GLOBAL IDENTIFIABILTY UNDER UNCORRELATED RESIDUALS

Leon L. Wegge

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Department of Economics
University of California
Davis, California 95616-8578

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by Leon L. Wegge
Economics, University of California, Davis 95676

Suppose in each equation, not counting covariance restrictions, we need one more restriction to meet the order condition. If we now add to each equation a restriction that its structural residual is uncorrelated with the residual of some other equation, is the parameter of the new model identifiable globally? That is the question.

In general the answer is no. The parameter could remain either not identifiable or is locally identifiable, possibly globally under additional inequality restrictions. In this paper we find families of models for which the answer to the question is yes without the help of inequalities.

The families share common characteristics. First, the sufficient condition for local identifiability must hold. Secondly, the string of zero correlations between residuals contains a closed cycle of length at least four. Thirdly, with the variables, equations and residuals all numbered as they are in the cycle, the odd numbered variables must satisfy a kinship relationship and lastly, the structural residuals can not all be uncorrelated. There are also differences in the families, but these come from the difference in the required kinship relationship.

When there are four or more equations containing external variables, the variety of models with uniquely identifiable parameter under a string of uncorrelated residuals is considerable. In particular, when correlated inverse demand shocks are uncorrelated with correlated supply shocks, our results show that many flexible inverse demand and supply equations reproducing exactly the observed price and quantity moments are members of the above families.

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Keywords: unique identifiability, uncorrelated shocks, cyclical covariance restrictions kinship, siblings, parental lists

1. Introduction.

Global identifiability under covariance restrictions is a delicate matter. For one reason, all equations have to be considered simultaneously and an equational perspective, as in Hausman & Taylor (1983) or in some parts of Bekker & Pollock (1986), reveals the links between the parts but not necessarily the wholeness of the system.

Consider the linear model $By=u$, $E(u)=0$, $E(uu')=\Sigma=(\sigma_{ij})$, where $B=(\beta_{ij})$, $\beta_{ii}=-1$, $i,j=1,..,G$. Let Z_k be the kth row of Z. The parameter (B,Σ) of recursive models when B is lower triangular and Σ diagonal is globally identifiable because (B_1,σ_{11}) is identifiable without any covariance restriction, $(B_2,\sigma_{12},\sigma_{22})$ is identifiable with the help of the restriction $\sigma_{12}=0$ and the last equation (B_G,Σ_G) is identifiable because of the $G-1$ restrictions on Σ_G. Extensions to other cases of global identifiability under covariance restrictions as stated in Koopmans (1950) and in Theorem 4 of Wegge (1965) all have the characteristic that a first equation is identifiable without any covariance restrictions and that the restriction $\sigma_{ij}=0$, $i<j$, can be applied unambiguously to the identifiability of $(B_j,\sigma_{1j},..,\sigma_{jj})$ given that $(B_i,\sigma_{1i},..,\sigma_{ii})$ is identifiable already without it.

In an effort to expand on the scope of globally identifiable model parameters, Mallela and Patil (1976), Mallela (1989) and Mallela, P., Porter-Hudak S. and Yoo S-H. (1993) considered models and restrictions of the type

$$B = \begin{pmatrix} -1 & \beta_{12} & 0 & 0 \\ 0 & -1 & \beta_{23} & 0 \\ \beta_{31} & 0 & -1 & 0 \\ 0 & \beta_{42} & 0 & -1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & \sigma_{34} \\ 0 & 0 & \sigma_{43} & \sigma_{44} \end{pmatrix},$$

in which not a single equation is identifiable without covariance restrictions. We will call this the model $\{[2312] \mid \sigma_{13}=\sigma_{32}=\sigma_{24}=\sigma_{41}=0\}$. They have obtained some fragmented results and their examples are the inspiration for the design of the more general models considered here.

2. The **model** $\{[j_1,...,j_G] \mid \sigma_{m_in_i}=0,\ i=1,..,H,\ G\leq H\}$.

Assume that in the ith row of B with $\beta_{ii}=-1$, all its other elements are zero except for one unknown element β_{ij_i}, $j_i \neq i$, in column j_i, $i=1....G$. Call j_i the parent of variable i. Including the normalization restriction there are only G-1 restrictions on B_i and additional restrictions essential for identifiability are covariance restrictions. To this end, assume in the structural covariance matrix Σ the elements $\sigma_{m_in_i}$, $m_i \neq n_i$, $i=1,..,H$, $G \leq H$ are required to be zero.

In this paper we seek to find the special cases of the model $\{[j_1,...,j_G] \mid \sigma_{m_in_i}=0,\ 1\ \ ,..H,\ G\leq H)$ that have a globally identifiable parameter $\alpha = ((\mathrm{vec}B)', (\mathrm{vec}\Sigma\backslash)')'$, where $\Sigma\backslash$ are all elements of Σ on or below the diagonal. The results are extended readily to the general model with external variables in which the parameter (B, Γ, Σ) is required to satisfy G-1 restrictions on $(B, \Gamma)_i$, $i=1,..,G$.

The analytic formulation is sirnple enough. For $G\times G$ nonsingular T, the parameter $(TB,T\Sigma T')$ is equivalent to the parameter (B,Σ) if and only if

$$T = I_G + (\ \tau_1(B^{-1})_{j_1}',\ \tau_2(B^{-1})_{j_2}',\ \dots,\ \tau_G(B^{-1})_{j_G}')',$$

with $\tau=(\tau_1,..,\tau_G)$ satisfying the set of bilinear equations $(T\Sigma T')_{m_in_i}=0$ i.e. τ is a solution of

$$0 = (j_{m_i},n_i)\ \tau_{m_i} + (j_{n_i},m_i)\ \tau_{n_i} + \omega_{j_{m_i}j_{n_i}}\tau_{m_i}\tau_{n_i},\quad m_i\neq n_i,\quad i=1,..,H, \quad (1)$$

where we often let $(m,n) \equiv (B^{-1}\Sigma)_{mn} \square ((\Omega B')_{mn}$ and $\Omega \square B^{-1}\Sigma B'^{-1} = (\omega_{ij})$, $i,j=1,..,G$, is the reduced form covariance matrix.

If τ satisfies (1) the alternative parameter $(TB,T\Sigma T')$ satisfies

$$(TB)_{ij} = \beta_{ij}+\delta_{jj_i}\tau_i,\quad (T\Sigma T')_{ij} = \sigma_{ij} + (j_i,j)\tau_i + (j_j,i)\tau_j + \omega_{j_ij_j}\tau_i\tau_j,$$

where δ_{jj_i} is the Kronecker delta. If (1) implies $\tau_i=0$, B_i is identifiable and if $\tau=0$, the parameter α is identifiable.

In general bilinear equations (1) have multiple solutions and therefore only local identifiability properties are expected to hold. If $\varphi(\beta,\sigma)=0$, with $\beta\equiv\text{vec}B$, $\sigma\equiv\text{vec}\Sigma\backslash$, is the list of restrictions, the local identifiability condition is that the Jacobian matrix $J(\alpha)$ of the system of restrictions $\varphi((\text{vec}TB)',(\text{vec}T\Sigma T'\backslash)')=0$ has rank G^2 at $T=I_G$. Or equivalently with B nonsingular

$$J(\alpha)[I_G \otimes (B')^{-1}] \equiv [\ \varphi_{\beta'}(I_G \otimes B') \ + \ 2\varphi_{\sigma'}Q_G(I_G \otimes \Sigma)\ ]\,[I_G \otimes (B')^{-1}] \quad (2)$$

has rank G^2. From (2), after deletion of rows and columns corresponding to the restrictions on B, the parameter α is locally identifiable if, and under constant rank conditions, only if the matrix of coefficients in the linear parts of the equations (1) has rank G. Local identifiability is equivalent to $\tau=0$ is an isolated solution of (1).

Whereas local identifiability is necessary, global identifiability results of any generality have to be based on conditions under which the solution to the system of equations (1) can be shown by algebraic manipulations to be unique. This involves much more than knowing the rank of the Jacobian matrix.

Identifiability can also be stated as the condition that the population moment estimator $\{(B^*,\Sigma^*)\,|\,B^*\Omega B^{*\prime}-\Sigma^*=0\}$ satisfying the restrictions $0=\sigma^*_{m_i n_i}$ is unique i.e. the system

$$0=\omega_{m_i n_i}-\omega_{j_{m_i} n_i}\beta^*_{m_i j_{m_i}}-\omega_{j_{n_i} m_i}\beta^*_{n_i j_{n_i}}+\omega_{j_{m_i} j_{n_i}}\beta^*_{m_i j_{m_i}}\beta^*_{n_i j_{n_i}}$$

$$=(j_{m_i},n_i)\,\tau^*_{m_i}+(j_{n_i},m_i)\,\tau^*_{n_i}+\omega_{j_{m_i} j_{n_i}}\tau^*_{m_i}\tau^*_{n_i}, \quad i=1,\dots,H, \qquad (3)$$

where $\tau^*_{m_i}=\beta^*_{m_i j_{m_i}}-\beta_{m_i j_{m_i}}$ and $\tau^*_{n_i}=\beta^*_{n_i j_{n_i}}-\beta_{n_i j_{n_i}}$, has a unique solution $\tau^*=0$. The system (1) defining alternative parameters through the linear transformation operaticn is identical to the system (3) defining alternative values of the population moment estimator. Identifiability means (B^*,Σ^*) is unique and consistent.

