

The Maximization of Submodular Functions: Old and New Proofs for the Correctness of the Dichotomy Algorithm

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Abstract

The first purpose of this paper is to make an old (Russian) theoretical results about the structure of local and global maxima of submodular functions, Cherenin's excluding rules and his Dichotomy Algorithm more accessible for Western community. The second purpose of this paper is to present our main result which can be stated as follows. For any pair of embedded subsets, the difference of their function values is a lower bound for the difference between the unknown(!) optimal values of the corresponding partition defined by these subsets. A simple justification of Cherenin's rules, the Dichotomy Algorithm and its generalization with the new branching rules from our main result are presented. The usefulness of our new branching rules is illustrated by means of a numerical example.

Keywords: submodular function, maximization, dichotomy algorithm

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1. Introduction

We will follow to the Western tradition and discuss the maximization of a submodular set function (see Frank[7]); Lee et al.[20], Lovasz[21], and Nemhauser and Wolsey[24]) instead of the minimization of a supermodular set function as originally was done by Cherenin[5] (see also Khachaturov[16], Goldengorin[13] and Goldengorin et al.[14]). Many combinatorial optimization problems have as an underlying model the minimization of a supermodular (or, equivalently, maximization of a submodular) function, among them being the simple plant location (SPL) problem, generalized transportation problems, the max-cut problem with nonnegative edge weights, set covering and other well known problems involving the minimization of Boolean functions; see Nemhauser et al.[23], Lovasz[21] and Barahona et al.[3]. Submodular functions play an important role not only as a general presentation of the goal function for the above mentioned classes of combinatorial optimization problems but also in matroid theory (see Frank[7]). For example, the rank function of a matroid is submodular and for two distinct matroids the rank functions are distinct. So, we may obtain information about the structure of the matroid by investigating the properties of rank function (see also J. Edmonds[6]).

Although the general problem of the maximization of a submodular function is known to be NP-hard, there has a sustained research effort aimed at developing practical procedures for solving medium and large-scale problems in this class. Often the approach taken has been problem specific, and submodularity of the underlying objective function has been only implicit to the analysis. For example, Barahona et al.[3] have addressed the max-cut problem from the point of view of polyhedral combinatorics and developed a branch and cut algorithm, suitable for applications in statistical physics and circuit layout design. Beasley[4] applies Lagrangean heuristics to several classes of location problems including SPL problem and reports results of extensive experiments on a Cray supercomputer. Recently, Lee et al.[20] have made a study of the quadratic cost partition problem of which max-cut with non-negative edge weights is a special case, again from the standpoint of polyhedral combinatorics.

There have been fewer published attempts to develop algorithms for maximization of a general submodular function. We believe that the earliest attempt to exploit supermodularity in an operations research context is the work of Petrov and Cherenin[25], who identified a supermodular structure in their study of railway timetabling. Their procedure was subsequently published by Cherenin in 1962 as the so called “method of successive calculations”. Their Preliminary Preservation (Dichotomy) algorithm however is not widely known in the West (see Babayev[1] and Frieze[8]) where, as far we are aware, the only general procedure to have been studied in depth is the greedy algorithm (see Nemhauser et al.[23]) and the algorithm for the maximization

of submodular functions subject to linear constraints by Nemhauser and Wolsey[24]. Another greedy approach, can be found in Minoux[22], where an efficient implementation is proposed, known as the “accelerated greedy algorithm” (see Robertazzi and Schwartz[26]); it uses a bound already formulated in Khachaturov[16] (see also Goldengorin et al.[14]). For solving the so called experimental optimal design problem, in Robertazzi and Schwartz[26] an accelerated greedy algorithm is applied, while in Ko et al.[18] an exact branch and bound type algorithm is developed, which is later improved in Lee[19]. In Genkin and Muchnik[9] an optimal algorithm is constructed with exponential time complexity for the well-known Shannon max-min problem. This algorithm is applied to the maximization of submodular functions subject to a convex set of feasible solutions, and to the problem of – what is called – decoding monotonic Boolean functions.

The Dichotomy Algorithm has been successfully used for constructing branch and bound type algorithms, and is applied in Petrov and Cherenin[25], Cherenin[5], Khachaturov[16, 17], Frieze[8], Goldengorin[10, 13], and Goldengorin et al.[14] for solving a number of NP-hard problems.

The main purpose of this paper is to present a generalization and a simple proof for the Cherenin-Khachaturov’s results, and in particular, for their Preliminary Preservation (Dichotomy) Algorithm in English, so it will be more accessible for the Western community.

The article is organized as follows. In Section 2 we describe the structure of local and global maxima of a submodular function and present Cherenin’s theorem about the quasiconcavity of a submodular function on every chain which contains a local maximum. The Excluding (Cherenin’s) Rules and an old (Khachaturov’s) proof of their correctness with nonstrict inequalities are presented in Section 3. We give also the proof of the so called prime rules (see Theorem 3.3) without using Theorem 2.2. The main result of this paper is Theorem 4.1 which gives a generalization of Cherenin’s rules (see Section 4). We extend the preservation rules in the case where the conditions of Corollary 4.2 are violated. Corollary 4.3 is an attempt to explain what we can do in the case when the preservation rules are not applicable. In Section 5 we describe the Dichotomy (Preliminary Preservation) algorithm. We use the Dichotomy algorithm for determining a relevant polynomial solvable class of a submodular functions (PP-functions). We show that PP-functions have exactly one component of local maxima on their domain. In Corollary 5.2 we present a tertiary partitioning (branching) for a subset of the domain which can easily be generalized to a m -ary branching. Section 6 gives a number of concluding remarks.

2. The Structure of Local and Global Maxima of Submodular Set Functions

In this section we present Cherenin-Khachaturov's (see Cherenin[5] and Khachaturov[16]) results which are hardly known in the Western literature (see Babayev[1]).

