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DIVISIBLE DESIGN GRAPHS

By Willem H. Haemers, Hadi Kharaghani, Maaike A. Meulenberg

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Divisible Design Graphs

Willem H. Haemers*
Department of Econometrics and Operations Research,
Tilburg University, The Netherlands

Hadi Kharaghani
Department of Mathematics and Computer Science,
University of Lethbridge, Canada

Maaike A. Meulenberg
Department of Public Health, Erasmus MC,
University Medical Center Rotterdam, The Netherlands

Abstract

A divisible design graph is a graph whose adjacency matrix is the incidence matrix of a divisible design. These graphs are a natural generalization of (v, k, λ)-graphs. In this paper we develop some theory, find many parameter conditions and give several constructions.

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1 Introduction

Any graph Γ can be interpreted as a design (or incidence structure), by taking the vertices of Γ as points, and the neighborhoods of the vertices as blocks. In other words, the adjacency matrix of Γ is interpreted as the incidence matrix of a design. Let us call such a design the *neighborhood design* of Γ.

Consider a k-regular graph Γ on v vertices with the property that any two distinct vertices have exactly λ common neighbors. Rudvalis [15] has called such a graph a (v, k, λ)-graph, because the neighborhood design of Γ is a (v, k, λ)-design (also known as a symmetric 2-(v, k, λ)-design). Conversely, a (v, k, λ)-design with a polarity with no absolute points (meaning that it has a symmetric incidence matrix with zero diagonal), can be interpreted as a (v, k, λ)-graph.

This interplay between graphs and designs turned out to be fruitful for both parts. For example, a very easy construction of a symmetric 2-$(16, 6, 2)$ design

*corresponding author, e-mail haemers@uvt.nl

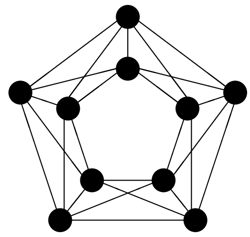

Figure 1: A proper divisible design graph

goes via the 4×4 grid, (that is, the line graph of the complete bipartite graph $K_{4,4}$), which is a $(16, 6, 2)$-graph.

In this paper we generalize the concept of a (v, k, λ)-graph, and introduce graphs with the property that the neighborhood design is a divisible design.

Definition 1.1 *A k-regular graph is a* divisible design graph *(DDG for short) if the vertex set can be partitioned into m classes of size n, such that two distinct vertices from the same class have exactly λ_1 common neighbors, and two vertices from different classes have exactly λ_2 common neighbors.*

For example the graph of Figure 1 (which is the strong product of K_2 and C_5) is a DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n) = (10, 5, 4, 2, 5, 2)$. Note that a DDG with $m = 1$, $n = 1$, or $\lambda_1 = \lambda_2$ is a (v, k, λ)-graph. If this is the case, we call the DDG *improper*, otherwise it is called *proper*.

The definition of a divisible design (often also called group divisible design) varies. We take the definition given in Bose [2].

Definition 1.2 *An incidence structure with constant block size k is a* (group) divisible design *whenever the point set can be partitioned into m classes of size n, such that two points from one class occur together in λ_1 blocks, and two points from different classes occur together in exactly λ_2 blocks.*

A divisible design D is said to have the *dual property* if the dual of D (that is, the design with the transposed incidence matrix) is again a divisible design with the same parameters as D. From the definition of a DDG it is clear that the neighborhood design of a DDG is a divisible design D with the dual property. Conversely, a divisible design with a polarity with no absolute points is the neighborhood design of a DDG.

A DDG is closely related to a strongly regular graph. We recall that a k-regular graph with $1 \leq k \leq v - 2$ is *strongly regular* with parameters (v, k, λ, μ), whenever any two adjacent vertices have exactly λ common neighbors, and any two distinct nonadjacent vertices have exactly μ neighbors in common. Thus, a (v, k, λ)-graph is a strongly regular graph with $\lambda = \mu$. It follows easily that a proper DDG is strongly regular if and only if the graph or the complement is mK_n, the disjoint union of m complete graphs of size n.

Deza graphs (see [8]) are k-regular graphs which are not strongly regular, and where the number of common neighbors of two distinct vertices takes just two values. So proper DDGs, which are not isomorphic to mK_n or the complement, are Deza graphs.

In this paper we obtain many necessary conditions for parameters of a DDG. We develop some structure theory, and present several constructions. Among these constructions there are many new Deza graphs and several new divisible designs. We also give a number of characterization results for DDGs.

A complication with the neighborhood design of a graph is that isomorphisms and automorphisms are defined differently for both structures. For example, the two non-isomorphic $(16, 6, 2)$-graphs, produce isomorphic 2-$(16, 6, 2)$ designs. Since in this paper we are mainly concerned with existence, this difficulty will not play an important role.

2 Eigenvalues

As usual, I_ℓ (or just I), and J_ℓ (or just J) are the $\ell \times \ell$ identity and all-ones matrix, respectively. We define $K = K_{(m,n)} = I_m \otimes J_n = \text{diag}(J_n, \ldots, J_n)$. Then we easily have that a graph Γ is a DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ if and only if Γ has an adjacency matrix A that satisfies:

$$A^2 = kI_v + \lambda_1(K_{(m,n)} - I_v) + \lambda_2(J_v - K_{(m,n)}). \tag{1}$$

Clearly $v = mn$, and taking row sums on both sides of Equation 1 yields

$$k^2 = k + \lambda_1(n-1) + \lambda_2 n(m-1).$$

So we are left with at most four independent parameters. Some obvious conditions are $1 \le k \le v-1$, $0 \le \lambda_1 \le k$, $0 \le \lambda_2 \le k-1$. From Equation (1) strong information on the eigenvalues of A can be obtained. (Throughout we write eigenvalue multiplicities as exponents.)

Lemma 2.1 *The eigenvalues of the adjacency matrix of a DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ are*

$$\left\{k^1,\ \left(\sqrt{k-\lambda_1}\right)^{f_1},\ \left(-\sqrt{k-\lambda_1}\right)^{f_2},\ \left(\sqrt{k^2-\lambda_2 v}\right)^{g_1},\ \left(-\sqrt{k^2-\lambda_2 v}\right)^{g_2}\right\},$$

where $f_1 + f_2 = m(n-1)$, $g_1 + g_2 = m-1$ and $f_1,\ f_2,\ g_1,\ g_2 \ge 0$.

Proof. The eigenvalues of $K_{(m,n)}$ are $\{0^{m(n-1)}, n^m\}$. Because I, J and K commute it is straightforward to compute the eigenvalues of A^2 from equation (1). They are

$$\{(k^2)^1,\ (k-\lambda_1)^{m(n-1)},\ (k^2-\lambda_2 v)^{m-1}\},$$

and must be the squares of the eigenvalues of A. □

Some of the multiplicities may be 0, and some values may coincide. In general, the multiplicities f_1, f_2, g_1 and g_2 are not determined by the parameters, but if we know one, we know them all because $f_1 + f_2 = m(n-1)$, $g_1 + g_2 = m-1$, and

$$\operatorname{trace} A = 0 = k + (f_1 - f_2)\sqrt{k - \lambda_1} + (g_1 - g_2)\sqrt{k^2 - \lambda_2 v}. \tag{2}$$

This equation leads to the following result.

