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On a Necessary and Sufficient Condition for Regularity in Constrained Optimization

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1 Introduction

The present paper aims to contribute to the problem of the existence of Lagrangian multipliers for constrained optimization. Given a solution of a constrained extremum problem, it consists in finding a vector of multipliers, associated to the constraints, in such a way that the pair solution-vector of multipliers be a stationary point for the Lagrangian function. This is equivalent to claim that a positive multiplier can be associated to the objective function. Classical results in this sense date back to the first half of 19th century and are due to W. Karush [9], F. John [8], H.W. Kuhn and A. W. Tucker [10].

In literature, a condition which guarantees that the multiplier associated with the objective function is positive, is called regularity condition or constraint qualification, according to whether the condition does or does not involve the objective function, respectively.

In this paper, a regularity condition will be established by means of the image space analysis [4] which has been shown to be a fundamental tool to study many topics in optimization theory. More precisely, since the optimality of a feasible point $\bar{x}$ can be proved by means of the linear separation between two suitable subsets of the image space, we begin the study by giving, in Section 2, a condition equivalent to the linear separation between a convex cone C and a generic set S in the Euclidean space $\mathbb{R}^n$. This condition can be called of "Helly-type" because if each subset of S of finite cardinality enjoys a separability property then S itself enjoys a separability property. In Section 3, we propose a regularity condition for the linear separation between C and S, under the assumption that such a separation holds. The regularity condition is given in terms of the tangent cone to a suitable approximation of the set, which allows us to include also the nonconvex case. In Section 4, given a constrained extremum problem, we consider in the image space a convex cone, which depends on the the kind of constraints (equalities or inequalities), and a set, which is the image of the domain of the given problem through the map of the constraining functions. Then, Theorem 4.1 is applied to achieve the existence of a regular separation hyperplane for the two above sets, and hence for the existence of John multipliers (if the second of the above sets is the linearization of the image) or of regular saddle-point multipliers (if such a set is precisely the image set). It is worth to mention that, even if separation arguments are developed in the finite dimensional image space, the regularity condition which we obtain holds also for the infinite-dimensional extremum problems having finite dimensional image, like for instance problems of isoperimetric type. The existing literature contains a lot of interesting regularity conditions, as those of Slater [14], Mangasarian-Fromovitz [11], Guignard [5], Penot [13], Clarke calmness [1] and Ioffe metric regularity [6]. In this paper, we begin the comparison among the present condition and each of the existing ones by investigating in Section 5 and Section 6 the connections with the calmness and the metric regularity, that are two concepts which have produced regularity conditions. Some comments and examples are given in the paper with the aim of showing the importance and the consequences of the condition.

We conclude this section by mentioning some notations which will be used in the sequel:

for any $x \in \mathbb{R}^n$, $x \geq 0$ means $x_i \geq 0$, $\forall i = 1, ..., n$;

$\mathbb{R}^n_+$ denotes $\{x \in \mathbb{R}^n : x \geq 0\}$;

O_n denotes the n-tuple, whose entries are zero; when there is no fear of confusion the subfix is omitted; for $n = 1$, the 1-tuple is identified with its element, namely, we set $O_1 = 0$;

$A \subseteq B$ means that the set A is contained in the set B;

$A \subset B$ means that the set A is contained in the set B, but $A \neq B$;

$A - B$ denotes vector difference between sets A and B;

$\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$.

Let $M \subseteq \mathbb{R}^n$, then:

dim M denotes the dimension of M;

aff M denotes the affine hull of M;

cl M denotes the closure of M;

conv M denotes the convex hull of M;

int M denotes the interior of M;

ri M denotes the relative interior of M;

$d(x; M)$:=inf $\{\|x - y\| \; y \in M\}$ denotes the distance from the point x to the set M.

If $M \neq \emptyset$ and $\bar{x} \in$cl M, then the set of $\bar{x} + x \in \mathbb{R}^n$ for which $\exists \{x^i\} \subseteq$cl M, with $\lim_{i \to +\infty} x^i = \bar{x}$, and $\exists \{\alpha_i\} \subset \mathbb{R}_+ \setminus \{0\}$ such that $\lim_{i \to +\infty} \alpha_i(x^i - \bar{x}) = x$ is called *tangent cone* to M at $\bar{x}$ and denoted by $TC(\bar{x}; M)$. We stipulate that $TC(\bar{x}; \emptyset) = \emptyset$. If $\bar{x} = O$, then the notation $TC(M)$ is used.

2 A Helly-type condition for linear separability between a cone and a set

Let $C \subset \mathbb{R}^n$ be a nonempty and convex cone with apex at $O \in$cl C such that $C +$ cl $C = C$. Let $S \subset \mathbb{R}^n$ be nonempty and let $s :=$ dim S. If $z \in \mathbb{R}^n$, then denote by *proj* z its projection on the orthogonal complement of C:

$$C^{\perp} = \{x \in \mathbb{R}^n : \langle x, k \rangle = 0, \forall k \in C\}.$$

Let $p :=$ dim $C^{\perp}$ and hence dim $C = n - p$.

In the following statement, if $p = 0$ we stipulate that (1)-(2) shrinks to (2). When $p > 0$ and it does not exist affinely independent $z^1, ..., z^{s+1} \in S$, such that

(1) is fulfilled, then, of course, condition (1)-(2) is meant to be satisfied. We stipulate that a singleton coincides with its relative interior.

Theorem 2.1 . If and only if for every set $\{z^1, ..., z^{s+1}\}$ of affinely independent vectors of S such that

$$\dim \text{ aff } (\text{ri conv}\{\text{proj } z^1, ..., \text{proj } z^{s+1}\}) = p \text{ and}$$

$$O \in \text{ri conv}\{\text{proj z}^1, ..., \text{proj z}^{s+1}\} \tag{1}$$

we have

$$(\text{ri C}) \cap \text{ri conv}\{\text{z}^1, ..., \text{z}^{s+1}\} = \emptyset \tag{2}$$

then C and S are (linearly) separable.

Proof. If. The proof will be split up into four parts.

(A) $s = 0$ or $p = 0$. If $s = 0$, then S is a singleton, say $\{\widehat{z}\}$; (1) and (2) become, respectively,

$$0 = \dim \text{ proj } \widehat{z} = p, \qquad O = \text{proj } \widehat{z}, \tag{1$'$}$$

and

$$(\text{ri } C) \cap \{\widehat{z}\} = \emptyset. \tag{2$'$}$$

$(1)'$ is satisfied or not, according to, respectively, $p = 0$ or $p > 0$; if $p = 0$, then $(2)'$ gives the thesis; if $p > 0$, then $(1)'$-$(2)'$ collapse to $(2)'$ and the thesis follows. When $p = 0$, then C is a convex body and thus, obviously, (2) implies linear separation (even proper) between C and S.

