A THEORY OF QUALITATIVE SIMILARITY

KLAUS NEHRING

Working Paper Series #97-10

Department of Economics
University of California
Davis, California 95616-8578

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Klaus Nehring

Working Paper Series No. 97-10
March, 1997

*The inspiration for this work came from a collaboration with Clemens Puppe'on a (quantitative) "theory of diversity" in which symmetric ternary relations play a central role.

1. INTRODUCTION

"A chimpanzee is more similar to a human being than a donkey is"; "multiplication is more similar to addition than division is"; "Canada is more similar to the **U.S.** than Belgium is"; "it is ambiguous whether Belgium is more similar to the U.S. than Russia is". Judgements such as these can be explicated naturally in terms of comparisons of sets of relevant features. The first judgment may be explicated, for instance, by taking the relevant features to be the membership in different biological taxa such as species, group, family, order: the chimpanzee has all features that a human and a donkey share (i.e. belongs to all taxa common to humans and donkeys, being a mammal) and shares the feature of being a primate with a human that the donkey does not have. Belgium and the U.S. share the features of being rich, members of NATO and the OECD, all shared by Canada as well, which in addition shares a lot of important features with the **U.S.** that Belgium does not share. On the other hand, while Belgium has some important features in common with the **U.S.** that Russia doesn't share, the converse holds as well, so that it seems natural to deem Belgium and Russia non-comparable in terms of overall similarity with the **U.S.**. In **this** paper, we will analyze judgments of ***qualitative*** *similarity* in terms of a ternary relation T "y is at least as similar to x than z is".[1] Its central results are two representation theorems which establish an isomorphism between sets of "attributes" (extensions[2] of features) and ternary relations with appropriate structure.

The analysis can also be motivated in purely mathematical terms in which the primitive object of study is a symmetric ternary relation T with the geometric interpretation of "betweenness" of points in some space. A ternary betweenness relation has been introduced into the axiomatic foundations of geometry by Pasch **(1882)** and

[1] **See example 3.3 and section 6 for cases in which a ternary relation is derived from a** distance-**function.**

[2] E.g. **the set of all mammals is the attribute corresponding to the feature "is a mammal"**; co-**extensive features are mapped into the same attribute.**

frequently employed since then[3]. Whereas this literature largely focuses on special cases such as betweenness on a line or in a lattice[4], the goal of this paper is to provide a general definition of "ordered betweenness". The key is to specify an appropriate ternary transitivity condition, resulting in the concept of a ternary preorder.

The representation of a ternary preorder on a set X by a family of subsets of X (referred to as a "convex topology") is of interest in a variety of ways. First of all, it yields a "semantics" (or "model") for betweenness, thereby confirming the appropriateness of the proposed transitivity condition and indicating the range of its applicability. Moreover, it leads to a perhaps surprisingly economic representation of ternary preorders in terms of collections of at most $\frac{n(n+1)}{2}$ subsets of the set X with cardinality n out of 2^n. This makes it significantly easier to visualize and work with ternary preorders; the fact that the axiomatic conditions defining a convex topology are much easier to apply than the 5-point transitivity condition also helps.

The paper is structured as follows. In section 2, the notions of a ternary preorder and of a convex topology are defined, and the central result of the paper, a pair of representation theorems, is proved. The concepts and result are illustrated by a plethora of examples in section **3.** Section 4 addresses uniqueness and minimality issues of the representation; it also points out that the class of convex topologies is closed under intersection, hence a lattice, as is the class of ternary preorders. In section 5, binary preorders are embedded as "effectively binary ternary preorders", and the associated class of convex topologies is characterized. It is shown that under effective binariness, ternary transitivity is equivalent to ordinary binary transitivity. Finally, the F'undamental Representation Theorem of section **2** is employed to obtain a version of Birkhoff's **(1933)** classic representation theorem for finite distributive lattices. In section *6*, we analyze taxonomic attribute hierarchies, a structure of central importance in the literatures on similarity and classification. These are shown

[3]See, for instance, Hilbert (1899), Suppes (1972), Fishburn (1985), ch. 4

[4]To the **best** of our limited acquaintance with it.

to be characterized by a strong connectedness condition on the ternary relation, which in effect makes it possible to view the ternary relation as an n-tuple of weak orders; ternary transitivity is shown to be equivalent in this context to (binary) transitivity of each of the weak orders. We then apply the Fundamental Representation Theorem to obtain a qualitative version of the classic characterization of "indexed hierarchies" by ultra-metric distances. All proofs are collected in the appendix.

2. THE FUNDAMENTAL REPRESENTATION THEOREM

Let **X** denote a universe of *objects.* Subsets of the power set 2^X of X will be called *attribute collections,* or simply *collections.*

A collection $\mathbf{A} \subseteq$ **2''** induces a ternary relation $T_{\mathcal{A}} \subseteq \mathbf{X} \times \mathbf{X} \times \mathbf{X} =: X^3$ according to

$$T_{\mathcal{A}} := \{(\mathbf{x}, y, z) \mid \{x, z\} \subseteq A \Rightarrow y \in \mathbf{A} \ \forall A \in \mathbf{A}).$$

The expression $(\mathbf{x},\mathbf{y},\mathbf{z}) \in T_{\mathcal{A}}$ can be read as " y is at least as similar to **x** as *z* is to x" or " y lies between **x** and *z* "; it is important to read these as weak rather than as strict relations. Say that the triple $(\mathbf{x},\mathbf{y},\mathbf{z})$ is *compatible with* the set **A** iff $\{\mathbf{x}, z\} \subseteq \mathbf{A} \Rightarrow \mathbf{y} \in \mathbf{A}.$ With this terminology,

$$T_{\mathcal{A}} = \{\tau \in X^3 \mid 7 \text{ is compatible with } \mathbf{A}, \text{ for all } \mathbf{A} \in \mathbf{A}\};$$

an $\mathcal{A} \subseteq 2^X$ such that $\mathrm{T} = T_{\mathcal{A}}$ is a *multi-attribute representation* of T. Conversely, any ternary relation $T \subseteq \mathbf{X}^3$ induces a collection

$$\mathcal{A}_T := \{\mathbf{A} \in \underset{/}{2^X} \mid \tau \text{ is compatible with } \mathbf{A}, \text{ for all } \tau \in \mathrm{T}).$$

A $T \subseteq X^3$ such that $\mathbf{A} = \mathcal{A}_T$ is a *ternary representation* of **A.**

It is easily verified that $T_{\mathcal{A}}$ satisfies the following properties for any **A** : [5]

[5] **For transitivity, see the proof of theorem** 1

Axiom 1 **T1** (Reflexivity) : $y \in \{x, z\} \Rightarrow (x, y, z) \in T$, *for all* $x, y, z \in X$.

T2 (Symmetry) : $(x, y, z) \in T \Rightarrow (z, y, x) \in T$, *for all* $x, y, z \in X$.

T3 (Transitivity): $(x, x', z) \in T$ & $(x, z', z) \in T$ & $(x', y, z') \in T \Rightarrow (x, y, z) \in T$, *for all* $x, x', y, z, z' \in X$.

Remarks on reflexivity can be found section 5, remarks on symmetry in section 6. Below and in the examples of section **3,** the emphasis of the discussion is on transitivity. Similarly, for every $T \in X^3$, $\mathcal{A}_T$ satisfies the following three properties:

Axiom 2 **A1** (Boundedness): $\mathcal{A} \supseteq \{\emptyset, X\}$.

A2 **(Intersection-Closedness):** $A_i \in \mathcal{A}$, *for all* $i \in I \Rightarrow \bigcap_{i \in I} A_i \in \mathcal{A}$.

A3 (Abstract Convexity): $A \in \mathcal{A}$ *whenever, for all* $x, y \in A$, *there exists* $B \in \mathcal{A}$ *such that* $\{x, y\} \subseteq B \subseteq A$.