Theorem 2.2 *Consider a proper DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n)$, and eigenvalue multiplicities (f_1, f_2, g_1, g_2).*
a. $k - \lambda_1$ or $k^2 - \lambda_2 v$ is a nonzero square.
b. If $k - \lambda_1$ is not a square, then $f_1 = f_2 = m(n-1)/2$.
c. If $k^2 - \lambda_2 v$ is not a square, then $g_1 = g_2 = (m-1)/2$.

Proof. If one of $k - \lambda_1$ and $k^2 - \lambda_2 v$ equals 0, then Equation (2) gives that the other one is a nonzero square. If $k - \lambda_1$ and $k^2 - \lambda_2 v$ are both non-squares, it follows straightforwardly that the square-free parts of these numbers are equal non-squares, hence Equation (2) has no solution. The second and third statement are obvious consequences of Equation (2). □

If $k - \lambda_1$, or $k^2 - \lambda_2 v$ is not a square, the multiplicities (f_1, f_2, g_1, g_2) can be computed from the parameters. The outcome must be a set of nonnegative integers. This gives a condition on the parameters, which is often referred to as the *rationality condition*. Only if $k - \lambda_1$ and $k^2 - \lambda_2 v$ are both squares (that is, all eigenvalues of A are integers), the parameters do not determine the spectrum. Then $0 \leq g_1 \leq m - 1$, so there are at most m possibilities for the set of multiplicities.

3 The quotient matrix

The vertex partition from the definition of a DDG gives a partition (which will be called the *canonical partition*) of the adjacency matrix

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} \end{bmatrix}.$$

We shall see that the canonical partition is equitable, which means that each block $A_{i,j}$ has constant row (and column) sum. For this, we introduce the $v \times m$ matrix S, whose columns are the characteristic vectors of the partition classes. Then S satisfies

$$S = I_m \otimes \mathbf{1}_n,\ S^\top S = nI_m,\ SS^\top = K_{(m,n)},$$

where $\mathbf{1}_n$ denotes the all-ones vector with n entries. Next we define $R = \frac{1}{n} S^\top AS$, which means that each entry $r_{i,j}$ of R is the average row sum of $A_{i,j}$. We will call R the *quotient matrix* of A.

Theorem 3.1 *The canonical partition of the adjacency matrix of a proper DDG is equitable, and the quotient matrix R satisfies*

$$R^2 = RR^\top = (k^2 - \lambda_2 v)I_m + \lambda_2 n J_m.$$

The eigenvalues of R are

$$\left\{k^1,\ \left(\sqrt{k^2-\lambda_2 v}\right)^{g_1},\ \left(-\sqrt{k^2-\lambda_2 v}\right)^{g_2}\right\}.$$

Proof. Equation (1) gives $(\lambda_1 - \lambda_2)K_{(m,n)} = A^2 - \lambda_2 J_v - (k - \lambda_1)I_v$. Clearly A commutes with the right hand side of this equation and therefore with $K_{(m,n)}$. Thus $ASS^\top = SS^\top A$. Using this we find:

$$SR = \tfrac{1}{n}SS^\top AS = \tfrac{1}{n}ASS^\top S = AS,$$

which reflects that the partition is equitable. Similarly,

$$R^2 = \tfrac{1}{n^2}S^\top ASS^\top AS = \tfrac{1}{n}S^\top A^2 S = (k^2 - \lambda_2 v)I_m + \lambda_2 n J_m,$$

where in the last step we used $k^2 = k + \lambda_1(n-1) + \lambda_2 n(m-1)$. From the formula for R^2 it follows that R has eigenvalues $\pm\sqrt{k^2 - \lambda_2 v}$, whose multiplicities add up to $m - 1$. If v is an eigenvector of R, then Sv is an eigenvector of A for the same eigenvalue. Therefore the multiplicity of an eigenvalue $\pm\sqrt{k^2 - \lambda_2 v}$ of R is at most equal to the multiplicity of the same eigenvalue of A. This implies that the multiplicities are the same. □

The above lemma can easily be generalized to divisible designs with the dual property. This more general version of the lemma is due to Bose [2] (who gave a much longer proof).

If one wants to construct a DDG with a given set of parameters, one first tries to construct a feasible quotient matrix. For this the following straightforward properties of R can be helpful:

Proposition 3.2 *The quotient matrix R of a DDG satisfies*

$$\begin{aligned} \textstyle\sum_i (R)_{i,j} &= k \text{ for } j = 1,\dots,m,\\ \textstyle\sum_{i,j}(R)^2_{i,j} &= \operatorname{trace}(R^2) = mk^2 - (m-1)\lambda_2 v,\\ 0 \le \operatorname{trace}(R) &= k + (g_1 - g_2)\sqrt{k^2-\lambda_2 v} \le m(n-1). \end{aligned}$$

In some cases these conditions lead to nonexistence or limited possibilities for R. The following result is essentially due to Bose [2] (though his formulation is different).

Theorem 3.3 *Consider a DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n)$. Write $k = mt + k_0$ for some integers t and k_0 with $0 \le k_0 \le m-1$. Then the entries of R take exactly one, or two consecutive values if and only if*

$$k_0^2 - mk_0 - k^2 + km + \lambda_1 m(n-1) = 0\ .$$

If this is the case then $R = tJ + N$, where N is the incidence matrix of a (possibly degenerate) (m, k_0, λ_0)-design with a polarity.

Proof. If each entry of R equals t or $t+1$, then in each row k_0 entries are equal to $t+1$ and $m-k_0$ entries are equal to t (because the row sums of R are k). Therefore,

$$mk_0(t+1)^2 + mt^2(m-k_0) = \mathrm{trace}(R^2) = mk^2 + (m-1)\lambda_2 v,$$

which leads to $k_0^2 - mk_0 - k^2 + km + \lambda_1 m(n-1) = 0$. Conversely, if the equation holds, then a matrix R with k_0 entries $t+1$ in each row, and all other entries equal to t satisfies the conditions of Equation 3.2. Moreover, any other solution to these equations has the same properties. (Indeed changing some entries to integer values different from t and $t+1$, such that the sum of the entries remains the same, increases the sum of the squares of the entries). Suppose $R = tJ + N$ for some incidence structure N, then $N = N^\top$, and Theorem 3.1 implies that $N^2 \in \langle J, I\rangle$, therefore N is the incidence matrix of a (m, k_0, λ)-design. □

Note that the number of absolute points of the polarity equals $\mathrm{trace}\, N = \mathrm{trace}\, R - mt = k + (g_1 - g_2)\sqrt{k^2 - \lambda_2 v} - mt$, which is equal to $k - mt = k_0$ if $k^2 - \lambda_2 v$ is not a square.

