

Risk & Sustainable Management Group

Risk and Uncertainty Working Paper: R10#1

Research supported by an Australian Research Council Federation Fellowship
http://www.arc.gov.au/grant_programs/discovery_federation.htm

The Computation of Perfect and Proper
Equilibrium for Finite Games via Simulated
Annealing

By

Stuart McDonald

and

Liam Wagner

Schools of Economics and Political Science
University of Queensland
Brisbane, 4072
rsmg@uq.edu.au
<http://www.uq.edu.au/economics/rsmg>



THE UNIVERSITY
OF QUEENSLAND
AUSTRALIA

The Computation of Perfect and Proper Equilibrium for Finite Games via Simulated Annealing

Stuart McDonald and Liam Wagner
The School of Economics
The University of Queensland
Brisbane QLD 4072 Australia

April 21, 2010

This paper exploits an analogy between the “trembles” that underlie the functioning of simulated annealing and the player “trembles” that underlie the Nash refinements known as perfect and proper equilibrium. This paper shows that this relationship can be used to provide a method for computing perfect and proper equilibria of n -player strategic games. This paper also shows, by example, that simulated annealing can be used to locate a perfect equilibrium in an extensive form game.

1 Introduction

This paper exploits an analogy between the “trembles” that underlie the functioning of simulated annealing and the player “trembles” that underlie the Nash refinements known as perfect and proper equilibrium. This paper shows that this relationship can be used to provide a method for computing perfect and proper equilibria of n -player strategic games. This paper also shows, by example, that simulated annealing can be used to locate a perfect equilibrium in an extensive form game. The approach that is used in this paper is to let the agent’s set of pure strategies become the nodes of a fully connected undirected graph. Movement from a candidate pure strategy to an alternative is then governed by the Markov chain that is generated by the algorithm. The mixed strategy equilibrium is then given by the stationary distribution of this Markov chain, with the players’ pure strategy space acting as the state space. The main contribution of this paper is to provide a proof demonstrating that the algorithm not only computes a Nash equilibrium for n -player finite strategy strategic games, but will locate the perfect and proper Nash equilibrium in these games should one exist. This paper also demonstrate, using the example of the three player game “Selten’s Horse”, that the algorithm can also be used to compute the perfect equilibrium for an extensive form game.

As such this paper represents a significant departure from approach generally followed in the literature on computation of Nash equilibria in non-cooperative game theory. Most of the existing algorithms for computing Nash equilibria utilize the underlying geometric properties of non-cooperative games to compute the Nash equilibria. For example Lemke and Howson’s [16] solution method for bimatrix games, generalized to n -person games by Rosenmüller [22] and Wilson [33], exploits the idea that many games can be expressed in the form of a linear complementarity problem (LCP). The algorithm then literally traces a path through the set of all feasible solutions for the LCP, to identify those solutions that are complementary and these will then coincide with the Nash equilibria for the game in question. Another example is the Scarf algorithm [23], which computes Nash equilibria as fixed points of convex sets by subdividing the product space of agents’ strategies and then identifying the equilibrium by method of triangulation. Much of the later research, such as the papers by van der Laan and Talman [28, 29, 30] and Doup and Talman [4, 5] have concentrated on making improvements on these Scarf algorithm.

In terms of algorithms that can be used specifically for computing “trembling hand” perfect Nash equilibria, the most well known are the linear and logarithmic tracing procedures developed by Harsanyi [8] and Harsanyi and Selten [9]. Both tracing procedures work similarly in that an equilibrium in a game is found by “tracing” a feasible path through a family of auxiliary games. The solution’s progress along the feasible path is intended to represent the way in which players adjust their expectations and predictions about the play of the game. The tracing procedure is different from the algorithms discussed above, because it sets out to compute and identify a refinement of Nash equilibria; “trembling hand” perfection [24, 25]. The algorithms discussed in the previous paragraphs will compute an arbitrary equilibrium point, with no indication of whether or not the equilibrium chosen is stable, or if the strategies chosen by game players are qualitatively rational.

One of the limitations of the tracing procedure is that logarithmic tracing procedures do not always trace a path to a perfect equilibrium. Harsanyi [8, p.69] has argued that this problem can be resolved by eliminating all dominated pure strategies before applying the tracing procedure. However, van Damme [3, p.77] points out that examples can be constructed without dominated pure strategies in which the tracing procedure yields a non-perfect equilibria. He has suggested that the problem lies in the logarithmic tracing procedure as it involves approximating a normal form game with games with logarithmic control costs. Games with control costs are normal form games in which players, in addition to choosing strategies, incur costs depending on how well they choose to control their actions. To address these problems, van den Elsen and Talman [26, 27] have constructed a homotopy approach, described as a “complementary pivoting” algorithm. This algorithm then traces a piecewise linear path from a given starting vector to an equilibrium.

Work by Koller and Megiddo [14] and Koller, Megiddo and von Stengel [15], have extended this approach to computing Nash equilibria of extensive game forms, with von Stengel, van den Elzen and Talman [32] devising an algorithm that will find the perfect equilibrium in extensive game forms. So far this procedure has only been applied to two player extensive game forms. It should be noted that these papers computes the Nash equilibrium of ex-

tensive form games by “collapsing” the game down into its equivalent strategic form game. By contrast, the approach adopted in this paper leads to equilibrium strategy selection by a pruning of the game tree of the extensive game so that the correct path is found. The reason why our algorithm avoids this problem is that it avoids applying simulated annealing directly to the unit simplex of players’ mixed strategy profiles, which is the path employed all algorithms using “geometric approaches. The problem with applying the direct approach to extensive game forms is that there is no immediate equivalent notion of a mixed strategy. However, the approach proposed in this paper the perfect equilibrium is then given by the stationary distribution of the Markov chain.

The structure of this paper is given as follows. The second section provides formal definitions of perfection and properness in finite strategy strategic games. The third section of this paper will provide a characterization of these algorithms in terms of the trembling hand of trembling hand perfection, introducing their application to the computation of perfect and proper equilibria of strategic game forms. This section shows how simulated annealing can be used to provide a sequence of perturbed mixed strategies that will eventually converge on perfect and proper equilibria, should they exist. The basic idea is to select a Markov chain and then use this to deliver a Nash equilibrium via simulated annealing. To do this we must show that it is possible to select a Markov chain that is best suited to deliver convergence for the sequence of completely mixed Nash equilibria of perturbed games to a perfect and proper equilibrium. This is the objective that is undertaken in this section. The fourth section provides an example of the use of simulated annealing for computing the Nash equilibrium of a finite game. We use “Selten’s horse”, a three-person extensive game which is known to have a unique perfect equilibrium, as well as another “non-rational” sub-game perfect Nash equilibrium.