(B) $1 \le s \le p - 1$. Let B_C and B_S be bases for aff C and aff S, respectively; dim aff $B_C = n - p$, dim aff $B_S = s$ and dim aff $(B_C \cup B_S) \le n - p + s \le n - 1$. This shows that there exists a hyperplane of $\mathbb{R}^n$ which contains C and is parallel to aff S, so that separation holds.

(C) $s \ge p \ge 1$ and (1) does not hold, in the sense that no set of affinely independent vectors of S verifies (1). Since $s \ge 1$, there exists at least one set of $s+1 \geq p+1$ affinely independent vectors of S; let $\{z^1, ..., z^{s+1}\}$ be one of such sets. Denote by proj $S \subset \mathbb{R}^n$ the projection of S into $C^\perp$. Since for every $(s+1)$-dimensional set of affinely independent vectors of S, relation (1) does not hold, then

$$O \notin \text{ri conv proj } S.$$

Otherwise, if $O \in$ ri conv proj S, then $\exists \alpha_1, ..., \alpha_{p+1} > 0$ with $\sum_{i=1}^{p+1} \alpha_i = 1$ and $\exists x^1, ..., x^{p+1} \in$ proj S affinely independent, such that $O = \sum_{i=1}^{p+1} \alpha_i x^i$. Thus, we will have $p+1$ affinely independent vectors of S such that $x^i =$ proj z^i, $i = 1, ..., p+1$ and $O = \sum_{i=1}^{p+1} \alpha_i \text{proj } z^i$. Since dim $S = s$, then the set $\{z^1, ..., z^{p+1}\}$ can be augmented to form a set $\{z^1, ..., z^{s+1}\}$ of affinely independent vectors of S which would satisfy (1), that is a contradiction with the initial assumption. Because of Hahn-Banach Theorem, $O \notin$ ri conv proj S implies the existence of $a \in C^\perp \setminus \{O\}$ such that

$$\text{conv proj } S \subseteq H^- := \{x \in \mathbb{R}^n : \langle a, x \rangle \leq 0\}.$$

By introducing the hyperplane $H^0 := \{y \in \mathbb{R}^n : \langle a, y \rangle = 0\}$, it is obvious that $C \subseteq H^0$. Because conv and proj are permutable, we get proj conv $S \subseteq H^-$. Let be $s \in$ conv S and s_p its projection on $C^\perp$. Since $a \in C^\perp$, then we have $\langle a, s - s_p \rangle = 0$, or $\langle a, s \rangle = \langle a, s_p \rangle \leq 0$, where the last inequality comes from $s_p \in$ proj conv $S \subseteq H^-$. Therefore conv $S \subseteq H^-$. Hence we can conclude that H^0 linearly separates C and S.

(D) $s \geq p \geq 1$ and (1) holds, in the sense that there exists a set $\{z^1, ..., z^{s+1}\}$ of affinely independent vectors of S which verifies (1). We prove that (2) implies

$$\text{ri } C \cap \text{ri conv } S = \emptyset. \tag{3}$$

Suppose that (3) does not hold, i.e. there exists $\bar{z} \in$ ri $C \cap$ri conv S. Because of a well-known Carathéodory Theorem, $\bar{z}$ can be expressed as a convex combination of $s+1$ affinely independent vectors of S, say $\{w^1, ..., w^{s+1}\}$. If this set verifies (1), then (2) is contradicted. Otherwise,

$$O \notin \text{ri conv } \{\text{proj } w^1, ..., \text{proj } w^{s+1}\}.$$

Set $H^0 := \{x \in \mathbb{R}^n : \langle a, x\rangle = 0\}$, where $a \in C^\perp \setminus \{O\}$ and such that conv $\{\text{proj } w^1, ..., \text{proj } w^{s+1}\} \subseteq H^-$. We have that also conv $\{w^1, ..., w^{s+1}\} \subseteq H^-$. On the other side, $\bar{z} \in \text{ri } C$ and $C \subseteq H^0$ imply $\langle a, \bar{z}\rangle = 0$, or $\sum_{i=1}^{s+1} \alpha_i \langle a, w^i\rangle = 0$, where $\alpha_i > 0$, $i = 1, ..., s+1$ and $\sum_{i=1}^{s+1} \alpha_i = 1$. Being $\langle a, w^i\rangle \geq 0$ and $\alpha_i > 0$, $i = 1, ..., s+1$, we have

$$\langle a, w^i\rangle = 0, \ i = 1, ..., s+1.$$

From here, conv $\{w^1, ..., w^{s+1}\} \subseteq H^0$ and thus $S \subseteq H^0$, in particular $\langle a, z^i\rangle = 0, \ i = 1, ..., s+1$. It follows $\langle a, \text{proj } z^i\rangle = 0, \ i = 1, ..., s+1$ and $a \neq O$ implies that

$$\dim \text{ aff } (\text{ri conv } \{\text{proj } z^1, ..., \text{proj } z^{s+1}\}) < p,$$

which contradicts (1). So (3) is true and implies separation (even proper) between C and S.

Only if. By assumption, $\exists a \in \mathbb{R}^n \setminus \{O\}$ and $b \in \mathbb{R}$, such that

$$\langle a, x\rangle \geq b, \ \forall x \in C \text{ and } \langle a, y\rangle \leq b, \ \forall y \in S.$$

Since $O \in \text{cl } C$, we can put $b = 0$. Set $H^0 := \{x \in \mathbb{R}^n : \langle a, x\rangle = 0\}, \ H^- := \{x \in \mathbb{R}^n : \langle a, x\rangle \leq 0\}$. Let us assume that there exists a set $\{z^1, ..., z^{s+1}\}$ of affinely independent vectors of S such that (1) holds (if no set of $s+1$ affinely independent vectors of S exists, such that (1) is satisfied, then the thesis is trivial) while (2) is not valid, i.e.

$$O \in \text{ri conv } \{\text{proj } z^1, ..., \text{proj } z^{s+1}\}, \tag{4}$$

and

$$(\text{ri } C) \cap \text{ri conv } \{z^1, ..., z^{s+1}\} \neq \emptyset. \tag{5}$$