4 Constructions

In this section we present the constructions of DDGs known to us.

4.1 (v, k, λ)-graphs and designs

We recall that the *incidence graph* of a design with incidence matrix N is the bipartite graph with adjacency matrix

$$\begin{bmatrix} O & N \\ N^\top & O \end{bmatrix}.$$

Construction 4.1 *The incidence graph of an (n, k, λ_1)-design with $1 < k \le n$ is a proper DDG with $\lambda_2 = 0$.*

Construction 4.2 *The disconnected graph for which each component is an (n, k, λ_1)-graph ($1 < k < n$), or the incidence graph of an (n, k, λ_1)-design ($1 < k \le n$), is a proper DDG with $\lambda_2 = 0$.*

Proposition 4.3 *For a proper DDG Γ the following are equivalent.*
a. Γ comes from Construction 4.1, or 4.2.
b. Γ is bipartite or disconnected.
c. $\lambda_2 = 0$.

Proof. It is clear that a bipartite or disconnected DDG has $\lambda_2 = 0$. Assume Γ is a DDG with $\lambda_2 = 0$. Then in every block row of the canonical partition of the adjacency matrix there is exactly one nonzero block (otherwise the neighborhood of a vertex contains vertices in different blocks which contradicts $\lambda_2 = 0$), and each nonzero block is the incidence matrix of a (n, k, λ_1)-design. If such a block is on the diagonal it is the adjacency matrix of a (n, k, λ_1)-graph with $1 < k < n$. If it is not on the diagonal the transposed block is on the transposed position, and together they make the bipartite incidence graph of a (n, k, λ_1)-design with $1 < k \le n$. $\square$

Construction 4.4 *If A' is the adjacency matrix of a (m, k', λ')-graph ($1 \le k' < m$), then $A' \otimes J_n$ is the adjacency matrix of a proper DDG with $k = \lambda_1 = nk'$, $\lambda_2 = n\lambda'$.*

Proposition 4.5 *For a proper DDG Γ the following are equivalent.*
a. Γ comes from Construction 4.4.
b. The adjacency matrix of Γ can be written as $A' \otimes J_n$ for some $m \times m$ matrix A'.
c. $\lambda_1 = k$.

Proof. The only nontrivial claim is that c implies a. Assume Γ is a DDG with $k = \lambda_1$. Then any two rows of the adjacency matrix belonging to the same class are identical. Since the blocks have constant row and column sum this implies that all blocks have only ones, or only zeros. Therefore the adjacency matrix has the form $A' \otimes J_n$, where A' is a symmetric $(0, 1)$-matrix with zero diagonal and row sum k/n. Moreover, any two distinct rows of A' have inner product λ_2/n. Therefore A' represents a (m, k', λ')-graph. $\square$

Construction 4.6 *Let $A_1, \ldots, A_m$ ($m \ge 2$) be the adjacency matrices of m (n, k', λ')-graphs with $0 \le k' \le n - 2$. Then $A = J - K + \text{diag}(A_1, \ldots, A_m)$ is the adjacency matrix of a proper DDG with $k = k' + n(m-1)$, $\lambda_1 = \lambda' + n(m-1)$), $\lambda_2 = 2k - v$.*

Proposition 4.7 *For a proper DDG Γ the following are equivalent.*
a. Γ comes from Construction 4.6.
b. The complement of Γ is disconnected.
c. $\lambda_2 = 2k - v$.

Proof. Let x and y be two vertices of Γ. Simple counting gives that the number of common neighbors is at most $2k - v$, and equality implies that x and y are adjacent. So, if $\lambda_2 = 2k - v$, then two vertices from different classes are adjacent, and hence the complement is disconnected. Conversely, suppose Γ is a DDG with disconnected complement $\overline{G}$ (say). Let x and y be vertices in different components of $\overline{G}$. Then x and y have no common neighbors in $\overline{G}$, and hence x and y are adjacent vertices in Γ with $2k - v$ common neighbors. Therefore $\lambda_2 = 2k - v$, and all vertices from different classes are adjacent. Finally, equivalence of a and b is straightforward. $\square$

Note that in the above constructions the used (v, k, λ)-graphs and designs may be degenerate. This means that the above constructions include the k-regular complete bipartite graph ($k \ge 2$), the $(k + 1)$-regular complete bipartite graph minus a perfect matching ($k \ge 2$), the disjoint union of m complete graphs K_n

($m \geq 2$, $n \geq 3$), the complete m-partite graph with parts of size n ($m \geq 2$, $n \geq 2$), and the complete m-partite graphs with parts of size n extended with a perfect matching of the complement ($m \geq 2$, $n \geq 4$, n even). So these DDGs exist in abundance, and we'll call them *trivial*.

4.2 Hadamard matrices

An $m \times m$ matrix H is a *Hadamard matrix* if every entry is 1 or -1, and $HH^\top = mI$. A Hadamard matrix H is called *graphical* if H is symmetric with constant diagonal, and *regular* if all row and column sums are equal (to ℓ say). Without loss of generality we assume that a graphical Hadamard matrix has diagonal entries -1. Consider a regular graphical Hadamard matrix H. It is well known (and easily proven; see [6]) that $\ell^2 = m$ and that $\frac{1}{2}(H+J)$ is the adjacency matrix of a $(m, (m+\ell)/2, (m+2\ell)/4)$-graph.

Construction 4.8 *Consider a regular graphical Hadamard matrix H of order $m \geq 4$ and row sum $\ell = \pm\sqrt{m}$. Let $n \geq 2$. Replace each entry with value -1 by $J_n - I_n$, and each $+1$ by I_n, then we obtain the adjacency matrix of a DDG with parameters $(mn, n(m-\ell)/2 + \ell, (n-2)(m-\ell)/2, n(m-2\ell)/4 + \ell, m, n)$.*

In terms of the adjacency matrix the construction becomes:

$$H \otimes I_n + \tfrac{1}{2}(J - H) \otimes J_n \ .$$

Using this, it is straightforward to check that Equation 1 is satisfied. We saw that the order m of a regular graphical Hadamard matrix is an even square. Such Hadamard matrices exit for example if $m = 4^t$ for $\ell = 2^t$ and $\ell = -2^t$, for all $t \geq 1$. But for many more values of m and ℓ such Hadamard matrices are known (see [6] for a survey, and [11] for some recent developments). The smallest regular graphical Hadamard matrices are

$$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix} .$$

For the first one, the DDG is the $4 \times n$ grid, that is, the line graph of $K_{4,n}$. The second one gives DDGs with parameters $(4n, 3n-2, 3n-6, 2n-2, 4, n)$; for $n = 2$ this is the complement of the cube. The DDGs of Construction 4.8 are improper whenever $\lambda_1 = \lambda_2$, which is the case if and only if $n = 4$.