In (2) and in the analysis below the elements of the matrix $B^{-1}\Sigma$ □ RB' □ $E(yu')$ play a crucial role. This is the matrix of covariances between variables and residuals. If $(i,j) = (B^{-1}\Sigma)_{ij} = B_j'\Omega_i$ □ $E(y_i u_j) = 0$, the variable y_i is an exogenous or instruments1 variable with respect to the jth equation, or the jth residual is an instrument in the ith equation. The latter interpretation is developed in Hausman and Taylor (1983) in the context of an equational analysis. In recursive models with B lower triangular. $B^{-1}\Sigma$ is lower triangular with $E(y_i u_j) = 0$, $i<j$, or variable i is exogenous in all equations $j>i$. In the type of models considered here, non-exogeneity is the rule and exogeneity relations are the exception.

3. Cycles of Uncorrelated Residuals.

To a single covariance restriction $\sigma_{k_1k_2} = 0$ corresponds in (1) an equation containing the two unknowns (τ_{k_1},τ_{k_2}) and corresponding to two restrictions $\sigma_{k_1k_2} = \sigma_{k_3k_4} = 0$ we have two equations containing three or four unknowns. To three or G_1 restrictions of the type

$$\sigma_{k_1k_2} = \sigma_{k_2k_3} = .. = \sigma_{k_{G_1}k_1} = 0, \quad 3 \leq G_1 \leq G,$$

corresponds in (1) a subsystem of G_1 equations in G_1 unknowns $\tau(G_1)$ □

$(\tau_{k_1},\tau_{k_2},..,\tau_{k_{G_1}})$.

Definitions:

1. With G the number of equations, the H restrictions

$$\sigma_{m_1n_1} =..= \sigma_{m_in_i} =..\square\ \sigma_{m_Hn_H} = 0, \quad m_i \neq n_i, \quad i = 1,..,H,$$

a) are *adequate* if $H \geq G$ and for each integer j there exists a different pair of subscripts (m_i,n_i), with $j \in (m_i,n_i)$, $j=1,..,G$.
b) are *connected* if adequate and $m_i \in \{m_1,..,m_{i-1},n_1,..,n_{i-1}\}$, $i=2,..,H$.
c) are disjoint if $\{m_i,n_i \mid i=1....G_1\} \cap \{m_i,n_i \mid i=G_1+1,...H\}$ is empty. $G_1<G$.

2. A G_1-cycle is a set of G_1 restrictions

$$c(G_1) \equiv \sigma_{k_1k_2} = \sigma_{k_2k_3} = .. = \sigma_{k_{G_1}k_1} = 0,$$

with k_i, $i=2,..,k_{G_1}$ distinct.

3. A G_1-cyclical G-tuple is the set of G restrictions

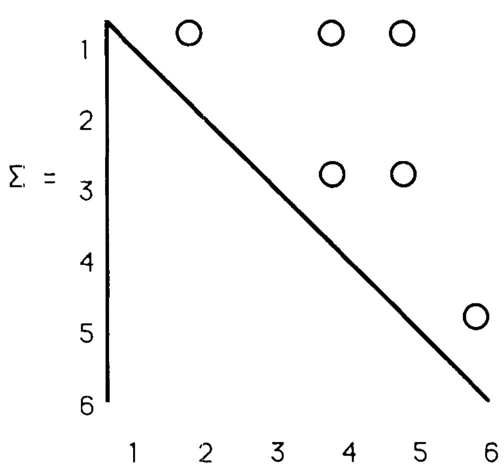
$$c(G_1)=0,\ \sigma_{k_{G_1+1}\ell_{G_1+1}} = .. = \sigma_{k_G\ell_G} = 0, \quad G_1 \leq G.$$

A disjoint G-tuple of covariance restrictions could be adequate, but is not connected. As is clear from (1), if $(j_{n_i},m_i)\neq0$ i.e. if the parent of variable n_i is not exogenous in the m_ith equation, connected restrictions have the property that the n_i-th equation is identifiable if the equations $\{m_1,..,m_{i-1}, n_1,..,n_{i-1}\}$ are identifiable.

Examples of G_1-cyclical G-tuples are rectangular G-tuples which are systems of G covariance restrictions containing the 4-cycle $\sigma_{k_1k_2}=\sigma_{k_2k_3}=\sigma_{k_3k_4}=\sigma_{k_4k_1}=0$. Every 4-cycle can be represented as a rectangle in the covariance matrix Σ by locating some $\sigma_{k_ik_{i+1}}$ at $\sigma_{k_{i+1}k_i}$ if necessary. Similarly 3-cycles, 5-cycles and 6-cycles can be represented as triangles, pentagons and sexagons in Σ.

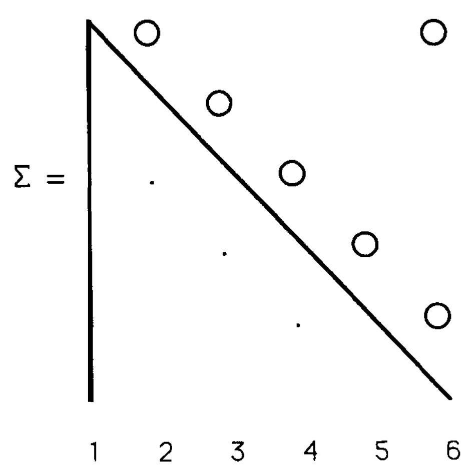
A connected 4-cyclical 6-tuple of restrictions

$\sigma_{14}=\sigma_{43}=\sigma_{35}=\sigma_{51}=\sigma_{12}=\sigma_{56}=0$



A 6-cycle of covariance restrictions

$\sigma_{12}=\sigma_{23}=\sigma_{34}=\sigma_{45}=\sigma_{56}=\sigma_{61}=0$



Our main result concerns the identifiability under a connected G_1-cyclical G-tuple of zero correlations. In considering the total number of G_1-cyclical G-tuples, the order in which the covariance restrictions are written does not matter. Their numbers for values of $G=4,5,6,..,G$ are stated in Table 1 where $G^{\#}=G(G-1)/2$ is the number of distinct off-diagonal elements in $G \times G$ **I**.

Table 1. Total Number of G_1-cyclical G-tuples in $G \times G$ **I**.

<u>Connected</u>	4-tuple	5-tuple	6-tuple	G-tuple
3-cyclical	12	150	2160	$G!G^{G-4}/2(G-3)!$
4-cyclical'	3	60	1080	$G!G^{G-5}/2(G-4)!$
5-cyclical	0	12	360	$G!G^{G-6}/2(G-5)!$
6-cyclical	0	0	60	$G!G^{G-7}/2(G-6)!$
G-cyclical	0	0	0	$(G-1)!/2$
<u>Not connected</u>	0	30	1345	
<u>Total</u>	15	252	5005	$G^{\#}!/G!(G^{\#}-G)!$

To see this, the total number of G-tuples is the number of different tuples of G elements that can be selected from $G^{\#}$ off-diagonal elements of Σ. Of these $(G-1)!/2$ are G-cycles since in the G-cycle

$$\sigma_{1k_2} = \sigma_{k_2k_3} = .. = \sigma_{k_G 1} = 0,$$

k_i in turn can be selected in $G-i+1$ different ways and reading the G-cycle backwards is the same G-cycle. When G=3 there is one 3-cycle, namely $\sigma_{12}=\sigma_{23}=\sigma_{31}=0$.

As shown in Appendix 1 in $G \times G$ Σ the total number of G_1-cycles is equal to $G!/2G_1(G-G_1)!$ and each G_1-cycle can be embedded in $G_1 G^{G-G_1-1}$ connected G-tuples. The product $G!G^{G-G_1-1}/2(G-G_1)!$ is the number of connected G_1-cyclical G-tuples. Riordan (1958) studies cycles and related constructions.

Among the 1345 not connected 6-tuples, 100 have all 6 integers present in their subscripts. Of the latter, ten are two disjoint triangles and ninety contain a rectangle, one element doubly joint and one disjoint element. The results of Theorem 1 below apply to the former but not to the latter. 1245 of the not connected 6-tuples have missing integers in their subscripts.

With these preliminaries the main results are now stated. Lemma 1 states that under local identifiability conditions and a G_1-cycle of uncorrelated residuals, a solution $\tau(G_1)\neq 0$ of (1) implies each component of $\tau(G_1)$ is not zero. It is proved in Appendix 2.