Let z be a real-valued function defined on the power set 2^N of $N = \{1, 2, \dots, n\}$; $n \geq 1$. For each $S, T \in 2^N$ with $S \subseteq T$, define

$$[S, T] = \{I \in 2^N \mid S \subseteq I \subseteq T\}.$$

Note that $[\emptyset, N] = 2^N$. Any *interval* $[S, T]$ is, in fact, a *subinterval* of $[\emptyset, N]$ if $\emptyset \subseteq S \subseteq T \subseteq N$; notation $[S, T] \subseteq [\emptyset, N]$. In this paper we mean by an interval always a subinterval of $[\emptyset, N]$. Throughout this paper, it is assumed that z attains a finite maximum value on $[\emptyset, N]$. The function z is called *submodular* on $[S, T]$ if for each $I, J \in [S, T]$ it holds that

$$z(I) + z(J) \geq z(I \cup J) + z(I \cap J).$$

Expressions of the form $S \setminus \{k\}$ and $S \cup \{k\}$ will shortly be written as $S - k$ and $S + k$. Let $k \in T \setminus S$ and $[S, T]$ be an interval.

A subset $L \in [\emptyset, N]$ is called a *local maximum* of z if for each $i \in N$

$$z(L) \geq \max\{z(L - i), z(L + i)\}.$$

A subset $S \in [\emptyset, N]$ is called a *global maximum* of z if $z(S) \geq z(I)$ for each $I \in [\emptyset, N]$. We will use the Hasse diagram (see e.g., Grimaldi[15]) as the ground graph $G = (V, E)$ in which $V = [\emptyset, N]$ and a pair (I, J) is an edge iff either $I \subset J$ or $J \subset I$, and $|I \setminus J| + |J \setminus I| = 1$. The graph $G = (V, E)$ is called *z -weighted* if the weight of each vertex $I \in V$ is equal to $z(I)$; notation $G = (V, E, z)$.

A sequence of subsets $I^t \in 2^N$ such that $|I^t| = t$ and

$$\emptyset = I^0 \subset I^1 \subset I^2 \subset \dots \subset I^t \subset \dots \subset I^{n-1} \subset I^n = N$$

will be called a *chain* and be denoted by Γ . A submodular function z is *nondecreasing* (*nonincreasing*) on the chain Γ if $z(I^l) \leq z(I^m)$ ($z(I^l) \geq z(I^m)$) for all l, m such that $0 \leq l \leq m \leq n$; concepts of *increasing*, *decreasing* and *constant* (signs, respectively, $<$, $>$, $=$) are defined in an obvious manner.

A local maximum $\underline{L} \in 2^N$ ($\overline{L} \in 2^N$) is called a *lower* (respectively, *upper*) *maximum* if there is no another local maximum L such that $L \subset \underline{L}$ (respectively, $\overline{L} \subset L$).

The following Cherenin's theorem shows the quasiconcavity property of a submodular function for which a maximal chain includes a local maximum.

Theorem 2.1 *Let z be a submodular function on 2^N and let L be a local maximum which belongs to a chain $\emptyset \subseteq \dots \subseteq L \subseteq \dots \subseteq N$. Then z is nondecreasing on each subchain $\emptyset \subseteq \dots \subseteq L$ of $[\emptyset, L]$, and nonincreasing on each subchain $L \subseteq \dots \subseteq N$ of $[L, N]$.*

PROOF. We show that z is nondecreasing on $[\emptyset, L]$. The proof of nonincreasing case is similar and left to the reader. If either $L = \emptyset$ (we obtain the nonincreasing case) or $|L| = 1$, the assertion is true, since L is a local maximum of z . So, let $|L| > 1$ and $I, J \in [\emptyset, L]$ such that $J = I + k, k \in N \setminus I$.

Note that $\emptyset \subseteq \dots \subseteq I \subset J \subseteq \dots \subseteq L$. The submodularity of z implies $z(J) + z(L - k) \geq z(I) + z(L)$, or $z(J) - z(I) \geq z(L) - z(L - k)$. Since L is a local maximum, $z(L) - z(L - k) \geq 0$. Hence $z(J) \geq z(I)$, and we have finished the proof of nondecreasing case. \square

Corollary 2.1 *Let z be a submodular function on 2^N and let L_1 and L_2 be local maxima with $L_1 \subseteq L_2$. Then z is constant on $[L_1, L_2]$.*

PROOF. Let us use Theorem 2.1 to a chain $\emptyset \subseteq \dots \subseteq L_1 \subseteq L_2 \subseteq \dots \subseteq N$, first with the single local maximum L_2 and second with the single local maximum L_1 . For the first case we obtain $z(\emptyset) \leq \dots \leq z(L_1) \leq \dots \leq z(I) \leq z(L_2)$. For any subchain of the interval $[L_1, L_2]$ we have $z(L_1) \leq \dots \leq z(L_2)$. By the same reasons for the second case we have $z(L_1) \geq \dots \geq z(L_2)$. Combining both sequences of inequalities we have finished the proof of corollary 2.1. \square

The following Khachaturov's theorem is an application of Cherenin's theorem on case of a nontrivial STC.

Theorem 2.2 *Let z be a submodular function on 2^N and let \underline{L} and \overline{L} be lower and upper maxima with $\underline{L} \subseteq \overline{L}$, both located in an STC. Then z is increasing on each subchain $\emptyset \subseteq \dots \subseteq \underline{L}$ of $[\emptyset, \underline{L}]$, constant on $[\underline{L}, \overline{L}]$, and decreasing on each subchain $\overline{L} \subseteq \dots \subseteq N$ of $[\overline{L}, N]$. Moreover, every $L \in [\underline{L}, \overline{L}]$ is a local maximum of z .*

PROOF. We first show that z is increasing on $[\emptyset, \underline{L}]$. The proof of decreasing case is similar and left to the reader. If either $\underline{L} = \emptyset$ (we obtain the decreasing case) or $|\underline{L}| = 1$, the assertion is true, since \underline{L} is a local maximum of z . So, let $|\underline{L}| > 1$ and $I, J \in [\emptyset, \underline{L}]$ such that $J = I + k, k \in N \setminus I$. Note that $\emptyset \subseteq I \subset J \subseteq \dots \subseteq \underline{L}$. The submodularity of z implies $z(J) + z(\underline{L} - k) \geq z(I) + z(\underline{L})$, or $z(J) - z(I) \geq z(\underline{L}) - z(\underline{L} - k)$. Since $\underline{L} \in V_0^j$ for some $j \in J_1$, $z(\underline{L}) - z(\underline{L} - k) > 0$. Hence

$z(J) > z(I)$, and we have finished the proof of increasing case.