Construction 4.9 *Consider a regular graphical Hadamard matrix H of order $\ell^2 \geq 4$ with diagonal entries -1 and row sum ℓ. The graph with adjacency matrix*

$$A = \begin{bmatrix} M & N & O \\ N & O & M \\ O & M & N \end{bmatrix} , \text{ where}$$

$$M = \frac{1}{2}\begin{bmatrix} J+H & J+H \\ J+H & J+H \end{bmatrix}, \text{ and } N = \frac{1}{2}\begin{bmatrix} J+H & J-H \\ J-H & J+H \end{bmatrix},$$

is a DDG with parameters $(6\ell^2, 2\ell^2+\ell, \ell^2+\ell, (\ell^2+\ell)/2, 3, 2\ell^2)$.

For the two Hadamard matrices presented above, this leads to DDGs with parameters $(24, 10, 6, 3, 3, 8)$ and $(24, 6, 2, 1, 3, 8)$, respectively.

4.3 Divisible designs

Here we examine known constructions of divisible designs that admit a symmetric incidence matrix with zero diagonal, and therefore correspond to DDGs. Clearly, we can restrict ourselves to divisible designs with the dual property. Many constructions for these kind of designs come from divisible difference sets. Such a construction uses a group $\mathcal{G}$ of order $v = mn$, together with a subset of $\mathcal{G}$ of order k, called the base block. The blocks of the design are the images of the base block under the group operation. Thus we obtain v blocks of size k (blocks may be repeated). This construction gives a divisible design if the group $\mathcal{G}$ has a normal subgroup $\mathcal{N}$ of order n and the base block is a so called divisible difference set relative to $\mathcal{N}$. It follows from the construction that such a divisible design has the dual property. Moreover, one can order the points and blocks such that the incidence matrix becomes symmetric, and it is also easy to find an ordering that gives a zero diagonal. The problem is to find an ordering that simultaneously provides a symmetric matrix and a zero diagonal. Such an ordering is not always possible. For example, consider the group $\mathcal{G} = C_4 = \{1, a, a^2, a^3\}$ with normal subgroup $\mathcal{N} = \{1, a^2\}$ and base block $\{1, a\}$. Then we obtain a divisible design with blocks $\{1, a\}$, $\{a^2, a^3\}$, $\{a, a^2\}$, $\{a^3, a\}$, and point classes $\{1, a^2\}$ and $\{a, a^3\}$. Some possible incidence matrices are:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

So symmetry as well as a zero diagonal can be achieved. However, there is no way to permute the rows (and columns) such that the matrix becomes symmetric with zero diagonal. Indeed, it would give a DDG with parameters $(4, 2, 0, 1, 2, 2)$ which is impossible by Theorem 2.2. For having a symmetric incidence matrix with zero diagonal, the divisible difference set should be reversible (or equivalently, it must have a strong multiplier -1). Several reversible relative difference sets are known. For example, for the group $\mathcal{G} = C_5 \times S_2 = \{1, a, a^2, a^3, a^4\} \times \{1, b\}$ the base block $\{(1, b), (a, 1), (a, b), (a^4, 1), (a^4, b)\}$ is a reversible difference set relative to $\mathcal{N} = S_2$, and hence gives a DDG. This DDG is the one given in Figure 1. In fact, several of the examples constructed so far can also be made with a reversible divisible difference set. These include all trivial examples and some of the ones from Construction 4.8. For more examples and information on reversible difference sets we refer to [1].

Another useful result on divisible designs is the construction and characterization of divisible designs with $k - \lambda_1 = 1$ given in [9]. We recall that the *strong product* of two graphs with adjacency matrices A and B, is the graph with adjacency matrix $(A + I) \otimes (B + I) - I$.

Construction 4.10 *Let Γ' be a strongly regular graph with parameters $(m, k', \lambda, \lambda+1)$. Then the strong product of K_2 with Γ' is a DDG with $n = 2$, $\lambda_1 = k-1 = 2k'$ and $\lambda_2 = 2\lambda + 2$.*

Checking the correctness of the construction is straightforward. There exist infinitely many strongly regular graphs with the required property. For example the Paley graphs (see [4]). But there are infinitely many others. It easily follows that the complement of a strongly regular graph with $\mu - \lambda = 1$ has the same property. Thus we can get two DDGs from one strongly regular graph with $\mu - \lambda = 1$, unless the strongly regular graph is isomorphic to the complement (which is the case for the Paley graphs). For example the Petersen graph and its complement lead to DDGs with parameters $(v, k, \lambda_1, \lambda_2, m, n) = (20, 7, 6, 2, 10, 2)$ and $(20, 13, 12, 8, 10, 2)$, respectively. The pentagon, which is a strongly regular graph with parameters $(5, 2, 0, 1)$, leads once more to the example of Figure 1. In fact, several graphs coming from Construction 4.10 can also be constructed by use of a reversible divisible difference set. This includes all Paley graphs.

Theorem 4.11 *Let Γ be a nontrivial proper DDG, then Γ comes from Construction 4.10 if and only if $k - \lambda_1 = 1$.*

Proof. Assume Γ is a DDG with $k - \lambda_1 = 1$. According to [9] the neighborhood design D, or its complement has incidence matrix $N = (A \otimes J_n) + I_v$, where one of the following holds: (i) $J - 2A$ is the core of s skew-symmetric Hadamard matrix (this means that $A + A^\top = J - I$, and $4AA^\top = (v+1)I + (v-3)J$). (ii) $n = 2$, and A is the adjacency matrix of a strongly regular graph with $\mu - \lambda = 1$, or (iii) $A = O$, or $A = J - I$. Case iii and its complement correspond to trivial DDGs. Case ii corresponds to Construction 4.10 (note that N has no zero diagonal, but interchanging the two rows in each class gives N the required property). Also the complement of Case ii corresponds to Construction 4.10. Indeed, $J_v - N = J_v - (A \otimes J_2) - I_v = (J_m - A) \otimes J_2 - I_v$, where A, and therefore also $J_m - A - I_m$ is the adjacency matrix of a strongly regular graph with $\mu - \lambda = 1$. Finally we will show that Case i is not possible for a DDG. Suppose $PN = P(A \otimes J) + P$, or $P(J - N)$ is symmetric with zero diagonal for some permutation matrix P, then P is symmetric and preserves the block structure. The quotient matrix Q of P is a symmetric permutation matrix such that QA is symmetric with zero diagonal. We have $A + A^\top = J - I$, so $J - Q = AQ + A^\top Q = AQ + QA$, and therefore $\mathrm{trace}(J - Q) = 2\,\mathrm{trace}(QA) = 0$, so $Q = I$, a contradiction. □

4.4 Distance-regular graphs

The main purpose of the section is to obtain DDGs from distance-regular graphs. We assume that the reader is familiar with the concept of a distance-regular graph (see [5]). We start with an observation on the diameter of a connected DDG.