4. Lemma 1.

Consider system (1) consisting of the G_1 cyclical covariance restrictions

$$(T\Sigma T')_{k_1k_2} = (T\Sigma T')_{k_2k_3} = .. = (T\Sigma T')_{k_{G_1}k_1} = 0,$$

and *either* $(j_{k_1},k_2)(j_{k_2},k_3)\times..\times(j_{k_{G_1}},k_1) \neq 0$

or $(j_{k_2},k_1)(j_{k_3},k_2)\times..\times(j_{k_1},k_{G_1}) \neq 0.$

If $\tau(G_1)=(\tau_{k_1},\tau_{k_2},..,\tau_{k_{G_1}})\neq 0$ is a solution of (1), $\tau_{k_1}\tau_{k_2}\times..\times\tau_{k_{G_1}}\neq 0.$

This means that if the model $\{[j_1....j_G], \sigma_{m_in_i}=0, i=1,..,H\}$ contains the G_1-cycle $\sigma_{k_1k_2}= .. = \sigma_{k_{G_1}k_1}=0$, the equations $(k_1,k_2,..,k_{G_1})$ are either all identifiable or none is identifiable when either the parent of variable k_i is not exogenous in the k_{i+1}-th equation or the parent of variable k_{i+1} is not exogenous in the k_i-th equation, $i=1,..,G_1$. Our main result is stated as Theorem 1 and it is proved in Appendix 4.

Theorem 1.

Given the model $\{[j_1....j_{G_1},j_{G_1+1}....j_G],\ \sigma_{m_i n_i}=0,\ i=1,..G\}$ assume:

(A.1) The covariance restrictions are a connected G_1-cyclical G-tuple with G_1 even, $4 \le G_1 \le G$ and its first G_1 restrictions are the G_1-cycle

$$\sigma_{k_1k_2} = \sigma_{k_2k_3} = .. = \sigma_{k_{G_1-1}k_{G_1}} = \sigma_{k_{G_1}k_1} = 0.$$

(A.2) $(B^{-1}\Sigma)_{j_{n_i}m_i} \neq 0,\ i=G_1+1,...,G.$

(A.3) In $\{1,2,..,G_1\}$ for some integer i, i even, and for all ℓ, ℓ odd, one of the following holds:

a) $j_{k_\ell} = k_i,$

b) $j_{k_\ell} = k_{i+1},\ \ell \neq i+1,$

c) $j_{k_\ell} = k_n,\quad \ell \neq i+1,\ n$ even, $n \neq \{i,i+2\}$ satisfies

$$j_{k_n} = j_{k_{i+1}},\quad \sigma_{k_ik_n} = \sigma_{k_{i+2}k_n} = 0,\quad j_{k_i} = j_{k_{i+2}},$$

d) $j_{k_\ell} = j_{k_n},\quad \ell \neq i+1,\ n \neq \{i,i+1,i+2\}$ satisfies

$$k_n = j_{k_{i+1}},\quad \sigma_{k_ik_n} = \sigma_{k_{i+2}k_n} = 0, \text{ and}$$

either $j_{k_i} = j_{k_{i+2}}$, or $k = j_{k_{i+2}}$, or $k_{i+2} = j_{k_i}$.

The parameter $\alpha = ((\text{vec}B)',(\text{vec}\Sigma\backslash)')'$ is globally identifiable if and only if $\Delta(G_1)\equiv z_1(G_1)-z_2(G_1) \neq 0$, where cyclically G_1+m stands for m and

$$z_1(G_1)\equiv \prod_{\ell=1}^{G_1} a_\ell,\quad z_2(G_1)\equiv \prod_{\ell=1}^{G_1} b_\ell,\quad a_\ell \equiv (B^{-1}\Sigma)_{j_{k_\ell}k_{\ell+1}},\quad b_\ell \equiv (B^{-1}\Sigma)_{j_{k_{\ell+1}}k_\ell}$$

In the proof the G_1 bilinear equations are reduced to $G_1/2$ linear homogeneous equations. Uniqueness of the solution $\tau=0$ then follows from standard rank conditions $\Delta(G_1)\neq 0$. (A.3) are the reduction permitting assumptions in the G_1-cycle.

5. Interpretation and implementation remarks.

1. The verbal understanding of (A.3) is that in a G_1-cycle of uncorrelated equations, the odd numbered variables are siblings. Variable k_{i+1} where i is an even number, is a possible exception. The common parent is

a) k_i under (A.3) a), with no exception,

b) k_{i+1} under (A.3) b), k_{i+1} itself is the exception,

c) k_n, a sibling of k_{i+1}, under (A.3) c), where k_n is an odd numbered variable that is the exception and provided the adjacent variables k_i and k_{i+2} are siblings with residuals that are uncorrelated with the residual of the common parent k_n,

d) k_n, a grandpprent of k_{i+1}, under (A.3) d), where k_n is an odd numbered variable that is the exception and provided the adjacent variables k_i and k_{i+2} are either siblings or direct descendants of each other with residuals that are uncorrelated with the residual of the common parent k_n.

Since a cycle can be traversed forwards or backwards, in Theorem 1 the variable k_{i+1} and its neighbors k_i and k_{i+2} could be replaced by k_{i-1} and its neighbors k_i and k_{i-2}.

Clearly under (A.3) a), c) and d) at least one variable is parent to at least two variables. As we will show next, this is also implied by (A.3) b) and local identifiability. Therefore under Theorem 1 at least one variable is not a parent i.e. B containing at least one unit column is reducible.

2. The inequality conditions (A.2) $(B^{-1}\Sigma)_{j_{n_i}m_i} \neq 0$, $i=G_1+1,..,G$, and $\Delta(G_1)\neq 0$ are equivalent to the local identifiability conditions (2). Define

$$\delta_1 \equiv \begin{vmatrix} (B^{-1}\Sigma)_{jk_2k_1} & (B^{-1}\Sigma)_{jk_2k_3} \\ (B^{-1}\Sigma)_{jk_4k_1} & (B^{-1}\Sigma)_{jk_4k_3} \end{vmatrix}, \qquad \delta_2 \equiv \begin{vmatrix} (B^{-1}\Sigma)_{jk_1k_2} & (B^{-1}\Sigma)_{jk_1k_4} \\ (B^{-1}\Sigma)_{jk_3k_2} & (B^{-1}\Sigma)_{jk_3k_4} \end{vmatrix}$$

Under (A.3) b) with either $k_3=j_{k_1}$ or $k_1=j_{k_3}$, (P.4) of Appendix 3 implies $\delta_2=0$ and $\Delta(4)= -(B^{-1}\Sigma)_{j_{k_1}k_2}(B^{-1}\Sigma)_{j_{k_3}k_4}\ \delta_1$. The condition $\Delta(4)\neq 0$ requires $\delta_1\neq 0$ and therefore we must have $j_{k_2}\neq j_{k_4}$, $k_2\neq j_{k_4}$, $k_4\neq j_{k_2}$. Therefore, when $k_3=j_{k_1}$, the same variable k_3 must also be either the parent of k_2 or of k_4. One of the latter two variables could be the parent of k_3 but then the other has no descendant and the assumption (A.3) b) together with $\Delta(G_1)\neq 0$ also imply that B is reducible, having at least one unit column.

If both B and Σ are conformably reducible, $B^{-1}\Sigma$ contains null submatrices and the local identifiability condition would fail. in particular, the parameter of a model satisfying (A.3) of Theorem 1 is riot locally identifiable when Σ is diagonal or also for $G_1=4$, when B has two columns that are unit vectors. For larger systems B may contain more unit vectors provided enough off-diagonal elements in Σ are different from zero so that $B^{-1}\Sigma$ does not contain null submatrices. More precisely and operationally speaking the local identifiability condition (2) has to be verified.

We now list the models with globally identifiable parameter defined in Theorem 1 when G=4, followed by G=5 and G=6.

6. Four equation models with identifiable parameter.

There are three 4-cycles when G=4. These are the restriction systems

$$S_1(4) \equiv \sigma_{13}=\sigma_{32}=\sigma_{24}=\sigma_{41}=0,$$
$$S_2(4) \equiv \sigma_{12}=\sigma_{23}=\sigma_{34}=\sigma_{41}=0,$$
$$S_3(4) \equiv \sigma_{12}=\sigma_{24}=\sigma_{43}=\sigma_{31}=0,$$

Let $[j_1,j_2,j_3,j_4]$ be the list of parents i.e. the column indices of the non zero unknown elements in the rows (1,2,3,4) of B. There are 81 different sets of parents for each restriction system. The parameter is globally identifiable in 24 cases under (A.3) a) and in 48 cases under (A.3) b). These cases are the following.

a). <u>24 Globally Identifiable Cases</u>: One variable is not a parent.