The property of z to be constant on $[\underline{L}, \overline{L}]$ follows from corollary 2.1.

For the ‘moreover’ part, assume to the contrary that there exists a $L \in [\underline{L}, \overline{L}]$ that is not a local maximum of z . Then either there is a $L - i \notin [\underline{L}, \overline{L}]$ with $z(L) < z(L - i)$ or there is a $L + i \notin [\underline{L}, \overline{L}]$ with $z(L) < z(L + i)$. For the first case we get according the definition of submodularity $z(L) + z(\overline{L} - i) \geq z(L - i) + z(\overline{L})$ or $z(L) - z(L - i) \geq z(\overline{L}) - z(\overline{L} - i) \geq 0$. This contradicts to $z(L) < z(L - i)$. For the second case a similar argument holds by using \underline{L} instead of \overline{L} . \square

Let V_0 be the subset of V corresponding to all local maxima of z . Let $H_0 = (V_0, E_0, z)$ be the subgraph of G induced by V_0 . This subgraph consists of at least one connected component. We denote the connected components by $H_0^j = (V_0^j, E_0^j, z)$, with $j \in J_0 = \{1, \dots, r\}$. Note that if L_1 and L_2 are vertices in the same component then $z(L_1) = z(L_2)$. We will use the following two types of local maxima; see also Khachaturov[16] and Goldengorin[13].

A component H_0^j is called a *component of strict local maxima* (shortly, STC) if for each $I \notin V_0^j$, for which there is an edge (I, L) with $L \in V_0^j$, it holds that $z(I) < z(L)$. A component H_0^j is called a *component of saddle vertices* (shortly, SDV) if for some $I \notin V_0^j$, for which there is an edge (I, L) with $L \in V_0^j$, it holds that $z(I) = z(L)$.

All vertices in a component H_0^j are local maxima of the same kind. Therefore, the index J_0 set of these components can be split into two subsets: J_1 being the index set of the STCs, and J_2 being the index set of the SDVs.

Lemma 2.1 *Let $L \in V_0^j$ for some $j \in J_1$, and let I satisfy $z(I) = z(L)$ and $(I, L) \in E$. Then $I \in V_0^j$ for the same $j \in J_1$.*

PROOF. Let $L \in V_0^j$ for some $j \in J_1$. If $I \notin V_0^j$, then $z(I) < z(L)$, since $(I, L) \in E$ and L is a local maximum of the STC. \square

In Khachaturov[16] it has been observed that any global maximum is in an STC.

Theorem 2.3 *Let S be a global maximum of the submodular function z defined on 2^N . Then $S \in V_0^j$ for some $j \in J_1$.*

PROOF. Suppose, to the contrary, that $S \in V_0^i$ with $i \in J_2$. Then there exists an $I \in V \setminus V_0$, adjacent to an $J \in V_0^i$. This I is not a local maximum. Hence, I has an adjacent vertex M with $z(M) > z(I)$. Combined we get $z(S) = z(J) = z(I) <$

$z(M)$, which contradicts that S is a global maximum of z . \square

Theorem 2.3 implies that we may restrict ourself to STCs when searching a global maximum of a submodular function z . Based on Lemma 2.1 and Corollary 2.1 we can present each component of local maxima as a maximal connected set of intervals whose end points are lower and upper maxima.

3. Excluding Rules: an Old Proof

There are two, so-called (see Petrov and Cherenin[25], and Frieze[8]), excluding rules, that can be used to exclude certain subsets from V when determining a global maximum of a submodular function. Babayev [1] has shown that Cherenin's excluding rules and Frieze's tests OP1 and OP2 are the same. By using the definitions of STC, chain, nondecreasing (nonincreasing) of a submodular function z , and Lemma 2.1 with Theorem 2.2 Cherenin and Khachaturov have proved the correctness of both excluding rules.

Theorem 3.1 *Let z be a submodular function on $[S, T] \subseteq [\emptyset, N]$ and for every $j \in J_1$, $V_0^j \cap [S, T] \neq \emptyset$. Then the following assertions hold.*

a. First Excluding Rule (FER).

If for some T_1 and T_2 with $S \subseteq T_1 \subset T_2 \subseteq T$ holds that $z(T_1) \geq z(T_2)$, then $V_0^j \cap [T_2, T] = \emptyset$ for all $j \in J_1$.

b. Second Excluding Rule (SER).

If for some S_1 and S_2 with $S \subseteq S_1 \subset S_2 \subseteq T$ holds that $z(S_1) \leq z(S_2)$, then $V_0^j \cap [S, S_1] = \emptyset$ for all $j \in J_1$.

PROOF. We prove the case (a) because a proof of the case (b) is similar. Let us consider a chain $\emptyset \subset \dots \subset S \subseteq T_1 \subset T_2 \subseteq L \subseteq T \subset \dots \subset N$ with $L \in V_0^j \cap [T_2, T] \neq \emptyset$ for some $j \in J_1$. Applying Theorem 2.2 to the subchain $\emptyset \subset \dots \subset S \subseteq T_1 \subset T_2 \subseteq L$ we have $z(\emptyset) < \dots < z(S) < z(T_1) < z(T_2) \leq z(L)$ which is a contradiction to $z(T_1) \geq z(T_2)$. \square

Note that, if we use Theorem 2.1 instead of Theorem 2.2 in the proof of Theorem 3.1 we can prove the following statement.