Lemma 4.12 *Let Γ be a connected proper DDG. Then Γ has diameter 2 or 3. If the diameter is 3, then $\lambda_1 = 0$, or $\lambda_2 = 0$.*

Proof. Clearly the diameter is not 1. Proposition 4.3 implies that $\lambda_2 = 0$ if and only if Γ is the bipartite incidence graph of a symmetric design, which has diameter 3. Suppose $\lambda_2 > 0$. Then vertices from different classes have distance at most 2. If, in addition $\lambda_1 > 0$, then the diameter of Γ is 2. Suppose $\lambda_1 = 0$. If every pair of vertices from the same class is adjacent, then $n = 2$ and Γ has diameter 2. Otherwise, two distinct nonadjacent vertices x and y from the same class C have no common neighbor, so the distance is at least 3. Take a vertex z adjacent to x but not in C. (If x has no neighbors outside C, then the same is true for all vertices of C, so the graph is disconnected.) Then z and y have $\lambda_2 > 0$ common neighbors, so x and y have distance 3, and Γ has diameter 3. □

Note that Construction 4.8 with $n = 2$ provides examples with diameter 2 for which $\lambda_1 = 0$. Distance-regular graphs of diameter 2 are precisely the connected strongly regular graphs. Improper DDGs are (v, k, λ)-graphs, so they are strongly regular. The proper nontrivial distance-regular DDGs have diameter 3, and therefore $\lambda_1 = 0$ or $\lambda_2 = 0$. The case $\lambda_2 = 0$ is characterized in Proposition 4.3. The next proposition gives DDGs with $\lambda_1 = 0$. We recall that for a distance-regular graph the parameters λ and μ give the number of common neighbors of a pair of vertices at distance 1, and 2, respectively. Moreover, a distance-regular graph of diameter d is called *antipodal* if being at distance distance d or 0 defines an equivalence relation on the vertices.

Proposition 4.13 *Suppose Γ is an antipodal distance-regular graph of diameter 3. If $\lambda = \mu$, then Γ is a proper DDG with parameters $(n(\mu n + 2), \mu n + 1, 0, \mu, \mu n + 2, n)$. If $\lambda = \mu - 2$, then the complement of Γ is a proper DDG with parameters $(\mu n^2, \mu n(n-1), \mu n(n-2), \mu(n-1)^2, \mu n, n)$.*

Proof. The parameters (intersection array) of an antipodal distance-regular graph of diameter 3 are given in [5], p.431. From this the first statement follows straightforwardly. The second statement follows from the simple observation that in a k-regular graph two vertices x and y with $\lambda_{x,y}$ common neighbors have $v - 2k + \lambda_{x,y}$ common neighbors in the complement if x and y are adjacent, and $v - 2k + \lambda_{x,y} - 2$ common neighbors in the complement if x and y are nonadjacent. □

There are (infinitely) many distance-regular graphs having one of the properties of the above proposition. For example the cube is antipodal with $\lambda = \mu - 2 = 0$, so the complement is a DDG with parameters $(8, 4, 0, 2, 4, 2)$ (this graph can also be constructed by Construction 4.8 or with a divisible difference set). Another example is the point graph of the generalized quadrangle $GQ(2,4)$ from which a spread has been deleted. This is an antipodal distance-regular graph of diameter 3 with $v = 27$, $k = 8$, $\lambda = \mu - 2 = 1$, so the complement is a DDG with parameters $(27, 18, 9, 12, 9, 3)$. The Klein graph is an antipodal distance-regular graph with 24 vertices, degree 7, and $\lambda = \mu = 2$. This gives a DDG with parameters $(24, 7, 0, 2, 8, 3)$.

Theorem 4.14 *A graph* Γ *is a distance-regular proper DDG if and only if* Γ *is one of the following.*
a. A complete multipartite graph,
b. The incidence graph of a (n, k, λ)*-design with* $1 < k \leq n$.
c. An antipodal distance-regular graph of diameter 3 *with* $\lambda = \mu$.

Proof. We saw that the graphs a, b and c are distance-regular graphs as well as proper DDGs. Suppose Γ is a distance-regular proper DDG, which is not complete multipartite. Then Γ has diameter 3, and $\lambda_2 = 0$ or $\lambda_1 = 0$ (Lemma 4.12). If $\lambda_2 = 0$, then Γ belongs to case b (Proposition 4.3). If $\lambda_1 = 0$, then Γ is a distance-regular graph with $d = 3$ and $\mu = \lambda_2$. Therefore $\lambda = 0$, or $\lambda = \mu$. If $\lambda = 0$, then being at distance 0, 1, or 3 defines an equivalence relation on the vertices. This means that Γ is an imprimitive distance-regular graph which is bipartite nor antipodal, which is impossible (see [5], p.140). Therefore $\lambda = \mu = \lambda_2$, and being at distance 0 or 3 defines an equivalence relation. This implies that Γ is antipodal, so we are in case c. □

4.5 Partial complements

The complement of a DDG is almost never a DDG again. If the partition classes are the same, then only the complete multipartite graph and its complement have this property. The cube (which is a bipartite DDG with two classes) and its complement (which is a DDG with four classes) is an example where the canonical partitions differ. However, if we only take the complement of the off-diagonal blocks it is more often the case that we get a DDG again. We call this the *partial complement* of the DDG. We have seen one such example in Construction 4.10, where the partial complement can be constructed in the same way, and hence produces no new examples. The following idea however can give new examples.

Proposition 4.15 *The partial complement of a proper DDG* Γ *is again a DDG if one of the following holds:*
a. The quotient matrix R *equals* $t(J - I)$ *for some* $t \in \{1, \ldots, n-1\}$.
b. $m = 2$.

Proof. We use Equation 1. In Case a, the partial complement has adjacency matrix $\widetilde{A} = J - K - A$. In Section 3 we saw that $AK = KA = ASS^\top = SRS^\top$. Since $R = t(J - I)$ this implies $AK \in \text{Span}\{J, K\}$. Therefore $\widetilde{A}^2 \in \text{Span}\{I, J, K\}$, and $\widetilde{A}$ represents a DDG.