Let $F_a(S_i(4))$ be the family of' models with B nonsingular, Σ positive definite and globally identifiable parameter α under the covariance restrictions $S_i(4)$ and (A.3) a) of Theorem 1. We have

$$F_a(S_1(4)) = \begin{pmatrix} [3312] & [3321] & [4412] & [4421] & \text{if } \sigma_{34}\neq 0 \\ [3422] & [4322] & [3411] & [4311] & \text{if } \sigma_{12}\neq 0 \end{pmatrix},$$

$$F_a(S_2(4)) = \begin{pmatrix} [2123] & [2321] & [4143] & [4341] & \text{if } \sigma_{24}\neq 0 \\ [2343] & [4323] & [2141] & [4121] & \text{if } \sigma_{13}\neq 0 \end{pmatrix},$$

$$F_a(S_3(4)) = \begin{pmatrix} [2142] & [2412] & [3143] & [3413] & \text{if } \sigma_{23}\neq 0 \\ [2443] & [3442] & [2113] & [3112] & \text{if } \sigma_{14}\neq 0 \end{pmatrix}.$$

Each family has eight members satisfying (A.3) a) and the local identifiability condition $\Delta(4)\neq 0$. There are two members in each of four groups:

i) $jk_1=jk_3=k_2,\ jk_2\neq jk_4,\ jk_2\neq k_4,\ jk_4\neq k_2,\qquad |\Delta| = \sigma_{k_2k_2}\sigma_{k_2k_4}\rho_{k_1k_3}/|B|^3\neq 0$

ii) $jk_1=jk_3=k_4,\ jk_2\neq jk_4,\ jk_2\neq k_4,\ jk_4\neq k_2,\qquad |A| = \sigma_{k_4k_4}\sigma_{k_4k_2}\rho_{k_3k_1}/|B|^3\neq 0$

iii) $jk_2=jk_4=k_3,\ jk_3\neq jk_1,\ jk_3\neq k_1,\ jk_1\neq k_3,\qquad |A| = \sigma_{k_3k_3}\sigma_{k_3k_1}\rho_{k_2k_4}/|B|^3\neq 0$

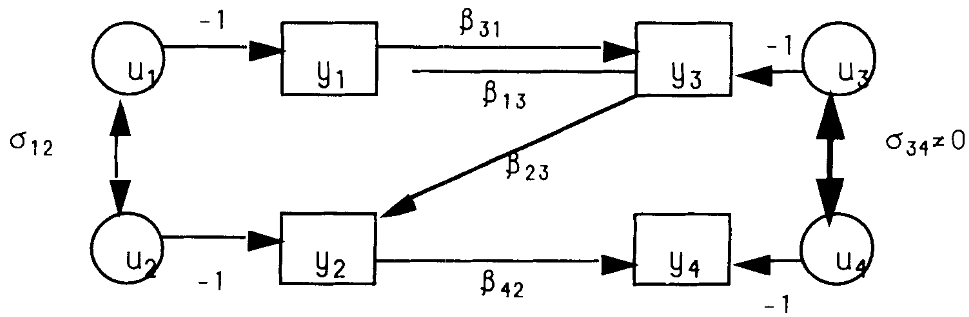
iv) $jk_2=jk_4=k_1,\ jk_3\neq jk_1,\ jk_3\neq k_1,\ jk_1\neq k_3,\qquad |A| = \sigma_{k_1k_1}\sigma_{k_1k_3}\rho_{k_2k_4}/|B|^3\neq 0$

where $\rho_{k_ik_j} = \sigma_{k_ik_i}\sigma_{k_jk_j} - \sigma_{k_ik_j}\sigma_{k_jk_i}$ and $|\Delta|$ is the absolute value of $\Delta(4)$.

The last two groups are obtained from the first two by rotating the subscripts of' the 4-cycle one place i.e. by placing the first restriction last.

In each case the hypotheses that the unrestricted off-diagonal coefficients of B or Σ are zero, are testable. However the coefficients needed to keep $\Delta(4)\neq 0$ are riot testable. In the graph below. $\sigma_{34}=0$ is not testable.

The graph of Model $\{[3312] | S_1(4)\}$



The graph gives a representation of the local identifiability conditions i.e. the residuals u_3 and u_4 must be correlated. The assumption (A.3) a) is shown by having y_1 and y_2 as siblings with y_3 their parent, and y_4 having no descendant.

b). <u>48 Globally Identifiable Caseg</u>: One variable is not a parent.

Let $F_b(S_i(4))$ be the family of models with B nonsingular, Σ positive definite and globally identifiable parameter α under the covariance restrictions $S_i(4)$ and (A.3) b) of Theorem 1. We have

$$F_b(S_1(4)) = \left(\begin{array}{llllll} [2312] & [2321] & \text{if } \beta_{23}\sigma_{34} \neq 0; & [2421] & [2412] & \text{if } \beta_{24}\sigma_{43} \neq 0 \\ [3121] & [3112] & \text{if } \beta_{13}\sigma_{34} \neq 0; & [4112] & [4121] & \text{if } \beta_{14}\sigma_{43} \neq 0 \\ [3441] & [4341] & \text{if } \beta_{41}\sigma_{12} \neq 0; & [4342] & [3442] & \text{if } \beta_{42}\sigma_{21} \neq 0 \\ [4313] & [3413] & \text{if } \beta_{31}\sigma_{12} \neq 0; & [3423] & [4323] & \text{if } \beta_{32}\sigma_{21} \neq 0 \end{array} \right)$$

$$F_b(S_2(4)) = \left(\begin{array}{llllll} [3123] & [3321] & \text{if } \beta_{32}\sigma_{24} \neq 0; & [3341] & [3143] & \text{if } \beta_{34}\sigma_{42} \neq 0 \\ [2311] & [2113] & \text{if } \beta_{12}\sigma_{24} \neq 0; & [4113] & [4311] & \text{if } \beta_{14}\sigma_{42} \neq 0 \\ [2441] & [4421] & \text{if } \beta_{41}\sigma_{13} \neq 0; & [4423] & [2443] & \text{if } \beta_{43}\sigma_{31} \neq 0 \\ [4122] & [2142] & \text{if } \beta_{21}\sigma_{13} \neq 0; & [2342] & [4322] & \text{if } \beta_{23}\sigma_{31} \neq 0 \end{array} \right)$$

$$F_b(S_3(4)) = \left(\begin{array}{llllll} [4142] & [4412] & \text{if } \beta_{42}\sigma_{23} \neq 0; & [4413] & [4143] & \text{if } \beta_{43}\sigma_{32} \neq 0 \\ [2411] & [2141] & \text{if } \beta_{12}\sigma_{23} \neq 0; & [3141] & [3411] & \text{if } \beta_{13}\sigma_{32} \neq 0 \\ [2313] & [3312] & \text{if } \beta_{31}\sigma_{14} \neq 0; & [2343] & [3342] & \text{if } \beta_{34}\sigma_{41} \neq 0 \\ [3122] & [2123] & \text{if } \beta_{21}\sigma_{14} \neq 0; & [2423] & [3422] & \text{if } \beta_{24}\sigma_{41} \neq 0 \end{array} \right)$$

Each family has eight members satisfying (A.3) b) and the local identifiability condition $\Delta(4)\neq 0$. There are four members in each of four groups:

i) $jk_1=k_3, jk_2\neq jk_4, jk_2\neq k_4, jk_4\neq k_2, jk_3\neq k_1,\quad |\Delta|=\beta_{k_3 j_{k_3}}\sigma_{jk_3 jk_3}\sigma_{k_2k_4}\rho_{k_1k_3}/|B|^3\neq 0$

ii) $jk_3=k_1, jk_2\neq jk_4, jk_2\neq k_4, jk_4\neq k_2, jk_1\neq k_3,\quad |\Delta|=\beta_{k_1 j_{k_1}}\sigma_{jk_1 jk_1}\sigma_{k_2k_4}\rho_{k_3k_1}/|B|^3\neq 0$

iii) $jk_2=k_4, jk_3\neq jk_1, jk_3\neq k_1, jk_1\neq k_3, jk_4\neq k_2,\quad |\Delta|=\beta_{k_4 j_{k_4}}\sigma_{jk_4 jk_4}\sigma_{k_3k_1}\rho_{k_2k_4}/|B|^3\neq 0$

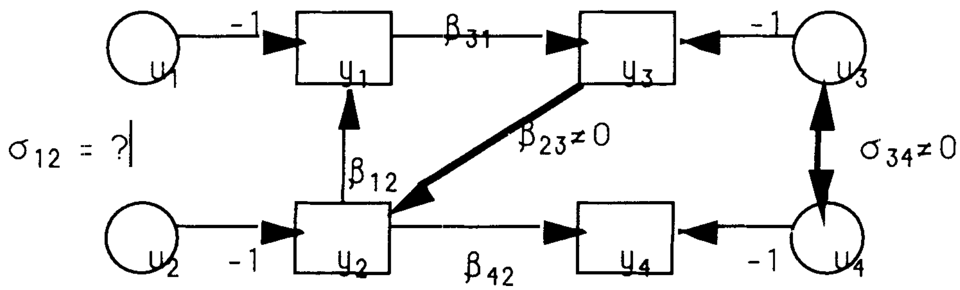
iv) $jk_4=k_2, jk_3\neq jk_1, jk_3\neq k_1, jk_1\neq k_3, jk_2\neq k_4,\quad |\Delta|=\beta_{k_2 j_{k_2}}\sigma_{jk_2 jk_2}\sigma_{k_3k_1}\rho_{k_4k_2}/|B|^3\neq 0$

where $\rho_{k_ik_j}=\sigma_{k_ik_i}\sigma_{k_jk_j}-\sigma_{k_ik_j}\sigma_{k_jk_i}$

The elements of (B,Σ) not required to be different from zero are testable. Thus in the model $\{[2312]|S_1(4)\}$ the restrictions $\beta_{12}=\beta_{31}=\beta_{42}=\sigma_{12}=0$ Or any subset are testable, but $\beta_{23}\sigma_{34}=0$ is not.