A ternary relation $T \subseteq X^3$ satisfying T1-T3 is called a *ternary preorder*; their class is denoted by TPO.

A collection $\mathcal{A} \subseteq 2^X$ satisfying A1-A3 is called *convex topology;* their class is denoted by CVT.

For any $\mathcal{A}$ satisfying A1 and A2, define a hull-operator $H_{\mathcal{A}} : X \times X \to 2^X$ by $H_{\mathcal{A}}(x, y) = \bigcap \{A \in \mathcal{A} \mid A \supseteq \{x, y\}\}$; $H_{\mathcal{A}}(x, y)$ is the smallest common attribute of $\{x, y\}$. A set S is $\mathcal{A}-$ convex if $H_{\mathcal{A}}(x, y) \subseteq S$ for all $x, y \in S$. With this terminology, abstract convexity can be read as the requirement that $\mathcal{A}$ contain all $\mathcal{A}$-convex sets.

It will also be useful to associate with the ternary relation T its correspondence $\mathbf{T} : X^2 \twoheadrightarrow 2^X$ defined by $\mathbf{T}(x, y) := \{z \in X \mid (x, z, y) \in T\}$; $\mathbf{T}(x, y)$ can be viewed as the "segment" between x and y. A set S is T – convex if $\mathbf{T}(x, y) \subseteq S$ for all $x, y \in S$. With this terminology, transitivity can be read as the requirement that all segments $\mathbf{T}(x, y)$ be T-convex sets.

Example 1 *Let* $\mathbf{X} = \mathrm{R}^m$ *and let* $[x, y]$ *denote the closed line segment connecting* x *and y.*

Define T^0 *by* $(x, y, z) \in T^0$ *iff* $y \in [x, z]$. *Likewise, let* $\mathcal{A}^0$ *denote the class of all convex subsets of* R^m. *One easily verifies the following facts:*

1. $H_{\mathcal{A}^0}(x, y) = \mathbf{T}^0(x, y) = [x, y]$; *by consequence, the notions of* $\mathcal{A}^0$– *and* T^0-*convexity both coincide with the ordinary Euclidean notion.*

2. T^0 *satisfies T1, T2 and T3 (line segments are convex sets).*

3. $\mathcal{A}^0$ *satisfies A1, A2 (intersections of convex sets are convex) and A3 (*$\mathcal{A}^0$ *contains all convex sets).*

4. $\mathcal{A}^0$ *represents* T^0, *i.e.* $T_{(\mathcal{A}^0)} = T^0$ *(a point y lies between two points x and z if and only if it cannot be separated from them by some convex set).*

5. T^0 *represents* $\mathcal{A}^0$ *(a set S is convex if and only if it contains all points on a line segment between any two points in S).* □

Theorem 1 **T** *has a multi-attribute representation* $\mathcal{A}$ *if and only if T is a TPO.*

There is a unique such representation that is a CVT; it is given by

$$A \in \mathcal{A} \Leftrightarrow A \text{ is } T-\text{convex.}$$

Theorem 2 $\mathcal{A}$ *has a ternary representation T if and only if* $\mathcal{A}$ *is a CVT.*

There is a unique such representation that is a TPO; it is given by

$$(x, y, z) \in T \iff y \in H_{\mathcal{A}}(x, z).$$

As indicated by these theorems, a key feature of the mutual representation of ternary preorders and convex topologies (and the guiding light to the proofs) is the coincidence of the segments defined by T and the smallest common attributes defined by $\mathcal{A}$. By consequence, a ternary preorder can be specified in terms of the (unordered) set of its segments.

In X is an infinite set, it will often be desirable to endow X with topological structure. This is done very easily here. Let $(X, \mathcal{T})$ be a topological space.

Definition 1 *i) A ternary relation T is* closed *if* $\mathbf{T}(x,z)$ *is closed for all* $x,z \in \mathbf{X}$.
ii) An attribute collection $\mathbf{A}$ *is* closed *if each* $\mathbf{A} \in \mathcal{A}$ *is closed.*

Using essentially identical proofs, one obtains the following topological version of the above results.

Theorem 3 *$\mathbf{T}$ has a* closed *multi-attribute representation* $\mathbf{A}$ *if and only if T is a* closed *TPO.*

There is a unique such representation that is a CVT; it is given by

$$\mathbf{A} \in \mathbf{A} \Leftrightarrow \mathbf{A} \text{ is } T-\text{convex and closed.}$$

Theorem 4 *A* closed $\mathcal{A}$ *has a ternary representation T if and only if $\mathcal{A}$ is a CVT.*

There is a unique such representation that is a TPO; it is closed *and given by*

$$(x,y,z) \in T \iff y \in H_{\mathcal{A}}(x,y).$$

3. EXAMPLES

1. Let X be finite. An attribute collection $\mathbf{A}$ is a *(taxonomic) hierarchy* if it satisfies the following axiom.

Axiom 3 (Hierarchy) *For all* $\mathbf{A}, B \in \mathbf{A}$: $\mathbf{A} \backslash \mathbf{B} \neq \emptyset \Longrightarrow \mathbf{A} \supseteq \mathbf{B}$

The following fact is well-known and easily verified:

Fact 1 *$\mathcal{A}$ is a hierarchy if and only if there exists a filtration* $\{\Pi_k\}_{1,\ldots,K}$ *(sequence of partitions ordered by refinement) such that* $\mathcal{A} \backslash \{\emptyset\} \subseteq \bigcup_{1,\ldots,K} \Pi_k$.

Note that the hierarchy property implies intersection-closedness as well as abstract convexity. The ternary preorders associated with hierarchical topologies are characterized in section 6.

2. Let A be any attribute-collection satisfying A1 and A2. Then by theorem 2, $T_{\mathcal{A}} \in$ TPO is given by $(x, y, z) \in T_{\mathcal{A}}$ if and only if $y \in H_{\mathcal{A}}(x, y)$. Sometimes, A will naturally be given as an abstract convex class, as in the hypercube-example #4, but in many cases it will not be, as in the following example.

3. Let (X,Γ) be a tree-graph, with the adjacency relation Γ being symmetric, acyclic and (graph-theoretically) connected. For any path ("walk without detour") π in (X,Γ), let **A,** denote the set of points reached by the path, and let

$$A := \{\emptyset, X\} \cup \{A_\pi \mid \pi \text{ is a path in } (X,\Gamma)\}.$$

A satisfies A1 and **A2.** The point y is $T_{\mathcal{A}}$−between the points x and z if and only if it lies on a path connecting x and z. $\mathcal{A}$ is a convex topology if and only if the tree is in fact a line; in the general case, $\mathcal{A}_{(T_{\mathcal{A}})}$ is the class of connected subsets of the tree.

4. Let F denote a set of features in the manner of Tversky's (1977) contrast model. An object is identified with the set of features it possesses, i.e. as an element of the hypercube $\{0,1\}^F =: X$. It is then natural to say that y is at least as similar to x as z is if and only if y shares every *feature* shared by x and z; formally, $(x, y, z) \in T$ if and only if $x_i = z_i$ implies $y_i = x_i$ for all $i \in F$. Here, $\mathcal{A}_T$ is given by the set of sub-hypercubes of $\{0,1\}^F$, i.e. by the sets of the form $\{0\}^{F_0} \times \{1\}^{F_1} \times \{0,1\}^{F \setminus (F_0 \cup F_1)}$ for disjoint $F_0, F_1 \subseteq F$. While it is easy to see that $\mathcal{A}_T$ represents T and that it is intersection-closed, it is non-trivial to show that $\mathcal{A}_T$ satisfies abstract convexity.