In Case b, the vertices can be ordered such that the partial complement has adjacency matrix $\widetilde{A} = J - K + DAD$, where $D = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$. The quotient matrix R is a symmetric 2×2 matrix with constant row sum, hence $R \in \mathrm{Span}\,\{I_2, J_2\}$, and therefore $AK = SRS^\top \in \mathrm{Span}\,\{K_{2,n}, J_v\}$, and also $DADK = DAK \in \mathrm{Span}\,\{K_{2,n}, J_v\}$. Moreover, $(DAD)^2 = DA^2D \in \mathrm{Span}\,\{I_v, J_v, K_{2,n}\}$, and hence $\widetilde{A}^2 \in \mathrm{Span}\,\{I, J, K\}$, which proves our claim. □

For example the antipodal distance-regular DDGs (Theorem 4.14,c) satisfy a of the above proposition. In particular the partial complement of the Klein graph is a DDG with parameters $(24, 14, 7, 8, 8, 3)$. Taking partial complements often gives improper DDGs. Conversely, the arguments also work if Γ is an improper DDG (that is, Γ is a (v, k, λ)-graph), provided Γ admits a nontrivial equitable partition that satisfies a or b. An equitable partition of a (v, k, λ)-graph that satisfies a is a so called Hoffman coloring (see [10]). Note that the diagonal blocks are zero, so the partition corresponds to a vertex coloring. Thus we have:

Construction 4.16 *Let Γ be a (v, k, λ)-graph. If Γ has a Hoffman coloring, or an equitable partition into two parts of equal size, then the partial complement is a DDG.*

Also this construction can give improper DDGs, but in many cases the DDG is proper. For example there exists a strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (40, 12, 2, 4)$ with a so called spread, which is a partition of the vertex set into cliques of size 4 (see [10]). The complement of Γ is a $(40, 27, 18)$-graph, and the spread of Γ is a Hoffman coloring in the complement. The partial complement is Γ with the edges of the cliques of the spread removed. This gives a DDG with parameters $(40, 9, 0, 2, 10, 4)$. By taking the union of five classes in this Hoffman coloring, we obtain an equitable partition into two parts of size 20. The partial complement with respect to this partition gives a DDG with parameters $(40, 17, 8, 6, 2, 20)$.

4.6 Symmetric balanced generalized weighing matrices

In this section we introduce DDGs that can be constructed from some more specialized combinatorial objects. The main ingredients for these construction methods consist of symmetric balanced weighing matrices with zero diagonal over a variety of cyclic groups [14], 2-designs with a symmetric circulant incidence matrix, and block nega-circulant Bush–type Hadamard matrices [12, 13]. For the positive integer t, let R_t be the back diagonal identity matrix, let $C_t = circ(0, 1, 0, \ldots, 0)$ be the circulant matrix of order t with the first row $(0, 1, 0, \ldots, 0)$ and let $N_t = negacirc(0, 1, 0, \ldots, 0)$ be the nega-circulant matrix of order t with the first row $(0, 1, 0, \ldots, 0)$. The cyclic group generated by C_t and N_t are of order t and $2t$ respectively. We denote these groups by $\mathcal{C}_t$ and $\mathcal{N}_{2t}$.

Theorem 4.17 *Let n be the number of points of a (n, k', λ)-design with a circulant incidence matrix, and let ℓ be a positive integer such that $q = 2\ell n + 1$*

is a prime power. Assume s is a nonnegative integer s, and define $m = 1 + q + q^2 + \cdots + q^{2s+1}$. Then there is a DDG with parameters

$$(mn,\ q^{2s+1}k',\ q^{2s+1}\lambda,\ 2q^{2s}k'^2\ell,\ m,\ n).$$

Proof. Let D be the incidence matrix of the (n, k', λ)-design, and let $W = [w_{ij}]$ be a symmetric balanced generalized weighing matrix with parameters $(m, q^{2s+1}k', q^{2s}(q-1))$ with zero diagonal over the cyclic group $\mathcal{C}_n$. Since $\frac{q-1}{n} = 2\ell$ is an even integer, the existence of W follows from [14], Corollary 3. The block matrix $A = [w_{ij}DR]$ is the incidence matrix of the desired DDG. To see that A is symmetric, note that each individual block is symmetric and w_{ij} and D are circulant and so commuting matrices for each i, j. There are q^{2s+1} blocks in each row of A and $DR(DR)^\top = DD^\top$. Thus $\lambda_1 = q^{2s+1}\lambda$. Noting the parameters of W, the set $\{w_{hj}w_{ij}^{-1} : 1 \leq j \leq n, w_{hj} \neq 0, w_{ij} \neq 0\}$ contains exactly $2q^{2s}\ell$ copies of every element of $\mathcal{C}_n$. Since $\sum_{g\in\mathcal{C}_n} g = J$, it follows that $\lambda_2 = 2q^{2s}k'^2\ell$. □

DDGs obtained from the above lemma are proper for all values of ℓ, except for $\ell = \frac{k-1}{n-k'}$. For example, if we take $s = 0$,

$$D = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

and $\ell = 1$, we get a $(40, 27, 18)$-graph. However, taking $s = 0$, $\ell = 1$, $n = 2$, $D = I_2$ and $\mathcal{C}_2$ for the group in Theorem 4.17, we get a DDG with parameters $(12, 5, 0, 2, 6, 2)$, that is, the dodecahedron. Taking $s = 0$, $\ell = 1$, $n = 3$, $D = circ(0, 1, 1)$ and C_3 for the group in 4.17, we get a DDG with parameters $(24, 14, 7, 2, 8, 3)$. Replacing the circulant matrix $D = circ(0, 1, 1)$ with $D = I_3$, will generate a DDG with parameters $(24, 7, 0, 8, 8, 3)$, which is the distance-regular Klein graph.

Theorem 4.18 *Assume $4h^2$ is the order of a block nega-circulant Bush–type Hadamard matrix. Let ℓ be a positive integer, such that $q = 8\ell h + 1$ is a prime power. Let s be a nonnegative integer, and define $m = 1 + q + q^2 + \cdots + q^{2s+1}$. Then there is a DDG with parameters*

$$(4h^2m,\ (2h^2 - h)q^{2s+1},\ (h^2 - h)q^{2s+1},\ 2\ell(2h - 1)(2h^2 - h)q^{2s},\ m,\ 4h^2).$$

Proof. Let H be a block nega-circulant Bush–type Hadamard matrix of order $4h^2$ and let $W = [w_{ij}]$ be a symmetric balanced generalized weighing matrix with parameters $(1+q+q^2+\cdots+q^{2m+1}, q^{2m+1}k', q^{2m}(q-1))$ with zero diagonal over the cyclic group $\mathcal{G}$ generated by $N_{2h} \otimes I_{2h}$. Let $M = H - K_{(2h,2h)}$ and define $Q = [Mw_{ij}]$. Let $P = J_{4h^2} - K_{(2h,2h)}$. Then we can split the matrix Q in two disjoint parts

$$A^+ = \tfrac{1}{2}[P|w_{ij}| + Qw_{ij}] \ \text{ and } \ A^- = \tfrac{1}{2}[P|w_{ij}| - Qw_{ij}],$$

where $|w_{ij}|$ denotes the matrix whose entries are the absolute values of the entries of the matrix w_{ij}. We need now to adjust the two matrices A^+ and A^- to make all the blocks symmetric. We do this by multiplying each of the block entries by the matrix $L = R_{2h} \otimes I_{2h}$. Each of the matrices A^+ and A^- are now the incidence matrices of a DDG with the above parameters. □

The pair of DDGs above is called a *twin* DDG. In fact, the construction satisfies the condition of Proposition 4.15(a), and the twin designs are the partial complements of each other. Noting that $\lambda_2 - \lambda_1 = (2\ell - h + 1)h$, all DDGs obtained from this theorem are proper for even values of h. However, for odd values of h, the lemma gives improper DDGs for $\ell = \frac{h-1}{2}$ and proper DDGs for all other values of ℓ.