Theorem 3.2 *Let z be a submodular function on $[S, T] \subseteq [\emptyset, N]$ and V_0^j with $j \in J_0$ be the components of local maxima. Then the following assertions hold.*

a. First Strict Excluding Rule (FSER).

If for some T_1 and T_2 with $S \subseteq T_1 \subset T_2 \subseteq T$ holds that $z(T_1) > z(T_2)$, then

$V_0^j \cap [T_2, T] = \emptyset$ for all $j \in J_0$.

b. *Second Strict Excluding Rule (SSER)*.

If for some S_1 and S_2 with $S \subseteq S_1 \subset S_2 \subseteq T$ holds that $z(S_1) < z(S_2)$, then $V_0^j \cap [S, S_1] = \emptyset$ for all $j \in J_0$.

PROOF. We prove the case (a) because a proof of the case (b) is similar. Let us consider a chain $\emptyset \subset \dots \subset S \subseteq T_1 \subset T_2 \subseteq L \subseteq T \subset \dots \subset N$ with $L \in V_0^j \cap [T_2, T] \neq \emptyset$ for some $j \in J_0$. Applying Theorem 2.1 to the subchain $\emptyset \subset \dots \subset S \subseteq T_1 \subset T_2 \subseteq L$ we have $z(\emptyset) \leq \dots \leq z(S) \leq z(T_1) \leq z(T_2) \leq z(L)$ which is a contradiction to $z(T_1) > z(T_2)$. \square

The last theorem shows that by strict excluding rules we can not exclude any local maximum. In Section 5 we will give an example of the SPL problem in which by application of an excluding rule we discard the local minimum $\{2, 4\}$ of the corresponding supermodular function. This local minimum is an analogue of the trivial SDV for the corresponding supermodular function.

By applying Theorem 3.1a (respectively, 3.1b) we can discard $2^{|T \setminus T_2|}$ (respectively, $2^{|S_1 \setminus S_1|}$) subsets of interval $[T_2, T]$ (respectively, $[S, S_1]$) because this interval does not include a local maximum of any STC from $[S, T]$. If $T_1 = S$ and $T_2 = S + i$ then in case of Theorem 3.1a the interval $[S + i, T]$ can be discarded. If $S_1 = T - i$ and $S_2 = T$ then in case of Theorem 3.1b the interval $[S, T - i]$ can be discarded. Based on the last special cases of excluding rules it is not difficult to construct the Dichotomy Algorithm (see Section 5) for the maximization of submodular functions. Before we present the Dichotomy Algorithm we give in Theorem 3.3 a proof of the correctness of these special cases of excluding rules which is based only on Lemma 2.1, the definition of a STC, and the property of submodularity of function z .

Theorem 3.3 *Let z be a submodular function on 2^N . Suppose that for $\emptyset \subseteq S \subset T \subseteq N$ and for every $j \in J_1$, $V_0^j \cap [S, T] \neq \emptyset$. Then the following assertions hold.*

a. *First Prime Excluding Rule (FPER)*.

If for some $i \in T \setminus S$ it holds that $z(S + i) \leq z(S)$, then $[S, T - i] \cap V_0^j \neq \emptyset$ for all $j \in J_1$.

b. *Second Prime Excluding Rule (SPER)*.

If for some $i \in T \setminus S$ it holds that $z(T - i) \leq z(T)$, then $[S + i, T] \cap V_0^j \neq \emptyset$ for all $j \in J_1$.

PROOF. We prove the part a. The proof of the part b is similar.

a. Let $z(S + i) \leq z(S)$ for some $i \in T \setminus S$ and let $G \in V_0^j \cap [S, T]$ for any $j \in J_1$.

Then $S \subset G$.

Case 1: $i \in G$. From the submodularity for $G - i$ and $S + i$

$$z(G - i) + z(S + i) \geq z(G \cup S + i) + z(S) \Rightarrow$$

$$z(G - i) - z(G \cup S + i) \geq z(S) - z(S + i) \geq 0 \Rightarrow$$

$$z(G - i) \geq z(G \cup S + i) = z(G) \Rightarrow (G \text{ is a local maximum})$$

$$z(G - i) = z(G). G \in V_0^j \Rightarrow (\text{Lemma 1}) G - i \in V_0^j \Rightarrow G - i \in V_0^j \cap [S, T - i] \Rightarrow V_0^j \cap [S, T - i] \neq \emptyset.$$

Case 2: $i \notin G$.

$$i \notin G \Rightarrow G \in V_0^j \cap [S, T - i] \Rightarrow V_0^j \cap [S, T - i] \neq \emptyset. \quad \square$$

Theorem 3.3a says that if $z(S + i) - z(S) \leq 0$ for some $i \in T \setminus S$, then by preserving the interval $[S, T - i]$ we preserve at least one strict local maximum from each STC, and hence we preserve at least one global maximum from each STC which includes a global maximum. Therefore, in this case it is possible to exclude exactly the whole interval $[S + i, T]$ of $[S, T]$ from consideration when we are searching a global maximum of the submodular function z on $[S, T] \subseteq [\emptyset, N]$.

The justification of both prime rules by Theorem 3.3 is using the following definitions: local maxima, STC of a submodular function with Lemma 2.1. In the next section we present a generalization and a simple justification of the same rules.

4. Preservation Rules: Generalizations and a Simple Justification

The maximum value of the function z on the interval $[S, T] \subseteq [\emptyset, N]$ is denoted by $z^*[S, T]$. The following Theorem 4.1 establishes a relationship between the unknown optimal values of z on the two parts of the partitioning $([S, T] \setminus [Q, T])$ and $[Q, T]$ of $[S, T]$ for the FER with some Q such that $S \subset Q \subset T$; and on the two parts of the partitioning $([S, T] \setminus [S, Q])$ and $[S, Q]$ of $[S, T]$ for the SER with some Q such that $S \subset Q \subset T$.