In the next paragraph, identify an object 1_S with its set of features S. The following is a somewhat simplified version of Tversky's "contrast model". Let there be two additive not necessarily unitary strictly positive measures on 2^S be given, λ and η. **A** non-symmetric distance-function $d(x, y)$ measuring the dissimilarity of x from y is given by defining

$$d(x, y) := \lambda(x \setminus y) + \eta(y \setminus x).$$

Note that d is symmetric if and only if $\lambda = \eta$, and that d is the Hamming distance if λ and η are the counting measure. Define $T^{(d)}$ as the additive component of d :

$$T^{(d)} := \{(x, y, z) \in X^3 \mid d(x, y) + d(y, z) = d(x, z)\} .$$

Then $T^{(d)}$ coincides with the betweenness relation T just defined[6]. Thus, we have here an important example in which a TPO can be recovered from a metric associated with it (see section 6 for another example).

5. **A** from the mathematical point of view potentially very interesting generalization of this example would be to the set of convex polytopes in which X is the set of extreme points of some polytope $P \subseteq R^m$, and the $A \in \mathcal{A}$ are the sets of extreme points of the faces of the polytope. The issue is whether $\mathcal{A}$ thus defined is a convex topology, i.e. whether it satisfies abstract convexity. If it did, the topology of convex polytopes might be representable/understandable in terms of its convex topology respectively ternary betweenness structure. What needs to be shown for this to be true is the following: if S is the convex hull in R^m of a subset S_o of X such that,,for any $x, y \in S_o$, S contains a face of the polytope which contains both x and y, then S itself must be a face of the polytope P.[7]

6. Let $(X, \geq)$ denote a partially ordered set. There are at least three natural ways to define a ternary relation in this context. The first is simply to "embed" as a ternary relation by setting

$$T := \{(x, y, z) \in X^3 \mid y \geq x \text{ or } y \geq z\}$$

This is explored in section 5.

7. The second is based on interpreting $\geq$ as a "polarity"-ordering (such as "right" versus "left" in a political context). Then T defined by

$$T := \{(x, y, z) \in X^3 \mid z \geq y \geq x \text{ or } x \geq y \geq z\}$$

[6](Here and elsewhere in this section we leave it to the reader to verify the asserted properties of the defined ternary relation).

[7]Is this known? Are there counterexamples?

defines a natural betweenness relation that is a TPO.

8. In an economic context in which X denotes a universe of choice-alternatives or consumption-bundles, and in which $\geq$ denotes a preference ordering over the alternatives, one can naturally define convexity (plus monotonicity) requirements on the preference relation in terms of a ternary order T describing the "convex topology" of the choice space by defining a preference-relation $\geq$ as T-convex if $(x, y, z) \in T$ implies $y \geq w$ for all w such that $x \geq w$ and $z \geq w$. (An alternative stronger definition would require that $(x, y, z) \in T$ implies $y \geq x$ or $y \geq z$; while equivalent if the preference relation is a weak order, it seems less attractive in the general case as it involves strong comparability assumptions). Important examples are (euclidean) convexity in general equilibrium theory and "single-peakedness" in voting theory.

Conversely, one may define the "convex structure" $T^{\geq} \in$ TPO of a preference relation $\geq$ by setting

$$T^{\geq} := \{(x, y, z) \in X^3 \mid y \geq w \text{ for all } w \text{ such that } x \geq w \text{ and } z \geq w\}$$

With this definition, convexity of preference with respect to a given T amounts to the requirement $T^{\geq} \supseteq T$.

9. If $(X, \geq)$ is a lattice (with $\wedge$ denoting the meet, and $\vee$ denoting the join), the definition of $T^{\geq}$ simplifies to

$$T^{\geq} := \{(x, y, z) \in X^3 \mid y \geq x \wedge z\} .$$

$T^{\geq}$ has an isomorphic dual

$$T^{\leq} := \{(x, y, z) \in X^3 \mid y \leq x \vee z\} .$$

Note that $T^{\leq}$ really is $T^{\geq'}$ for the converse relation $\leq = \geq'$. As an example, if $(X, \supseteq)$ denotes the set of linear subspaces of a given linear space ordered by set-inclusion, the subspace y lies $T^{\subseteq}$–between the subspaces x and z if and only if it lies in their span. The "spanning relations" $T^{\geq}$ and $T^{\leq}$ generate a particularly simple convex topology.

Fact 2 $\mathcal{A}_{(T^{\geq})} \backslash \{\emptyset\} = \{\uparrow x \mid x \in X\}$, *with* $\uparrow x := \{y \in X \mid y \geq x\}$.

10. Let P denote a finite set of propositional variables. Then the set of well-formed formulae x, y, .., identified under logical equivalence, can be represented truth-functionally as $X = \{0,1\}^P$. Then (X,+=) defines a lattice in which $>$ is given by inverse implication $\Leftarrow$, and in which the meet and join are given by logical con- and disjunction.

The wff y is $T^{\Leftarrow}-$between x and z if and only if y is logically entailed by the conjunction of x and z (i.e. $x \& z \Rightarrow$ y). Fact 2 implies that the CVT representing the ternary entailment relation is given by the class of wffs of the form $\uparrow$ x, i.e. of sets of wffs jointly implied by a single mother-wff x. The attributes $\uparrow x$ are exactly those sets of wffs that are closed under implication and conjunction. They can therefore be interpreted as potential "epistemic states". On this interpretation, x and z entail y if and only if y is believed (part of an epistemic state) whenever x and z are jointly believed.

11. Finally, let (X,T) be any space ordered by the TPO T, and let $\mathcal{A}_T$ be its associated convex topology. $(\mathcal{A}_T, \supseteq)$ is a lattice ordered by set-inclusion. Endow $\mathcal{A}_T$ with the TPO $T^{\subseteq}$, and let $h(x) := H_{\mathcal{A}_T}(x,x)$. Then h : X $\rightarrow \mathcal{A}_T$ defines an order-preserving mapping from the ordered space (X,T) to the ordered space $(\mathcal{A}_T, T^{\subseteq})$, i.e.

$$(h(x), h(y), h(z)) \in T^{\subseteq} \iff (\mathrm{x},\mathrm{y},z) \in \mathrm{T}.$$

Thus, in particular, every ordered space (X,T) can be embedded in an order-preserving manner in a lattice (Y,>) endowed with the TPO $T^{\leq}$.

4. UNIQUENESS AND MINIMALITY ASPECTS OF THE REPRESENTATION

Evidently, **TPO** is closed under intersection (in $2^{(X^3)}$). Somewhat less evident, and *very* pleasant for the development of the theory, is the fact that **CVT** as well is closed under intersection (in $2^{(2^X)}$)

Proposition 1 **CVT** is $\cap-$*closed.*

By consequence, the "TPO closure" T^* of T and the "CVT closure" $\mathcal{A}^*$ of $\mathcal{A}$ are well-defined as follows.

Definition 2 $T^* := \cap\{T' \in \mathbf{TPO} \mid T' \supseteq T\}$;
$\mathcal{A}^* := \cap\{\mathcal{A}' \in \mathbf{CVT} \mid \mathcal{A}' \supseteq \mathcal{A}\}$.

The following theorem shows that the mutual representation of TPOs and CVTs constitutes an order-isomorphism. Moreover, ternary relations have the same multi-attribute representation if and only if their transitive symmetric closure agrees; similarly, attribute-collections have the same ternary representation if and only if their CVT closure agrees.

Theorem 5 *i) The mapping* $\mathcal{A}_\bullet : T \longmapsto \mathcal{A}_T$ *defines an order-inverting bijection between* **TPO** *and* **CVT** *whose inverse is given by* $T_\bullet : \mathcal{A} \mapsto T_{\mathcal{A}}$.
ii) For any $T \subseteq X^3 : \mathcal{A}_T = \mathcal{A}_{T^*}$; *for any* $\mathcal{A} \subseteq 2^X : T_{\mathcal{A}} = T_{\mathcal{A}^*}$.