2 Perfection and Properness in Strategic Games

Consider an n -person game in strategic form $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ in which $N = \{1, \dots, n\}$ is the player set, each player i has a finite set of pure strategies $S_i = \{s_{i1}, \dots, s_{ik_i}\}$ and a payoff function $u_i : \times_{i=1}^n S_i \rightarrow \mathbb{R}$ mapping the set of pure strategy profiles $\times_{i=1}^n S_i$ into the real number line. In the strategic game G , for each player $i \in N$ there is a set of probability measures Δ_i that can be defined over the pure strategy set S_i , this is player i ’s mixed strategy set. The elements of the set Δ_i are of the form $p_i : S_i \rightarrow [0, 1]$ where $\sum_{j=1}^{k_i} p_{ij} = 1$, with $p_{ij} = p(s_{ij})$, i.e. Δ_i is isomorphic to the unit simplex.

The elements of the space of mixed strategy profiles $\times_{i=1}^n \Delta_i$ are denoted by $p = (p_1, \dots, p_n)$, where $p_i = (p_{i1}, \dots, p_{ik_i}) \in \Delta_i$. As is the convention, the following short-hand notation is used for the mixed strategy profile $p = (p_i, p_{-i})$, where p_{-i} denotes the other components of p . For each player i , the payoff function $u_i : \times_{i=1}^n S_i \rightarrow \mathbb{R}$ can be extended to the domain of mixed strategy profiles $\times_{i=1}^n \Delta_i$. The payoff function for each player i will be defined as follows

$$u_i(p_i, p_{-i}) = \sum_{j=1}^{k_i} p_{ij} u_i(s_{ij}, p_{-i}). \quad (2.1)$$

A mixed strategy $p \in \times_{i=1}^n \Delta_i$ is *Nash equilibrium* of the strategic game G , if for all players $i \in N$ and all $p'_i \in \Delta_i$

$$u_i(p_i, p_{-i}) \geq u_i(p'_i, p_{-i}). \quad (2.2)$$

Suppose that as well as there being a positive probability p_{ij} of a player i selecting a pure strategy $s_{ij} \in S_i$, there is a small probability ε_{ij} that player will mistakenly employ the j th pure strategy s_{ij} . Given that player i selects his j th pure strategy s_{ij} by mistake, the probability of doing so is given by \tilde{p}_{ij} . The total probability of player i selecting a pure strategy $s_{ij} \in S_i$ is then given by

$$\hat{p}_{ij} = (1 - \varepsilon_{ij}) p_{ij} + \varepsilon_{ij} \tilde{p}_{ij}. \quad (2.3)$$

It can be seen that in this case, the total probability of player i selecting a pure strategy $s_{ij} \in S_i$ will be bounded below by

$$\hat{p}_{ij} \geq \varepsilon_{ij} \tilde{p}_{ij}. \quad (2.4)$$

Equating $\eta_{ij} = \varepsilon_{ij} \tilde{p}_{ij}$ we can see that this condition can be rewritten as

$$\hat{p}_{ij} \geq \eta_{ij} \quad \forall s_{ij} \in S_i \text{ and } i \in N, \quad (2.5)$$

with

$$\sum_{j=1}^{k_i} \eta_{ij} < 1 \quad \forall i \in N. \quad (2.6)$$

This leads to the definition of a perturbed game (G, η) as a finite strategic game derived from the strategic game G , in which each player i 's mixed strategy set is the set of completely mixed strategies for player i constrained by the probability of making an error

$$\Delta_i(\eta_i) = \left\{ p_i = (p_{i1}, \dots, p_{ik_i}) \in \Delta_i; p_{ij} \geq \eta_{ij} \text{ and } \sum_{j=1}^{k_i} \eta_{ij} < 1 \right\}. \quad (2.7)$$

A mixed strategy combination $p \in \times_{i=1}^n \Delta_i(\eta_i)$ is a Nash equilibrium of the perturbed game (G, η) if and only if the following condition is satisfied

$$u_i(s_{ij}, p_{-i}) < u_i(s_{il}, p_{-i}) \text{ then } p_{ij} = \eta_{ij}, \quad \forall s_{ij}, s_{il} \in S_j. \quad (2.8)$$

This implies that a mixed strategy profile p is a Nash equilibrium of the perturbed game (G, η) if and only if no single player has the incentive to deviate from his current strategy to a different strategy among the set of constrained strategies defined by (2.7).

A mixed strategy profile $p \in \times_{i=1}^n \Delta_i$ is a *perfect equilibrium* in the strategic game G if there exists a sequence of completely mixed strategy profiles $\{p^k\}_{k=1}^{\infty}$ where $\lim_{k \rightarrow \infty} p^k = p$, and for every player $i \in N$ and for every $p'_i \in \Delta_i$

$$u_i(p_i, p_{-i}^k) \geq u_i(p'_i, p_{-i}^k) \quad \forall k = 1, 2, \dots \quad (2.9)$$

In terms of our definition of a perturbed game, a mixed strategy is a perfect equilibrium if and only if there exists some sequences $\{\eta^k = (\eta_1^k, \dots, \eta_n^k)\}_{k=1}^{\infty}$ and $\{p^k = (p_1^k, \dots, p_n^k)\}_{k=1}^{\infty}$ such that

1. each $\eta^k > 0$ and $\lim_{k \rightarrow \infty} \eta_k = 0$,
2. each p^k is a Nash equilibrium of a perturbed game equilibrium (G, η^k) , and
3. $\lim_{k \rightarrow \infty} p^k = p$ where for every player $i \in N$ and for every $p'_i \in \Delta_i$

$$u_i(p_i, p_{-i}^k) \geq u_i(p'_i, p_{-i}^k) \quad \forall k = 1, 2, \dots \quad (2.10)$$

An alternative definition of perfection was made by Myerson [20, pp 75–76] and is based on the idea that every pure strategy in a player's set of pure strategies has associated with it a small positive probability of at least $\varepsilon > 0$. However, on strategies that are best responses the associated probabilities greater than ε . More formally, for any player $i \in N$ a mixed strategy $p_i \in \Delta_i$ is an ε -*perfect equilibrium* if and only if it is completely mixed and there exists some sequences $\{\varepsilon^k\}_{k=1}^{\infty}$

$$\text{if } u_i(s_{ij}, p_{-i}) < u_i(s_{il}, p_{-i}) \text{ then } p_{ij} \leq \varepsilon^k, \quad \forall s_{ij}, s_{il} \in S_j. \quad (2.11)$$

Unlike Nash equilibria of perturbed games, the ε -perfect equilibria of a game G will not necessarily be one of its Nash equilibria. However, Myerson does show that $p = (p_1, \dots, p_n) \in \times_{i=1}^n \Delta_i$ will be a perfect equilibrium if and only if

1. each $\varepsilon^k > 0$ and $\lim_{k \rightarrow \infty} \varepsilon^k = 0$,
2. each p^k is an ε^k -perfect equilibrium of the game G , and
3. $\lim_{k \rightarrow \infty} p_i^k = p_i$ for every player $i \in N$.