Theorem 4.1 *Let z be a submodular function on the interval $[S, T] \subseteq [\emptyset, N]$. Then the following assertion hold.*

For any Q such that $S \subset Q \subset T$,

$$a. z^*([S, T] \setminus [Q, T]) - z^*[Q, T] \geq z(S) - z(Q).$$

$$b. z^*([S, T] \setminus [S, Q]) - z^*[S, Q] \geq z(T) - z(Q).$$

PROOF. (a) We prove only the case (a) because the proof of case (b) is similar. Let $z^*[Q, T] = z(Q \cup J)$ with $J \subseteq T \setminus Q$. Define $I = S \cup J$. Then $I \in [S, T] \setminus [Q, T]$ since $Q \setminus S \not\subseteq I$. We have that $z^*([S, T] \setminus [Q, T]) - z(S) \geq z(I) - z(S) = z(S \cup$

$J) - z(S)$. From the submodularity of z we have $z(S \cup J) - z(S) \geq z(Q \cup J) - z(Q)$. Therefore, $z^*([S, T] \setminus [Q, T]) - z(S) \geq z^*[Q, T] - z(Q)$. \square

Theorem 4.1 is a generalization of Cherenin-Khachaturov's rules saying that the difference of values of a submodular function on any pair of embedded subsets is a lower bound for the difference between the optimal values of z on the two parts of the partition which is defined by this pair of embedded subsets. The theorem can be used to decide in which part of the partition $([S, T] \setminus [Q, T])$ and $[Q, T]$ of $[S, T]$ a global maximum of z is located.

It is easy to present the partition of interval $[S, T]$ from Theorem 4.1 by means of its proper subintervals as follows:

$$(a)[S, T] \setminus [Q, T] = \bigcup_{i \in Q \setminus S} [S, T - i]$$

and

$$(b)[S, T] \setminus [S, Q] = \bigcup_{i \in T \setminus Q} [S + i, T].$$

If, in Theorem 4.1, we replace Q by $S+k$ in part (a), and Q by $T-k$ in part (b), we get the following Corollary which is related to Theorem 3.3 by means of the Dichotomy Algorithm (see Section 5)

Corollary 4.1 *Let z be a submodular function on the interval $[S, T] \subseteq [\emptyset, N]$ and let $k \in T \setminus S$. Then the following assertions hold.*

- a. $z^*[S, T - k] - z^*[S + k, T] \geq z(S) - z(S + k)$.
- b. $z^*[S + k, T] - z^*[S, T - k] \geq z(T) - z(T - k)$.

In fact, Corollary 4.1 as well as Theorem 4.1 are useful equivalent formulations for the submodularity property of function z . This can be seen easily, if we substitute in Corollary 4.1 $z^*[S, T - k]$ by $z(S_{T-k})$ for some $S_{T-k} \in [S, T - k]$ and $z^*[S + k, T]$ by $z(T_{S+k})$ for some $T_{S+k} \in [S + k, T]$.

Then, Corollary 4.1a can be read as follows: $z(S_{T-k}) - z(T_{S+k}) \geq z(S) - z(S+k)$ or, in case of Corollary 4.1b, $z(S_{T-k}) + z(S+k) \geq z(T_{S+k}) + z(S)$ where $T_{S+k} = S_{T-k} \cup S+k$ and $S = S_{T-k} \cap S+k$.

By adding the condition $z(S) - z(S+k) \geq 0$ to part (a) and the condition $z(T) - z(T-k) \geq 0$ to part (b) of Corollary 4.1 we obtain another form (see Corollary 4.2) of two prime rules from Theorem 3.3 for preserving subintervals containing at least one global maximum of z on $[S, T]$.

Corollary 4.2 *Let z be a submodular function on the interval $[S, T] \subseteq [\emptyset, N]$ and $k \in T \setminus S$. Then the following assertions hold.*

a. *First Preservation (FP) Rule.*

If $z(S) \geq z(S + k)$, then $z^[S, T] = z^*[S, T - k] \geq z^*[S + k, T]$.*

b. *Second Preservation (SP) Rule.*

If $z(T) \geq z(T - k)$, then $z^[S, T] = z^*[S + k, T] \geq z^*[S, T - k]$.*

PROOF. a. From Corollary 4.1a we have $z^*[S, T - k] - z^*[S + k, T] \geq z(S) - z(S + k)$. By assumption $z(S) - z(S + k) \geq 0$. Hence, $z^*[S, T] = z^*[S, T - k] \geq z^*[S + k, T]$.

b. The proof is similar. □

From calculation point of view these rules are the same as in Theorem 3.1 but in Theorem 3.3 more has been proven than in Corollary 4.2. In Theorem 3.3 we preserve at least one strict local maximum from each STC, and hence one global maximum from each STC that contains global maxima. In Corollary 4.2 we preserve at least one global maximum. However, we can use Corollary 4.2 for constructing some extension of the preservation rules and, consequently, excluding rules (see Corollary 4.3).

For $\varepsilon \geq 0$, the problem of ε -maximizing a submodular function z on $[S, T]$ is to find an element $J \in [S, T]$ such that $z^*[S, T] \leq z(J) + \varepsilon$; J is called an ε -maximum of z on $[S, T]$. In the following Corollary 4.3 we present an extension of the rules from Corollary 4.2, appropriate to ε -maximization.

