Assessing Credit with Equity: A CEV Model with Jump to Default

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Abstract

Unlike in structural and reduced-form models, we use equity as a liquid and observable primitive to analytically value corporate bonds and credit default swaps. Restrictive assumptions on the firm's capital structure are avoided. Default is parsimoniously represented by equity value hitting the zero barrier. Default can be either predictable, according to a CEV process that yields a positive probability of diffusive default and enables the leverage effect, or unpredictable, according to a Poisson-process jump that implies non-zero credit spreads for short maturities. Easy cross-asset hedging is enabled. By means of a carefully specified pricing kernel, we also enable analytical credit-risk management under possibly systematic jump-to-default risk.

JEL-Classification: G12, G33.

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1 Introduction

For individual firms in segments of the market with high default risk there is a clear link between default risk and equity returns and default risk appears to be systematic (Vassalou and Xing (2004)). Investors and credit-risk managers seem to have taken notice. Investors have been showing appetite for models that simultaneously handle credit and equity instruments, which is important in managing a portfolio of these two instruments. Indeed, cross-asset trading of credit risk has been gaining momentum[1] among hedge funds and banks. In their effort of assessing objective probabilities of default, credit-risk managers have been courting credit-risk models that focus on equity data[2] and that, given the systematic nature of default risk, could explicitly treat the relationship between the objective probability measure and the pricing measure(s).

Reduced-form models (see for example Duffie (1999) and the excellent reviews in Lando (2004) and Schönbucher (2003)) are not of great help, as they miss the direct linkage to the firm's capital structure. Structural models are driven by the value evolution in firm's assets. The assets-value evolution is often assumed to be diffusive so that the default can be seen predictably coming by observing changes in the capital structure of the firm (see the seminal papers of Merton (1974) and Black and Cox (1976) and the reviews in Lando (2004) and Schönbucher (2003)). While appealing,

[1] The rise of capital structure arbitrage is a good example (see Yu (2004)).

[2] KMV output is strongly driven by equity-value data. The observation that, for non-investment-grade reference entities, prices in credit default swap, corporate bond, and equity markets tend to adjust simultaneously (see Schaefer and Strebulaev (2003) and D'Ecclesia and Tompkins (2005)) impacts credit-risk management by affecting the assessment of the objective probability of default (see D'Ecclesia and Tompkins (2005)).

structural models suffer when it comes to applications. The underlying (the sum of firm's liabilities and equity) is illiquid and often non-tradable. Obtaining accurate asset volatility forecasts and reliable capital structure leverage data is difficult. Predictability of the default event implies the counterfactual prediction of zero credit spreads for short maturities[3] and, last but not least, arbitrary use of the structural default barrier is often a temptation hard to resist−endogenous barriers[4] are impractical because of the capital-structure assumptions under which they are derived are not fully realistic.

We propose a parsimonious credit risk model that does look at the firm's balance sheet but avoids the application mishaps of structural models. We take as underlying the most liquid and observable corporate security: Equity. This modelling choice brings in hedging viability and the possibility of reliable model calibration-leverage information from book values can be circumvented. We parsimoniously represent default as equity value hitting the zero barrier either diffusively or with a jump. The presence of an equity-value drop to zero has its credit-risk foundation in the incompleteness of accounting information (see Duffie and Lando (2001)), rules out default predictability, and embeds the concept of unexpected default, typical of reduced-form models, within a credit-risk model that is directly based on equity. We assume that the continuous-path part of equity value is a Constant-Elasticity-of-Variance (CEV) diffusion[5], which enables a pos-

[3] Zhou (1997) posits assets-value jumps to overcome default predictability. Duffie and Singleton (2001) explain such jumps with the presence of incomplete accounting information.

[4] See for example Leland and Toft (1996), Acharya and Carpenter (2002), and references therein.

[5] The CEV process has been first introduced to finance by Cox (1975). Among others,

itive probability of absorption at zero and fits the stylized fact of a negative link between equity volatility and equity price (the so-called 'leverage effect'), and that the jump to default is driven by an independent Poisson process. Such distributional assumptions prompt us to obtain closed forms for Corporate Bond (CB) prices and Credit Default Swap (CDS) fees, from which hedge ratios can be easily calculated. Those assumptions and a careful specification of the state-price density also empower analytical credit-risk management−we provide a closed form for the objective default probabilities in the presence of possibly systematic jump-to-default risk.

Albanese and Chen (2004) and Campi and Sbuelz (2004) also use a CEV-equity model to price credit instruments but they disregard the default predictability issue. In deriving closed-form values, we build upon a CEV result in Campi and Sbuelz (2004). Brigo and Tarenghi (2004), Naik, Trinh, Balakrishnan, and Sen (2003) and Trinh (2004) introduce a hybrid debt-equity model that considers equity as primitive but that, like structural models, necessitates a free default barrier, which is then left to potentially *ad-hoc* uses−equity value is assumed to be a geometric Brownian motion, except in Brigo and Tarenghi (2004)[6]. Das and Sundaram (2003) have proposed an equity-based model that accounts for default risk, interest risk, and equity risk using a lattice framework. As such, they do not seek hedger-friendly analytical solutions. Numerical credit risk pricing based on equity has also

the CEV-based asset-pricing literature includes the works of Albanese, Campolieti, Carr, and Lipton (2001), Beckers (1980), Boyle and Tian (1999), Cox and Ross (1976), Davydov and Linetsky (2001), Emanuel and MacBeth (1982), Forde (2005), Goldenberg (1991), Leung and Kwok (2005), Lo, Hui, Yuen (2000), Lo, Hui, and Yuen (2001), Lo, Tang, Ku, and Hui (2004), Sbuelz (2004), and Schroder (1989).

[6] Brigo and Tarenghi (2004) and Hui, Lo, and Tsang (2003) employ a flexible time-varying default barrier. Hui, Lo, and Tsang (2003) do not take equity as the underlying.

been suggested by the convertible bond[7] literature (see, for example, Andersen and Andreasen (2000), Andersen and Buffum (2003), and Tsiveriotis and Fernandes (1998); McConnell and Schwartz (1986) ignore the possibility of bankruptcy). In Cathcart and El-Jahel (2003), default occurs when a geometric-Brownian-motion signaling variable, interpreted as the credit quality of the reference entity, hits a lower default barrier or according to a hazard rate process, so that both expected and unexpected defaults are accomodated in a single framework. However, the signaling variable can hardly be identified with equity value (the default barrier is above the inaccessible zero level and there is no 'leverage effect') and the problem of a possibly freewheeling default barrier remains.