In analogy to the description of a lattice by its join-/meet-irreducible elements, it seems natural to look for minimal subsets that yield an equivalent representation. We carry out the analysis for CVTs.

It is helpful to define separate operators for intersection- and convex closure denoted by $\bar{\cdot}$ and $\hat{\cdot}$. Let $\mathcal{J}(2^X) := \{\mathcal{B} \subseteq 2^X \mid \mathcal{B}$ is $\cap-$ closed and reflexive$\}$.

Definition 3 *i)* $\overline{\mathcal{A}} := \cap\{\mathcal{B} \subseteq 2^X \mid \mathcal{J}(2^X), \mathcal{B} \supseteq \mathcal{A}\}$.
ii) For $\mathcal{A} \in \mathcal{J}(2^X) : \widehat{\mathcal{A}} := \{S \subseteq 2^X \mid S$ *is* $\mathcal{A}-$*convex*$\}$.

Part i) of the following lemma shows in particular that the mapping $\mathcal{A} \mapsto \widehat{\mathcal{A}}$ defines a genuine closure operator; part ii) is the key to the following analysis, in that it characterizes the convex-intersection closure as a simple composition of intersection-closure followed by convex closure.

Lemma 1 *i) For all* $\mathcal{A} \in \mathcal{J}(2^X) : \widehat{\mathcal{A}} = \mathcal{A}^*$

ii) *For all* $\mathcal{A} \subseteq 2^X : \widehat{\overline{\mathcal{A}}} = \mathcal{A}^*$.

The search for minimal collections of attributes closely related to that for %re-ducible" elements (attributes), i.e. for those attributes that are not generated from others via ·*-closure.

Definition 4 *i) For* $\mathcal{A} \subseteq 2^X$: $\mathcal{A}_* := \{S \in \mathcal{A} \mid S \notin (\mathcal{A} \backslash \{S\})^*\}$.

ii) For $\mathcal{A} \subseteq 2^X$: $\underline{\mathcal{A}} := \{S \in \mathcal{A} \mid S \notin \overline{\mathcal{A} \backslash \{S\}}\}$.

Also, let $\mathcal{H}_{\mathcal{A}} := \{H_{\mathcal{A}}(x,y) \mid x,y \in X\}$.

The following theorem summarizes the minimality results available for CVTs.

Theorem 6 *Let* $\mathcal{A} \in CVT$.

i) $\overline{\mathcal{H}}_{\mathcal{A}}$ *is the unique minimal set* $\mathcal{B} \in \mathcal{J}(2^X)$ *such that* $\widehat{\mathcal{B}} = \mathcal{A}$.

ii) $\mathcal{A}_* = \underline{\mathcal{H}}_{\mathcal{A}} \cap \underline{\mathcal{A}}$.

For iii) and iv), suppose that X is finite.

iii) $\underline{\mathcal{H}}_{\mathcal{A}}$ *is a minimal set* $\mathcal{B} \subseteq 2^X$ *such that* $\mathcal{B}^* = \mathcal{A}$.

iv) The following four statements are equivalent:

a) There exists a unique minimal collection $\mathcal{B} \in \mathcal{J}(2^X)$ *such that* $\mathcal{B}^* = \mathcal{A}$.

b) $(\mathcal{A}_*)^* = \mathcal{A}$.

c) $\underline{\mathcal{H}}_{\mathcal{A}} \subseteq \underline{\mathcal{A}}$.

d) $\underline{\mathcal{H}}_{\mathcal{A}} = \mathcal{A}_*$.

Examples: 1. In the hypercube-example **3.4**, with $\#F = m, \underline{\mathcal{H}}_{\mathcal{A}} = \underline{\mathcal{A}}$ consists of the 2m (m – 1)–dimensional sub-hypercubes.

2. If $\mathcal{A}$ is a taxonomic hierarchy, $\underline{\mathcal{H}_{\mathcal{A}}} = \underline{\mathcal{A}} = \mathcal{A}$.

3. If $\mathcal{A} = 2^{(2^X)}$, the "discrete topology", $\underline{\mathcal{H}_{\mathcal{A}}} = \{S \mid \#S = 2\}$, whereas $\underline{\mathcal{A}} = \{X \backslash \{x\} \mid x \in X\}$.

4. If $\mathcal{A}$ is the set of closed convex subsets of $\mathbb{R}^m$ (as in Euclidean betweenness), $\underline{\mathcal{H}_{\mathcal{A}}}$ is empty, whereas $\underline{\mathcal{A}}$ consists of all closed half-spaces. The role of $\underline{\mathcal{H}_{\mathcal{A}}}$ is taken by the set of closed half-lines.

5. BINARY PREORDERS EMBEDDED

With any ternary relation T, one can naturally associate a binary relation $R_T \subseteq X \times X$ as follows:

Definition 5 $yR_Tx :\Leftrightarrow (x,y,x) \in T$

Fact 3 i) *R_T is reflexive / transitive whenever T is.*

ii) If T is reflexive and transitive, then $yR_Tx \Leftrightarrow [(x,y,z) \in T \;\&\; (z,y,x) \in T \;\forall z \in X]$

Fact 3, ii) suggests then following definition of the "effective binariness" of a ternary relation T.

Axiom 4 (**Binariness**) $(x,y,z) \in T \Rightarrow (x,y,x) \in T$ *or* $(z,y,z) \in T$.

For a given reflexive binary relation $R \subseteq X^2$, there is more than one ternary relation T such that $R = R_T$; it proves convenient to single out the largest one given by $\{(x,y,z) \in X^3 \mid yRx \text{ or } yRz\} =: T_R$. Note that trivially $R_{(T_R)} = R$ for any $R \subseteq X^2$.

To precisely describe the interrelation between binary and ternary relations, the following two properties are helpful.

Axiom 5 **i)** **T** is simply reflexive if $(x,x,x) \in T$ *for all* $x \in X$.

ii) T is regular if $(x,y,z) \in T$ *implies* $(x,y,z) \in T$ *and* $(z,y,x) \in T$, *for all* $x, y, z \in X$.

Remark: Regularity is a substantive restriction. To see this, consider an operator $\circ : (x,y) \mapsto x \circ z$ with the properties $x \circ x = x$ and $x \circ z = z \circ x$ for all $x, z \in X$. Let $(x,y,z) \in T^{(\circ)}$ iff $y = x \circ z$. $T^{(\circ)}$ is simply reflexive, symmetric and vacuously transitive but fails to be regular. One can accommodate regularity by defining $(x,y,z) \in T^{[\circ]}$ iff $y \subseteq \{x \circ z, x, z\}$. Then $T^{[\circ]}$ defines a TPO if and only if $\circ$ satisfies, in addition to the above, the property that $x \circ (x \circ z) = x \circ z$ for all $x, z \in X$. This property rules out "averaging operators", for instance.

The following implications are easily established.

Fact 4 i) T is simply reflexive if it is reflexive.

ii) T is reflexive if it is simply reflexive and regular.

iii) T is regular if it is reflexive and transitive.

iv) A simply reflexive and transitive T is reflexive if and only if it is regular.

The next proposition characterizes the embedding $R \mapsto T_R$ and some of its properties; its last part anticipates the subsequent theorem **7.**

Proposition *2 i) For any relation $R \subseteq X \times X$, T_R is the unique regular effectively binary relation $T \subseteq X^3$ such that $R_T = R$.*

ii) T_R is always symmetric; it is reflexive whenever R is.

iii) T_R is a ternary preorder if and only if R is a preorder.

Binary preorders can be viewed as effectively binary ternary preorders.