Corollary 4.3 *Let z be a submodular function on the interval $[S, T] \subseteq [\emptyset, N]$, and $k \in T \setminus S$. Then the following assertions hold.*

a. *First θ -Preservation (θ -FP) Rule.*

If $z(S) - z(S + k) = \theta < 0$, then $z^[S, T] - z^*[S, T - k] \leq -\theta$, which means that $[S, T - k]$ contains a $|\theta|$ -maximum of $[S, T]$.*

b. *Second η -Preservation (η -SP) Rule.*

If $z(T) - z(T - k) = \eta < 0$, then $z^[S, T] - z^*[S + k, T] \leq -\eta$, which means that $[S + k, T]$ contains a $|\eta|$ -maximum of $[S, T]$.*

PROOF. The proof of part (a) is as follows. Case 1. If $z^*[S, T] = z^*[S, T - k]$ then $z^*[S, T - k] - z^*[S, T - k] \leq -\theta$ or $z^*[S, T] - z^*[S, T - k] \leq -\theta$. Case 2. If $z^*[S, T] = z^*[S + k, T]$, then from Theorem 3.3a follows that $z^*[S, T - k] - z^*[S + k, T] \geq \theta$ or $z^*[S, T - k] - z^*[S, T] \geq \theta$. Hence $z^*[S, T] - z^*[S, T - k] \leq -\theta$. The proof of (b) is similar. □

5. The Dichotomy (Preliminary Preservation) Algorithm

By means of Corollary 4.2 it is often possible to exclude a large part of $[\emptyset, N]$ from consideration when determining a global maximum of z on $[\emptyset, N]$. The so called *Preliminary Preservation (PP) algorithm* (see Goldengorin et al.,[14]) determines a subinterval $[S, T]$ of $[\emptyset, N]$ that certainly contains a global maximum of z , whereas $[S, T]$ cannot be made smaller by using the preservation rules of Corollary 4.2.

We call the PP-algorithm the *dichotomy algorithm* because in every successful step it halves the current domain of a submodular function.

Let $[S, T]$ be an interval. For each $i \in T \setminus S$, define $\delta^+(S, T, i) = z(T) - z(T - i)$ and $\delta^-(S, T, i) = z(S + i) - z(S)$; moreover, define $\delta_{\max}^+(S, T) = \max\{\delta^+(S, T, i) \mid i \in T \setminus S\}$, $r^+(S, T) = \min\{r \mid \delta^+(S, T, r) = \delta_{\max}^+(S, T)\}$. Similarly, for $\delta^-(S, T, i)$ define $\delta_{\max}^-(S, T) = \max\{\delta^-(S, T, i) \mid i \in T \setminus S\}$, $r^-(S, T) = \min\{r \mid \delta^-(S, T, r) = \delta_{\max}^-(S, T)\}$. If no confusion is likely, we shortly write $r^-, r^+, \delta^-, \delta^+$ instead of $r^-(S, T), r^+(S, T), \delta_{\max}^-(S, T)$, and $\delta_{\max}^+(S, T)$ respectively.

The Dichotomy (Preliminary Preservation) Algorithm

Procedure PP(U, W, S, T)

Input: A submodular function z on the subinterval $[U, W]$ of $[\emptyset, N]$

Output: A subinterval $[S, T]$ of $[U, W]$ such that $z^*[S, T] = z^*[U, W]$,
 $z(S) < z(S + i)$ and $z(T) < z(T - i)$ for each $i \in T \setminus S$.

begin

$S \leftarrow U; \quad T \leftarrow W;$

Step 1: **if** $S = T$

then goto Step 4;

Step 2: Calculate δ^+ and r^+ ;

if $\delta^+ \geq 0$ (Corollary 4.2b)

then begin call PPA($S + r^+, T; S, T$)
goto Step 4

end;

Step 3: Calculate δ^- and r^- ;

if $\delta^- \leq 0$ (Corollary 4.2a)

then begin call PPA($S, T - r^-; S, T$)
goto Step 4

end;

Step 4:

end;

Each time S or T are updated during the execution of the PPA, the conditions of Corollary 4.2 remain satisfied, and therefore the invariant $z^*[S, T] = z^*[U, W]$ remains valid at each step of the PPA. At the end of the algorithm we have that $\max\{\delta^+, \delta^-\} < 0$, which shows that $z(S) < z(S + i)$ and $z(T) < z(T - i)$ for each $i \in T \setminus S$. Hence Corollary 4.2 cannot be applied for further reduction of the interval $[S, T]$ without violation $z^*[S, T] = z^*[U, W]$. Note that this remark shows the correctness of the procedure $PP(\cdot)$.

If we replace in the PPA the rules of Corollary 4.2 by rules of Corollary 4.3 we obtain an ε -maximization variant of the PPA. In this case the output of the ε -PPA will be presented by a subinterval $[S, T]$ of $[U, W]$ such that $z^*[U, W] - z^*[S, T] \leq \varepsilon$ with postconditions $z(S) + \varepsilon < z(S + i)$ and $z(T) + \varepsilon < z(T - i)$ for each $i \in T \setminus S$.

The following theorem can also be found in Goldengorin (1982). It provides an upper bound for the worst case complexity of the PPA; the complexity function is taken only dependent of the number of comparisons of values for $z(I)$.

Theorem 5.1 *The time complexity of the PP algorithm procedure is at most $O(n^2)$.*

PROOF. In the steps 2 and 3 at most $2m$ comparisons are made. If the comparisons do not result in an update of either S or T , then the algorithm stops. Each time the procedure is executed, the number of elements in $T \setminus S$ is decreased by at least one. The PP algorithm starts with $N = \{1, 2, \dots, n\}$, so that the number of comparisons is bounded from above by $(2)[n + (n - 1) + \dots + 1] = (n)(n + 1)$. Hence the time complexity of the algorithm is at most $O(n^2)$. \square

Note that if the PP algorithm terminates with $S = T$, then S is a global maximum of z . Any submodular function z on $[U, W]$ for which the PP algorithm returns a global maximum for z is called a *PP-function*.

In the following example z is a *PP-function*; and we use it for illustrating the working of the PP algorithm. Let $N = \{1, 2, 3\}$; the values of z are given in Table 5.1.