Corollary 4.19 *If $2h$ is the order of a Hadamard matrix and $q = 8h\ell + 1$ is a prime power, then there exist a DDG with the parameters of Theorem 4.18.*

Proof. In this case, existence of the required nega-cyclic Bush–type Hadamard matrix follows from the construction in [13] with some obvious modification. □

As an example, for $h = \ell = 1$, we have a DDG with parameters $(40, 9, 0, 2, 10, 4)$ from 4.18. There are only two known block nega-circulant Bush–type Hadamard matrix of order $4h^2$, h odd, the trivial one of order 4 and the non-trivial of order 36, see [12].

4.7 Sporadic constructions

In this section we construct DDGs with parameter sets $(12, 6, 2, 3, 3, 4)$ and $(18, 9, 6, 4, 6, 3)$. We didn't see how to generalize these constructions, so we call them *sporadic*. Verifying correctness of the two constructions is straightforward.

Construction 4.20 *The line graph of the octahedron is a DDG with parameters $(12, 6, 2, 3, 3, 4)$. Each class consist of four edges of the octahedron forming a quadrangle.*

Construction 4.21 *The following matrix A is the adjacency matrix of a DDG with parameters $(18, 9, 6, 4, 6, 3)$.*

$$A = \begin{bmatrix} O & J & J & I & I & I \\ J & O & J & I & P & Q \\ J & J & O & I & Q & P \\ I & I & I & O & J & J \\ I & Q & P & J & O & J \\ I & P & Q & J & J & O \end{bmatrix}, \text{ where } P = Q^\top = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

5 More conditions

5.1 Hasse-Minkowski

The Hasse-Minkowski theory provides necessary condition for divisible designs, that push the condition from Theorem 2.2 a little further.

Theorem 5.1 *If there exits a DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n)$, then both of the diophantine equations*

$$(k - \lambda_1)X^2 + (-1)^{m(n-1)/2} n^m Y^2 = Z^2 \quad \textit{and}$$
$$(k^2 - \lambda_2 v)X^2 + (-1)^{(m-1)/2} n\lambda_2 Y^2 = Z^2$$

have an integral solution $(X, Y, Z) \neq (0, 0, 0)$.

Proof. The Hasse-Minkowski conditions for square divisible designs have been worked out (see [3], or [5], p.23). Combining this with the results from Theorem 2.2 we obtain the conditions above. □

For example, a DDG with intersection array $(25, 8, 4, 2, 5, 5)$ does not exist because $14X^2 + 10Y^2 = Z^2$ has no nonzero integral solution (consider the equation modulo 7). Note that, as soon as one of the coefficients is a square, there is a nonzero solution. So the above theorem gives no condition if all eigenvalues are integral.

5.2 The quotient matrix

We already mentioned that sometimes one can prove nonexistence of the quotient matrix R. In case $m = 3$ we can make a general statement.

Proposition 5.2 *If $m = 3$ and $k^2 - \lambda_2 v$ is not a square, then the following system of equations has an integral solution.*

$$X + Y + Z = k,$$
$$X^2 + Y^2 + Z^2 = k^2 - 2\lambda_2 v/3,$$
$$X^3 + Y^3 + Z^3 = 3XYZ + k(k^2 - \lambda_2 v)\,.$$

Proof. The quotient matrix R is a symmetric 3×3 matrix with all row and column sums equal to k and, since $k^2 - v\lambda_2$ is not a square, also $\mathrm{trace}(R) = k$. This implies

$$R = \begin{bmatrix} X & Y & Z \\ Y & Z & X \\ Z & X & Y \end{bmatrix} ,$$

so $\mathrm{trace}(R^2) = 3(X^2 + Y^2 + Z^2) = k^2 + 2(k^2 - \lambda_2 v)$. The third equation comes from $\det R = -k(k^2 - \lambda_2 v)$. □

For example a DDG with parameters $(21, 12, 8, 6, 3, 7)$ does not exist because $X^2 + Y^2 + Z^2 = 60$ has no integral solution. Note that Construction 4.9 gives infinitely many DDGs that satisfy the condition of the above proposition.

Proposition 5.3 *There exists no DDG with parameters* $(14, 10, 6, 7, 7, 2)$*, and* $(20, 11, 2, 6, 10, 2)$.

Proof. In both cases $n = 2$, so $\operatorname{trace} R \leq m$. For the first parameter set this gives a contradiction, because $\operatorname{trace} R = k = 10$ and $m = 7$. For the second parameter set, Theorem 3.3 implies that $R = J + P$ for some symmetric permutation matrix P. Therefore $\operatorname{trace} R = 10$, P has zero diagonal, and the spectrum of R is $\{11, 1^4, -1^5\}$. This implies that the adjacency matrix has eigenvalues 11, 3^{f_1}, -3^{f_2}, 1^4 and -1^5 where $f_1 + f_2 = 10$. This is impossible. □

5.3 Four eigenvalues

E.R. van Dam and E. Spence [7] studied regular graphs with four distinct eigenvalues. In particular, for all feasible spectra up to 27 vertices they decided on existence on a graph with that spectrum. In many cases they used a computer search to find all graphs with the required spectrum. For several feasible parameter sets for DDGs, nonexistence of the graph follows because it would be a graph with four distinct eigenvalues, which doesn't exist according to Van Dam and Spence. Of course, a graph with the required spectrum doesn't have to be a DDG. For example, [7] gives ten graphs with the spectrum $\{7^1, -1^7, \sqrt{7}^{\,8}, -\sqrt{7}^{\,8}\}$, of a DDG with parameters $(24, 7, 0, 2, 8, 3)$. But only one is a DDG, the distance-regular Klein graph. This example shows that being a DDG cannot be deduced from the spectrum.