Linetsky (2005) builds upon the convertible bond literature to assess zero-coupon CB prices within a geometric-Brownian-motion model with jump-like bankruptcy where the hazard rate of bankruptcy is a negative power of the share price. The dependence of the hazard rate on the share price strongly complicates the analysis[8]. In a recent independent work, Carr and Linetsky (2005) take the stock price to follow a CEV diffusion, punctuated by a possible jump to zero. To capture the possible positive link between default and volatility, they assume that the hazard rate of default is an increasing affine function of the instantaneous variance of returns on the underlying stock. Carr and Linetsky (2005) pursue a risk-neutral pricing analysis without showing the existence of some equivalent martingale measure in their incomplete-markets setting−with CEV-like complete markets,

[7] See Nelken (2000) for a review of hybrid debt-equity instruments.

[8] The valuation formulae in Linetsky (2005) are spectral expansions that embed single integrals with respect to the spectral parameter and calculations imply the use of numerical-integration routines.

Delbaen and Shirakawa (2002) show existence for a given lower bound on the CEV parameter. Also, no study of the pricing-kernel-based choice of an equivalent martingale measure is attempted.

By contrast, the (possibly) systematic nature of CEV-like diffusive risk as well as of jump-to-default risk is carefully and parsimoniously treated in our work. In particular, we prove that our parametric pricing kernel[9] does support equivalent martingale measures. In doing so, we extend the existence result of Delbaen and Shirakawa (2002) to any negative value of the CEV parameter.

The rest of the work is organized as follows. Section 2 describes the underlying equity value process. Section 3 provides analytical results for CBs and CDSs. Section 4 specifies a pricing kernel that permits analytical objective default probabilities. After the conclusions (Section 5), an Appendix gathers lengthy proofs, analytical formulae, and details about model-based hedging.

[9] Since the jump to default is not a stopping time of the filtration generated by the continuous-path part of the stock price, our chosen Radon-Nikodym derivative is similar to the one coming from dynamic asset pricing theory with uncertain time-horizon, Blanchet-Scaillet, El Karoui, and Martellini (2005), Proposition 2. Bellamy and Jeanbleanc (2000) analyze the incompleteness of markets driven by a mixed diffusion, construct a similar Radon-Nikodym derivative, and, among other contingent claims, study American contracts. Both Blanchet-Scaillet, El Karoui, and Martellini (2005) and Bellamy and Jeanbleanc (2000) assume bounded local volatility for the stock returns, which is not our CEV case. They also refrain from considering default-driven time-horizon uncertainty.

2 The equity value

Under an equivalent martingale measure[10] $\mathbb{Q}$, the reference entity's share-price process $\{S\}$ has the following pre-default jump-diffusion dynamics:

$$\frac{dS_t}{S_{t-}} = (r-q)\,dt + \sigma S_{t-}^{\rho-1} dz_t - (dN_t - \lambda dt)\,.$$

Here below we list the main objects appearing in the dynamics of $\{S\}$:

(i) $S_0 = S$ (current share price),

(ii) $S_{t-} \equiv \lim_{\varepsilon \searrow 0} S_{t-\varepsilon}$ (left time limit),

(iii) $\rho - 1 < 0$ (constant elasticity of the diffusive volatility),

(iv) $N_t \geq 0$ (first-jump-stopped Poisson process),

(v) $\tau \equiv \inf\{t : N_t = 1\}$ (time of jump-like default),

(vi) $E_0^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau > T\}}\right] = \exp(-\lambda T)$ (chance of surviving to jump-like default),

(vii) $T > 0$ (finite maturity, in years),

(viii) $\lambda \geq 0$ (jump-to-default intensity),

where r is the constant riskfree rate, q is the constant dividend yield[11], σ ($\sigma > 0$) is a constant scale factor for the diffusive volatility, and dz is the increment of a Wiener process under $\mathbb{Q}$. The processes $\{z\}$ and $\{N\}$ are

[10] Given our incomplete-markets setting, see Section 4 for a discussion of a tractable relationship between admissible $\mathbb{Q}$s and the objective measure $\mathbb{P}$.

[11] We consider the case $r - q + \lambda > 0$. For stocks, the cost of carry is typically positive.

assumed to be independent. The assumed absence of interest rate risk is unlikely to be restrictive for non-investment-grade reference entities, as the interest-rate sensitivity of credit instruments (mainly CBs) related to those entities is low (see Cornell and Green (1991) and Guha and Sbuelz (2003)).

According to the boundary classification, an inverse relationship between volatility and share price ($\rho - 1 < 0$) is necessary to have absorption at zero with positive probability mass in the absence of jumps. Such an assumption of inverse relationship not only enables predictable default at the zero barrier, but it is also consistent with much empirical evidence on the negative correlation between stock returns and their volatilities. Realized stock volatility is negatively related to stock price. This 'leverage effect' was first discussed in Black (1976) and its various patterns have been documented by many empirical studies, for example, Christie (1982), Nelson (1991), and Engle and Lee (1993).

The time of absorption at zero in the absence of jumps is ξ, that is

$$\xi \equiv \inf\left\{t : S_t = 0, N_t = 0\right\},$$

whereas the time of absorption at zero *tout court* is the minimum between τ and ξ, that is

$$\tau \wedge \xi = \inf\left\{t : S_t = 0\right\}.$$

We take the point 0 to be the absorbing state of the share-price process $\{S\}$, so that, once default has occurred, the share price remains at zero,

$$S_t \;=\; 0, \quad \forall t \geq \tau \wedge \xi.$$

We also introduce the time of absorption at zero of the continuous part $\{S^c\}$ of $\{S\}$, that is,

$$\xi^c \;\equiv\; \inf \{t : S_t^c = 0\},$$

where

$$\frac{dS_t^c}{S_t^c} \;=\; (r - q + \lambda)dt + \sigma(S_t^c)^{\rho-1} dz_t,$$

so that ξ^c and τ are clearly independent.

3 Analytical results for CBs and CDSs

Let $T > 0$ be a finite maturity (in years) and let $V^{\mathbb{Q}}(S, T, y)$ be the T-truncated Laplace transform of $\tau \wedge \xi$'s probability density function under $\mathbb{Q}$ ($\mathbb{Q}$-p.d.f.) with Laplace parameter y ($y \geq 0$),

$$V^{\mathbb{Q}}(S, T, y) \;\equiv\; E_0^{\mathbb{Q}} \left[\exp\left(-y(\tau \wedge \xi)\right) \mathbf{1}_{\{\tau \wedge \xi \leq T\}} \right].$$

Such a quantity is of great importance, as it is the building block for the analytical pricing of CBs and CDSs (with maturity T). $V^{\mathbb{Q}}(S,T,r)$ represents the fair present value of 1 unit of currency at the reference entity's default if default occurs within T, while $V^{\mathbb{Q}}(S,T,0)$ represents the risk-neutral probability of default within T.