This definition by Myerson is important, not only as an alternative route to perfection, but also as a means of further constraining the rationality requirements of this refinement by weighting more heavily the best responses from the set of player strategies. This leads to the definition of a proper equilibrium [20, pp 77–78]. Myerson begins by defining an ε -*proper equilibrium* for a game G as any completely mixed strategy $p = (p_1, \dots, p_n) \in \times_{i=1}^n \Delta_i$ such that for some sequence $\{\varepsilon^k\}_{k=1}^{\infty}$,

$$\text{if } u_i(s_{ij}, p_{-i}) < u_i(s_{il}, p_{-i}) \text{ then } p_{ij} \leq \varepsilon^k p_{il}, \quad \forall s_{ij}, s_{il} \in S_j. \quad (2.12)$$

A completely mixed strategy $p = (p_1, \dots, p_n) \in \times_{i=1}^n \Delta_i$ is then said to be a *proper equilibrium* of a game G if and only if

1. each $\varepsilon^k > 0$ and $\lim_{k \rightarrow \infty} \varepsilon^k = 0$,
2. each p^k is an ε^k -proper equilibrium of the game G , and
3. $\lim_{k \rightarrow \infty} p_i^k = p_i$ for every player $i \in N$.

Based on Myerson's definition of perfection, it can be seen that for any game, all proper equilibria must be contained inside the set of perfect equilibria.

3 Computing Perfect and Proper Equilibria in Strategic Games

Simulated annealing (Černý [2], Kirkpatrick et al. [11]) uses random perturbations in order to create opportunities for the algorithm to escape from the local minima trap. More formally, let E be a finite set and $C : E \rightarrow R$ be the cost function to be minimized such that there exists an $i_0 \in E$ where

$$C(i_0) \leq C(i), \quad \forall i \in E. \quad (3.1)$$

The algorithm works by defining a neighborhood structure on the E ,

$$N = \{N(i); i \in E\}. \quad (3.2)$$

This neighborhood structure is a fully connected graph on the elements of E with no self-loops, in which the nodes directly connected to i constitute the set

$$N(i) = \{j \in E; j \neq i\}. \quad (3.3)$$

There is a family of proposal distributions $Q = \{q_{ij}\}$, which is defined over all states contained in E has its support defined on $N(i)$, that gives the probability of proposing a transition from to state j from state i . There is also a control parameter T , such that for all values of T and all states i and $j \in E$, there is a probability of acceptance

$$\alpha_{ij}(T) = \min \{1, e^{-(C(j)-C(i))/T}\}. \quad (3.4)$$

When this type of acceptance rule is employed, then this algorithm is equivalent to the Metropolis-Hastings Algorithm. The transition probability of shifting between states is then given by

$$p_{ij} = \alpha_{ij}(T) q_{ij} \quad (3.5)$$

for the off-diagonal terms of the transition probability matrix.

Note that when the matrix of proposal probabilities Q is symmetric, then the stationary distribution is given by

$$\pi_i(T) = \frac{e^{-C(i)/T}}{\sum_{k \in E} e^{-C(k)/T}} \quad (3.6)$$

If the set of global minima is defined by

$$H = \{i \in E; C(i) \leq C(j) \quad \forall j \in E\}, \quad (3.7)$$

then $\pi_i(T)$ is maximal on this set. Furthermore, as $T \rightarrow 0$

$$\pi_i(T) = \begin{cases} \frac{1}{|H|} & \text{if } i \in H \\ 0 & \text{otherwise;} \end{cases} \quad (3.8)$$

i.e. as $T \rightarrow 0$, either $e^{-(C(i)-m)/T} \rightarrow 0$ if $C(i) > m$ or to 1 if $C(i) = m$, where m is the global minimum.

The starting basis for this algorithm for calculating perfection will be to follow Myerson [20] by constructing a sequence of ε -perfect equilibria for the strategic game G . As stated above, we know that for the strategic game G , $p \in \times_{i=1}^n \Delta_i$ is an ε -perfect equilibrium if and only if for each player $i \in N$, $p_i \in \Delta_i$ is a completely mixed strategy and

$$u_i(s_{ij}, p_{-i}) < u_i(s_{il}, p_{-i}) \text{ then } p_{ij} \leq \varepsilon, \quad \forall s_{ij}, s_{il} \in S_j. \quad (3.9)$$

Following Myerson [20, p 79], the set of mixed strategies is defined for each player i as follows:

$$\Delta_i^* = \{p_i \in \Delta_i; p_{ij} \geq \delta \quad \forall s_{ij} \in S_i\}, \quad (3.10)$$

where

$$\delta = \frac{1}{m} \varepsilon^m, \quad 0 < \varepsilon < 1 \quad (3.11)$$

with $m = \max_{i \in N} |S_i|$. For each player i the family of completely mixed ε -perfect equilibrium strategies contained in Δ_i^* is define in terms of the following correspondence

$$F_i(p_1, \dots, p_n) = \{p_i^* \in \Delta_i^*; u_i(s_{ij}, p_{-i}) < u_i(s_{il}, p_{-i}) \text{ then } p_{ij} \leq \varepsilon, \quad \forall s_{ij}, s_{il} \in S_j\}. \quad (3.12)$$