I	$\{\emptyset\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$z(I)$	10	10	12	20	12	8	12	7

Table 5.1: An example of a PP-function.

After the first execution of Step 3, we have that $[S, T] = [\{\emptyset\}, \{2, 3\}]$, because $\delta^- = z(\emptyset) - z(\{1\}) = 0$, and $r^- = 1$. After the second execution of Step 2 we have that $[S, T] = [\{3\}, \{2, 3\}]$, because $\delta^+ = z(\{2, 3\}) - z(\{2\}) = 0$ and $r^+ = 3$. Finally, after

the third execution we have that $[S, T] = \{3\}$, because $\delta^- = z(\{3\}) - z(\{2, 3\}) = 8$, and $r^- = 2$. So, $S = T$, and hence z is a *PP*-function.

In the following Corollary 5.1 we describe in terms of STCs some properties of the variables S and T during the iterations of the PP-algorithm. A local maximum $L_1^j \in V_0^j$ with $j \in J_1$ which will be preserved through all iterations during the execution of the PP-algorithm by FPER ($L_1^j \in V_0^j \cap [S, T - i] \neq \emptyset$ with $j \in J_1$) or SPER ($L_1^j \in V_0^j \cap [S + i, T] \neq \emptyset$ with $j \in J_1$) is called a *representative* of STC H_0^j with $j \in J_1$ (see Theorem 3.3).

Corollary 5.1 *If z is a submodular PP-function on $[U, W] \subseteq [\emptyset, N]$, then at each iteration of the PP algorithm $S \subseteq \bigcap_{j \in J_1} L_1^j$ and $T \supseteq \bigcup_{j \in J_1} L_1^j$.*

PROOF. Theorem 3.3a says that if $z(S + i) - z(S) \leq 0$ for some $i \in T \setminus S$, then by preserving the interval $[S, T - i]$ we preserve at least one representative L_1^j from each STC H_0^j , and hence $i \notin L_1^j$. In case of Theorem 3.3b we preserve representatives L_1^j such that $i \in L_1^j$ for all STCs in $[S, T]$. Therefore, $i \in S \subseteq \bigcap_{j \in J_1} L_1^j$ and $T \supseteq \bigcup_{j \in J_1} L_1^j$. \square

The following theorem has been proven in Goldengorin[11] and gives a property of PP-functions in terms of STCs.

Theorem 5.2 *If z is a submodular PP-function on $[U, W] \subseteq [\emptyset, N]$, then $[U, W]$ contains exactly one STC.*

PROOF. From $\bigcap_{j \in J_1} L_1^j \supseteq S = T \supseteq \bigcup_{j \in J_1} L_1^j$ we obtain $\bigcap_{j \in J_1} L_1^j = \bigcup_{j \in J_1} L_1^j$ or $L_1^j = L$ for all $j \in J_1$. \square

Note that not each submodular function with exactly one STC on $[\emptyset, N]$ is a PP-function. For example, let $N = \{1, 2, 3\}$ and $z(I) = 2$ for any $I \in [\emptyset, \{1, 2, 3\}] \setminus (\{\emptyset\} \cup \{1, 2, 3\})$ and $z(I) = 1$ for $I \in (\{\emptyset\} \cup \{1, 2, 3\})$. Thus the vertex set of the unique STC defined by this submodular function can be presented as $\{\{1\}, \{1, 2\}\} \cup \{\{1\}, \{1, 3\}\} \cup \{\{2\}, \{1, 2\}\} \cup \{\{2\}, \{2, 3\}\} \cup \{\{3\}, \{1, 3\}\} \cup \{\{3\}, \{2, 3\}\}$, and the PP-algorithm terminates with $[S, T] = [\emptyset, \{1, 2, 3\}]$. So, z is not a PP-function.

Usually in branch and bound type algorithms we use a *binary* branching rule by which the original set $[S, T]$ of feasible solutions will be splitted by element k into two subsets $[S+k, T]$ and $[S, T-k]$. Let us consider an interval $[S, T]$ for which the postconditions of the PPA algorithm are satisfied, i.e., $z(S) < z(S + i)$ and $z(T) < z(T - i)$ for each $i \in T \setminus S$. Hence the PPA cannot made the interval $[S, T]$ smaller. By using

the Corollary 5.2 we can sometimes find two subintervals $[S, T - k_1]$ and $[S, T - k_2]$ such that the postconditions of the PPA algorithm for each of these intervals are violated.

Corollary 5.2 *Let z be a submodular function on the interval $[S, T] \subseteq [\emptyset, N]$ and let $k_1, k_2 \in T \setminus S$. Then the following assertions hold.*

- a. $\max\{z^*[S, T - k_1], z^*[S, T - k_2]\} - z^*[S + k_1 + k_2, T] \geq z(S) - z(S + k_1 + k_2)$.
b. $\max\{z^*[S + k_1, T], z^*[S + k_2, T]\} - z^*[S, T \setminus \{k_1, k_2\}] \geq z(T) - z(T \setminus \{k_1, k_2\})$.

PROOF. We prove only the part (a) and leave the proof of the part (b) to the reader. Replace in Theorem 4.1a Q by $S + k_1 + k_2$. Then, $z^*([S, T] \setminus [Q, T]) - z^*[Q, T] = z^*(\bigcup_{i \in Q \setminus S} [S, T - i]) - z^*[Q, T] = z^*([S, T - k_1] \cup [S, T - k_2]) - z^*[S + k_1 + k_2, T] = \max\{z^*[S, T - k_1], z^*[S, T - k_2]\} - z^*[S + k_1 + k_2, T] \geq z(S) - z(Q) = z(S) - z(S + k_1 + k_2)$. \square

In case of $z(S) - z(S + k_1 + k_2) \geq 0$ we can discard the interval $[S + k_1 + k_2, T]$ and continue the searching for an optimal solution by applying the PPA separately to each of remaining intervals $[S, T - k_1]$ and $[S, T - k_2]$, each of which are obtained by subtraction an element k_i from T . The symmetrical case will be obtained if $z(T) - z(T \setminus \{k_1, k_2\}) \geq 0$. Corollary 5.2 can easily be generalized to the case of m -ary branching by elements k_1, k_2, \dots, k_m with $m \leq |T \setminus S|$.