Let $\bar{z}$ belong to the left-hand side of (5); thus there exists $\alpha_i \geq 0$, $i = 1, ..., s+1$ with $\sum_{i=1}^{s+1} \alpha_i = 1$ such that $\bar{z} = \sum_{i=1}^{s+1} \alpha_i z^i \in$ ri C. From $\bar{z} \in$ri C we have proj $\bar{z} = O$ and from (4) we have proj $z^i \neq O$ for $i \in J \subseteq \{1, ..., s+1\}$ such that $|J| = p+1$. Therefore, it results proj $\bar{z}$ = proj $\sum_{i=1}^{s+1} \alpha_i z^i = \sum_{i=1}^{s+1} \alpha_i \text{proj } z^i = \sum_{i \in J} \alpha_i \text{proj } z^i = O$. Since $z^1, ..., z^{s+1} \in S$, then $\langle a, z^i \rangle \leq 0$, $i = 1, ..., s+1$ and hence $\langle a, \sum_{i=1}^{s+1} \alpha_i z^i \rangle \leq 0$. On the other side, $\bar{z} \in$ ri C and thus $\langle a, \sum_{i=1}^{s+1} \alpha_i z^i \rangle \geq 0$. It follows $\bar{z} \in H^0$. From $\bar{z} \in$ ri C and C convex, we have that $\exists \beta_i > 0$, $i = 1, ..., n-p+1$ with $\sum_{i=1}^{n-p+1} \beta_i = 1$ and $\exists k^i \in C$, $i = 1, ..., n-p+1$ affinely independent, such that $\bar{z} = \sum_{i=1}^{n-p+1} \beta_i k^i$. Since $\bar{z} \in H^0$, then $\sum_{i=1}^{n-p+1} \beta_i \langle a, k^i \rangle = 0$, which implies $\langle a, k^i \rangle = 0$, $i = 1, ..., n-p+1$. Thus, conv $\{k^1, ..., k^{n-p+1}\} \subseteq H^0$ and, consequently, $C \subseteq H^0$. It follows that $a \in C^\perp$ and therefore from $S \subseteq H^-$ we have proj $S \subseteq H^-$. Using O = proj $\bar{z}$, we obtain

$$\langle a, O \rangle = \langle a, \text{proj } \bar{z} \rangle = \langle a, \sum_{i=1}^{s+1} \alpha_i \text{proj } z^i \rangle = \sum_{i=1}^{s+1} \alpha_i \langle a, \text{proj } z^i \rangle.$$

Since $\alpha_i > 0$, $i = 1, ..., s+1$, we get $\langle a, \text{proj } z^i \rangle = 0$, $i = 1, ..., s+1$; hence we have also $\{\text{proj } z^1, ..., \text{proj } z^{s+1}\} \subseteq H^0$ and, obviously, conv $\{\text{proj } z^1, ..., \text{proj } z^{s+1}\} \subseteq H^0$. Let us denote by $B(O_n, \varepsilon)$ an open ball of center O_n and radius $\varepsilon > 0$ in $\mathbb{R}^n$ such that dim aff $B(O_n, \varepsilon) = p$. From (4) we have that $\exists\ \bar{\varepsilon} > 0$ such that

$$B(O, \bar{\varepsilon}) \subseteq \text{conv } \{\text{proj } z^1, ..., \text{proj } z^{s+1}\} \subseteq H^0,$$

i.e. $\langle a, y \rangle = 0$, $\forall y \in B(O, \bar{\varepsilon})$. By assumption $a \neq O$; hence, for $\gamma := \frac{\bar{\varepsilon}}{\|a\|} > 0$, it turns out $\bar{y} := \gamma a \in B(0, \bar{\varepsilon})$. Consequently, we have

$$0 = \langle a, \bar{y} \rangle = \gamma \langle a, a \rangle = \gamma \|a\|^2,$$

which contradicts the assumption $a \neq O$. $\square$

3 A condition for regular separability

Let us consider Theorem 2.2.7 of [4].

Theorem 3.1 . Let $C \subseteq \mathbb{R}^n$ be a nonempty and convex cone with apex at $O \notin C$ such that $C + \text{cl } C = C$ and F be any face of C. Let $S \subseteq \mathbb{R}^n$ be nonempty with $O \in \text{cl } S$ and such that $S - \text{cl } C$ is convex. F is contained in every hyperplane which separates C and S if and only if

$$F \subseteq TC(S - \text{cl } C),$$

where $TC(S - \text{cl } C)$ is the tangent cone to $S - \text{cl } C$ at O.

Theorem 3.1 assumes the convexity of $S - \text{cl } C$. The following example shows that if we remove such an assumption, then the necessity in the theorem does not hold.

Example 3.1. Let C be the following convex cone in $\mathbb{R}^3$:

$$C = \{x \in \mathbb{R}^3 : x_1 > 0, x_2 = 0, x_3 = 0\}$$

and

$$S = \{x \in \mathbb{R}^3 : x_1 = x_2 \geq 0, x_3 = -x_1^2 - x_2^2\} \cup$$

$$\cup \{x \in \mathbb{R}^3 : x_1 = -x_2 \geq 0, x_3 = -x_1^2 - x_2^2\}.$$

Choose $F = C$. Obviously S and $S - \text{cl } C$ are not convex. The plane $H^0 = \{x \in \mathbb{R}^3 : x_3 = 0\}$ is the unique plane which separates C and S and it contains the face F, nevertheless F is not contained in $TC(S - \text{cl } C)$.

In order to extend Theorem 3.1 to nonconvex case, we have to consider $TC(\text{conv } (S - \text{cl } C))$ in place of $TC(S - \text{cl } C)$ and hence we have the following result.

Theorem 3.2 . Let $C \subset \mathbb{R}^n$ be a nonempty convex cone with apex at $O \notin C$ such that $C + \text{cl } C = C$ and F be any face of C. Let $S \subseteq \mathbb{R}^n$ be nonempty with $O \in \text{cl } S$. F is contained in every hyperplane which separates C and S (if any), if and only if $F \subseteq TC(\text{conv } (S - \text{cl } C))$.

Before proving Theorem 3.2, let us state some preliminary properties by means of the following lemma.

Lemma 3.1 . Under the same hypotheses of Theorem 3.2, the following statements, where H^0 denotes a generic hyperplane of $\mathbb{R}^n$, are equivalent:
(i) H^0 separates C and S;
(ii) H^0 separates C and $S - \text{cl } C$;
(iii) H^0 separates C and conv $(S - \text{cl } C)$;
(iv) H^0 separates C and $TC(\text{conv } (S - \text{cl } C))$.