Theorem 7 *An effectively binary relation $T \subseteq X^3$ is a ternary preorder if and only if R_T is a preorder and T is regular.*

We now study the implication of the representation theory for effectively binary relations. The first result shows that effective binariness of T is equivalent to $\mathcal{A}_T$

being union-closed. Given a binary relation R_T, $Y \subseteq X$ is an "up-set" if $x \in Y$ and yR_Tx imply $y \in Y$; let $\mathcal{U}_{(R_T)}$ denote their class. The "up-set" of x $\{y \mid yR_Tx\}$ is denoted by $\uparrow x$.

Theorem 8 *A ternary pre-order T is effectively binary if and only if $\mathcal{A}_T$ is closed under arbitrary (or finite) union. Moreover, $\mathcal{A}_T = \mathcal{U}_{(R_T)}$.*

Remark. An interesting aspect of the theorem is the conclusion that for convex topologies closedness under finite and closedness under arbitrary unions coincides; indeed, this is easily seen to be a direct implication of abstract convexity. As a result, the intersection of the class of "convex topologies" with that of ordinary topologies is minimal[8]; in particular, a convex topology describes the closed sets of a Hausdorff topology if and only if it is the discrete topology given by $\mathcal{A} = 2^{(2^X)}$.

The set $A \in \mathcal{A}$ is said to be *join-irreducible* if $A \neq \emptyset$, and for no $B, B' \in \mathcal{A} \setminus \{A\}$, $A = B \cup B'$. Let $\mathcal{A}^{irr}$ denote their class. For any pre-order R, let $\langle\langle X;R\rangle\rangle$ denote the quotient relation (partial order) induced by the equivalence relation $E : xEy \Leftrightarrow xRy$ & yRx.

We have the following corollary:

Corollary 1 *$\langle \mathcal{A};\subseteq\rangle$ is a sublattice of the distributive lattice $\langle 2^X;\subseteq\rangle$ if and only if $\mathcal{A} = \mathcal{U}_R$ for some pre-order $R \subseteq X \times X$. Moreover, if X is finite, then the mapping $f : \langle\langle X;R\rangle\rangle \rightarrow \langle \mathcal{A}^{irr};\subseteq\rangle$ is an order-isomorphism.*

In view of the well known and easily established fact that any finite distributive lattice on X is order-isomorphic to a sub-lattice of $\langle 2^X;\subseteq\rangle$, corollary 1 can be viewed as a version of Birkhoff's classic representation theorem for finite distributive lattices (Birkhoff (1933), Davey and Priestley (1989, p. 171)).

[8] Do the topologies $\mathcal{U}_{(R_T)}$ have a name?

6. TAXONOMIC HIERARCHIES

A ternary relation T can be viewed as an X–tuple of binary relations $\mathbf{T} = (T_{(x)})_{x \in X}$ defined by $yT_{(x)}z :\Leftrightarrow (x, y, z)$, which is read as " y is at least as close to x than z is". We will identify properties of **T** with those of T. Thus, for example, **T** is simply reflexive iff each $T_{(x)}$ is reflexive, and T is reflexive iff x is $T_{(x)}$ - maximal for all $x \in X$. From this point of view, it is of particular interest to study relations with the property that each $T_{(x)}$ is connected.

Axiom 6 (2-Connectedness) *For all* $x, y, z \in X$, $(x, y, z) \in T$ *or* $(x, z, y) \in T$.

2-connected ternary preorders will be shown to correspond to the hierarchical topologies of example 3.1 . First, however, we note that under 2-connectedness, (ternary) transitivity simplifies to transitivity of each $T_{(x)}$.

Axiom 7 (2-Transitivity) *For all* $x, y, z, z' \in X$, $(x, y, z') \in T$ *and* $(x, z', z) \in T$ *imply* $(x, y, z) \in T$.

Proposition 3 *i) Transitivity and reflexivity imply 2-transitivity.*

ii) 2-Transitivity, symmetry and 2-connectedness imply transitivity.

Theorem 9 *The ternary preorder* T *is 2-connected if and only if* $\mathcal{A}_T$ *is a hierarchy.*

Combining theorem 9, proposition **3** and theorem 5, one obtains the following result about X-tuples of weak orders as a corollary.[9]

Theorem 10 *Let* **T** *be a tuple of weak orders* $(T_{(x)})_{x \in X}$ *such that* $xT_{(x)}y$ *for all* $x, y \in X$. *Then* **T** *is symmetric if and only if there exists a hierarchy* $\mathcal{A}$ *such that, for all* $x, y, z \in X$, $yT_{(x)}z \Leftrightarrow [z \in A \Rightarrow y \in A] \quad \forall A \in \mathcal{A} : A \ni x$.

[9] This result had been obtained directly prior to the work on this paper in collaboration with Clemens Puppe.

Theorem 10 is written in such a way as to highlight the role of symmetry in ensuring the existence of *one* $\mathcal{A}$ (independent of x) representing all $T_{(x)}$ simultaneously. The result may be viewed as a qualitative analogue to well-known theorems on the representation of ultrametric distances by "indexed hierarchies" or weighted tree-graphs.

Definition 6 *A function* $d: X \times X \to \mathbf{R}_+$ *is an* ultra(pseudo)metric *if*

i) $d(x,x) = 0 \ \forall x \in X$, *and*

ii) $d(x,y) \leq max\{d(x,z), d(y,z)\} \ \forall x,y,z \in X$.

Note that an ultrametric is necessarily symmetric (put $z = x$) and satisfies the triangle inequality.

Definition 7 *A function* $\nu : \mathcal{A} \to \mathbf{R}_+$ *is an* index *of the hierarchy* $\mathcal{A}$ *if*

i) $\inf\{\nu(A) \mid A \in \mathcal{A} : A \ni x\} = 0 \quad \forall x \in X$, *and*

ii) $A \subset B \Rightarrow \nu(A) < \nu(B)$, $\forall A, B \in \mathcal{A} \setminus \{\emptyset\}$.

The pair $(\mathcal{A}, \nu)$ *is an* indexed hierarchy.

The following is a standard result[10].

Theorem 11 (Johnson, Benzecri) *A function* $d : X \times X \to \mathbf{R}_+$ *is an ultrametric if and only if there exists an indexed hierarchy* $(\mathcal{A}, \nu)$ *such that* $d(x,y) = \inf\{\nu(A) \mid A \in \mathcal{A} : A \supseteq \{x,y\}\}$, $\forall x, y \in X$.

With any given ultra-metric, one can associate a 2-connected ternary preorder $T^{[d]}$:

$$(x,y,z) \in T^{[d]} :\Leftrightarrow d(x,y) \leq d(x,z);$$

to verify that $T^{[d]} \in TP\acute{O}$ as claimed, we need to check its symmetry, i.e. that $d(x,y) \leq d(x,z)$ implies $d(y,x) \leq d(x,z)$, which is immediate from ultrametricity.

[10] The result has been proved independently by Johnson (1967), Benzecri (1973) and others; for an extensive treatment of the representation of proximity measures by taxonomic hierarchies and trees, see the monograph of Barthélemy and Guénoche (1991).

By consequence, theorem *11* can be obtained as a corollary of theorem 10: simply take $\mathcal{A}_{T[d]}$ from that theorem, and define $\nu(A) = \sup\{d(x,y) \mid \{x,y\} \subseteq A\}$, for $A \in \mathcal{A}_{T[d]} \setminus \{\emptyset\}$.

Conversely, one can obtain theorem *10* from theorem *11* by associating with a given 2-connected ternary preorder T an ultrametric $d^{[T]}$ defined by

$$d^{[T]}(x,y) := \#\{z \mid (x,z,y) \in T\} - 1.$$

This is a straightforward consequence of the following lemma.