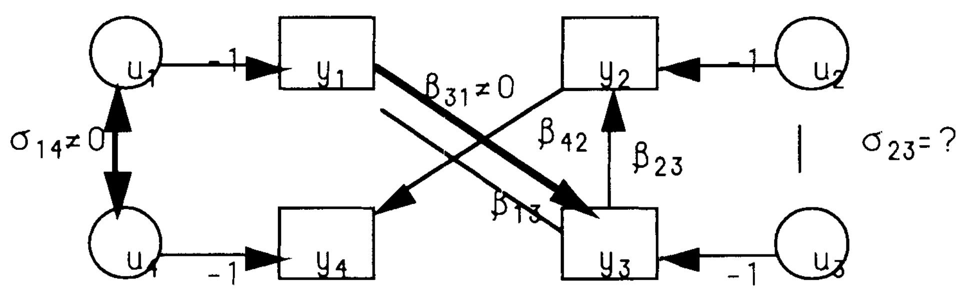
In all cases one and only one of the four variables has no descendant and one has two. The twenty-four sets of parents listed under (A.3) a) reappear under (A.3) b) with a different 4-cycle of covariance restrictions. Graphically there are two types of models. Above we listed the cases with an example of one type followed by an example of the second type. In the first type, three variables stand in a triangular relationship and one of the three variables has the fourth variable as direct descendant. The four structural errors fall into two uncorrelated pairs. The second pair contains the fourth error that must be intracorrelated with that of its partner in the pair. This causes the latter's structural error to have an effect on the residual and on the parent of the fourth variable, where they collide.. A graphical illustration with the correlated error u_3 and the fourth variable y_4 is this:

The graph of Model $\{[2312] | S_1(4)\}$



In the second type of models the triangular relation is replaced by a direct two-way relation between two of the variables. Again one variable is not a parent, but its residual and its grandparent are influenced by its partner's residual. A graphic illustration with y_4 as fourth variable and u_1 as its partner's residual is this:

The graph of Model $\{[3312] | S_3(4)\}$



c) With $G_1=4$, (A.3) c) is empty.

d). <u>24 Not Locally Identifiable Cases</u> under any 4-cyclical restrictions

A. Under (A.3) d) when i=2, n=l we have the group of cases having $k_1=j_{k_3}$, $j_{k_2}=j_{k_4}$ and therefore $\Delta(4)=0$. In this group are each one of the 12 cases with one variable having 3 descendants i.e.

[2111], [3111], [4111], [2122], [2322], [2422],
[3313], [3323], [3343], [4441], [4442], [4443],

under each 4-cycle of restrictions after suitable permutation.

B. The 12 cases with two variables having two descendants are

[2112], [2121], [3113], [2323], [2442], [3311],
[3322], [3443], [4141], [4411], [4343], [4422].

In all these cases $\Delta(4)=0$, either because the model is equivalent to a model under $S_1(4)$ with (B,Σ) conformably reducible satisfying

$$k_m = j_{k_{m+n}} = j_{k_{m+1+n}}, \quad k_{m+2} = j_{k_{m+2+n}} = j_{k_{m+3+n}}, \quad m\varepsilon(1,2),\ n\varepsilon(1,2),$$

or it satisfies for some value $m \varepsilon (0,1,2,3)$, either

$$k_{1+m} = j_{k_{2+m}} = j_{k_{3+m}}, \quad k_{2+m} = j_{k_{1+m}} = j_{k_{4+m}}, \quad \text{or}$$

$$k_{1+m} = j_{k_{2+m}} = j_{k_{4+m}}, \quad k_{2+m} = j_{k_{1+m}} = j_{k_{3+m}}$$

To see this, under the former conditions with m=0, we have

$$\begin{aligned}\Delta(4) &= (k_2,k_2)(k_1,k_3)(k_2,k_1)(k_1,k_4) - (k_1,k_2)(k_2,k_3)(k_1,k_1)(k_2,k_4) \\ &= (k_2,k_2)(k_1,k_3)(k_1,k_1)(k_2,k_4)(\beta_{k_2k_1}\beta_{k_1k_2}-\beta_{k_1k_2}\beta_{k_2k_1}) = 0,\end{aligned}$$

using the relation $B_{k_i}(B^{-1}\Sigma)^{k_j} \square \sigma_{k_ik_j} \square -(k_i,k_j) + \beta_{k_ij_{k_i}}(j_{k_i},k_j)$.

The problem occurs with the imposition of vanishing covariance restrictions, not if covariances have known values that are not zero.

e). <u>9 Locally Identifiable</u> Cases: Each variable is a parent.

For completeness sake we record the remaining parental lists and cycles of zero correlations under which the parameter is locally identifiable. With each variable having one descendant, these models do not satisfy (A.3) and the parameter is not globally identifiable. These models are:

{[3421], [4312]	$S_1(4)\}$.	$\sigma_{12}=0$, $\sigma_{34}=0$ are testable locally.
{[2341], [4123]	$S_2(4)\}$.	$\sigma_{13}=0$, $\sigma_{24}=0$ are testable locally.
{[2413], [3142]	$S_3(4)\}$.	$\sigma_{14}=0$, $\sigma_{23}=0$ are testable locally.

{[3412], [4321]	$S_1(4)$, $\sigma_{12}\neq 0\}$. $\sigma_{34}=0$ is testable locally.
{[3412], [4321]	$S_1(4)$, $\sigma_{34}\neq 0\}$. $\sigma_{12}=0$ is testable locally.
{[2143], [4321]	$S_2(4)$, $\sigma_{13}\neq 0\}$. $\sigma_{24}=0$ is testable locally.
{[2143], [4321]	$S_2(4)$, $\sigma_{24}\neq 0\}$. $\sigma_{13}=0$ is testable locally.
{[2143], [3412]	$S_3(4)$, $\sigma_{14}\neq 0\}$. $\sigma_{23}=0$ is testable locally.
{[2143], [3412]	$S_3(4)$, $\sigma_{23}\neq 0\}$. $\sigma_{14}=0$ is testable locally.

An important application of Case [3412] is the inverse demand-supply system

$$\begin{pmatrix} -1 & 0 & \beta_{13} & 0 \\ 0 & -1 & 0 & \beta_{24} \\ \beta_{31} & 0 & -1 & 0 \\ 0 & \beta_{42} & 0 & -1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{pmatrix} + \Gamma x = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & \sigma_{34} \\ 0 & 0 & \sigma_{43} & \sigma_{44} \end{pmatrix},$$

with

$$\Delta = (\sigma_{33}\sigma_{44}\sigma_{12}^2 - \sigma_{11}\sigma_{22}\sigma_{34}^2)/|B|^2, \qquad |B| = (1-\beta_{13}\beta_{31})(1-\beta_{24}\beta_{42}),$$

$$\tau_1 = \frac{1-\beta_{13}\beta_{31}}{\beta_{31}}, \quad \tau_2 = \frac{1-\beta_{24}\beta_{42}}{\beta_{42}}, \quad \tau_3 = \frac{1-\beta_{13}\beta_{31}}{\beta_{13}}, \quad \tau_4 = \frac{1-\beta_{24}\beta_{42}}{\beta_{24}}.$$

The parameter is locally identifiable if either $\sigma_{12}\neq 0$ or $\sigma_{34}\neq 0$. It is moment equivalent to one alternative parameter α^* of the system $B^*y+\Gamma^*x=u^*$, with $\beta_{13}^*=1/\beta_{31}$, $\beta_{24}^*=1/\beta_{42}$, $\beta_{31}^*=1/\beta_{13}$, $\beta_{42}^*=1/\beta_{24}$. Under the inequality constraints $\beta_{13}<0$, $\beta_{24}<0$, $\beta_{31}>0$, $\beta_{42}>0$, this alternative parameter (B^*,Γ^*,Σ^*) is not admissible. The parameter α is not locally identifiable if Σ is diagonal. The assumptions on Σ are justified under the theory that inverse demand and supply disturbances are uncorrelated. Below we seek alternative specifications with .identifiable parameter.