Proof. $(i) \Rightarrow (ii)$ Suppose that the hyperplane H^0, whose equation is $\langle a, x\rangle = b$, $a \neq O$ separates C and S. Since $O \in \text{cl } C$, we can set $b = 0$. Then $C \subseteq H^+$ and $S \subseteq H^-$, where H^+ and H^- are the halfspaces identified by $\langle a, x\rangle \geq 0$ and by $\langle a, x\rangle \leq 0$, respectively. Ab absurdo, suppose that $\exists \widehat{x} \in S - \text{cl } C$ such that $\langle a, \widehat{x}\rangle > 0$. From $\widehat{x} \in S - \text{cl } C$ we get the existence of $x^1 \in S$ and $x^2 \in \text{cl } C$ such that $\widehat{x} = x^1 - x^2$. Therefore $\langle a, \widehat{x}\rangle > 0$ implies

$$0 \geq \langle a, x^1\rangle > \langle a, x^2\rangle \geq 0,$$

where the first inequality is implied by $x^1 \in S \subseteq H^-$ and the third by $x^2 \in \text{cl } C \subseteq H^+$.
$(ii) \Rightarrow (iii)$ Suppose that the hyperplane H^0, whose equation is $\langle a, x\rangle = 0$, separates C and $S - \text{cl } C$, i.e. $C \subseteq H^+$ and $S - \text{cl } C \subseteq H^-$. Let z be any element of conv $(S - \text{cl } C)$. From Carathéodory's Theorem we have the existence of $z^1, ..., z^{n+1} \in S - \text{cl } C$ and $\alpha_i \in [0, 1]$, $i = 1, ..., n + 1$ with $\sum_{i=1}^{n+1} \alpha_i = 1$,

such that $z = \sum_{i=1}^{n+1} \alpha_i z^i$. From $z^1, ..., z^{n+1} \in S - \text{cl } C$ we have $\langle a, z^i \rangle \leq 0$, $\forall i = 1, ..., n+1$, and hence $\langle a, \alpha_i z^i \rangle \leq 0$, $\forall i = 1, ..., n+1$. Therefore it follows $\langle a, \sum_{i=1}^{n+1} \alpha_i z^i \rangle \leq 0$ or $\langle \text{a}, \text{z} \rangle \leq 0$.
$(iii) \Rightarrow (iv)$ Suppose that the hyperplane H^0, whose equation is $\langle a, x \rangle = 0$, separates C and conv $(S - \text{cl } C)$, i.e. $C \subseteq H^+$ and conv $(S - \text{cl } C) \subseteq H^-$. Now we will prove that conv $(S - \text{cl } C) \subseteq H^-$ implies $TC(\text{conv } (S - \text{cl } C)) \subseteq H^-$. Let $t \in TC(\text{conv } (S - \text{cl } C))$; then there exist a sequence $\{x^n\} \subseteq \text{conv } (S - \text{cl } C)$ with $\lim_{n \to +\infty} x^n = 0$ and a sequence $\{\alpha_n\} \subset \mathbb{R}_+ \setminus \{0\}$ such that $\lim_{n \to +\infty} \alpha_n x^n = t$. Since $x^n \in \text{conv } (S - \text{cl } C)$, $\forall n \geq 0$, then $\langle a, x^n \rangle \leq 0$, and hence $\langle a, \alpha_n x^n \rangle \leq 0$, $\forall n \geq 0$. Letting $n \to +\infty$ we obtain $\langle a, t \rangle \leq 0$ and thus $TC(\text{conv } (S - \text{cl } C)) \subseteq H^-$.
$(iv) \Rightarrow (i)$ This is an obvious consequence of the inclusions $S \subseteq S - \text{cl } C \subseteq \text{conv } (S - \text{cl } C) \subseteq TC(\text{conv } (S - \text{cl } C))$. $\square$

Proof of Theorem 3.2. Only if. Since $S \neq \emptyset$ and $O \in \text{cl } S$, then $O \in \text{cl conv } (S - \text{cl } C)$ and thus we can consider $TC(\text{conv } (S - \text{cl } C))$. Now, ab absurdo, suppose $F \not\subseteq \text{cl cone conv } (S - \text{cl } C)$ or, equivalently, that $\exists f^0 \in F$ such that $f^0 \notin TC(\text{conv } (S - \text{cl } C))$. Since $TC(\text{conv } (S - \text{cl } C))$ is closed and convex, then there exists a hyperplane H^0 of equation $\langle a, x \rangle = b$ with $a \in \mathbb{R}^n \setminus \{O\}$ such that

$$\langle a, x \rangle \leq b < \langle a, f^0 \rangle, \ \forall x \in TC(\text{conv } (S - \text{cl } C)).$$

Because of $O \in TC(\text{conv } (S - \text{cl } C))$, we can set $b = 0$ and thus we have

$$\langle a, x \rangle \leq 0 < \langle a, f^0 \rangle, \ \forall x \in TC(\text{conv } (S - \text{cl } C)). \tag{6}$$

The inclusion $S - \text{cl } C \subseteq TC(\text{conv } (S - \text{cl } C))$ implies that $\langle a, x \rangle \leq 0$, $\forall x \in S - \text{cl } C$. Now we prove that $\langle a, x \rangle \geq 0$, $\forall x \in C$. Ab absurdo, suppose that $\exists k \in C$ such that $\langle a, k \rangle < 0$ and let $s \in S$. Then we have $s - \alpha k \in S - \text{cl } C$, $\forall \alpha \in \mathbb{R}_+$ so that $\lim_{\alpha \to +\infty} \langle a, s - \alpha k \rangle = +\infty$, which contradicts $\langle a, x \rangle \leq 0$, $\forall x \in S - \text{cl } C$.

Therefore H^0 separates C and $S - \text{cl } C$. Because of Lemma 3.1, H^0 separates also C and S. Due to the assumption, we have $F \subseteq H^0$ and therefore $\langle a, f^0 \rangle = 0$, which contradicts (6).