Lemma 2 $d^{[T]}$ **is** *ultrametric for any 2-connected ternary preorder* T

APPENDIX: PROOFS

SECTION 2

Proof of Theorem 1 (up to uniqueness):

I. For any A : $T_{\mathcal{A}} \in$ **TPO.**

The necessity of T1 and T2 is trivial.

To verify *T3,* take any $x, x', y, z, z' \in X$ such that

$$\{(x,x',z),(x,z'.z),(x',y,z')\} \subseteq T_{\mathcal{A}} \tag{1}$$

and any $A \in \mathcal{A}$ such that $A \supseteq \{x,z\}$.

By assumption *(1)* then $A \supseteq \{x', z'\}$, exploiting the definition of $T_{\mathcal{A}}$ twice. Hence $A \ni y$ by *(1)* again ; A being arbitrary, this shows $(x,y,z) \in T_{\mathcal{A}}$. It follows that $T_{\mathcal{A}}$ satisfies T3. □ /

II. If **T** $\in$ **TPO** there exists **A** $\in$ CVT such that $T = T_{\mathcal{A}}$.

Lemma 3 **If T** $\in$ **TPO** , *then there exists* $A \in$ **CVT** *such that* $\mathbf{T}(.) = H_{\mathcal{A}}(.)$.

Proof. Let $\mathcal{A} := \{A \in 2^X \mid \forall x, y \in A : T(x,y) \subseteq A\}$; $\mathcal{A}$ is the class of all T-convex sets.

1. $\mathcal{A}$ is $\cap$-closed.

Take $\{A_i\}_{i \in I} \subseteq \mathcal{A}$. For any $x, y \in \bigcap_{i \in I} A_i$, $T(x,y) \subseteq A_i$ for any $i \in I$, by the definition of $\mathcal{A}$, hence $T(x,y) \subseteq \bigcap_{i \in I} A_i$. It follows that $\bigcap_{i \in I} A_i \in \mathcal{A}$

2. By definition, $H_{\mathcal{A}}(x,y) \supseteq T(x,y)$, for all $x, y \in X$.

3. By **T3**, for all $x, y \in X$ and $x', y' \in T(x,y), T(x',y') \subseteq T(x,y)$ (segments are T-convex), which implies $T(x,y) \in \mathcal{A}$, and thus $H_{\mathcal{A}}(x,y) \subseteq T(x,y)$. Together with 2., this shows $H_{\mathcal{A}} = T$.

4. Finally, in view of 3., A3 is immediate from the definition of $\mathcal{A}$. A1 is trivially satisfied. □

Lemma 4 If $\mathcal{A} \in CVT$, *then* $\mathbf{T}_{\mathcal{A}} = H_{\mathcal{A}}$.

Proof.

1. By definition $z \in \mathbf{T}_{\mathcal{A}}(x,y)$ and $A \in \mathcal{A} : A \supseteq \{x,y\} \Rightarrow A \ni z$.

Take any $x, y \in X$ and $z \in \mathbf{T}_{\mathcal{A}}(x,y)$ Since by definition of $H_{\mathcal{A}}$, $\{x,y\} \subseteq H_{\mathcal{A}}(x,y) \in \mathcal{A}$, it follows from the definition of $\mathbf{T}_{\mathcal{A}}$ that $H_{\mathcal{A}}(x,y) \ni z$.

This shows that $\mathbf{T}_{\mathcal{A}} \subseteq H_{\mathcal{A}}$.

2. Consider now x, y, z such that $z \notin \mathbf{T}_{\mathcal{A}}(x,y)$; by definition of $\mathbf{T}_{\mathcal{A}}$, there exists $A \in \mathcal{A}$ such that $A \supseteq \{x, y\}$ but $z \notin A$. Since $A \supseteq H_{\mathcal{A}}(x,y)$ by A3, $z \notin H_{\mathcal{A}}(x,y)$. It follows that $\mathbf{T}_{\mathcal{A}}(x,y) \supseteq H_{\mathcal{A}}(x,y)$ for all $x, y \in X$.

The lemma results from combining 1. and 2. . □

Lemmas 1 and 2 yield the desired result. ■

<u>Proof of Theorem 2.</u>

I. For any $T \subseteq X^3 : \mathcal{A}_T \in CVT$.

1. A1 is trivial.

2. (A2) Take $\{A_i\}_{i \in I} \subseteq \mathcal{A}_T$. Then, for any $(x,y,z) \in T$ and all $i \in I$, $A_i \supseteq$

$\{x, z\} \Rightarrow A_i \ni$ y. Thus also : $\left(\bigcap_{i\in I} A_i\right) \supseteq \{x, z\} \Rightarrow \left(\bigcap_{i\in I} A_i\right) \ni$ y , which shows that (x, y, z) is compatible with $\left(\bigcap A_i\right)$, thus verifying A2.

3. (A3) Take any $\mathbf{A} \in 2$ such that $H_{\mathcal{A}_T}(x', y') \subseteq$ A for all $x', y' \in \boldsymbol{A}$ and any $(x, y, z) \in$ T. We need to show that (x,y, z) is compatible with ***A***. Assume thus $A \supseteq \{x, z\}$.

Since $H_{\mathcal{A}_T}$ (x,y) $\in \mathcal{A}_T$ by A2 and $H_{\mathcal{A}} \supseteq \{x, z\}$ by A1, $H_{\mathcal{A}_T}$ (x,y) $\ni$ y.

Since $\boldsymbol{A} \supseteq H_{\mathcal{A}_T}$ (x,y) by assumption,,one obtains $\boldsymbol{A} \ni y$ as desired. □

II. Sufficiency follows immediately from the following lemma.

Lemma 5 If $\boldsymbol{A} \in$ CVT, $\boldsymbol{A} = \mathcal{A}_{(T_{\mathcal{A}})}$.

Proof. Take any A $\notin \boldsymbol{A}$. By A3, there exist x, y, z such that $\{x, y\} \subseteq$ A but $z \in H_{\mathcal{A}}(\{x, y\}) \setminus$ A. By lemma 2, $z \in \mathbf{T}_{\mathcal{A}}(\{x, y\}) \setminus$ A, in other words : ***A*** is incompatible with (x, z, y), which shows that $\boldsymbol{A} \subseteq \mathcal{A}_{(T_{\mathcal{A}})}$. Since on the other hand A $\supseteq \mathcal{A}_{(T_{\mathcal{A}})}$ from the respective definitions, one obtains A $= \mathcal{A}_{(T_{\mathcal{A}})}$. □ ■

SECTION 3

Proof of Fact 2.

In view of theorem 1, we need to show that $\mathcal{U} := \{\uparrow x \mid x \in X\}$ coincides with the set of non-empty $T^{\geq}$–convex sets. That $\mathcal{U}$ is contained in this class is straightforward. Conversely, suppose that S is a $T^{\geq}$–convex set. Thus, for all $x, y \in S : \uparrow (x \wedge y) \subseteq S$, and in particular $x \wedge y \in$ S. A simple inductive argument shows that therefore also $\left(\bigwedge_{x\in S} x\right) \in S$. Since on the other hand by definition $\uparrow \left(\bigwedge_{x\in S} x\right) \supseteq S$, one must in fact have $\uparrow \left(\bigwedge_{x\in S} x\right) = S$. ■

SECTION 4

Proof of Theorem 5.

Lemma **6** If T $\in$ TPO, *then* $T_{(\mathcal{A}_T)} =$ T.

Proof. Take T $\in$ TPO, and take $\mathcal{A} \in$ CVT such that T $= T_{\mathcal{A}}$, whose existence is assured by lemma 3. By lemma 5, thus also A $= \mathcal{A}_{(T_{\mathcal{A}})} = \mathcal{A}_T$. It follows that T $= T_{\mathcal{A}} = T_{(\mathcal{A}_T)}$. □

In view of lemmas 5 and 6, A. is a bijection; it is evidently order-preserving with respect to set-inclusion. This completes the proof of part i) of the theorem. □

To demonstrate the second claim of the theorem, take any T $\subseteq X^3$.

