiter, *The boundedness of the feasible set without convexity assumption.* International Econom. Review **14** (1973), 580–586.
[50] S.Y. Husseini, J.M. Lasry and M. Magill, *Existence of equilibrium with incomplete markets.* J. Math. Econom. **19** (1990), 39–67.
[51] E. Jouini, *Existence of equilibria in nonconvex economies without free-disposal.* Economics Letters **38** (1992), 37–42.
[52] K. Kamiya, *Existence and uniqueness of equilibria with increasing returns.* J. Math. Econom. **17** (1988), 149–178.
[53] S. Kakutani, *A generalization of Brouwer's fixed point theorem.* Duke Math. J. **8** (1941), 457–459.
[54] M. Magill and W. Shafer, *Incomplete markets* In Handbook of Mathematical Economics, vol. 4 (W. Hildenbrand and H. Sonnenschein eds.) (1990), 167–194.
[55] R. Mantel, *Equilibrio con rendimientos crecientes a escala.* Anales de la Asociacion Argentina de Economia Politica **1** (1979), 271–283.
[56] E. Michael, *Continuous selections. I.* Ann. of Math. **63** (1956), 361–382.

[57] H. Nikaid, *On the classical multilateral exchange problem.* Metroeconomica **8** (1956), 135–145.
[58] R. Radner, *Existence of equilibrium of plans, prices, and price expectations.* Econometrica **40** (1972), 289–303.
[59] W. Shafer, *Equilibrium in abstract economies without ordered preferences or free-disposal.* J. Math. Econom. **(**3) (1976), 135-137.
[60] W. Shafer and H. Sonnenschein, *Some theorems on the existence of competitive equilibrium*, J. of Econom. Theory **11** (1975), 83–93.
[61] W. Shafer and H. Sonnenschein, *Equilibrium in abstract economies without ordered preferences.* J. Math. Econom. **(**2) (1975), 345–348.
[62] H. Uzawa, *Walras' existence theorem and Brouwer's fixed point theorem* Economic Studies Quarterly **13** (1962), 59-62.
[63] A Villar, *Equilibrium with nonconvex technologies.* Econom. Theory **4** (1994), 629–638.
[64] A. Villar, *Existence and efficiency of equilibrium in economies with increasing returns to scale: An exposition.* Investigaciones Económicas **18** (1994), 205–243.
[65] R. Vohra, *On the existence of equilibria in economies with increasing returns.* J. Math. Econom. **17** (1988), 179–192.
[66] R. Vohra, *Marginal cost pricing under bounded marginal returns.* Econometrica **60** (1992), 859–876.
[67] J. Werner, *Equilibrium in economies with incomplete financial markets.* J. Econom. Theory **36** (1985), 110–119.
[68] N.C. Yannelis and N.D. Prabhakar, *Existence of maximal elements and equilibria linear topological spaces.* J. Math. Econom. **15** (1983), 85–110.

Hichem Ben-El-Mechaiekh
Department of Mathematics, Brock University, L2S 3A1 St. Catharines, Canada
hbenelmechaiekh@brocku.ca

Philippe Bich
Centre d'Economie de la Sorbonne (CNRS–Université Paris 1)
106-112 bd. de l'Hôpital, 75647 Paris cedex 13
Philippe.bich@univ-paris1.fr

Monique Florenzano
Centre d'Economie de la Sorbonne (CNRS–Université Paris 1)
106-112 bd. de l'Hôpital, 75647 Paris cedex 13
monique.florenzano@univ-paris1.fr