If. Suppose that there exists a hyperplane H^0, whose equation is $\langle a, x \rangle = 0$, which separates C and S. Because of Lemma 3.1, H^0 separates also C and $TC(\text{conv } (S - \text{cl } C))$ or, equivalently,

$$\langle a, x \rangle \leq 0 \leq \langle a, y \rangle, \ \forall x \in TC(\text{conv } (S - \text{cl } C)), \ \forall y \in C. \tag{7}$$

These inequalities can be written as $TC(\text{conv } (S - \text{cl } C)) \subseteq H^-$ and $C \subseteq H^+$, where H^- and H^+ are the halfspaces identified by $\langle a, x \rangle \leq 0$ and by $\langle a, x \rangle \geq 0$, respectively. The assumption $F \subseteq TC(\text{conv } (S - \text{cl } C))$ and the inclusion $TC(\text{conv } (S - \text{cl } C)) \subseteq H^-$ imply $F \subseteq H^-$. Besides, since $F \subseteq \text{cl } C$, then from (7) we obtain $F \subseteq H^+$. It follows $F \subseteq H^- \cap H^+ = H^0$.□

Notice that in Theorem 3.2 the tangent cone $TC(\text{conv } (S - \text{cl } C))$ can be replaced by cl cone conv $(S - \text{cl } C)$; in fact, if A is a convex set, then $TC(A) =$ cl cone A. Moreover, observe that in Theorem 3.2 it is not possible to replace $TC(\text{conv } (S - \text{cl } C))$ by conv $TC(S - \text{cl } C))$; in such a case the necessity of Theorem 3.2 does not hold, due to the fact that, without the convexity assumption, it may exist a hyperplane which separates C and $TC(S - \text{cl } C)$ but does not separate C and $S - \text{cl } C$. This situation is illustrated by the following example.

Example 3.2. Let C be the following convex cone in $\mathbb{R}^3$:

$$C = \{x \in \mathbb{R}^3 : x_1 > 0, x_2 = 0, x_3 = 0\} \text{ and}$$

$$S = \{x \in \mathbb{R}^3 : x_1 = x_2 \geq 0, x_3 \leq 0, x_3 = (x_1 - 1)^2 + (x_2 - 1)^2 - 2\} \cup$$

$$\cup \{x \in \mathbb{R}^3 : x_1 = -x_2 \geq 0, x_3 \leq 0, x_3 = (x_1 - 1)^2 + (x_2 + 1)^2 - 2\}.$$

Choose $F = C$. Obviously S and $S - \text{cl } C$ are not convex. The plane $H^0 = \{x \in \mathbb{R}^3 : x_3 = 0\}$ is the unique plane which separates C and S and it contains the face F. It results:

$$TC(S - \text{cl } C) = \{x \in \mathbb{R}^3 : x_1 = x_2, x_3 \leq 0, x_3 \leq -4x_1\} \cup$$

$$\cup \{x \in \mathbb{R}^3 : x_1 = -x_2, x_3 \leq 0, x_3 \leq -4x_1\}.$$

$TC(S - \text{cl } C)$ is not convex and we have that $F \not\subseteq \text{conv } TC(S - \text{cl } C)$. Moreover, every plane $H_a^0 = \{x \in \mathbb{R}^3 : ax_1 + x_3 = 0\}$, with $0 < a \leq 4$, separates C and $TC(S - \text{cl } C)$ (and hence also C and $\text{conv } TC(S - \text{cl } C)$), but does not separate C and S and does not contain the face F.

Remark. Observe that both in Example 3.1 and 3.2 we have $\text{int } C = \emptyset$. It is possible to give similar examples with $\text{int } C \neq \emptyset$, by putting for instance $C = \{x \in \mathbb{R}^3 : x_1 \geq 0, -10x_1 \leq x_2 \leq 0,\ 0 \leq x_3 \leq 10x_1\}$ and choosing $F \subset C$, $F = \{x \in \mathbb{R}^3 : -10x_1 \leq x_2 \leq 0, x_3 = 0\}$.

4 A regularity condition for constrained optimization

Let us consider the particular case of a constrained extremum problem. For this, assume we are given the integers m and p with $m \geq 0$ and $0 \leq p \leq m$, the nonempty subset X of a Banach space B and the functions $f : X \to \mathbb{R}$, $g_i : X \to \mathbb{R}$, $i \in \mathcal{I} := \{1, ..., m\}$. Let us consider the following constrained extremum problem

$$\min f(x), \quad \text{s.t.} \tag{8}$$

$$g_i(x) = 0, i \in \mathcal{I}^0 := \{1, ..., p\},$$

$$g_i(x) \geq 0, i \in \mathcal{I}^+ := \{p+1, ..., m\}, x \in X.$$

We stipulate that if $p = 0$ then $\mathcal{I}^0 = \emptyset$, if $m = 0$ then $\mathcal{I}^+ = \emptyset$, while when $m = 0$ we have $\mathcal{I} = \mathcal{I}^0 \cup \mathcal{I}^+ = \emptyset$. The feasible region of (8) is the set

$$R := \{x \in X : g(x) \in D\},$$

where $g(x) := (g_1(x), ..., g_m(x))$, $D := O_p \times \mathbb{R}_+^{m-p}$.

Suppose $\bar{x} \in R$ and set $f_{\bar{x}}(x) := f(\bar{x}) - f(x)$; introduce the following sets:

$\mathcal{H} := \{(u, v) \in \mathbb{R} \times \mathbb{R}^m : u > 0, v \in D\}$

$\mathcal{H}_u := \{(u, v) \in \mathbb{R} \times \mathbb{R}^m : u > 0, v = 0\}$

$\mathcal{K}_{\bar{x}} := \{(u, v) \in \mathbb{R} \times \mathbb{R}^m : u = f_{\bar{x}}(x), v = g(x), x \in X\}$

$\mathcal{E}(\mathcal{K}_{\bar{x}}) := \mathcal{K}_{\bar{x}} - \text{cl}\, \mathcal{H}.$

The set $\mathcal{K}_{\bar{x}}$ is called the *image* of the problem (8) and the space $\mathbb{R}^{1+m}$, where both $\mathcal{H}$ and $\mathcal{K}_{\bar{x}}$ lay, is called *image space*.

It is quite immediate to prove the following result [4].

Proposition 4.1 .(i) $\bar{x} \in R$ is a global minimum point of (8) $\Leftrightarrow$ the system (in the unknown x)

$$f_{\bar{x}}(x) > 0, \ g(x) \in D, \ x \in X \tag{9}$$

is impossible or, equivalently,

$$\mathcal{H} \cap \mathcal{K}_{\bar{x}} = \emptyset. \tag{10}$$

(ii) $\mathcal{H} \cap \mathcal{K}_{\bar{x}} = \emptyset \Leftrightarrow \mathcal{H} \cap \mathcal{E}(\mathcal{K}_{\bar{x}}) = \emptyset$.