7. **Five equation models with identifiable parameter.**

There are sixty connected 4-cyclical quintuplets when G-5, there being fifteen 4-cycles and each can have anyone of four different restrictions from the missing column (row) as fifth element. These are the restriction systems

$R_1(5) \equiv \sigma_{13}=\sigma_{32}=\sigma_{24}=\sigma_{41}=\sigma_{m5}=0,$ $\quad R_{21}(5)\equiv\sigma_{12}=\sigma_{23}=\sigma_{34}=\sigma_{41}=\sigma_{m5}=0,$

$R_5(5) \equiv \sigma_{13}=\sigma_{32}=\sigma_{25}=\sigma_{51}=\sigma_{m4}=0,$ $\quad R_{25}(5)\equiv\sigma_{12}=\sigma_{23}=\sigma_{35}=\sigma_{51}=\sigma_{m4}=0,$

$R_9(5) \equiv \sigma_{13}=\sigma_{35}=\sigma_{54}=\sigma_{41}=\sigma_{m2}=0,$ $\quad R_{29}(5)\equiv\sigma_{12}=\sigma_{25}=\sigma_{54}=\sigma_{41}=\sigma_{m3}=0,$

$R_{13}(5)\equiv\sigma_{15}=\sigma_{52}=\sigma_{24}=\sigma_{41}=\sigma_{m3}=0,$ $\quad R_{33}(5)\equiv\sigma_{15}=\sigma_{53}=\sigma_{34}=\sigma_{41}=\sigma_{m2}=0,$

$R_{17}(5)\equiv\sigma_{53}=\sigma_{32}=\sigma_{24}=\sigma_{45}=\sigma_{m1}=0,$ $\quad R_{37}(5)\equiv\sigma_{52}=\sigma_{23}=\sigma_{34}=\sigma_{45}=\sigma_{m1}=0,$

$R_{41}(5) \equiv \sigma_{12}=\sigma_{24}=\sigma_{43}=\sigma_{31}=\sigma_{m5}=0,$

$R_{45}(5) \equiv \sigma_{12}=\sigma_{24}=\sigma_{45}=\sigma_{51}=\sigma_{m3}=0,$

$R_{49}(5) \square \sigma_{12}=\sigma_{25}=\sigma_{53}=\sigma_{31}=\sigma_{m4}=0,$

$R_{53}(5) \equiv \sigma_{15}=\sigma_{54}=\sigma_{43}=\sigma_{31}=\sigma_{m2}=0,$

$R_{57}(5) \equiv \sigma_{52}=\sigma_{24}=\sigma_{43}=\sigma_{35}=\sigma_{m1}=0,$

where σ_{mk} is a covariance with $m \in \{1,..,5\}$, $m \neq k$.

Any model with identifiable parameter when $G=4$ could be augmented with a parent j_5 and a restriction $\sigma_{m5}=0$, $m\in\{1....4\}$, to constitute a model with identifiable parameter when $G=5$, provided $(B^{-1}\Sigma)_{j_5m}\neq0$, $j_5\in\{1,..,4\}$. This would hold under the local identifiability condition (2).

Examples of families of models $F_a(R_k(5))$ with B nonsingular and Σ positive definite having identifiable parameter under the covariance restrictions $R_k(5)$ and (A.3) a) of Theorem 1 are

$$F_a(R_1(5)) = \begin{pmatrix} [3312\lambda_1]\ [3321\lambda_2]\ [4412\lambda_3]\ [4421\lambda_4] \text{ if } \sigma_{34}\neq0 \\ [3422\lambda_5]\ [4322\lambda_6]\ [3411\lambda_7]\ [4311\lambda_8] \text{ if } \sigma_{12}\neq0 \end{pmatrix},\ (B^{-1}\Sigma)_{\lambda_n 1}\neq0,$$

$$F_a(R_{21}(5)) = \begin{pmatrix} [2123\mu_1]\ [2321\mu_2]\ [4143\mu_3]\ [4341\mu_4] \text{ if } \sigma_{24}\neq0 \\ [2343\mu_5]\ [4323\mu_6]\ [2141\mu_7]\ [4121\mu_8] \text{ if } \sigma_{13}\neq0 \end{pmatrix},\ (B^{-1}\Sigma)_{\mu_n 1}\neq0,$$

$$F_a(R_{41}(5)) = \begin{pmatrix} [2142\upsilon_1]\ [2412\upsilon_2]\ [3143\upsilon_3]\ [3413\upsilon_4] \text{ if } \sigma_{23}\neq0 \\ [2443\upsilon_5]\ [3442\upsilon_6]\ [2113\upsilon_7]\ [3112\upsilon_8] \text{ if } \sigma_{14}\neq0 \end{pmatrix},\ (B^{-1}\Sigma)_{\upsilon_n 1}\neq0$$

$$F_a(R_{60}(5)) = \begin{pmatrix} [o_1 5422]\ [o_2 4522]\ [o_3 5433]\ [o_4 4533] \text{ if } \sigma_{23}\neq0 \\ [o_5 4432]\ [o_6 4423]\ [o_7 5532]\ [o_8 5523] \text{ if } \sigma_{54}\neq0 \end{pmatrix},\ (B^{-1}\Sigma)_{o_n 5}\neq0,$$

where

$$\lambda_n \varepsilon \{1,2,3,4\}, \quad \mu_n \varepsilon \{1,3,2,4\}, \quad \upsilon_n \varepsilon \{1,3,4,2\}, \quad o_n \varepsilon \{5,3,4,2\}$$

and $F_a(R_{21}(5))$ is obtained from $F_a(R_1(5))$ by interchanging rows and columns two and three, where $F_a(R_{41}(5))$ is obtained from $F_a(R_{21}(5))$ by interchanging rows and columns three and four and where $F_a(R_{60}(5))$ is obtained from $F_a(R_{41}(5))$ by interchanging rows and columns one and five.

Examples of families of models $F_b(R_k(5))$ with B nonsingular and Σ positive definite having identifiable parameter under the covariance restrictions $R_k(5)$ and (A.3) b) of Theorem 1 are

$$F_b(R_1(5)) = \begin{pmatrix} [2312\lambda_1] & [2321\lambda_2] & \text{if } \beta_{23}\sigma_{34}\neq 0; & [2421\lambda_3] & [2412\lambda_4] & \text{if } \beta_{24}\sigma_{43}\neq 0 \\ [3121\lambda_5] & [3112\lambda_6] & \text{if } \beta_{13}\sigma_{34}\neq 0; & [4112\lambda_7] & [4121\lambda_8] & \text{if } \beta_{14}\sigma_{43}\neq 0 \\ [3441\lambda_9] & [4341\lambda_{10}] & \text{if } \beta_{41}\sigma_{12}\neq 0; & [4342\lambda_{11}] & [3442\lambda_{12}] & \text{if } \beta_{42}\sigma_{21}\neq 0 \\ [4313\lambda_{13}] & [3413\lambda_{14}] & \text{if } \beta_{31}\sigma_{12}\neq 0; & [3423\lambda_{15}] & [4323\lambda_{16}] & \text{if } \beta_{32}\sigma_{21}\neq 0 \end{pmatrix}$$

$$F_b(R_{21}(5)) = \begin{pmatrix} [3123\mu_1] & [3321\mu_2] & \text{if } \beta_{32}\sigma_{24}\neq 0; & [3341\mu_3] & [3143\mu_4] & \text{if } \beta_{34}\sigma_{42}\neq 0 \\ [2311\mu_5] & [2113\mu_6] & \text{if } \beta_{12}\sigma_{24}\neq 0; & [4113\mu_7] & [4311\mu_8] & \text{if } \beta_{14}\sigma_{42}\neq 0 \\ [2441\mu_9] & [4421\mu_{10}] & \text{if } \beta_{41}\sigma_{13}\neq 0; & [4423\mu_{11}] & [2443\mu_{12}] & \text{if } \beta_{43}\sigma_{31}\neq 0 \\ [4122\mu_{13}] & [2142\mu_{14}] & \text{if } \beta_{21}\sigma_{13}\neq 0; & [2342\mu_{15}] & [4322\mu_{16}] & \text{if } \beta_{23}\sigma_{31}\neq 0 \end{pmatrix}$$

$$F_b(R_{41}(5)) = \begin{pmatrix} [4142\upsilon_1] & [4412\upsilon_2] & \text{if } \beta_{42}\sigma_{23}\neq 0; & [4413\upsilon_3] & [4143\upsilon_4] & \text{if } \beta_{43}\sigma_{32}\neq 0 \\ [2411\upsilon_5] & [2141\upsilon_6] & \text{if } \beta_{12}\sigma_{23}\neq 0; & [3141\upsilon_7] & [3411\upsilon_8] & \text{if } \beta_{13}\sigma_{32}\neq 0 \\ [2313\upsilon_9] & [3312\upsilon_{10}] & \text{if } \beta_{31}\sigma_{14}\neq 0; & [2343\upsilon_{11}] & [3342\upsilon_{12}] & \text{if } \beta_{34}\sigma_{41}\neq 0 \\ [3122\upsilon_{13}] & [2123\upsilon_{14}] & \text{if } \beta_{21}\sigma_{14}\neq 0; & [2423\upsilon_{15}] & [3422\upsilon_{16}] & \text{if } \beta_{24}\sigma_{41}\neq 0 \end{pmatrix}$$

$$F_b(R_{60}(5)) = \begin{pmatrix} [o_1 5424] & [o_2 4524] & \text{if } \beta_{42}\sigma_{23}\neq 0; & [o_3 4534] & [o_4 5434] & \text{if } \beta_{43}\sigma_{32}\neq 0 \\ [o_5 4552] & [o_6 5452] & \text{if } \beta_{52}\sigma_{23}\neq 0; & [o_7 5453] & [o_8 4553] & \text{if } \beta_{53}\sigma_{32}\neq 0 \\ [o_9 3532] & [o_{10} 3523] & \text{if } \beta_{35}\sigma_{54}\neq 0; & [o_{11} 3432] & [o_{12} 3423] & \text{if } \beta_{34}\sigma_{45}\neq 0 \\ [o_{13} 5223] & [o_{14} 5232] & \text{if } \beta_{25}\sigma_{54}\neq 0; & [o_{15} 4232] & [o_{16} 4223] & \text{if } \beta_{24}\sigma_{45}\neq 0 \end{pmatrix}$$

where

$\lambda_k \varepsilon \{1,2,3,4\}$ with $(B^{-1}\Sigma)_{\lambda_k 1} \neq 0$, $\mu_k \varepsilon \{1,3,2,4\}$ with $(B^{-1}\Sigma)_{\mu_k 1} \neq 0$,,