If for each player $i \in N$, a mixed strategy is defined by

$$p_{ij}^* = \frac{e^{|N_{ij}|}}{\sum_{l=1}^{k_i} e^{|N_{il}|}}, \quad (3.13)$$

where

$$|N_{ij}| = |\{s_{il} \in S_i; u_i(s_{ij}, p_{-i}) < u_i(s_{il}, p_{-i}) \text{ and } p \in \times_{i=1}^n \Delta_i^*\}|, \quad (3.14)$$

then it can be seen that $p_i^* \in F_i(p_1, \dots, p_n)$ will be non-empty. As each $F_i(p_1, \dots, p_n)$ will a finite collection of linear inequalities, they will also be closed convex sets. In addition, each $F_i(p_1, \dots, p_n)$, by the continuity of the payoff function $u_i(s_{ij}, \cdot)$, will also be upper semi-continuous. As a consequence the mapping $F : \times_{i=1}^n \Delta_i^* \rightarrow \times_{i=1}^n \Delta_i^*$ satisfies all the conditions of the Kakutani Fixed Point Theorem. In other words, there exists some completely mixed strategy $p_\varepsilon \in \times_{i=1}^n \Delta_i^*$ such that p_ε is an ε -perfect equilibrium of G . As $\times_{i=1}^n \Delta_i$ is compact, the sequence ε -perfect equilibria $p_\varepsilon \rightarrow p$ as $\varepsilon \rightarrow 0$, where p is the perfect equilibrium of G .

An alternative route to the same result can be arrived at by using an argument based on the convergence properties of Markov chains.

Theorem 3.1. *For any normal form game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, it is possible to define a MCMC algorithm such that its transition probabilities will converge to a perfect equilibrium as long as the following conditions hold:*

1. *if $u_i(s_{ij}, p_{-i}^k) - u_i(s_{il}, p_{-i}^k) \geq 0$ then accept, where p_{-i}^k is the tuple mixed strategies selected on the k th iteration;*
2. *otherwise, accept if $\exp\left(\frac{u_i(s_{il}, p_{-i}^k) - u_i(s_{ij}, p_{-i}^k)}{T}\right) > \varepsilon$, where $\varepsilon \sim U[0, 1]$; and*

3. in addition it can be seen that for all s_{ij} and $s_{il} \in S_i$ such that $u_i(s_{ij}, p_{-i}^k) < u_i(s_{il}, p_{-i}^k)$, $\alpha_{jl}^i(T) \rightarrow 0$ as $T \rightarrow 0$.

Proof. For each player $i \in N$, there will be a collection these subsets

$$N_{ij} = \{s_{il} \in S_i; u_i(s_{ij}, p_{-i}) < u_i(s_{il}, p_{-i}) \text{ and } p \in \times_{i=1}^n \Delta_i^*\} \quad (3.15)$$

of i 's pure strategy space S_i . The collection of these sets will referred to as player i 's local neighborhood structure. What we would like to do is for any two pure strategies $s_{ij}, s_{il} \in S_i$ define a path from s_{ij} to s_{il} such that

$$s_{ij_1} \in N_{ij}, s_{ij_2} \in N_{ij_1}, \dots, s_{il} \in N_{ij_m}. \quad (3.16)$$

In order to do this, we observe that the point-set mapping defined by the set

$$F_i(p_1, \dots, p_n) = \{p_i^* \in \Delta_i^*; u_i(s_{ij}, p_{-i}) < u_i(s_{il}, p_{-i}) \text{ then } p_{ij} \leq \varepsilon, \quad \forall s_{ij}, s_{il} \in S_i\}. \quad (3.17)$$

is a collection of homogenous transition probabilities S_i

$$p_{jl}^i(k) = \Pr \{s_i(k) = s_{il} | s_i(k-1) = s_{ij}\} = \Pr \{s_{il} | s_{ij}\}. \quad (3.18)$$

Furthermore we can see that these transition probabilities have the Markov property, i.e. given the path from s_{ij} to s_{il} such that

$$s_{ij_1} \in N_{ij}, s_{ij_2} \in N_{ij_1}, \dots, s_{il} \in N_{ij_m}. \quad (3.19)$$

the conditional probability

$$\Pr \{s_{il} | s_{ij_1}, s_{ij_2}, \dots, s_{ij_m}, s_{ij}\} = \Pr \{s_{il} | s_{ij_m}\} \Pr \{s_{ij_m} | s_{ij_{m-1}}\} \dots \Pr \{s_{ij_2} | s_{ij_1}\}. \quad (3.20)$$

We define the following generating probability for the Markov chain for each player $i \in N$

$$g_{jl}^i = \begin{cases} \frac{1}{|N_{ij}|}, & \text{if } s_{il} \in N_{ij} \\ 0, & \text{otherwise,} \end{cases} \quad (3.21)$$

where

$$|N_{ij}| = |\{s_{il} \in S_i; u_i(s_{ij}, p_{-i}) < u_i(s_{il}, p_{-i}) \text{ and } p \in \times_{i=1}^n \Delta_i^*\}|. \quad (3.22)$$

We now introduce the following acceptance probability

$$\alpha^i(T) = \min \left\{ 1, \exp \left(\frac{u_i(s_{ij}, p_{-i}^{k-1}) - u_i(s_{il}, p_{-i}^{k-1})}{T} \right) \right\}, \quad (3.23)$$

where $T > 0$ is a control parameter. This last condition implies that

1. if $u_i(s_{ij}, p_{-i}^k) - u_i(s_{il}, p_{-i}^k) \geq 0$ then accept, where p_{-i}^k is the tuple mixed strategies selected on the k th iteration;

2. otherwise, accept if $\exp\left(\frac{u_i(s_{ij}, p_{-i}^k) - u_i(s_{il}, p_{-i}^k)}{T}\right) > \varepsilon$, where $\varepsilon \sim U[0, 1]$; and
3. in addition it can be seen that for all s_{ij} and $s_{il} \in S_i$ such that $u_i(s_{ij}, p_{-i}^k) < u_i(s_{il}, p_{-i}^k)$, $\alpha_{jl}^i(T) \rightarrow 0$ as $T \rightarrow 0$.

Given these three conditions we can now see that the following will hold:

- We know that under this acceptance criterion as $k \rightarrow \infty$ The transition probability matrix p_i^k of the homogenous Markov chain generated by the game G will converge on a stationary distribution $\pi(T)$ as $k \rightarrow \infty$.

$$p_i^k \rightarrow \pi_i(T) = \frac{e^{(u_i(s_{ij}, p_{-i}^k) - u_i(s_{il}, p_{-i}^k))/T}}{\sum_{l=1}^{k_i} e^{(u_i(s_{ij}, p_{-i}^k) - u_i(s_{il}, p_{-i}^k))/T}}, \quad (3.24)$$

where $C(i)$ and as $T \rightarrow \infty$

$$\pi_i(T) = \begin{cases} \frac{1}{|N_{ij}|} & \text{if } i \in H \\ 0 & \text{otherwise} \end{cases} \quad (3.25)$$

where

$$N_{ij} = \{s_{il} \in S_i; u_i(s_{ij}, p_{-i}) < u_i(s_{il}, p_{-i}), p_i = 0\}. \quad (3.26)$$

(See van Laarhoven and Aarts [31, p.22–25] for the proof of this last statement.)