The direct proof of (10) is, in general, impracticable; therefore a separation approach has been introduced in [4] which consists in finding a functional such that $\mathcal{H}$ and $\mathcal{K}_{\bar{x}}$ lie in opposite level sets of the functional.

Given a nonempty subset K of $\mathbb{R}^{1+m}$, we say that $\mathcal{H}$ and K are linearly separable if and only if $\exists(\theta, \lambda) \neq O_{1+m}$ such that

$$\theta u + \langle \lambda, v \rangle \geq 0, \forall (u, v) \in \mathcal{H} \text{ and} \tag{11}$$

$$\theta u + \langle \lambda, v \rangle \leq 0, \forall (u,v) \in K \tag{12}$$

where the separation hyperplane H^0 is the zero level set of the functional: $H^0 := \{(u,v) \in \mathbb{R} \times \mathbb{R}^m : \theta u + \langle \lambda, v \rangle = 0\}$. Conditions (11) and (12) are equivalent to saying that $\exists \theta \geq 0$, $\exists \lambda \in D^*$, $(\theta, \lambda) \neq O_{1+m}$ such that $\theta u + \langle \lambda, v \rangle \leq 0, \forall (u,v) \in K$ or, if $K = \mathcal{K}_{\bar{x}}$, such that

$$\theta f_{\bar{x}}(x) + \langle \lambda, g(x) \rangle \leq 0, \forall x \in X. \tag{13}$$

Definition 4.1 .A condition assuring the existence of $\theta > 0$ (or equivalently, after normalization, $\theta = 1$) in (12) is called a *regularity condition.*

Combining Theorem 2.1 and Theorem 3.2 for the particular case of the extremum problem (8), we obtain a general regularity condition, where the set K can play the role of both the image set and its linearization or homogenization [3]. The former part of (i) of the following theorem, i.e. conditions (14)-(15), guarantees the existence of a separation hyperplane, while the latter one, i.e. condition (16), guarantees that at least one of the existing separation hyperplanes has gradient (θ, λ) with $\theta = 1$.

Theorem 4.1 . Let $K \subset \mathbb{R}^{1+m}$ be any nonempty subset of the image space and $s :=$dim K. The following conditions are equivalent:

(i) For every set $\{z^1, ..., z^{s+1}\}$ of affinely independent vectors of K such that

$$\text{dim aff(ri conv } \{\text{proj } z^1, ..., \text{proj } z^{s+1}\}) = p \quad \text{and}$$

$$O \in \text{ri conv } \{\text{proj } z^1, ..., \text{proj } z^{s+1}\} \tag{14}$$

we have

$$(\text{ri } \mathcal{H}) \cap \text{ri conv } \{z^1, ..., z^{s+1}\} = \emptyset. \tag{15}$$

Moreover,

$$\mathcal{H}_u \cap TC(\text{conv } \mathcal{E}(K)) = \emptyset. \tag{16}$$

(ii) $\theta = 1$ in (12).

Proof.$(i) \Rightarrow (ii)$ Consider Theorem 2.1 in the particular case $n = m+1$, $C = \mathcal{H}$ and $S = K$; conditions (14)-(15) imply linear separation between $\mathcal{H}$ and K and thus the existence of a separation hyperplane

$$H^0 = \{(u,v) \in \mathbb{R} \times \mathbb{R}^m : \theta u + \langle \lambda, v \rangle = 0\},\ (\theta, \lambda) \neq O_{1+m}.$$

Ab absurdo, suppose $\theta = 0$. Then the separation hyperplane becomes $H^0 = \{(u,v) \in \mathbb{R} \times \mathbb{R}^m : \langle \lambda, v \rangle = 0\},\ \lambda \neq O_m$. We observe that $\mathcal{H}_u \subseteq H^0$; hence from Theorem 3.2 it follows that $\mathcal{H}_u \subseteq TC(\text{conv } \mathcal{E}(K))$, which contradicts (16).
$(ii) \Rightarrow (i)$ Suppose that we have regular separation, i.e. $\theta = 1$ in (12). Obviously, from Theorem 2.1 conditions (14)-(15) hold. If, ab absurdo, (16) does not hold, i.e. $\mathcal{H}_u \subseteq TC(\text{conv } \mathcal{E}(K))$, then from Theorem 3.2 it results that $\mathcal{H}_u$ is contained in every hyperplane which separates $\mathcal{H}$ and K, that is $\theta u + \langle \lambda, v \rangle = 0$, $\forall (u,v) \in \mathcal{H}_u$, or $\theta u = 0$, $\forall u > 0$. This implies $\theta = 0$ and hence the thesis follows.□

5 Comparison with calmness

Let us recall the definition of calmness which was introduced in [1].

Definition 5.1 .Problem (8) is said to be *calm* at a local solution $\bar{x}$ if and only if there exist two real numbers $\rho > 0$ and $\varepsilon > 0$ such that for all $\xi \in B(O_m, \varepsilon)$ and for all $x \in R_\varepsilon(\xi) := \{x \in X \cap B(\bar{x}, \varepsilon) : g(x) + \xi \in D\} \neq \emptyset$ we have:

$$f(x) - f(\bar{x}) + \rho \|\xi\| \geq 0. \tag{17}$$

Analyzing Definition 5.1, we can see that the notion of calmness is a local notion with respect to $\bar{x}$, not only because $\bar{x}$ is a local solution of the problem, but mostly because in the definition of $R_\varepsilon(\xi)$ it is required that x belong to the neighbourhood $B(\bar{x}, \varepsilon)$. Hence, in order to compare the notion of calmness with the regularity condition (16), we have to remove the condition $x \in B(\bar{x}, \varepsilon)$ in the definition of $R_\varepsilon(\xi)$ or, alternatively, to consider the regularity condition (16) in a local form.

In the next theorem we will make the comparison from a local point of view. First of all, observe that condition (16) is equivalent to

$$e_u \notin TC(\text{conv } \mathcal{E}(K)), \tag{18}$$

where $e_u := (1, O_m) \in \mathbb{R}^{1+m}$.

Now, consider the following local regularity condition

$$e_u \notin TC(\text{conv } (\mathcal{K}^\varepsilon_{\bar{x}} - \text{cl } \mathcal{H})), \tag{19}$$

where $\mathcal{K}^\varepsilon_{\bar{x}} := \{(u, v) \in \mathbb{R}^{1+m} : u = f_{\bar{x}}(x), v = g(x), x \in X \cap B(\bar{x}, \varepsilon)\}$.

Theorem 5.1 .Let us consider problem (8), where f is supposed to be continuous at the local solution $\bar{x}$. If the condition (19) holds then problem (8) is calm at $\bar{x}$.