The next proposition is a neat and useful result stemming from the independence between $\{z\}$ and $\{N\}$. It gives an analytical characterization of $V^{\mathbb{Q}}(S,T,y)$. It states that the quantity of interest is the linear convex combination of the adjusted risk-neutral probability of default within T (with weight $\frac{\lambda}{y+\lambda}$) and of the $(y+\lambda)$-discounted value of 1 unit of currency at the diffusive default within T (with weight $\frac{y}{y+\lambda}$). The latter is the T-truncated Laplace transform of ξ^c's $\mathbb{Q}$-p.d.f. with Laplace parameter $y+\lambda$,

$$E_0^{\mathbb{Q}}\left[\exp\left(-\left(y+\lambda\right)\xi^c\right)\mathbf{1}_{\{\xi^c\leq T\}}\right],$$

and its closed form[12] has been recently derived by Campi and Sbuelz (2004). The closed form is provided in the Appendix.

Proposition 1 *Under the above assumptions, the T-truncated Laplace trans-*

[12] Davidov and Linetsky (2001), see pp. 953 and 956, point out that the T-truncated Laplace transform of ξ^c's $\mathbb{Q}$-p.d.f. with Laplace parameter $y+\lambda$ can be obtained by numerically inverting the closed-form non-truncated Laplace transform

$$\frac{1}{a}E_0^{\mathbb{Q}}\left[\exp\left(-\left(y+\lambda+a\right)\xi^c\right)\right],$$

where the inversion parameter is $a>0$.

form of $\tau \wedge \xi$'s $\mathbb{Q}$-p.d.f. with Laplace parameter y is

$$
\begin{aligned}
V^{\mathbb{Q}}(S,T,y) &= \frac{\lambda}{y+\lambda}\left[1-\exp\left(-\left(y+\lambda\right)T\right)\left(1-E_0^{\mathbb{Q}}\left[\mathbf{1}_{\{\xi^c \le T\}}\right]\right)\right] \\
&\quad + \frac{y}{y+\lambda}E_0^{\mathbb{Q}}\left[\exp\left(-\left(y+\lambda\right)\xi^c\right)\mathbf{1}_{\{\xi^c \le T\}}\right].
\end{aligned}
$$

Proof. See the Appendix. ■

Proposition 1 empowers analytical pricing of CBs and CDSs. Consider a reference entity's CB that has face value F and pays an (annualized) coupon C at regular $\frac{1}{k}$-spaced dates T_j up to its maturity T (k is a positive integer). For the sake of simplifying notation, we take the maturity T to be a rational number of the type $\frac{n}{k}$, $n \in N$.

Proposition 2 *Given the recovery rate R at default and given the assumption of Recovery of Face Value at Default (RFV), the fair CB price is*

$$
\begin{aligned}
P_{CB}(S,T,r) &= \sum_{j=1}^{kT}\frac{1}{k}\exp\left(-rT_j\right)\left[1-V^{\mathbb{Q}}\left(S,T_j,0\right)\right]C \\
&\quad + \exp\left(-rT\right)\left[1-V^{\mathbb{Q}}\left(S,T,0\right)\right]F \\
&\quad + V^{\mathbb{Q}}\left(S,T,r\right)\cdot R\cdot F.
\end{aligned}
$$

Proof. The result comes from taking the $\mathbb{Q}$-expectation of CB's discounted payoffs. RFV bears the value $V^{\mathbb{Q}}(S,T,r) \cdot R \cdot F$ for CB's defaultable part. ■

R is a fixed historical data input in applications. Under RFV, CB holders receive the same fractional recovery R of the face value F at default for CBs issued by the reference entity regardless of maturity. Guha and Sbuelz (2003) show that the RFV recovery form is consistent with typical bond indenture language (for example, the claim acceleration clause), defaulted bond price data, and relevant stylized facts of non-defaulted bond price data (the mentioned low duration of high-yield bonds; see Cornell and Green (1991)).

Consider a CDS related to the CB just described. It offers a protection payment of $(1-R)F$ in exchange for an (annualized) fee f_{CDS} paid at regular $\frac{1}{k}$-spaced dates up to the contract's maturity.

Proposition 3 *The fair CDS fee is*

$$f_{CDS}(S,T,r) = \frac{V^{\mathbb{Q}}(S,T,r)(1-R)}{\sum_{j=1}^{kT} \frac{1}{k} \exp(-rT_j)\left[1 - V^{\mathbb{Q}}(S,T_j,0)\right]}.$$

Proof. Under $\mathbb{Q}$, the fee $f_{CDS}(S,T,r)$ zeroes the CDS' net present value. ■

The holder of a CB can achieve total recouping of the face value F at default by being long a CDS. Being short $\frac{\partial}{\partial S} P_{CB}(S,T,r)$ shares Delta-hedges[13]

[13] We already remarked that interest-rate sensitivity of bonds issued by non-high-credit-quality entities is low. However, parallel shifts of the (flat) term structure of the interest rates can be hedged by selling a portfolio of default-free bonds that has interest-rate

against the pre-default price shocks driven by diffusive news. Recent evidence shows that such equity-based hedges perform reasonably well for high-yield CBs (see Naik, Trinh, Balakrishnan, and Sen (2003) and Schaefer and Strebulaev (2003)). Given analytical CB prices, an easy and effective measure of the Delta-hedge ratio is

$$\frac{\partial}{\partial S} P_{CB}(S,T,r) \simeq \frac{P_{CB}(S+\varepsilon,T,r) - P_{CB}(S-\varepsilon,T,r)}{2\varepsilon},$$

for a small ε. More details on model-based CB hedging are in the Appendix.

Tables 1 and 2 exhibit, across different maturities and levels of the parameter ρ, the yield spread of a semiannual-coupon 7% CB and the fee of a CDS with quarterly installments. The left-hand (right-hand) panel fixes $\lambda = \frac{1}{20}$ ($\lambda = \frac{1}{10}$), that is, it refers to a situation in which, on $\mathbb{Q}$-average, there is one chance of jump-like default every 20 (10) years. Positive levels of the the risk-neutral jump intensity exert the remarkable pricing impact (for short maturities in particular) that is known from pure reduced-form models. The 'leverage effect' is quite important in boosting CB spreads and CDS fees, especially at low levels of the risk-neutral jump intensity.

Table 1: The CB spread (promised yield to maturity minus r, %)

The input values are $C = 7\%$, $F = \$100$, $R = 50\%$, $S = \$1$, $k = 2$, $r = 5\%$, $q = 2\%$, and $\sigma = 35\%$.