6 Small parameters

We have generated all putative parameters sets $(v, k, \lambda_1, \lambda_2, m, n)$ for DDGs on at most 27 vertices that survive the eigenvalue conditions given in Section 2. The outcome is presented in the table below. The parameter sets with $\lambda_2 = 0$, $\lambda_1 = k$ and $\lambda_2 = 2k - v$, which have been characterized in Section 4.1, are omitted. For each parameter set we computed the eigenvalues different from the degree: $\vartheta_1 = \sqrt{k - \lambda_1}$, $\vartheta_2 = -\sqrt{k - \lambda_1}$, $\vartheta_3 = \sqrt{k^2 - \lambda_2 v}$, $\vartheta_4 = -\sqrt{k^2 - \lambda_2 v}$. If possible, we also computed the respective multiplicities f_1, f_2, g_1, g_2, and denote them in the table as exponents. If the multiplicities are not determined we only give the eigenvalues, but be aware that in this case a multiplicity may be equal to 0. The table gives fifty parameter sets. For each set we tried to decide on existence or nonexistence using the results from this article. Only in ten cases we don't know the answer.

v	k	λ_1	λ_2	m	n	$\vartheta_1^{f_1}$	$\vartheta_2^{f_2}$	$\vartheta_3^{g_1}$	$\vartheta_4^{g_2}$	existence	reference
8	3	0	1	4	2	$\sqrt{3}^{2}$	$-\sqrt{3}^{2}$	-	-1^{3}	no	[7]
8	4	0	2	4	2	2^{1}	-2^{3}	0^{3}	-	yes	4.8
10	5	4	2	5	2	-	-1^{5}	$\sqrt{5}^{2}$	$-\sqrt{5}^{2}$	yes	4.10
12	5	0	2	6	2	$\sqrt{5}^{3}$	$-\sqrt{5}^{3}$	-	-1^{5}	yes	4.13,4.17
12	5	1	2	4	3	2	-2	1	-1	yes	4.8
12	6	2	3	3	4	2^{3}	-2^{6}	0^{2}	-	yes	4.20
12	7	3	4	4	3	2	-2	1	-1	yes	4.8
14	10	6	7	7	2	2^{1}	-2^{6}	$\sqrt{2}^{3}$	$-\sqrt{2}^{3}$	no	5.3
15	4	0	1	5	3	2	-2	1	-1	yes	4.13
16	4	0	1	4	4	2^{5}	-2^{7}	0^{3}	-	no	[7]
16	7	0	3	8	2	$\sqrt{7}^{4}$	$-\sqrt{7}^{4}$	-	-1^{7}	no	[7]
16	12	8	9	4	4	2^{3}	-2^{9}	0^{3}	-	no	[7]
18	9	6	4	6	3	$\sqrt{3}^{6}$	$-\sqrt{3}^{6}$	3^{1}	-3^{4}	yes	4.21
18	9	8	4	9	2	-	-1^{9}	3^{4}	-3^{4}	yes	4.10
18	10	6	5	3	6	2^{5}	-2^{10}	$\sqrt{10}^{1}$	$-\sqrt{10}^{1}$	no	5.2
20	7	3	2	4	5	2	-2	3	-3	yes	4.8
20	7	6	2	10	2	-	-1^{10}	3^{5}	-3^{4}	yes	4.10
20	9	0	4	10	2	3	-3	1	-1	yes	4.13
20	11	2	6	10	2	3	-3	1	-1	no	5.3
20	13	9	8	4	5	2	-2	3	-3	yes	4.8
20	13	12	8	10	2	-	-1^{10}	3^{4}	-3^{5}	yes	4.10
21	12	8	6	3	7	2^{6}	-2^{12}	$\sqrt{18}^{1}$	$-\sqrt{18}^{1}$	no	5.2
24	5	0	1	6	4	$\sqrt{5}^{9}$	$-\sqrt{5}^{9}$	-	-1^{5}	no	[7]
24	6	2	1	3	8	2^{9}	-2^{12}	$\sqrt{12}^{1}$	$-\sqrt{12}^{1}$	yes	4.9
24	7	0	2	8	3	$\sqrt{7}^{8}$	$-\sqrt{7}^{8}$	-	-1^{7}	yes	4.13,4.17
24	8	4	2	4	6	2	-2	4	-4	yes	4.8
24	9	4	3	6	4	$\sqrt{5}^{9}$	$-\sqrt{5}^{9}$	3^{1}	-3^{4}		
24	9	6	3	12	2	$\sqrt{3}^{6}$	$-\sqrt{3}^{6}$	3^{4}	-3^{7}		
24	10	2	4	12	2	$\sqrt{8}^{6}$	$-\sqrt{8}^{6}$	2^{3}	-2^{8}		
24	10	3	4	8	3	$\sqrt{7}^{8}$	$-\sqrt{7}^{8}$	2^{1}	-2^{6}		
24	10	6	3	3	8	2^{8}	-2^{13}	$\sqrt{28}^{1}$	$-\sqrt{28}^{1}$	yes	4.9
24	11	0	5	12	2	$\sqrt{11}^{6}$	$-\sqrt{11}^{6}$	-	-1^{11}	no	[7]
24	14	6	8	12	2	$\sqrt{8}^{6}$	$-\sqrt{8}^{6}$	2^{2}	-2^{9}		
24	14	7	8	8	3	$\sqrt{7}^{8}$	$-\sqrt{7}^{8}$	-	-2^{7}	yes	4.15,4.17
24	14	10	7	3	8	2^{7}	-2^{14}	$\sqrt{28}^{1}$	$-\sqrt{28}^{1}$	no	5.2
24	15	10	9	6	4	$\sqrt{5}^{9}$	$-\sqrt{5}^{9}$	-	-3^{5}	no	[7]
24	15	12	9	12	2	$\sqrt{3}^{6}$	$-\sqrt{3}^{6}$	3^{3}	-3^{8}		
24	16	12	10	4	6	2	-2	4	-4	yes	4.8
24	18	14	13	3	8	2^{6}	-2^{15}	$\sqrt{12}^{1}$	$-\sqrt{12}^{1}$	no	5.2
25	8	4	2	5	5	2^{8}	-2^{12}	$\sqrt{14}^{2}$	$-\sqrt{14}^{2}$	no	5.1
25	12	8	5	5	5	2^{7}	-2^{13}	$\sqrt{19}^{2}$	$-\sqrt{19}^{2}$		
26	9	0	3	13	2	3^{5}	-3^{8}	$\sqrt{3}^{6}$	$-\sqrt{3}^{6}$		
26	13	12	6	13	2	-	-1^{13}	$\sqrt{13}^{6}$	$-\sqrt{13}^{6}$	yes	4.10
27	6	3	1	9	3	$\sqrt{3}^{9}$	$-\sqrt{3}^{9}$	3^{3}	-3^{5}		
27	8	4	2	9	3	2^{7}	-2^{11}	$\sqrt{10}^{4}$	$-\sqrt{10}^{4}$	no	5.1
27	12	6	5	9	3	$\sqrt{6}^{9}$	$-\sqrt{6}^{9}$	3^{2}	-3^{6}	no	5.1
27	16	12	8	3	9	2^{8}	-2^{16}	$\sqrt{40}^{4}$	$-\sqrt{40}^{4}$	no	5.2
27	16	12	9	9	3	2^{5}	-2^{13}	$\sqrt{13}^{4}$	$-\sqrt{13}^{4}$		
27	18	9	12	9	3	3^{6}	-3^{12}	0^{8}	-	yes	4.13
27	20	16	14	3	9	2^{7}	-2^{17}	$\sqrt{22}^{4}$	$-\sqrt{22}^{4}$	no	5.2

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