Proof. Ab absurdo, suppose that (8) is not calm at $\bar{x}$. Then, if we set $\rho = n$ and $\varepsilon = \frac{1}{n}$, $\forall n \geq 1$, we obtain the existence of $\xi_n \in B(O, \frac{1}{n})$ and of $x_n \in R_\varepsilon(\xi_n)$, in particular $\|x_n - \bar{x}\| < \frac{1}{n}$, such that

$$f_{\bar{x}}(x_n) = f(\bar{x}) - f(x_n) > n\|\xi_n\|. \tag{20}$$

From $g(x_n) + \xi_n \in D$ it follows the existence of $d_n \in D$ such that $g(x_n) - d_n = -\xi_n$, $n \geq 1$. Since $\|x_n - \bar{x}\| < \frac{1}{n}$ and $\|\xi_n\| < \frac{1}{n}$, $\forall n \geq 1$, we have that $\lim_{n \to +\infty} x_n = \bar{x}$ and $\lim_{n \to +\infty} g(x_n) - d_n = O$; hence, from the continuity of f at $\bar{x}$, it results $\lim_{n \to +\infty} f_{\bar{x}}(x_n) = 0$. Moreover, it is obvious that $(f_{\bar{x}}(x_n), g(x_n) - d_n) \in \mathcal{K}^\varepsilon_{\bar{x}} - \text{cl } \mathcal{H}$, $\forall n \geq 1$. Taking $\alpha_n := \frac{1}{f_{\bar{x}}(x_n)}$ (observe that (20) implies $f_{\bar{x}}(x_n) > 0$, $\forall n \geq 1$), then we get

$$\lim_{n\to+\infty} \alpha_n(f_{\bar{x}}(x_n), g(x_n) - d_n) = \lim_{n\to+\infty} \alpha_n(f_{\bar{x}}(x_n), -\xi_n) = (1, O)$$

or, equivalently, that

$$(1, O) \in TC(\mathcal{K}_{\bar{x}}^{\varepsilon} - (O \times \text{cl } D)). \tag{21}$$

Since $\mathcal{K}_{\bar{x}}^{\varepsilon} - (O \times \text{cl } D) \subset \mathcal{K}_{\bar{x}}^{\varepsilon} - \text{cl } \mathcal{H} \subseteq \text{conv } (\mathcal{K}_{\bar{x}}^{\varepsilon} - \text{cl } \mathcal{H})$, from (21) and from the isotonicity of the tangent cone we have $(1, O) \in TC(\text{conv } (\mathcal{K}_{\bar{x}}^{\varepsilon} - \text{cl } \mathcal{H}))$ which contradicts the assumption (19).□

The following example shows that the converse statement of Theorem 5.1 does not hold.

Example 5.1. Let us consider problem (8) with the following positions: $p = m = 2$; $X = \mathbb{R}$, $D = \{O_2\}$, $f(x) = -|x|$, $g_1(x) = x$, $g_2(x) = -2x^2$. Obviously $\bar{x} = 0$ is the (unique) optimal solution to problem (8). It will be shown that the problem is calm at $\bar{x} = 0$. Set $\xi = (\xi_1, \xi_2)$; $g(x) + \xi \in D$ is equivalent to $x + \xi_1 = 0$, $-2x^2 + \xi_2 = 0$, so that:

$$R_\varepsilon(\xi) = \left\{ x \in \mathbb{R} : |x| < \varepsilon \;\text{ and either } x = -\xi_1 = \sqrt{\frac{\xi_2}{2}} \text{ or } x = -\xi_1 = \sqrt{\frac{\xi_2}{2}} \right\},\ \xi_2 \geq 0.$$

Condition (17) becomes $|x| \leq \rho\sqrt{\xi_1^2 + \xi_2^2}$; being $x = -\xi_1$, the last inequality is either an identity (if $\xi_1 = 0$) or it is equivalent to $1 \leq \rho\sqrt{1 + 4\xi_1^2}$, which is verified, if $\rho \geq 1$ and $\varepsilon > 0$. Hence Definition 5.1 is fulfilled.

However, the problem is not regular. Its image set is

$$\mathcal{K}_0^{\varepsilon} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = |v_1|, v_2 = -2v_1^2, |v_1| < \varepsilon\},$$

and is formed by two parabolic arcs having the bisectors of quadrants (u, v_1) and $(u, -v_1)$ as tangents at O. We notice that in this case $\mathcal{H} = \mathcal{H}_u$ and hence

$$TC(\text{conv } (\mathcal{K}_0^{\varepsilon} - \text{cl } \mathcal{H})) = \{(u, v_1, v_2) \in \mathbb{R}^3 : v_2 \leq 0\}.$$

The unique plane which separates $\mathcal{H}$ and $\mathcal{K}_0^\varepsilon$ is $H^0 = \{(u, v_1, v_2) \in \mathbb{R}^3 : v_2 = 0\}$ and the regularity condition (19) is not satisfied.

6 Comparison with metric regularity

Definition 6.1 .Let us consider problem (8). Let $\bar{x} \in X$. The mapping g is said to be *metrically regular* at $\bar{x}$ with respect to R if and only if there exist two real numbers $L > 0$ and $\varepsilon > 0$ such that

$$d(x; R) \leq Ld(g(x); D), \ \forall x \in X \cap B(\bar{x}, \varepsilon). \tag{22}$$

Let us suppose that problem (8) is convex; i.e., the functions f and $-g_i$, $i \in \mathcal{I}^+$, are convex, and the functions g_i, $i \in \mathcal{I}^0$, are linear. In what follows, we shall prove that under these assumptions, the metric regularity implies the regularity condition (18) with $K = \mathcal{K}_{\bar{x}}$. The convexity of problem (8) imply the convexity of $\mathcal{E}(\mathcal{K}_{\bar{x}})$ and therefore condition (18) becomes

$$e_u \notin TC(\mathcal{E}(\mathcal{K}_{\bar{x}})). \tag{23}$$

Theorem 6.1 .Let $\bar{x} \in X$ be a local solution to problem (8), where f and $-g_i$, $i \in \mathcal{I}^+$, are convex, and g_i, $i \in \mathcal{I}^0$, are linear. If f is locally Lipschitz at $\bar{x}$ and g is metrically regular at $\bar{x}$, then the regularity condition (23) holds.