---

sensitivity equal to $\frac{\partial}{\partial r} P_{CB}(S,T,r)$. Such a hedge ratio can be easily calculated in our model as $\frac{P_{CB}(S,T,r+\varepsilon) - P_{CB}(S,T,r-\varepsilon)}{2\varepsilon}$ for a small ε.

$\lambda = \frac{1}{20}$	$T = 2$	$T = 5$	$T = 10$	$\lambda = \frac{1}{10}$	$T = 2$	$T = 5$	$T = 10$
$\rho - 1 = -0.75$	02.70	03.18	03.20		05.05	05.13	04.79
$\rho - 1 = -2.00$	05.30	04.40	03.68		07.06	05.97	05.14

Table 2: The CDS fee (%)

The input values are $R = 50\%$, $S = \$1$, $k = 4$, $r = 5\%$, $q = 2\%$, and $\sigma = 35\%$.

$\lambda = \frac{1}{20}$	$T = \frac{1}{2}$	$T = 2$	$T = 5$	$\lambda = \frac{1}{10}$	$T = \frac{1}{2}$	$T = 2$	$T = 5$
$\rho - 1 = -0.75$	02.53	02.71	03.25		05.09	05.22	05.57
$\rho - 1 = -2.00$	03.88	05.48	04.74		06.28	07.49	06.78

4 The objective default probability

Our equity-based model contributes also to credit risk management by being conducive to closed forms for the objective default probability[14], $V^{\mathbb{P}}(S, T, 0)$,

[14] For example, the New Basel Capital Accord allows the use of model-based objective probabilities of default to determine the appropriate level of reserves to support credit risky activities.

with

$$V^{\mathbb{P}}(S,T,y) \equiv E_0^{\mathbb{P}}\left[\exp\left(-y(\tau\wedge\xi)\right)_{\{\tau\wedge\xi\leq T\}}\right],$$

where $\mathbb{P}$ is the objective probability measure. A parsimonious and closed-form-conducive way of specifying the dynamics of the share price process $\{S\}$ under the objective measure is the following:

$$\frac{dS_t}{S_{t-}} = \mu_{\mathbb{P}}dt + \sigma S_{t-}^{\rho-1}dz_t^{\mathbb{P}} - \left(dN_t^{\mathbb{P}} - \lambda_{\mathbb{P}}dt\right),$$

where

(i) $\mu_{\mathbb{P}} \equiv r - q + \theta\sigma + E^{\mathbb{P}}[(\exp(\zeta) - 1)]\lambda_{\mathbb{P}}$,

(ii) $\theta\sigma \geq 0$ (premium for the diffusive risk),

(iii) $E^{\mathbb{P}}[(\exp(\zeta) - 1)]\lambda_{\mathbb{P}} \geq 0$ (premium for the jump-like default risk).

ζ is a random variable independent from $\{z^{\mathbb{P}}\}$ and $\{N^{\mathbb{P}}\}$, which are assumed to be independent[15]. Such a terse specification of $\{S\}$'s $\mathbb{P}$-dynamics makes a neat account of systematic jump-like default risk. The risk-neutral

[15] The underlying filtration is that generated by $\{z^{\mathbb{P}}\}$, $\{N^{\mathbb{P}}\}$, and $\left\{\zeta\mathbf{1}_{\{\tau<t\}}\right\}$.

jump-to-default intensity λ maintains a simple link to the objective jump-to-default intensity $\lambda_{\mathbb{P}}$ ($\lambda_{\mathbb{P}} > 0$):

$$\lambda \quad = \quad E^{\mathbb{P}} \left[\exp \left(\zeta \right) \right] \lambda_{\mathbb{P}}.$$

If the jump-like default risk disappears ($\lambda_{\mathbb{P}} \searrow 0$), its premium shrinks to zero and the risk-neutral jump-to-default intensity does so as well. In the case of a jump to default ($\tau \wedge \xi = \tau$), the state-price-density process $\{\pi\}$ that backs the measure $\mathbb{Q}$ jumps from $\pi_{\tau-}$ to π_{τ},

$$\pi_{\tau} \quad = \quad \pi_{\tau-} \exp \left(\zeta \right).$$

Since π_{τ} provides the fair present value of 1 unit of currency received at the time of jump-like default per unit probability of such a dislikeable event, it is reasonable to impose the restriction that π_{τ} must always be at least as much as $\pi_{\tau-}$ is. Such restriction is granted by a non-negative ζ, which forces the risk premium $E^{\mathbb{P}}[(\exp(\zeta) - 1)]\lambda_{\mathbb{P}}$ to be non-negative. This is in line with the finding of Vassalou and Xing (2004) that high default risk firms earn higher equity returns than low default risk firms. The criterion of parameter parsimony suggests to take for ζ a one-parameter non-negative distribution. One such distribution is the discrete Poisson distribution with parameter ϕ ($\phi > 0$) and with support $\{0, 1, 2, ...\}$, so that the expectation

$E^{\mathbb{P}}[\exp(\zeta)]$ admits a concise closed form,

$$\begin{aligned} E^{\mathbb{P}}[\exp(\zeta)] &= \exp(\phi(e-1)) > 1, \\ E^{\mathbb{P}}[\zeta] &= \phi, \\ Var^{\mathbb{P}}[\zeta] &= \phi. \end{aligned}$$

As long as jump-like default risk is systematic (ϕ is well above 0), the jump-to-default intensity under $\mathbb{Q}$ is always greater than its level under $\mathbb{P}$ ($\lambda > \lambda_{\mathbb{P}}$). If the state-price density does not jump in the case of a jump to default ($\phi \searrow 0$, that is, $\zeta = 0$ $\mathbb{P}$-almost surely), the systematic nature of the jump-like default risk is washed away so that risk-neutral and objective jump-to-default intensities tend to coincide ($\lambda \searrow \lambda_{\mathbb{P}}$).

As far as diffusive risk is concerned, if its premium faints, it is either because such a risk is not priced ($\theta \searrow 0$) or because the risk is dimming ($\sigma \searrow 0$).

The above specification of $\{S\}$'s $\mathbb{P}$-dynamics forces $\{\pi\}$'s $\mathbb{P}$-dynamics to be as follows.