We finish our paper with an example of the SPL problem of which the data are presented in Table 5.2. This example is borrowed from Boffey[2].

Location		Delivery cost to site				
i	r_i	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
1	7	7	15	10	7	10
2	3	10	17	4	11	22
3	3	16	7	6	18	14
4	6	11	7	6	12	8

Table 5.2: The data of the SPL problem

For solving the SPL problem it suffices to solve the problem $\min\{z(I) \mid I \in [\emptyset, N]\} =$

$z^*[\emptyset, N] = z(G)$ with $N = \{1, 2, 3, 4\}$, $n = 5$ and

$$z(I) = \sum_{i \in I} r_i + \sum_{j=1}^n \min_{i \in I} c_{ij}.$$

As usual for the SPL problem, r_i is the fixed cost of opening a plant at location i , c_{ij} is the cost of satisfying the demand of customer j by plant i , and $z(I)$ is a supermodular function. We use this example for illustrating that the supermodular function defined by data from Table 5.2 is not a PP-function. Of course, here we mean the corresponding definition of PP-function which can be obtained by correct changing all signs together with definitions of local, global maxima of submodular function to local, global minima of supermodular function. It is easy to check that this supermodular function has two trivial analogues of STCs: $\{1, 4\}$, $\{1, 3\}$ and one trivial analogue of SDV: $\{2, 4\}$.

After the first execution of Step 3 of the PP-algorithm, we have that $[S, T] = [\{1\}, \{1, 2, 3, 4\}]$, because $\delta^+ = z(\{1, 2, 3, 4\}) - z(\{2, 3, 4\}) = 0$ and $r^+ = 1$. Together with interval $[\{\emptyset\}, \{2, 3, 4\}]$ the PP-algorithm has discarded the trivial SDV $\{2, 4\}$. After the second execution of Steps 2 and 3 the PP-algorithm terminates with interval $[S, T] = [\{1\}, \{1, 2, 3, 4\}]$, because all postconditions of the PP-algorithm are satisfied. Hence, this function is not a PP-function. A global minimum of this SPL problem can be found by application the following analogue of the inequality from Corollary 5.2b:

$$\min\{z^*[S + k_1, T], z^*[S + k_2, T]\} - z^*[S, T \setminus \{k_1, k_2\}] \leq z(T) - z(T \setminus \{k_1, k_2\}).$$

Let us substitute all possible pairs of $\{k_1, k_2\}$ to the right side of this inequality with $S = \{1\}$ and $T = \{1, 2, 3, 4\}$. Then, we have that only $z(\{1, 2, 3, 4\}) - z(\{1, 2, 3, 4\} - \{3, 4\}) = 52 - 53 < 0$. Hence, we can discard the interval $[\{1\}, \{1, 2, 3, 4\} - \{3, 4\}]$ and we may continue the solving of the problem $z^*[\{1\}, \{1, 2, 3, 4\}]$ by solving the two remained subproblems $z^*[S + k_1, T] = z^*[\{1, 3\}, \{1, 2, 3, 4\}]$ and $z^*[S + k_2, T] = z^*[\{1, 4\}, \{1, 2, 3, 4\}]$. Each of these subproblems can be solved by the corresponding analogue of the PP-algorithm.

6. Conclusions

We presented to the Western community the Cherenin's theorem about the quasi-concavity of a submodular function on any maximal chain which passes through a component of the local maxima. By using this result the structure of a submodular function can be described in terms of components of graphs of local maxima. Each

component of the graph of local maxima is a maximal connected set of intervals whose end points are lower and upper local maxima.

Our theorem 3.3 can be considered as a more easy way for proving the correctness of the prime excluding rules without using Cherenin's and Khachaturov's theorems 2.1 and 2.2, respectively.

Our main result of the paper is Theorem 4.1 which can be stated as follows. For any pair of embedded subsets, the difference of their function values is a lower bound for the difference between the unknown (!) optimal values of the corresponding partition defined by these subsets. We have successfully applied a special case of this theorem (see Corollary 4.2) for constructing a Data-Correcting algorithm (see Goldengorin et al.[14]) for more efficient solving of the instances from Lee et al.[20] for the quadratic cost partition problem. By using the different presentations (a) and (b) of the partition for any pair of embedded subsets of the domain we have proved the correctness of the Dichotomy algorithm (see Corollary 4.2) and have given the basis of a generalization of the Dichotomy algorithm for the case of ε -maximization of submodular functions (see Corollary 4.3).

In Theorem 5.2 we prove that the functions that belong to an algorithmically defined (by the Dichotomy algorithm) polynomially solvable class of submodular functions (the PP-functions) contains exactly one component of local maxima. So, the number of subproblems created in a branch and bound type algorithm which is based on the Dichotomy algorithm can be used as an upper bound for the number of the STCs of local maxima. By the same way, an upper bound for the number of all local maxima (STCs and SDVs) by using the strict excluding rules (see Theorem 5) can be calculated.

Another way for constructing branch and bound type algorithms including the data-correcting algorithms by a new possibility to split a subset of the domain of a submodular function into more than two parts such that each of which can be obtained by only subtracting a single element k_i from the top of the interval $[S, T]$ (see Corollary 5.2a) or by only adding a single element k_i to the bottom of the interval $[S, T]$ (see Corollary 5.2b) is presented. The last possibility of branching can be used for reducing the values of δ^+ or δ^- in the branch and bound type algorithms which are based on the Dichotomy algorithm. An interesting subject for future research is the investigation of the computational efficiency of m -ary branching rules for specific problems which can be reduced to the maximization of submodular functions.

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