Since T* $\supseteq$ T, $\mathcal{A}_{T^*} \subseteq \mathcal{A}_T$, and thus also $T_{(\mathcal{A}_{T^*})} \supseteq T_{(\mathcal{A}_T)}$.

Since T* $= T_{(\mathcal{A}_{T^*})}$ by lemma 6, and $T_{(\mathcal{A}_T)} \supseteq$ T by the definition of $\mathcal{A}_\bullet$, it follows that T*$\supseteq T_{(\mathcal{A}_T)} \supseteq T$.

Since $T_{(\mathcal{A}_T)} \in$ TPO by theorem 1, from the monotonicity of the *-operator one obtains $T_{(\mathcal{A}_T)} = \left(T_{(\mathcal{A}_T)}\right)^* \supseteq T^*$, and thus in fact T* $= T_{(\mathcal{A}_T)}$.

Since $\mathcal{A}_T \in$ CVT by theorem 2, one can infer from lemma 5 that $\mathcal{A}_{(T^*)} =$ A$_{\left(T_{(\mathcal{A}_T)}\right)} = \mathcal{A}_T$.

The proof of the claim for T. is analogous. ■

Proof of Proposition **1.**

Consider any family $\{\mathcal{A}_i\}_{i\in I} \subseteq 2^{(2^X)}$; Let A $:= \bigcap_{i\in I} \mathcal{A}_i$. Satisfaction of A1 and A2 by A is trivial. For A3, consider any S that is A-convex . Since $H_{\mathcal{A}}(.) \supseteq H_{\mathcal{A}_i}(.)$, for all i $\in$ I, (due to $\mathcal{A} \subseteq \mathcal{A}_i$). S is $\mathcal{A}_i$-convex , for all i $\in$ I. Since each $\mathcal{A}_i$ satisfies A3, $S \in \cap \mathcal{A}_i = \mathcal{A}$. □

Proof of Lemma **1.**

For steps 1 through 5, assume throughout that the sets A, B are in $\mathcal{J}\left(2^X\right)$. We begin by showing in steps 1 through **3** that $\hat{\cdot}$ is a closure operator.

1. $\widehat{\mathcal{A}} \supseteq \mathcal{A}$.

Since $\mathcal{A}$ is $\cap$–closed, for any $A \in \mathcal{A}$ arid $x, y \in A, H_{\mathcal{A}}(x,y) \subseteq A$; thus any $A \in \mathcal{A}$ is $\mathcal{A}$-convex .

2. $\widehat{\widehat{\mathcal{A}}} = \widehat{\mathcal{A}}$ (Idempotence).

Since by construction, for all $x, y \in X$ and all $A \in \widehat{\mathcal{A}}$: $A \supseteq H_{\mathcal{A}}(x,y)$, and since $H_{\mathcal{A}}(x,y) \in \widehat{\mathcal{A}}$ by step 1, it follows that, for all $x, y \in X$, $H_{\mathcal{A}}(x,y) = H_{\widehat{\mathcal{A}}}(x,y)$. Thus, a set A is $\widehat{\mathcal{A}}$-convex exactly if it is $\mathcal{A}$-convex . It follows that $\widehat{\widehat{\mathcal{A}}} = \widehat{\mathcal{A}}$.

3. $\mathcal{B} \supseteq \mathcal{A} \Longrightarrow \widehat{\mathcal{B}} \supseteq \widehat{\mathcal{A}}$ (Monotonicity).

$\mathcal{B} \supseteq \mathcal{A}$ implies $H_{\mathcal{B}}(.) \subseteq H_{\mathcal{A}}(.)$, which in turn implies $\widehat{\mathcal{B}} \supseteq \widehat{\mathcal{A}}$.

4. $\widehat{\mathcal{A}} \in \mathcal{J}\left(2^X\right)$.

Satisfaction of A1 is straightforward. To verify the intersection-closedness of $\widehat{\mathcal{A}}$, take an arbitrary collection of $\mathcal{A}$-convex sets $\{S_i\}_{i \in I}$. Choose any $x, y \in S := \bigcap_i S_i$. By assumption, $H_{\mathcal{A}}(x,y) \subseteq S_i$ for each S_i, hence also $H_{\mathcal{A}}(x,y) \subseteq S$. It follows that S itself is $\mathcal{A}$-convex , i.e. that $S \in \widehat{\mathcal{A}}$.

5. $\widehat{\mathcal{A}} = \mathcal{A}^*$.

By step 2, $\widehat{\mathcal{A}}$ satisfies A3, and thus one obtains from step 4 $\widehat{\mathcal{A}} \in$ CVT. It follows that $\mathcal{A}^* \subseteq \widehat{\mathcal{A}}$. On the other hand, since by the definition of $\mathcal{A}^*, \mathcal{A}^* = \widehat{\mathcal{A}^*}$, step **3** implies $\widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}^*} = \widehat{\mathcal{A}}$. This completes the proof of part i).

6. Part ii) follows from part i) via the identities $\widehat{\overline{\mathcal{A}}} = \left(\overline{\mathcal{A}}\right)^* = \mathcal{A}^*$. ■

Proof of Theorem 6.

I. Lemmas 7 and 8 demonstrate the first part of the theorem.

Lemma 7 **For** any $\mathcal{A} \in$ CVT : $\widehat{\overline{\mathcal{H}}}_{\mathcal{A}} = \mathcal{A}$.

Proof. From $\overline{\mathcal{H}}_{\mathcal{A}} \supseteq \mathcal{H}_{\mathcal{A}}$, one obtains $H_{(\overline{\mathcal{H}}_{\mathcal{A}})}(.) \subseteq H_{\mathcal{A}}(.)$ which implies $\widehat{\overline{\mathcal{H}}}_{\mathcal{A}} \supseteq \mathcal{A}$. Since also by the monotonicity of the $\widehat{\ }$-operator $\mathcal{A} = \widehat{\mathcal{A}} \supseteq \widehat{\overline{\mathcal{H}}}_{\mathcal{A}}$, one infers $\widehat{\overline{\mathcal{H}}}_{\mathcal{A}} = \mathcal{A}$

Lemma 8 *For* any $B \in \mathcal{J}(2^X)$ such *that* $\widehat{\mathcal{B}} = A$ $(\in \text{CVT})$: $B \supseteq \mathcal{H}_{\mathcal{A}}$.

Proof. Suppose $\mathcal{H}_{\mathcal{A}} \setminus \mathcal{B} \ni H$. Since $H \in \widehat{\mathcal{B}}$, H is $\mathcal{B}$-convex, i.e. $H = \bigcup_{x,y} H_{\mathcal{B}}(x,y)$, with $H_{\mathcal{B}}(x,y) \subset H$ (strictly!) for all $x, y \in H$. Since $\mathcal{B} \subseteq A$, $H_{\mathcal{B}}(.) \supseteq H_{\mathcal{A}}(.)$. Hence any B-convex set is A-convex. Thus $H_{\mathcal{A}}(x,y) \subset H$ for all $x, y \in H$. However $H = H_{\mathcal{A}}(x', y')$ for some $x', y' \in H$ (since $H \in \mathcal{H}_{\mathcal{A}}$), a contradiction. □

II. The inclusion $\mathcal{A}_* \subseteq \underline{\mathcal{H}}_{\mathcal{A}} \cap \underline{\mathcal{A}}$ is essentially straightforward.