$\upsilon_k \varepsilon \{1,3,4,2\}$ with $(B^{-1}\Sigma)_{\upsilon_k 1} \neq 0$, $o_k \varepsilon \{5,3,4,2\}$ with $(B^{-1}\Sigma)_{o_k 5} \neq 0$, $k=\{1,...,12\}$.

With G=5, Theorem 1 only contains results when the restrictions contain a 4-cycle. Pentagona! restrictions do not imply unique identifiability.

8. Six equation models with identifiable parameter.

When G=6 we could have identifiability under anyone of the 1080 4-cyclical sixtuples. Anyone of the 45 4-cycles can be augmented with a pair of restrictions from 24 different possible pairs, each pair containing one element in each of the two columns not included in the 4-cycle. For example from $S_1(4)$ we can construct the 24 connected sixtuples

$$R_k(6) = \{S_1(4) = \sigma_{\ell 5} = \sigma_{m6} = 0,\ \ell \ \varepsilon \ \{1,..,4\},\ m \ \varepsilon \ \{1,..,5\}\}, \qquad k= 1,..,20,$$
$$R_k(6) = \{S_1(4) = \sigma_{\ell 6} = \sigma_{56} = 0,\ \ell \ \varepsilon \ \{1,..,4\}\ \} \qquad k=21,..,24.$$

The parameter of the model is identifiable under $R_k(6)$, $k=1,..,20$, under the conditions of Theorem 1 on $\{j_1,j_2,j_3,j_4\}$ provided $(B^{-1}\Sigma)_{j_5 \ell} \neq 0$ and $(B^{-1}\Sigma)_{j_6 m} \neq 0$. Letting k=1 when $\ell = m = 1$, examples of families of identifiable parameters are

$$F_a(R_1(6)) = \begin{pmatrix} [3312\lambda_1\mu_1] & [3321\lambda_2\mu_2] & [4412\lambda_3\mu_3] & [4421\lambda_4\mu_4] & \text{if } \sigma_{34}\neq 0 \\ [3422\lambda_5\mu_5] & [4322\lambda_6\mu_6] & [3411\lambda_7\mu_7] & [4311\lambda_8\mu_8] & \text{if } \sigma_{12}\neq 0 \end{pmatrix},$$

$F_b(R_1(6))=$

$$\begin{pmatrix} [2312\upsilon_1 o_1] & [2321\upsilon_2 o_2] & \text{if } \beta_{23}\sigma_{34}\neq 0; & [2421\upsilon_3 o_3] & [2412\upsilon_4 o_4] & \text{if } \beta_{24}\sigma_{43}\neq 0 \\ [3121\upsilon_5 o_5] & [3112\upsilon_6 o_6] & \text{if } \beta_{13}\sigma_{34}\neq 0; & [4112\upsilon_7 o_7] & [4121\upsilon_8 o_8] & \text{if } \beta_{14}\sigma_{43}\neq 0 \end{pmatrix}$$
$$\begin{pmatrix} [3441\upsilon_9 o_9] & [4341\upsilon_{10} o_{10}] & \text{if } \beta_{41}\sigma_{12}\neq 0; & [4342\upsilon_{11} o_{11}] & [3442\upsilon_{12} o_{12}] & \text{if } \beta_{42}\sigma_{21}\neq 0 \\ [4313\upsilon_{13} o_{13}] & [3413\upsilon_{14} o_{14}] & \text{if } \beta_{31}\sigma_{12}\neq 0; & [3423\upsilon_{15} o_{15}] & [4323\upsilon_{16} o_{16}] & \text{if } \beta_{32}\sigma_{21}\neq 0 \end{pmatrix},$$

with

$\lambda_n \ \varepsilon\{1....4\}$, $\mu_n \varepsilon\{1....4\}$, with $(B^{-1}\Sigma)_{\lambda_n 1}(B^{-1}\Sigma)_{\mu_n 1} \neq 0$, $n=1....8$,

$\upsilon_n \ \varepsilon\{1....4\}$, $o_n \varepsilon\{1....4\}$, With $(B^{-1}\Sigma)_{\upsilon_n 1}(B^{-1}\Sigma)_{o_n 1} \neq 0$, $n=1,...,16$.

When G=6 we could have identifiability also with a sexagon of restrictions of which there are 60. With k_i, $i=1,..,6$, all distinct in the set $\{1,..,6\}$, define

$$S_k(6) = \sigma_{k_1k_2} = \sigma_{k_2k_3} = \sigma_{k_3k_4} = \sigma_{k_4k_5} = \sigma_{k_5k_6} = \sigma_{k_6k_1} = 0.$$

From Theorem 1 we have the families of models $[j_{k_1}, j_{k_2}, j_{k_3}, j_{k_4}, j_{k_5}, j_{k_6}]$ with identifiable parameters

$F_a(S_k(6),i) = \{ [k_i, j_{k_2}, k_i, j_{k_4}, k_i, j_{k_6}]$, $(j_{k_2},k_3)(j_{k_4},k_5)(j_{k_6},k_1) \neq (j_{k_2},k_1)(j_{k_4},k_3)(j_{k_6},k_5)$, $(k_i,k_2)(k_i,k_4)(k_i,k_6) \neq 0$, i even$\}$

$F_b(S_k(6),i=2) = \{[k_3, j_{k_2}, j_{k_3}, j_{k_4}, k_3, j_{k_6}]$, $(k_3,k_2)(j_{k_2},k_3)(j_{k_3},k_4)(j_{k_4},k_5)(j_{k_6},k_1) \neq (j_{k_2},k_1)(j_{k_3},k_2)(j_{k_4},k_3)(k_3,k_4)(j_{k_6},k_5)$, $(k_3,k_6) \neq 0$ $\}$

$F_c(S_k(6),i=2,n=6) = \{ [k_6, j_{k_2}, j_{k_3}, j_{k_2}, k_6, j_{k_3}]$, $(k_6,k_2)(j_{k_3},k_4)(j_{k_2},k_5)(j_{k_3},k_1) \neq (j_{k_2},k_1)(j_{k_3},k_2)(k_6,k_4)(j_{k_3},k_5)$, $(j_{k_2},k_3)(k_6,k_6) \neq 0$, $\sigma_{k_2k_6}=\sigma_{k_4k_6}=0\}$

$F_d(S_k(6),i=2,n=1) = \{ [j_{k_1} j_{k_2}, k_1, j_{k_2}, j_{k_1}, j_{k_6}]$, $(j_{k_1},k_2)(k_1,k_4)(j_{k_2},k_5)(j_{k_6},k_1) \neq (j_{k_2},k_1)(k_1,k_2)(j_{k_1},k_4)(j_{k_6},k_5)$, $(j_{k_1},k_6)(j_{k_2},k_3) \neq 0$, $\sigma_{k_4k_1}=0\}$.

For the 6-cycle $S_1(6) \equiv \sigma_{13}=\sigma_{32}=\sigma_{24}=\sigma_{45}=\sigma_{56}=\sigma_{61}=0$, examples with Σ positive definite and nonsingular B with the parents written in the order $[j_1, j_2, j_3, j_4, j_5, j_6]$ are

1) [331231] ε $F_a(S_1(6),2)$, if $\sigma_{12}(\sigma_{25}+\beta_{23}\sigma_{35}) \neq \sigma_{22}(\sigma_{15}+\beta_{13}\sigma_{35})$, $|B|=1-\beta_{13}\beta_{31}$.

2) [231221] ε $F_b(S_k(6),i=2)$ if $\beta_{23}(\beta_{31}\sigma_{14}+\sigma_{34}) \neq 0$, $\sigma_{12}(\sigma_{25}+\beta_{23}\sigma_{35}) \neq \sigma_{22}\sigma_{15}$, $|B|=1-\beta_{12}\beta_{23}\beta_{31}$.