- The transition probability matrix p_i^k satisfies Myerson's definition of an ε -perfect equilibria and as Myerson has shown, the fixed point that this sequence converges on is also a perfect equilibrium. \square

As a slight extension to this algorithm, we show that by changing the acceptance criteria a proper equilibrium can now be computed. Following Myerson [20, p.78], the following criteria is used to determine properness. Myerson begins by defining an ε -proper equilibrium for a game G to any completely mixed strategy $p = (p_1, \dots, p_n) \in \times_{i=1}^n \Delta_i$ such that

$$\text{if } u_i(s_{ij}, p_{-i}) < u_i(s_{il}, p_{-i}) \text{ then } p_{ij} \leq \varepsilon p_{il}, \quad \forall s_{ij}, s_{il} \in S_j. \quad (3.27)$$

A completely mixed strategy $p = (p_1, \dots, p_n) \in \times_{i=1}^n \Delta_i$ is then said to be a *proper equilibrium* of a game G if and only if

1. each $\varepsilon^k > 0$ and $\lim_{k \rightarrow \infty} \varepsilon^k = 0$,
2. each p^k is an ε^k -proper equilibrium of the game G , and
3. $\lim_{k \rightarrow \infty} p_i^k = p_i$ for every player $i \in N$.

The approach to defining transition probabilities would be quite similar to the procedure outlined above. Firstly, for each player $i \in N$ we define the a set of completely mixed strategies

$$\Delta_i^* = \{p_i \in \Delta_i; p_{ij} \geq \delta \ \forall s_{ij} \in S_i\}, \quad (3.28)$$

where

$$\delta = \frac{1}{m} \varepsilon^m, \quad 0 < \varepsilon < 1 \quad (3.29)$$

with $m = \max_{i \in N} |S_i|$. We then define a point-to-set mapping $F_i : \times_{i=1}^n \Delta_i^* \rightarrow \Delta_i^*$ to be the family of completely mixed strategies contained in Δ_i^* for which player i desires no deviation

$$F_i(p_1, \dots, p_n) = \{p_i^* \in \Delta_i^*; u_i(s_{ij}, p_{-i}) < u_i(s_{il}, p_{-i}) \text{ then } p_{ij} \leq \varepsilon p_{il}, \quad \forall s_{ij}, s_{il} \in S_j\}. \quad (3.30)$$

In this way, for each player $i \in N$, by defining a mixed strategy

$$\hat{p}_{il}^* = \frac{e^{|N_{ij}|}}{\sum_{l=1}^{k_i} e^{|N_{ij}|}}, \quad (3.31)$$

where

$$|N_{ij}| = |\{s_{il} \in S_i; u_i(s_{ij}, p_{-i}) < u_i(s_{il}, p_{-i}) \text{ and } p \in \times_{i=1}^n \Delta_i^*\}|, \quad (3.32)$$

then it can be shown that the set of mixed strategies $F_i(p_1, \dots, p_n)$ will be a non-empty finite collection of linear inequalities; by their definition these sets will also be closed convex sets. In addition each $F_i(p_1, \dots, p_n)$, by the continuity of the payoff function $u_i(s_{ij}, \cdot)$, will also be upper semi-continuous. As a consequence, the mapping $F : \times_{i=1}^n \Delta_i^* \rightarrow \times_{i=1}^n \Delta_i^*$ satisfies all the conditions of the Kakutani Fixed Point Theorem. In other words there exists some completely mixed strategy $p_\varepsilon \in \times_{i=1}^n \Delta_i^*$ such that p_ε is an ε -perfect equilibrium of G . As $\times_{i=1}^n \Delta_i$ is compact, the sequence ε -perfect equilibria $p_\varepsilon \rightarrow p$ as $\varepsilon \rightarrow 0$, where p is the proper equilibrium of G .

By a similar argument it can be shown the probabilities given in the set $F_i(p_1, \dots, p_n)$ characterize the transition probabilities of a homogenous Markov chain, which fulfills the requirements of stationarity - recursiveness and irreducibility. It can be noted that criteria

$$u_i(s_{ij}, p_{-i}) < u_i(s_{il}, p_{-i}) \text{ then } p_{ij} \leq \varepsilon p_{il} \quad (3.33)$$

resembles quite closely the acceptance criteria employed by the Metropolis-Hastings algorithm.

Theorem 3.2. *For any normal form game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, it is possible to define a MCMC algorithm that will provide a proper equilibrium.*

Proof. Firstly, for each player $i \in N$ a set of completely mixed strategies is defined

$$\Delta_i^* = \{p_i \in \Delta_i; p_{ij} \geq \delta \ \forall s_{ij} \in S_i\}, \quad (3.34)$$

where

$$\delta = \frac{1}{m} \varepsilon^m, \quad 0 < \varepsilon < 1 \quad (3.35)$$

with $m = \max_{i \in N} |S_i|$. A point-to-set mapping $F_i : \times_{i=1}^n \Delta_i^* \rightarrow \Delta_i^*$ is then defined as follows,

$$F_i(p_1, \dots, p_n) = \{p_i^* \in \Delta_i^*; u_i(s_{ij}, p_{-i}) < u_i(s_{il}, p_{-i}) \text{ then } p_{ij} \leq \varepsilon p_{il}, \quad \forall s_{ij}, s_{il} \in S_j\}. \quad (3.36)$$

This mapping contains all completely mixed ε -proper strategies for player i . In this way, for each player $i \in N$, by defining the following generating and transition probabilities

$$g_i(s_{ih}|s_{ij}) = \frac{1}{|N_{ij}|} \quad (3.37)$$

where

$$|N_{ij}| = |\{s_{il} \in S_i; u_i(s_{ij}, p_{-i}) < u_i(s_{il}, p_{-i}) \text{ and } p \in \times_{i=1}^n \Delta_i^*\}|, \quad (3.38)$$

and

$$q_i(s_{il}|s_{ih}) = \frac{\exp(u_i(s_{il}, p_{-i}))}{\sum_{h=1}^{k_i} \exp(u_i(s_{ih}, p_{-i}))}, \quad (3.39)$$

we are defining the generating probability of moving from one neighborhood to another.