For the converse, consider any $S \in \underline{\mathcal{A}} \cap (\mathcal{A} \setminus \mathcal{A}_*)$. We need to show $S \notin \underline{\mathcal{H}}_{\mathcal{A}}$. Let $B := \mathcal{A} \setminus \{S\}$. By assumption, we have $\mathcal{B}^* = \mathcal{A}$ and $\mathcal{B} = \overline{\mathcal{B}}$. From lemma 1 one can infer that $\widehat{\mathcal{B}} = \widehat{\widehat{\mathcal{B}}} = A$. Since $\mathcal{B} \in \mathcal{J}(2^X)$, part i) yields $\overline{\mathcal{B}} \supseteq \overline{\mathcal{H}}_{\mathcal{A}} \supseteq \underline{\mathcal{H}}_{\mathcal{A}}$. □

III. It is (must be?) a standard result that $\underline{(\overline{\mathcal{B}})} = \underline{\mathcal{B}}$ for any $\mathcal{B} \subseteq 2^X$ and finite X. Using lemma 7, this yields applied to $\mathcal{H}_{\mathcal{A}}$: $(\underline{\mathcal{H}}_{\mathcal{A}})^* = A$.

To verify minimality, consider any B strictly contained in $\mathcal{H}_{\mathcal{A}}$. Then $\overline{\mathcal{B}} \subset \overline{\mathcal{H}}_{\mathcal{A}}$, and thus $\mathcal{B}^* \subset A$ by part i) of the theorem. □

IV. The equivalence of a) and b) follows from the monotonicity of the ·*-operator.

The equivalence of c) and d) is straightforward from part ii), while that of a) and d) follows directly from part iii. □ ■

SECTION 5

Proof of Fact **3.**

i) Straightforward from the definition.

ii) Take $x, y, z \in X$ such that yR_Tx. By reflexivity of T, $(x, x, z) \in T$. Application of transitivity to the triple $((x,x,z), (x,x,z), (x,y,z)) \in T^3$ yields $(x,y,z) \in T$. An analogous inference yields $(z,y,z) \in T$. ■

Proof of Fact **4.**

i) and ii) are trivial, iii) is analogous to the proof of fact 3,ii), and part iv) follows from combining i), ii) and iii). ■

Proof of Proposition 2.

i) It is clear that T_R has the asserted properties. Consider my regular effectively binary T such that $R_T = R$. Then $(x,y,z) \in T$ implies yRx or yRz by binariness, while the converse follows from regularity; hence $T = T_R$.

ii) Symmetry of T_R follows from its definition, its reflexivity (given the reflexivity of R) from fact 4, ii).

iii) The final claim follows from theorem 7 below. ■

Proof of Theorem 7.

Necessity follows from facts 3 and 4.

For sufficiency, we need to verify the transitivity of T.

Consider any $x, x', y, z, z' \in X$ such that $(x, x', z) \in T$, $(x, z', z) \in T$, and $(x', y, z') \in T$. By binariness, yR_Tx' or yR_Tz'; assume yR_Tx' w.l.o.g. By binariness again, $x'R_Tz$ or $x'R_Tx$. Hence by the transitivity of R_T, yR_Tz or yR_Tz'. Moreover, by the assumption on T and binariness, $z'R_Tz$ or $z'R_Tx$. By the transitivity of R_T, one obtains yR_Tz or yR_Tx, and thus $(z, y, z) \in T$ or $(x, y, x) \in T$. Thus, by the regularity of T, $(x, y, z) \in T$. ■

Proof of Theorem 8.

Suppose that T is effectiyely binary and consider any A compatible with T (i.e. $A \in \mathcal{A}_T$). Take any $x \in A$ and y such that yR_Tx, i.e. $(x,y,x) \in T$. By compatibility, $y \in A$. A is thus an upset, from which it follows that $\mathcal{A}_T \subseteq \mathcal{U}_{(R_T)}$.

Conversely, take any $A \in \mathcal{U}_{(R_T)}$ and any $(x, y, a) \in T$ such that $\{x, z\} \subseteq A$. By the effective binariness of T, $(x, y, x) \in T$ or $(z, y, z) \in T$. Thus, w.l.o.g. yR_Tx. Since A is an up-set of R_T, we have $y \in A$, verifying the compatibility of A with T. This shows that $\mathcal{A}_T = \mathcal{U}_{(R_T)}$.

It is straightforward to verify that $\mathcal{U}_{(R_T)}$ is closed with respect to arbitrary unions.

Finally, suppose that $\mathcal{A}_T$ is closed under finite union. Consider any $(x,y,z) \in X^3$ such that neither $(x,y,x) \in T$ nor $(z,y,z) \in T$. Since $T = T_{(\mathcal{A}_T)}$, there must exists $A, B \in \mathcal{A}_T$ such that $x \in A$, $y \notin A$, $z \in B$ and $y \notin B$. By union-closedness, $A \cup B \in \mathcal{A}_T$. Since $\{x,z\} \subseteq A \cup B$ but $y \notin A \cup B$ by construction, we have $(x, y, z) \notin T$, thus verifying the effective binariness of T. ■

Proof of Corollary **1.**

The first part follows from theorem 8 by setting $R = R_{(T_{\mathcal{A}})}$.

The second part follows from, noting that $\mathcal{U}_R^{irr} = \{\uparrow x \mid x \in X)$, and that $xRy \Leftrightarrow \uparrow x \subseteq \uparrow y$. ■

SECTION 6

Proof of Proposition **3.**

i) Take $x, y, z, z' \in X$ such that $(x,y,a') \in T$ and $(x,a',z) \in T$.By reflexivity, $(x,x,z) \in T$. Set $x' = x$ and apply transitivity to obtain $(x,y,a) \in T$.

ii) Take $x, x', y, z, z' \in X$ such that $(x,x',a) \in T$, $(x,z',z) \in T$ and $(x,y,z) \in T$. By 2-connectedness, $(x,x',z') \in T$ or $(x,a',x') \in T$; w.l.o.g. assume $(x,x',z') \in T$.

By symmetry, $(z',x',x) \in T$ as well. as $(a',y,x') \in T$. By 2-transitivity therefore $(z',y,x) \in T$, whence by symmetry, $(x,y,z') \in T$. Finally, by 2-transitivity again, $(x,y,z) \in T$. ■

Proof of **Theorem 9.**

Suppose that $\mathcal{A}_T$ is not a hierarchy, i.e. that there exist $A, B \in \mathcal{A}_T$ and $x, y, z \in X$

such that $y \in A \setminus B$, $z \in B \setminus A$ and $x \in A \cap B$. By construction, neither $(x,y,z) \in T$ nor $(x,z,y) \in T$; T is therefore not 2-connected.

Conversely, suppose that T is not %-connected, i.e. that neither $(x,y,z) \in T$ nor $(x,z,y) \in T$ for some $x,y,z \in X$. Since $T = T_{(\mathcal{A}_T)}$ by theorem 5, there exist $A, B \in \mathcal{A}_T$ such that $\{x, z\} \subseteq A$, $y \notin A$ and $\{x, y\} \subseteq B$, $z \in B$. Since $A \cap B \ni x$, $A \setminus B \ni z$ **and** $B \setminus A \ni y$, $\mathcal{A}_T$ is not a hierarchy. ■

<u>Proof of Lemma 2.</u>

Note first that $d^{[T]}$ is symmetric due to the symmetry of T.

Consider any $x,y,z \in X$ such that $d^{[T]}(x,z) < d^{[T]}(x,y)$; from the definition of $d^{[T]}$, $(x,z,y) \in T$ and $(x,y,z) \notin T$. By symmetry, the latter implies $(z,y,x) \notin T$, and thus $(z,x,y) \in T$ by 2-connectedness. In turn, this yields $(y,x,z) \in T$ by symmetry, hence $d^{[T]}(y,z) \geq d^{[T]}(y,x) = d^{[T]}(x,y)$ by the symmetry of $d^{[T]}$, thus verifying ultrametricity. ■

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