Proposition 4 *For $t < \tau \wedge \xi$, the $\mathbb{P}$-dynamics of the state-price-density* *process* $\{\pi\}$ *is*

$$\begin{aligned}\frac{d\pi_t}{\pi_{t-}} &= -rdt \\ &\quad -\theta S_{t-}^{1-\rho} dz_t^{\mathbb{P}} \\ &\quad + \Big((\exp(\zeta) - 1)\, dN_t^{\mathbb{P}} - [\exp(\phi(e-1)) - 1]\, \lambda_{\mathbb{P}} dt \Big),\end{aligned}$$

and, for $t \geq \tau \wedge \xi$,

$$\pi_t = \pi_{\tau \wedge \xi} \exp(-r(t - \tau \wedge \xi)).$$

Proof. If the process $\{\pi\}$ has the stated $\mathbb{P}$-dynamics, then there are no arbitrage opportunities. By virtue of Itô's Formula, the π-deflated gain processes generated by holding one share and by holding one unit of currency in the money-market account are local $\mathbb{P}$-martingales,

$$E_t^{\mathbb{P}} [d(\pi_t \cdot S_t \exp(qt))] = 0, \qquad E_t^{\mathbb{P}} [d(\pi_t \cdot \exp(rt))] = 0,$$

and, hence, the market is arbitrage-free[16]. ∎

[16] This indeed rules out arbitrage opportunities involving $S_t \exp(qt)$ and $\exp(rt)$, under natural conditions on dynamic trading strategies. See, for example, Appendix B.2 in Pan (2000).

We can even say more. Given finite values for θ and ϕ, our chosen state-price-density process does support an equivalent martingale measure $\mathbb{Q}$.

Proposition 5 *Let π_t be defined as above and let $T > 0$ be any finite time horizon. Then, the local $\mathbb{P}$-martingale process $\{e^{rt}\pi_t\}$, is a $\mathbb{P}$-martingale over $[0, T]$.*

Proof. See the Appendix. ∎

The previous proposition can be rephrased as follows: since the π-deflated gain process generated by holding one unit of currency in the money-market account is also a $\mathbb{P}$-martingale, its T-time level represents the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$, $\pi_T \exp(rT) = \frac{d\mathbb{Q}}{d\mathbb{P}}$.

Given our choice of the pricing kernel, the quantity $V^{\mathbb{P}}(S, T, y)$ admits an analytical expression and, as soon as diffusive risk and/or jump-to-default risk are systematic, it is always smaller than the quantity $V^{\mathbb{Q}}(S, T, y)$ for any y. In particular, systematic risk makes the $\mathbb{P}$-probability of default smaller than the $\mathbb{Q}$-probability of default.

Proposition 6 *The quantity $V^{\mathbb{P}}(S, T, y)$ has the following closed form:*

$$
\begin{aligned}
V^{\mathbb{P}}(S, T, y) \quad = \quad & \frac{\lambda_{\mathbb{P}}}{y + \lambda_{\mathbb{P}}} \left[1 - \exp\left(- \left(y + \lambda_{\mathbb{P}} \right) T \right) \left(1 - E_0^{\mathbb{P}} \left[\mathbf{1}_{\{\xi^c \leq T\}} \right] \right) \right] \\
& + \frac{y}{y + \lambda_{\mathbb{P}}} E_0^{\mathbb{P}} \left[\exp\left(- \left(y + \lambda_{\mathbb{P}} \right) \xi^c \right) \mathbf{1}_{\{\xi^c \leq T\}} \right],
\end{aligned}
$$

Proof. Since the objective drift $\mu_{\mathbb{P}} + \lambda_{\mathbb{P}}$ is constant, arguments similar to those behind Proposition 1 lead to the result. ∎

The T-truncated Laplace transform of ξ^{c}'s $\mathbb{P}$-p.d.f. with Laplace parameter $y+\lambda_{\mathbb{P}}$ is analytical (see Campi and Sbuelz (2004)). Its closed form is provided in the Appendix.

Table 3 exhibits, across different maturities and levels of the parameter ρ, the probabilities of default $V^{\mathbb{Q}}(S,T,0)$ and $V^{\mathbb{P}}(S,T,0)$. The equity premium is fixed at $\mu_{\mathbb{P}}-(r-q)=12\%$ by choosing a pricing kernel that, given $\lambda=\frac{1}{10}$, implies on average one chance of jump-like default every 16.67 years under the objective probability measure $\mathbb{P}$ ($\lambda_{\mathbb{P}}=\frac{1}{16.67}$). A greater 'leverage effect' clearly inflates the probabilities of default, which, even if the drifts $r-q$ and $\mu_{\mathbb{P}}+\lambda^{\mathbb{P}}$ are positive, remain non-defective (they approach 1 as T goes to infinity) under both $\mathbb{Q}$ and $\mathbb{P}$ as long as jump-like default has a non-zero chance to occur (λ and $\lambda^{\mathbb{P}}$ are positive).

Table 3: The probability of default under $\mathbb{Q}$ and $\mathbb{P}$ (%)

The input values are $S=\$1$, $r=5\%$, $q=2\%$, $\sigma=35\%$, $\lambda=\frac{1}{10}$, and θ and ϕ such that the risk premia $\theta\sigma$ and $[\exp(\phi(e-1))-1]\lambda^{\mathbb{P}}$ are 8% and 4%, respectively. This implies that $\mu^{\mathbb{P}}=15\%$ and $\lambda^{\mathbb{P}}=\frac{1}{16.67}$.

$V^{\mathbb{Q}}(S,T,0)$	$T=\frac{1}{2}$	$T=2$	$T=5$
$\rho-1=-0.75$	04.88	18.56	42.51
$\rho-1=-2.00$	05.97	25.45	47.51

$V^{\mathbb{P}}(S,T,0)$	$T=\frac{1}{2}$	$T=2$	$T=5$
	02.96	11.58	27.89
	03.85	16.87	32.05

In summary, we achieve analytical objective probabilities of default by augmenting the original parameter set $\{r, q, \sigma, \rho, \lambda\}$ with two risk-pricing parameters only, θ for the diffusive risk and ϕ for the jump-like default risk.

5 Conclusions

We present an equity-based credit risk model that, by taking as primitive the most liquid and observable part of a firm's capital structure, overcomes many of the problems suffered by structural models in pricing and hedging applications. Our parsimonious model avoids any assumption on the firm's liabilities. It empowers the analytical pricing of CBs and CDSs and it can match non-zero short-maturity spreads. Cross-asset hedging is viable and easy to implement. A careful specification of the state price density enables analytical credit-risk management in the presence of systematic jump-to-default risk.