It can be shown that the probabilities given in the set $F_i(p_1, \dots, p_n)$ characterize the transition probabilities of a homogenous Markov chain, which fulfills the requirements of stationarity - recursiveness and irreducibility. It can be noted that criteria

$$u_i(s_{ij}, p_{-i}) < u_i(s_{il}, p_{-i}) \text{ then } p_{ij} \leq \varepsilon p_{il} \quad (3.40)$$

resembles quite closely the acceptance criteria employed by the Metropolis-Hastings algorithm.

It can be shown that the set of mixed strategies $F_i(p_1, \dots, p_n)$ will be a non-empty finite collection of linear inequalities; by their definition these sets will also be closed convex sets. In addition each $F_i(p_1, \dots, p_n)$, by the continuity of the payoff function $u_i(s_{ij}, \cdot)$, will also be upper semi-continuous. As a consequence the mapping $F : \times_{i=1}^n \Delta_i^* \rightarrow \times_{i=1}^n \Delta_i^*$ satisfies all the conditions of the Kakutani Fixed Point Theorem. In other words there exists some completely mixed strategy $p_\varepsilon \in \times_{i=1}^n \Delta_i^*$ such that p_ε is an ε -perfect equilibrium of G . As $\times_{i=1}^n \Delta_i^*$ is compact, the sequence ε -perfect equilibria $p_\varepsilon \rightarrow p$ as $\varepsilon \rightarrow 0$, where p is the proper equilibrium of G . \square

4 An Application to Extensive Game Forms: Selten's Horse

As was shown by Selten [24], the perfect equilibria of a game's strategic and extensive forms need not coincide. However, Selten showed that an equivalence relationship holds between the equilibria of any extensive game and its associated *agent normal form*. This is because the agent normal form of any game views each node of the game tree, of the extensive form of the game, as a player in the game. As a consequence each player represents an information set held by the player and will have an identical payoff function to the player.

We let Γ^e define an finite extensive game consisting of a set of n players $N = \{0, 1, \dots, n\}$ and one random player denoted by 0, and a finite game tree $K = (T, \sigma)$ consisting of a set of nodes T and a predecessor function $\sigma : D \rightarrow D$. The set of nodes is partitioned into the set of terminal nodes Z and a set of non-terminal decision nodes $D = T - Z$. The predecessor function σ is meant to define a path of successive nodes, and has the following properties:

- $\sigma(d^o) = \emptyset$, where d^o is the origin of the game tree K , and
- for $\{d_1, \dots, d_m\} \subset D$ such that $\sigma(d) = d_m$, then $\sigma(d_k) = d_{k-1}$.

The non-terminal nodes of the game tree in X are partitioned into player sets $\{P_0, P_1, \dots, P_n\}$. Each player set P_i , $i = 1, \dots, n$, contains the non-terminal decision nodes associated with that particular player. The player set P_0 is the set of non-terminal nodes that are associated with the random player. For each player $i \in N$, we can define subsets $I_i \in P_i$ called information sets, such that all nodes within an information set $I_i \in P_i$ have the same number of immediate successors, and all paths intersect an information set at most once.

We can then collect all of these information sets $I_i \in P_i$ that are associated with a particular player $i \in N$ and group them into a collection of information sets belonging to that player i ,

$$\mathcal{I}_i = \{I_{i1}, \dots, I_{im_i}\}. \quad (4.1)$$

The tuple $\mathcal{I} = (\mathcal{I}_1, \dots, \mathcal{I}_n)$ is the information partition of the game Γ^e . The implication is that if the information set $I_i \in \mathcal{I}_i$ is eligible, i.e. it contains at least one node $x \in P_i$, player i will not be able to distinguish other nodes contained in this information set based on information possessed when attempting the move to node x .

With the information partition \mathcal{I} a choice set $C = \{C_I : I \in \cup_{i=1}^n \mathcal{I}_i\}$ can be defined, where each C_I is a partition of the union of sets of successors of node x $S(x) = \{y; x \in P(y)\}$. The interpretation is that if player i takes the choice $c \in C_I$ at information set $I_{ij} \in \mathcal{I}_i$, then if i is at $x \in I$, the next node reached is the element of $S(x)$ contained in c . For the random player, p_{0j} is the probability distribution associated with the choices for each $I_{0j} \in \mathcal{I}_0$.

A probability distribution b_i is assigned on C_I to each information set $I \in \mathcal{I}_i$. This distribution b_i is a behavioral strategy, with the set of all these strategies for player i defined by B_i . The profile of all the players' behavioral strategies is denoted by $b \in B := \times_{i=1}^n B_i$, where B is the set of all behavioral strategy combinations. The probability of a particular realization of the game Γ^e is denoted by $p_b(z)$.

The payoffs of the game are associated with the set of terminal points Z of the game tree are denoted by the n -tuple $h = (h_1, \dots, h_n)$, where each player i 's payoff is a function of the terminal points $h_i(z)$, $z \in Z$. The payoff profile h is an n -tuple, where the i th element is defined as

$$h_i(z) = \sum_{z \in Z_1} p_b(z) h_i(z), \quad \forall z \in Z, \text{ and } i \in N.$$

A pure strategy $s_{ij} \in S_i$ for a player $i \in N$ in the extensive game Γ^e is a sequence of moves $s_{ij} = \{s_{ij}^k\}$, where each $s_{ij}^k \in I_{ij} \in \mathcal{I}_i$. A mixed strategy $p_i \in \Delta_i$ attaches a probability

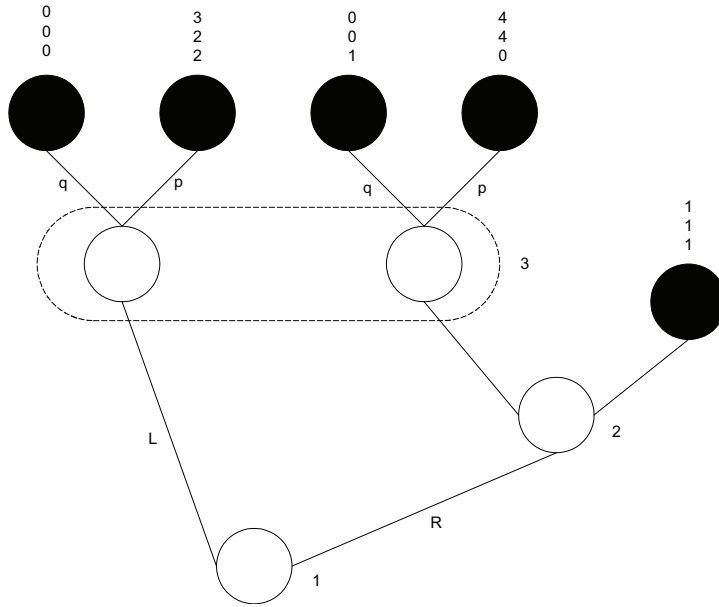


Figure 4.1: Game tree for Selten's Horse.