Proof. Since f is locally Lipschitz at $\bar{x}$, we can apply Theorem 5.1 from [2] which proves our assertion. We want to remark that in the mentioned theorem of [2] it is not needed g_i, $i = 1, ..., m$ to be locally Lipschitz at $\bar{x}$. $\square$

Removing the convexity assumption in Theorem 6.1, the metric regularity is no more sufficient for regularity condition (18). For this, consider again Example 5.1.

Example 5.1(continuation). Recall that in this case the sets R and D are $R = \{0\}$ and $D = \{O_2\} = \{(0,0)\}$, respectively. Thus, for a given $\varepsilon > 0$, we have $d(x; R) = |x|$, $\forall x \in B(0, \varepsilon)$. On the other hand, it turns out that

$$d(g(x); D) = \|g(x)\| = \sqrt{x^2 + 4x^4} = |x|\sqrt{1 + 4x^2}, \ \forall x \in B(0, \varepsilon).$$

Setting $L = 1$, relation (22) becomes obvious at $x = 0$, while if $x \neq 0$ we have

$$1 \leq \sqrt{1 + 4x^2}, \ \forall x \in B(0, \varepsilon).$$

This means that the metric regularity condition holds, but, as we have seen, the problem is not regular.

The following example shows that also the locally Lipschitz condition cannot be removed in the above theorem.

Example 6.1. Let problem (8) be given with $p = 1, m = 1$; $X = [0, +\infty)$, $D = \{0\}$; $f(x) = -\sqrt{x}$ and $g(x) = x$. We have $R = \{0\}$. The function f is convex but not locally Lipschitz at $\bar{x} = 0$, which is the (unique) optimal solution to problem (8).

It results

$$\mathcal{K}_{\bar{x}} = \{(u, v) \in \mathbb{R}^2 : u = \sqrt{v}, v \geq 0\}$$

and

$$TC(\text{conv } \mathcal{E}(\mathcal{K}_{\bar{x}})) = \{(u, v) \in \mathbb{R}^2 : v \geq 0\}.$$

One obtains $d(x; R) = |x|$ and $d(g(x); D) = |x|$, $\forall x \in X$. Thus the metric regularity condition holds but, as it can be easily seen, the regularity condition (23) does not.

The following example shows that the converse statement of Theorem 6.1 does not hold.

Example 6.2. Let us consider problem (8) with the following positions: $p = 0$, $m = 1$; $X = \mathbb{R}$, $D = [0, +\infty)$; $f(x) = x^4$ and $g(x) = -x^2$. We have $R = \{0\}$. Obviously, f and $-g$ are convex functions and $\bar{x} = 0$ is the (unique) optimal solution to problem (8).

We find

$$\mathcal{K}_{\bar{x}} = \{(u, v) \in \mathbb{R}^2 : u = -v^2, v \leq 0\}$$

and

$$TC(\text{conv } \mathcal{E}(\mathcal{K}_{\bar{x}})) = \{(u, v) \in \mathbb{R}^2 : u \leq 0, v \leq 0\}.$$

Therefore the regularity condition (23) holds.

On the other hand it results $d(x; R) = |x|$ and $d(g(x); D) = x^2$, $\forall x \in \mathbb{R}$. Condition (22) becomes $|x| \leq Lx^2$; $\forall L > 0$ and in every neighborhood of $\bar{x} = 0$ this inequality is not fulfilled.

7 Futher developments

Further investigations will deal, first of all, with comparison of the present results with the existing regularity conditions and with the well-posedness. We consider interesting to extend such a result to optimality conditions of higher order, to extremum problems having infinite dimensional image and to vector extremum problems.

References

[1] Clarke F.H., *Optimization and Nonsmooth Analysis*, J. Wiley, New York, 1983; reprint, SIAM Classics in Applied Mathematics, Vol.5, Philadelphia, 1994.

[2] Dien P.H., Mastroeni G., Pappalardo M. and Quang P.H., *Regularity Conditions for Constrained Extremum Problems via Image Space*, Journ. of Optim. Theory and Appl., Vol.80, pp.19–37, 1994.

[3] Giannessi F., *Semidifferentiable Functions and Necessary Optimality Conditions*, Journ. of Optim. Theory and Appl., Vol.60, pp.191–214, 1989.

[4] Giannessi F., *Constrained Optimization and Image Space Analysis.* Vol.1: Separation of Sets and Optimality Conditions, Springer, 2005.

[5] Guignard M., *Generalized Kuhn-Tucker Conditions for Mathematical Programming Problems in a Banach Space*, SIAM Journ. Control, Vol.7, pp.232–241, 1969.

[6] Ioffe A.D., *Regular Points of Lipschitz Functions*, Trans. Amer. Math. Soc., Vol.251, pp.533–548, 1994.

[7] Jourani A., *Constraint Qualifications and Lagrange Multipliers in Nondifferentiable Programming Problems*, Journ. of Optim. Theory and Appl., Vol.81, pp.19–37, 1994.

[8] John F., *Extremum Problems with Inequalities as Subsidiary Conditions*, in K.O. Friedrichs, O.E. Neugebauer and J.J. Stoker (eds.), "Studies and Essays: Courant Anniversary Volume", Interscience Publishers, New York, pp. 187–204, 1948.

[9] Karush W., *Minima of Functions of Several Variables with Inequalities as Side Conditions*, Master's Thesis, University of Chicago, Chicago, 1939.

[10] Kuhn H.W. and Tucker A.W., *Nonlinear Programming*, in J. Neyman (ed.), "Proceedings of the Second Berkeley Symposium on Mathematical Statistics

and Probabilty", University of California Press, Berkeley, California, pp. 481–492, 1951.

[11] Mangasarian O.L. and Fromovitz S., *The Fritz John Necessary Optimality Conditions in the Presence of Equality and Inequality Constraints*, Journ. of Math. Analysis and Appl., Vol.17, pp. 37–47, 1967.

[12] Mangasarian O.L., *Nonlinear Programming*, McGraw-Hill, New York, 1969; reprint, SIAM Classics in Applied Mathematics, Vol.10, Philadelphia, 1994.

[13] Penot J. P., *A New Constraint Qualification*, Journ. of Optim. Theory and Appl., Vol. 48, pp. 459–468, 1986.

[14] Slater M., *Lagrange Multipliers Revisited: A Contribution to Nonlinear Programming*, Cowles Commission Discussion Paper, Mathematics 403, 1950.

[15] Ye J.J., *Multiplier Rules under Mixed Assumptions of Differentiability and Lipschitz Continuity*, SIAM Journ. Control Optim., Vol.39, pp.1441–1460, 2001.