6 Appendix

Proof of Proposition 1

We have that

$$\begin{aligned}
E_0^{\mathbb{Q}}\left[_{\{\tau\wedge\xi>s\}}\right] &= E_0^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau>s\}}\mathbf{1}_{\{\xi>s\}}\right] \\
&= E_0^{\mathbb{Q}}[\mathbf{1}_{\{\tau>s\}}E_0^{\mathbb{Q}}\left[\mathbf{1}_{\{\xi>s\}}|N_u=0,u\leq s\right]] \\
&= E_0^{\mathbb{Q}}[\mathbf{1}_{\{\tau>s\}}E_0^{\mathbb{Q}}\left[\mathbf{1}_{\{\xi^c>s\}}|N_u=0,u\leq s\right]] \\
&= E_0^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau>s\}}\right]E_0^{\mathbb{Q}}\left[\mathbf{1}_{\{\xi^c>s\}}\right],
\end{aligned}$$

where the last equality follows from the independence between ξ^c and τ. Hence, the time-s-evaluated $\mathbb{Q}$-p.d.f. of the stopping time $\tau\wedge\xi$ is

$$\begin{aligned}
f_{\tau\wedge\xi}(s) &= -\frac{d}{ds}E_0^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau\wedge\xi>s\}}\right] \\
&= -\frac{d}{ds}\left(E_0^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau>s\}}\right]E_0^{\mathbb{Q}}\left[\mathbf{1}_{\{\xi^c>s\}}\right]\right) \\
&= f_\tau(s)\,E_0^{\mathbb{Q}}\left[\mathbf{1}_{\{\xi^c>s\}}\right]+f_{\xi^c}(s)\,E_0^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau>s\}}\right] \\
&= \lambda\exp(-\lambda s)\,E_0^{\mathbb{Q}}\left[\mathbf{1}_{\{\xi^c>s\}}\right]+f_{\xi^c}(s)\exp(-\lambda s).
\end{aligned}$$

The T-truncated Laplace transform of $\tau\wedge\xi$'s $\mathbb{Q}$-p.d.f. with Laplace parameter y is

$$\begin{aligned} V^{\mathbb{Q}}(S,T,y) &= E_0^{\mathbb{Q}}\left[\exp\left(-y(\tau\wedge\xi)\right)\mathbf{1}_{\{\tau\wedge\xi\leq T\}}\right] \\ &= \int_0^T \exp\left(-ys\right) f_{\tau\wedge\xi^c}\left(s\right) ds \\ &= \lambda Y_1 + Y_2, \end{aligned}$$

$$\begin{aligned} Y_1 &= \int_0^T \exp\left(-\left(y+\lambda\right)s\right) E_0^{\mathbb{Q}}\left[\mathbf{1}_{\{\xi^c>s\}}\right] ds, \\ Y_2 &= \int_0^T \exp\left(-\left(y+\lambda\right)s\right) f_{\xi^c}\left(s\right) ds. \end{aligned}$$

Y_2 is the T-truncated Laplace transform of ξ^c's $\mathbb{Q}$-p.d.f. with Laplace parameter $y+\lambda$,

$$Y_2 = E_0^{\mathbb{Q}}\left[\exp\left(-\left(y+\lambda\right)\xi^c\right)\mathbf{1}_{\{\xi^c\leq T\}}\right].$$

Its closed form has been derived by Campi and Sbuelz (2004) and it can be found below after this proof. An integration by parts gives

$$\begin{aligned} Y_1 &= \frac{-1}{y+\lambda}\exp\left(-\left(y+\lambda\right)s\right) E_0^{\mathbb{Q}}\left[\mathbf{1}_{\{\xi^c>s\}}\right]\Big|_0^T \\ &\quad -\int_0^T \frac{-1}{y+\lambda}\exp\left(-\left(y+\lambda\right)s\right)\left(-f_{\xi^c}\left(s\right)\right) ds \\ &= \frac{1}{y+\lambda}\left[1-\exp\left(-\left(y+\lambda\right)T\right) E_0^{\mathbb{Q}}\left[\mathbf{1}_{\{\xi^c>T\}}\right]\right] - \frac{1}{y+\lambda}Y_2. \end{aligned}$$

This completes the proof.

Proof of Proposition 5

We will use the following auxiliary result:

Lemma 7 *Let $\rho < 1$, so possibly taking negative values, S^c be the continuous part of S as previously defined and let η_t be defined as follows:*

$$\eta_t \equiv \mathcal{E}\left(-\theta \int_0^{\cdot} (S_u^c)^{\rho-1} dz_u^{\mathbb{P}}\right)_t, \quad t \geq 0.$$

Then, for any $0 < T < \infty$, $\{\eta\}$ is a true $\mathbb{P}$-martingale over $[0,T]$. In particular, $E_0^{\mathbb{P}}[\eta_T] = 1$.

Proof. Following the proof of Theorem 2.3 in Delbaen and Shirakawa (2002), the crucial argument for η_t to be a true $\mathbb{P}$-martingale is that the integral $\int_0^T (S_u^c)^{2(1-\rho)} du$ is finite a.s.. Delbaen and Shirakawa (2002) show that this is the case for $\rho \in (0,1)$. We notice that the integral $\int_0^T (S_u^c)^{2(1-\rho)} du$ remains finite a.s. even for $\rho \leq 0$. Indeed, S^c has continuous trajectories so that the integral cannot explode. ■

To simplify the notation, we set $\widetilde{\pi}_t := e^{rt}\pi_t$. From the dynamics of π_t follows that

$$d\widetilde{\pi}_t = \widetilde{\pi}_{t-}[-\theta S_{t-}^{1-\rho} dz_t^{\mathbb{P}} + ((e^{\zeta} - 1)dN_t^{\mathbb{P}} - E^{\mathbb{P}}{}_0[e^{\zeta} - 1]\lambda_{\mathbb{P}})dt], \quad t < \tau \wedge \xi$$

and $\widetilde{\pi}_t = \widetilde{\pi}_{\tau\wedge\xi}$ for $t \geq \tau \wedge \xi$. The initial condition is of course $\widetilde{\pi}_0 = 1$. We can write the process $\widetilde{\pi}_t$ as a Doléans-Dade stochastic exponential (see, e.g., Protter (1990), p. 78) in the following way:

$$\widetilde{\pi}_t = \mathcal{E}\left(-\int_0^{\cdot} \theta S_{u-}^{1-\rho} dz_u^{\mathbb{P}}\right)_{t\wedge\tau\wedge\xi} Y_{t\wedge\tau\wedge\xi},$$

where we set

$$
\begin{aligned}
Y_t \quad &\equiv \quad \exp\left\{\int_0^t (e^{\zeta}-1)dN_u^{\mathbb{P}} - \int_0^t E_0^{\mathbb{P}}[e^{\zeta}-1]\lambda_{\mathbb{P}}du\right\} \\
&\quad\ \times \prod_{u\le t}(1+(e^{\zeta}-1)\Delta N_u^{\mathbb{P}})e^{-(e^{\zeta}-1)\Delta N_u^{\mathbb{P}}} \\
&= \quad \exp\left\{\sum_{u\le t}\ln\left(1+(e^{\zeta}-1)\Delta N_u^{\mathbb{P}}\right) - \int_0^t E_0^{\mathbb{P}}[e^{\zeta}-1]\lambda_{\mathbb{P}}du\right\}.
\end{aligned}
$$