$p_{ij} = p_i(s_{ij})$ to each $s_{ij} \in S_i$ such that $\sum_{s_{ij} \in S_i} p_{ij} = 1$. A payoff function can now be defined for each player $i \in N$

$$h_i(p_i, p_{-i}) = \sum_{s_{ij} \in S_i} p_{ij} h_i(s_{ij}, p_{-i}). \quad (4.2)$$

Kuhn [13] has shown that for arbitrary mixed strategies in games of complete information, it is possible to find a unique behavioral strategy that is payoff invariant. Therefore it does not really matter whether player strategies are behavioral or mixed.

Let Γ^e be an extensive game and I_1, \dots, I_n be the information sets of players in Γ^e , the agent normal form G of Γ^e associates a player with each information set. In other words, for each player $i \in N$, let ϕ_i be the set of all choices C_{I_i} at I_i , then a strategic game $G = (N, (\phi_i)_{i \in N}, (u_i)_{i \in N})$ can be defined, where ϕ_i become the pure strategy sets of each player $i \in N$ and the payoff functions $u_i : \times_{i=1}^n \phi_i \rightarrow \mathcal{R}$. Noting that for every player $i \in N$

$$u_i(\phi_i, \phi_{-i}) = h_i(p_i, p_{-i}), \quad (4.3)$$

where p_i is the mixed strategy that assigns a probability to a choice c made at information set I_i .

A perturbed game of G is defined by (G, η) , where η is a mapping that assigns to every choice in Γ a positive number η_c such that

$$\sum_{c \in C_u} \eta_c < 1$$

for every information set u . An equilibrium point b of the strategic game Γ is a perfect equilibrium if b is a limit point of a sequence $\{b(\eta)\}$ as $\eta \rightarrow 0$, where each $b(\eta)$ is an

equilibrium points of the associated perturbed game (Γ, η) . Proofs of convergence would then follow as a natural corollary from the work in the section four.

The game tree of three person extensive game is shown in Figure 4.1 [6, p. 50]. This example is based on the three player extensive form game used by Selten [24] to illustrate the existence of perfect equilibrium. It is known that this game possesses both a perfect equilibrium as well as a “non-rational” subgame perfect equilibria. The perfect equilibrium for this extensive form game is defined via the perturbed payoff functions:

$$\begin{aligned} R_1 &= \gamma_1(1 - \varepsilon_2 - 3\varepsilon_3 + 4\varepsilon_2\varepsilon_3) + 3\varepsilon_3 \\ R_2 &= 2\varepsilon_3(2 - \varepsilon_1) + \gamma_2(1 - \varepsilon_1 - 4\varepsilon_3 + 4\varepsilon_1\varepsilon_3) \\ R_3 &= 1 - \varepsilon_1 + \gamma_3(2\varepsilon_1 - \varepsilon_2 + \varepsilon_1\varepsilon_2), \end{aligned}$$

where the γ_i are being used to define the agents’ the mixed strategies and ε_i are errors defined for each player $i = 1, 2, 3$. Letting the errors approach zero, it can be seen that perfect equilibrium is defined by $(1, 1, 0)$.

Using these payoffs, an algorithm is constructed that is based on the simulated annealing algorithm found in van Laarhoven and Aarts [31, p. 10]. The pseudocode for this algorithm is given below.

```

begin
Initialize;
M := 0;
repeat
  repeat
    Perturb(config.i → j, ΔR1) for player 1;
    if (ΔR1 ≥ 0) then accept
      elseif (exp( $\frac{-\Delta R_1}{c}$ ) > rand[0, 1)) then accept;
    if accept then Update(config.j);
    Perturb(config.i → j, ΔRn) for player n;
    if (ΔRn ≥ 0) then accept
      elseif (exp( $\frac{-\Delta R_n}{c}$ ) > rand[0, 1)) then accept;
    if accept then Update(config.j);
  until equilibrium is approached sufficiently closely;
  cM+1 := f(cM);
  M := M + 1;
until stop criterion = true;
end

```

The results of the simulation are shown below in Figure 4.2 and indicate convergence to the trembling hand perfect equilibrium.

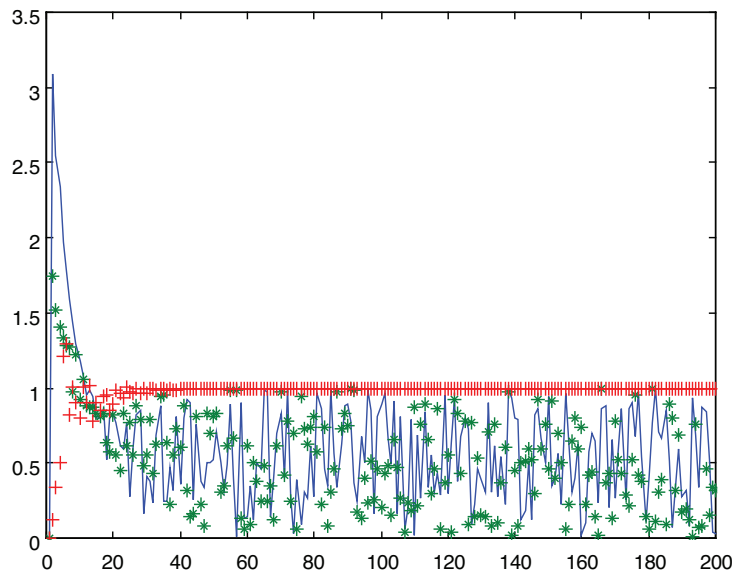


Figure 4.2: Simulation results for Selten's Horse

5 Conclusion

This paper has concentrated on some of the underlying theoretical mechanics of simulated annealing and how they relate to the trembling hand perfect refinement of Nash equilibrium. It has been argued that the trembles that underlie global optimization by simulated annealing are analogous to the “mistakes” of trembling hand perfection, in that they present a means of moving from local ε -perfect equilibria towards a perfect Nash equilibrium. The main contribution of this paper has been to demonstrate that simulated annealing can be employed to compute the perfect and proper refinement of Nash equilibrium. In addition, by using the example of “Selten's Horse”, we demonstrate that simulated annealing can also be used to compute the Nash equilibrium of extensive form games, via its agent normal form.