Fix a finite time horizon $T > 0$. We first prove that the process

$$
\mathcal{E}\left(-\int_0^{\cdot} \theta S_{u-}^{1-\rho} dz_u^{\mathbb{P}}\right)_{t\wedge\tau\wedge T} Y_{t\wedge\tau\wedge T}, \quad t \ge 0, \tag{1}
$$

is a $\mathbb{P}$-martingale. To do so, we observe that, being (1) a strictly positive local $\mathbb{P}$-martingale, it is a $\mathbb{P}$-supermartingale too[17]. To show that it is a $\mathbb{P}$-martingale, it suffices to prove that $E_0^{\mathbb{P}}[\mathcal{E}(-\int \theta S_{u-}^{1-\rho} dz_u^{\mathbb{P}})_{\tau\wedge T} Y_{\tau\wedge T}] = 1$[18].

[17] This comes from the following well-known fact from martingale theory: let $M = (M_t)_{t\ge 0}$ be a local martingale defined on a given filtered probability space $(\Omega, (\mathcal{F}_t)_{t\ge 0}, \mathcal{F}, P)$ and bounded from below by a constant $a > 0$, i.e. $M_t \ge -a$ for each t. Then, M is a supermartingale. Indeed, let τ_n be a localizing sequence of stopping times for M_t, i.e. $\tau_n \uparrow +\infty$ a.s. and every stopped process $(M_{t\wedge\tau_n})_{t\ge 0}$ is a true martingale, for each n. Fix two instants $s \le t$. Fatou's lemma gives

$$
E[M_t|\mathcal{F}_s] = E[\liminf_{n\to\infty} M_{t\wedge\tau_n}|\mathcal{F}_s] \le \liminf_{n\to\infty} E[M_{t\wedge\tau_n}|\mathcal{F}_s] = M_s.
$$

[18] Indeed, let $0 < T < \infty$ and let $M = (M_t)_{t\in[0,T]}$ be a supermartingale defined on a given filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{F}, P)$ where $\mathcal{F}_0$ is trivial, such that $E[M_T] = M_0$. Then M is a martingale. To prove this, observe that, since $E[M_T] \le E[M_t] \le M_0$ for each $t \in [0,T]$, the condition $E[M_T] = M_0$ is equivalent to $E[M_t] = M_0$ for all $t \in [0,T]$. This implies that

$$
E[M_s - E[M_t|\mathcal{F}_s]] = E[M_s] - E[M_t] = 0,
$$

for every couple of instants $s \le t \le T$. Since the supermartingale property gives that $M_s - E[M_t|\mathcal{F}_s] \ge 0$, we have $E[M_t|\mathcal{F}_s] = M_s$.

Indeed, note that, in the stochastic exponential, we can replace the process S with its continuous part S^c, which is independent of $N^{\mathbb{P}}$ and ζ by construction. Conditioning with respect to τ and ζ gives

$$
\begin{aligned}
E_0^{\mathbb{P}}\left[\mathcal{E}\left(-\int_0^{\cdot}\theta S_{u-}^{1-\rho}dz_u^{\mathbb{P}}\right)_{\tau\wedge T}Y_{\tau\wedge T}\right] &= E_0^{\mathbb{P}}\left[E_0^{\mathbb{P}}\left[\mathcal{E}\left(-\int_0^{\cdot}\theta(S_{u-}^{c})^{1-\rho}dz_u^{\mathbb{P}}\right)_{\tau\wedge T}|\tau,\zeta\right]Y_{\tau\wedge T}\right] \\
&= E_0^{\mathbb{P}}\left[E_0^{\mathbb{P}}\left[\mathcal{E}\left(-\int_0^{\cdot}\theta(S_{u-}^{c})^{1-\rho}dz_u^{\mathbb{P}}\right)_t\right]_{|t=\tau\wedge T}Y_{\tau\wedge T}\right] \\
&= E_0^{\mathbb{P}}[Y_{\tau\wedge T}].
\end{aligned}
$$

The first equality is due to the fact that

$$
\begin{aligned}
Y_{\tau\wedge T} &= \exp\left\{\ln\left(1+(e^{\zeta}-1)\mathbf{1}_{\{\tau\leq T\}}\right)-E_0^{\mathbb{P}}[e^{\zeta}-1]\lambda_{\mathbb{P}}(\tau\wedge T)\right\} \\
&= (1+(e^{\zeta}-1)\mathbf{1}_{\{\tau\leq T\}})e^{-E_0^{\mathbb{P}}[e^{\zeta}-1]\lambda_{\mathbb{P}}(\tau\wedge T)},
\end{aligned}
$$

so that it depends only on τ and ζ. The second equality follows from the independence of τ, ζ and S^c and the third one from Proposition 7, stating in particular that $E_0^{\mathbb{P}}[\mathcal{E}\left(-\int_0^{\cdot}\theta(S_{u-}^{c})^{1-\rho}dz_u^{\mathbb{P}}\right)_{t\wedge T}]=1$.

It remains to compute $E_0^{\mathbb{P}}[Y_{\tau\wedge T}]$. To do so, recall that τ is exponentially distributed with parameter $\lambda_{\mathbb{P}}$, so that $\mathbb{P}[\tau>T]=e^{-\lambda_{\mathbb{P}}T}$. Then, being ζ and τ independent by assumption, we have

$$
\begin{aligned}
E_0^{\mathbb{P}}[Y_{\tau\wedge T}] &= E_0^{\mathbb{P}}[e^{\zeta}e^{-E_0^{\mathbb{P}}[e^{\zeta}-1]\lambda_{\mathbb{P}}\tau}\mathbf{1}_{\{\tau\leq T\}}]+E_0^{\mathbb{P}}[e^{-E_0^{\mathbb{P}}[e^{\zeta}-1]\lambda_{\mathbb{P}}T}\mathbf{1}_{\{\tau>T\}}] \\
&= E_0^{\mathbb{P}}[e^{\zeta}]E_0^{\mathbb{P}}[e^{-E_0^{\mathbb{P}}[e^{\zeta}-1]\lambda_{\mathbb{P}}\tau}\mathbf{1}_{\{\tau\leq T\}}]+e^{-E_0^{\mathbb{P}}[e^{\zeta}-1]\lambda_{\mathbb{P}}T}\mathbb{P}[\tau>T] \\
&= E_0^{\mathbb{P}}[e^{\zeta}]\int_0^T\lambda_{\mathbb{P}}e^{-E_0^{\mathbb{P}}[e^{\zeta}]\lambda_{\mathbb{P}}t}dt+e^{-E_0^{\mathbb{P}}[e^{\zeta}]\lambda_{\mathbb{P}}T} \\
&= 1-e^{-E_0^{\mathbb{P}}[e^{\zeta}]\lambda_{\mathbb{P}}T}+e^{-E_0^{\mathbb{P}}[e^{\zeta}]\lambda_{\mathbb{P}}T}=1.
\end{aligned}
$$