Acknowledgments: A previous version of this paper was presented at the IEEE Congress on Evolutionary Computation 2003 [17] and we are especially grateful to the editor of the proceedings and two anonymous referees for their useful input on this paper. The research contained in this paper was partially funded by the Australian Research Council Centre for Complex Systems.

References

- [1] Besag, J. (1974) Spatial interaction and the statistical analysis of lattice systems (with discussion). *Journal of the Royal Statistical Society Series B* 36, 192–236.
- [2] Černý, V. (1985) Thermodynamical approach to the travelling salesman problem: an efficient simulation algorithm. *Journal of Optimization Theory and its Applications* 45, 41–51.
- [3] van Damme, E. (1991) *Stability and Perfection of Nash Equilibria (2nd ed. rev. enl.)*. Springer-Verlag, Berlin.
- [4] Doup, T.M. and Talman, A.J.J. (1987a) A new simplicial variable dimension algorithm to find equilibria on the product space of unit simplices. *Mathematical Programming*, 37, 319–355.
- [5] Doup, T.M. and Talman, A.J.J. (1987b) A continuous deformation algorithm on the product space of unit simplices. *Mathematics of Operations Research*, 12, 485–521.
- [6] Friedman, J.W. (1991) *Game Theory with Applications to Economics*. Oxford University Press, Oxford.
- [7] Gilks, W.R. (1996) Full conditional distributions. In Gilks, W.R., Richardson, S. Spiegelhalter, D.J. (Eds.) *Markov Chain Monte Carlo in Practice*, 75–88. Chapman and Hall, London.
- [8] Harsanyi, J.C. (1975) The tracing procedure: a Bayesian approach to defining a solution for n -person non-cooperative games. *International Journal of Game Theory* 4, 61–94.
- [9] Harsanyi, J.C. and Selten, R. (1988) *A General Theory of Equilibrium Selection in Games*. MIT Press, Cambridge, MA.
- [10] Hastings, W.K. (1970) Monte Carlo sampling methods using Markov chains and their application. *Biometrika* 57, 97–109.
- [11] Kirkpatrick, S., Gelatt, C.D. and Vecchi, M.P. (1983) Optimization by Simulated Annealing. *Science* 220, 671–680.
- [12] Kreps, D.M. and Wilson, R. (1982) Sequential equilibrium. *Econometrica* 50, 863–894.
- [13] Kuhn, H.W. (1953) Extensive games and the problem of information. In Kuhn, H.W. and Tucker, A.W. *Contributions to the Theory of Games Vol I*. Princeton University Press, Princeton N.J. pp. 193–216.
- [14] Koller, D. and Megiddo, N. (1992) Finding mixed strategies with small supports in extensive form games. *International Journal of Game Theory* 25, 73–92.
- [15] Koller, D., Megiddo, N. and von Stengel, B. (1992) Efficient computation of equilibria for two-person games. *Games and Economic Behavior* 14, 247–259.
- [16] Lempke, C.E. and Howson, J.T. (1964) Equilibrium points of bimatrix games. *SIAM Journal on Applied Mathematics* 12, 413–423.

- [17] McDonald, S. and Wagner L.D., (2003) Using Simulated Annealing to Calculate the Trembles of Trembling Hand Perfection. *Proceedings of the IEEE Congress on Evolutionary Computation 2003* vol.4, 2482-2489.
- [18] McKelvey, R.D. and McLennan A. (1996) Computation of Equilibria in Finite Games. In Amman, H.M., Kendrick, D.A. and Rust, J. (Eds.) *Handbook of Computational Economics Vol. 1*. Elsevier Science B.V., Amsterdam.
- [19] Metropolis, N., Rosenbluth, A.W., Rosenbluth, M.N., Teller, A.H., Teller, E., (1953) Equations of state calculations by fast computing machines. *Journal of Chemistry Physics* 21, 1087–1091.
- [20] Myerson, R.B. (1978) Refinements of the concept of Nash equilibrium. *International Journal of Game Theory* 7, 73–80.
- [21] Roberts, G.O. (1996) Markov chain concepts related to sampling algorithms. In Gilks, W.R., Richardson, S. Spiegelhalter, D.J. (Eds.) *Markov Chain Monte Carlo in Practice*, 45–57. Chapman and Hall, London.
- [22] Rosenmüller, J. (1971) On a generalization of the Lemke-Howson algorithm to noncooperative N -person games. *Econometrica* 33, 520–534.
- [23] Scarf, H.E. (1973) *Computation of Economic Equilibria*. Yale University Press, New Haven, Conn.
- [24] Selten, R. (1975) Reexamination of the Perfectness Concept for Equilibrium Concepts in Extensive Form Games. *International Journal of Game Theory* 4, 25–55.
- [25] Selten, R. (1978) The Chain Store Paradox. *Theory and Decision* 9, 127–159.
- [26] van den Elsen, A.H. and Talman, A.J.J. (1991) A procedure for finding the Nash equilibria in bi-matrix games. *ZOR – Methods and Models of Operations Research* 35, 27–43.
- [27] van den Elsen, A.H. and Talman, A.J.J. (1991) An algorithmic approach towards the tracing procedure for bi-matrix games. *Games and Economic Behavior* 14, 220–246.
- [28] van der Laan, G. and Talman, A.J.J. (1979), A restart algorithm for computing fixed points without an extra dimension. *Mathematical Programming* 17, 74–84.
- [29] van der Laan, G. and Talman, A.J.J. (1980) On the computation of fixed points in the product space of unit simplices and an application to non-cooperative N -person games. *Mathematics of Operations Research*, 12, 377–397.
- [30] van der Laan, G. and Talman, A.J.J. (1982) On the computation of fixed points in the product space of unit simplices and an application to non-cooperative N -person games. *Mathematics of Operations Research*, 12, 377–397.
- [31] van Laarhoven, P.J.M. and Aarts, E.H.L. (1987) *Simulated Annealing: Theory and Applications*. D. Reidel Publishing, Dordrecht, Holland.
- [32] von Stengel, B., van den Elsen, A.H. and Talmand, A.J.J. (2002) Computing normal form perfect equilibria for extensive two-person games. *Econometrica* 70, 693–715.

- [33] Wilson, R. (1971) Computing Equilibria of N -Person Games. *SIAM Journal on Applied Mathematics* 21, 80–87.