This yields that $\mathcal{E}(-\int \theta S_{u-}^{1-\rho} dz_u^{\mathbb{P}})_{t\wedge\tau\wedge T} Y_{t\wedge\tau\wedge T}$ is a $\mathbb{P}$-martingale. Doob's optional sampling theorem applies (e.g., Theorem 18 in Protter (1990)) and gives that the process $\widetilde{\pi}_t$ is a $\mathbb{P}$-martingale over the time interval $[0, T]$. Being T arbitrary, the proof is now complete.

The discounted value of cash at ξ^c within T

The T-truncated Laplace transform of ξ^c's $\mathbb{Q}$-p.d.f. with Laplace parameter w $(w \geq 0)$ has been shown by Campi and Sbuelz (2004) to be

$$E_0^{\mathbb{Q}}\left[\exp\left(-w\cdot\xi^c\right)\mathbf{1}_{\{\xi^c\leq T\}}\right] = \lim_{\epsilon\searrow 0}\sum_{n=0}^{\infty} a_n\left(A,B\right)\left(\frac{x}{2}\right)^n \frac{\Gamma(\nu-n,\frac{x}{2K},\frac{x}{2\epsilon})}{\Gamma(\nu)},$$

where

$$\Gamma(\nu) \equiv \int_0^{+\infty} u^{\nu-1}e^{-u}du \qquad \text{(Gamma Function)},$$

$$\Gamma\left(\nu-n,\frac{x}{2K},\frac{x}{2\epsilon}\right) \equiv \int_{\frac{x}{2K}}^{\frac{x}{2\epsilon}} u^{-n}u^{\nu-1}e^{-u}du \qquad \text{(Generalized Incomplete Gamma Function)},$$

$$a_n\left(A,B\right) \equiv (-1)^n C\left(B,n\right)A^n,$$

$$C\left(B,n\right) \equiv \frac{\prod_{k=1}^n\left(B-(k-1)\right)}{n!}\mathbf{1}_{\{n\geq 1\}}+\mathbf{1}_{\{n=0\}},$$

$$x \equiv S^{2(1-\rho)}, \quad \nu\equiv\frac{1}{2(1-\rho)},$$

$$A \equiv \frac{2\left(r-q+\lambda\right)}{\sigma^2(1-\rho)}, \quad K\equiv\frac{\sigma^2(1-\rho)}{2\left(r-q+\lambda\right)}\left(1-e^{-2T(r-q+\lambda)(1-\rho)}\right),$$

$$B \equiv \frac{w}{2\left(r-q+\lambda\right)\left(1-\rho\right)}.$$

The Generalized Incomplete Gamma Function, the Incomplete Gamma Function, and the Gamma function are built-in routines in many computing software like MATLAB and Mathematica, which makes the above expressions fully viable.

Model-based CB hedging

Full dynamic hedging of a long position in a CB implies being short η units of stocks as well as being long ξ units of CDSs with given fee f (for recovery rate Z and notional X), where η and ξ are adapted processes that satisfy the following system of risk-nullifying equations:

$$
\begin{aligned}
0 = \frac{\partial}{\partial S} P_{CB} \quad &- \quad \eta + \xi \frac{\partial}{\partial S} \left(\begin{array}{c} V^{\mathbb{Q}}(S,T,r)(1-Z)X \\ -\sum_{j=1}^{kT} \frac{1}{k} \exp(-rT_j)\left[1 - V^{\mathbb{Q}}(S,T_j,0)\right] f \end{array} \right), \\
0 = R \cdot F \quad &- \quad P_{CB}(S,T,r) \\
& \quad -\eta(0-S) \\
& \quad +\xi(1-Z)X \\
& \quad -\xi \left(\begin{array}{c} V^{\mathbb{Q}}(S,T,r)(1-Z)X \\ -\sum_{j=1}^{kT} \frac{1}{k} \exp(-rT_j)\left[1 - V^{\mathbb{Q}}(S,T_j,0)\right] f \end{array} \right).
\end{aligned}
$$

Our model also states that, in the case of a jump to default ($\tau \wedge \xi = \tau$), pure Delta hedging recoups a fraction

$$
\frac{\frac{\partial}{\partial S} P_{CB}(S_{\tau-}, T-\tau, r) S_{\tau-}}{P_{CB}(S_{\tau-}, T-\tau, r) - R \cdot F}
$$

of the CB loss suffered at default.

The objective probability of default at ξ^c within T

The replacement of the risk-neutral intensity-added drift $r - q + \lambda$ with the objective intensity-added drift $\mu_{\mathbb{P}} + \lambda_{\mathbb{P}}$ implies that the T-truncated Laplace transform of ξ^c's $\mathbb{P}$-p.d.f. with Laplace parameter w ($w \geq 0$) has this analytical expression:

$$E_0^{\mathbb{P}}\left[\exp\left(-w\xi^c\right)\mathbf{1}_{\{\xi^c \leq T\}}\right] = \lim_{\epsilon \searrow 0} \sum_{n=0}^{\infty} a_n\left(A_{\mathbb{P}}, B_{\mathbb{P}}\right)\left(\frac{x}{2}\right)^n \frac{\Gamma(\nu - n, \frac{x}{2K_{\mathbb{P}}}, \frac{x}{2\epsilon})}{\Gamma(\nu)},$$

for

$$A_{\mathbb{P}} \equiv \frac{2\left(\mu_{\mathbb{P}} + \lambda_{\mathbb{P}}\right)}{\sigma^2(1-\rho)},$$

$$K_{\mathbb{P}} \equiv \frac{\sigma^2(1-\rho)}{2\left(\mu_{\mathbb{P}} + \lambda_{\mathbb{P}}\right)}\left(1 - e^{-2T(\mu_{\mathbb{P}}+\lambda_{\mathbb{P}})(1-\rho)}\right),$$

$$B_{\mathbb{P}} \equiv \frac{w}{2\left(\mu_{\mathbb{P}} + \lambda_{\mathbb{P}}\right)(1-\rho)}.$$

The analytical expression of the objective probability of diffusive default within time T is retrieved by taking $w = 0$.

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