3) [632263] ε $F_c(S_k(6),i=2,n=6)$ if $\beta_{63}\sigma_{21}\sigma_{34}\sigma_{35} \neq 0$, $|B|=1-\beta_{23}\beta_{32}$, $\sigma_{36}=\sigma_{46}=0$.

4) [315531] ε $F_d(S_k(6), i=2, n=1)$ if $\sigma_{52}\sigma_{36}\beta_{13}\sigma_{34}(\sigma_{33}+\beta_{35}\sigma_{53})\neq 0$,
$\sigma_{11}\sigma_{55}-\sigma_{15}\sigma_{51}\neq\sigma_{35}(\sigma_{51}\beta_{13}-\sigma_{11}\beta_{53})$, $|B|=1-\sigma_{35}\sigma_{53}$, $\sigma_{14}=0$.

The above inequalities are the conditions that $\Delta(6)\neq 0$ and correspond to the sufficient rank conditions (2) for local identifiability.

For general models their satisfaction almost everywhere can be checked by calculating $B^{-1}\Sigma$ for some numerically chosen matrices (B,Σ) that satisfy the equality restrictions. This is due to the fortunate fact that if all the coefficients in the system (1) are different from zero, the reciprocals $(1/\tau_{k_i})$ satisfy a linear system. Under (A.3) and $\Delta(G_1)\neq 0$ of Theorem 1, $1/\tau_{k_\ell}=0$ and under (A.3) b) $1/\tau_{k_{i+1}} = -1/\beta_{k_{i+1}j_{k_{i+1}}}$. The nonexistence of a solution for the system (1) translates into a zero value for some of the reciprocals.

If some coefficients in (1) are zero, a more careful procedure with subsets of equations is needed.

All null-hypotheses under which the inequality $\Delta(6)\neq 0$ continues to hold are testable. But by imposition of additional over-identifying restrictions, especially covariance restrictions that make Σ into a diagonal matrix, we could lose identifiability.

For G=7, Theorem 1 describes models with identifiable parameter either under 4-cyclical or 6-cyclical septuples. When G=8 we also have cases identifiable under 8-cyclical or two disjoint 4-cyclical octuples.

9. Extensions to Models having External Variables.

If the structural model is the standard $By+\Gamma x=u$, $u \approx i.i.\mu(0,\Sigma)$, with $\Pi=(\pi_{ij})=-B^{-1}\Gamma$, $\Omega=(\omega_{ij})=B^{-1}\Sigma B'^{-1}$, the results of this paper hold if there are $G-1$ restrictions on each row $(B,\Gamma)_\ell$, $\ell=1,..G$. If $n(\ell)+1$ elements of B_ℓ are unknown, $n(\ell)$ elements in Γ_ℓ are zero, $0 \leq n(\ell) \leq G-2$, where $\beta_{\ell\ell}=-1$. We need some new notation.

Let $(\beta_{\ell j_\ell}, \beta_{\ell j_{\ell 1}}, .. ,\beta_{\ell j_{\ell n(\ell)}})$ be the unknown elements in B_ℓ standing in the columns $(j_{\ell 0},j_{\ell 1},...,j_{\ell n(\ell)})$ and let $(\gamma_{\ell m_{\ell 1}},...,\gamma_{\ell m_{\ell n(\ell)}})=0$ be the zero elements in Γ_ℓ standing in the columns $(m_{\ell 1},..,m_{\ell n(\ell)})$. The parameter $(TB,T\Gamma,T\Sigma T')$ is equivalent to (B,Γ,Σ) under a cycle of covariance restrictions if and only if

$$T = I_G + PB^{-1}, \text{ where } P_\ell=(0 \; .. \; \tau_{\ell 0} \; .. \; 0 \; .. \; \tau_{\ell 1} \; ..0 \; .. \; \tau_{\ell n(\ell)}) \; .. \; 0),$$

with the unknowns $(\tau_{\ell 0},\tau_{\ell 1},...,\tau_{\ell n(\ell)})$ in row P_ℓ standing in its columns $(j_{\ell 0},j_{\ell 1}, .. ,j_{\ell n(\ell)})$ and satisfying

$$0=(T\Gamma)_{\ell m_{\ell j}} = (P\Pi)_{\ell m_{\ell j}}, \qquad j=1,..,n(\ell), \tag{4}$$

$$0=(T\Sigma T')_{k_i k_{i+1}}=(PB^{-1}\Sigma)_{k_i k_{i+1}}+(PB^{-1}\Sigma)_{k_{i+1} k_i}+(P\Omega P')_{k_i k_{i+1}}, \quad i=1,..,G. \tag{5}$$

With $c_{j_{\ell k}}$, $k=0,..,n(\ell)$, the signed determinant of order $n(\ell)$ in

$$R_\ell \equiv \begin{pmatrix} \pi_{j_{\ell 0} m_{\ell 1}} & .. & \pi_{j_{\ell 0} m_{\ell n(\ell)}} \\ \pi_{j_{\ell 1} m_{\ell 1}} & .. & \pi_{j_{\ell 1} m_{\ell n(\ell)}} \\ .. & .. & .. \\ \pi_{j_{\ell n(\ell)} m_{\ell 1}} & .. & \pi_{j_{\ell n(\ell)} m_{\ell n(\ell)}} \end{pmatrix} \qquad n(\ell) \geq 1,$$

after deleting its row $(\pi_{j_{\ell k} m_{\ell 1}}, .. ,\pi_{j_{\ell k} m_{\ell n(\ell)}})$, the equations (4) imply

$$(\tau_{\ell 0}, \tau_{\ell 1}, .. ,\tau_{\ell n(\ell)}) = \tau_\ell\, (c_{j_{\ell 0}}, c_{j_{\ell 1}}, .. ,c_{j_{\ell n(\ell)}}). \tag{6}$$

If there are no restrictions on Γ_ℓ, put $c_{j_{n0}}$-1. Substituting into (5) we have $(TB,T\Gamma,T\Sigma T')$ is equivalent to (B,Γ,Σ) under a G-cycle of covariance restrictions if and only **if** $\tau=(\tau_1,\tau_2,..,\tau_G)$ satisfies the system

$$0 = \bar{a}_i\, \tau_{k_i} + \bar{b}_i\, \tau_{k_{i+1}} + \bar{d}_i\, \tau_{k_i}\tau_{k_{i+1}}, \quad i=1,..,G, \tag{7}$$

where $\bar{a}_i \equiv (CB^{-1}\Sigma)_{k_i k_{i+1}}$, $\bar{b}_i \equiv (CB^{-1}\Sigma)_{k_{i+1} k_i}$, $\bar{d}_i \equiv (C\Omega C')_{k_i k_{i+1}}$ and

$$C_\ell = (\ ..\ 0\ ..\ c_{j_{\ell 0}}\ ..\ 0\ ..\ c_{j_{\ell 1}}\ ..\ 0\ ..\ c_{j_{\ell n(\ell)}}\ ..\ 0\ ..)\ .$$

Again (7) is a system of G bilinear equations in G unknowns and if its solution $\tau=0$ is unique, the parameter (B,Γ,Σ) is globally identifiable.

Special Cases

A. As in the proof of Theorem 1, the G bilinear equations (7), with G even, can be reduced to G/2 linear equations if

$$(CB^{-1}\Sigma)_{k_{i-1}k_i}(C\Omega C')_{k_i k_{i+1}} = (CB^{-1}\Sigma)_{k_{i+1}k_i}(C\Omega C')_{k_{i-1}k_i}, \text{ for } i \text{ even.} \tag{8}$$

This condition holds if $C_{k_{i-1}} = C_{k_{i+1}}$ i.e. if $j_{k_{i-1}\ell} = j_{k_{i+1}\ell}$, for $\ell =$ 0,1, .. , $n(k_{i-1}) = n(k_{i+1})$ and $R_{k_{i-1}} = R_{k_{i+1}}$. This is the case **if** the rows k_{i-1} and k_{i+1} have unknown elements in the same columns of B and zero elements in the same columns of Γ.

When G=4 there is only the condition that $C_{k_1} = C_{k_3}$. Any model in $F_a(S_1(4))$ can be modified into a model satisfying this condition. For example $\{[3312] \mid S_1(4), \sigma_{34}\neq 0\} \in F_a(S_1(4))$ can be modified into the model

$$(B,\Gamma,\Sigma) = \begin{pmatrix} -1 & 0 & \beta_{13} & \beta_{14} & 0 & \sigma_{11} & \sigma_{12} & 0 & 0 \\ 0 & -1 & \beta_{23} & \beta_{24} & 0 & \sigma_{21} & \sigma_{22} & 0 & 0 \\ \beta_{31} & 0 & -1 & 0 & \gamma_{31} & 0 & 0 & \sigma_{33} & \sigma_{34} \\ 0 & \beta_{42} & 0 & -1 & \gamma_{41} & 0 & 0 & \sigma_{43} & \sigma_{44} \end{pmatrix}, \qquad (9)$$

in which $(j_{10},j_{11})=(j_{20},j_{21})=(3,4)$, $n(1)=n(2)=1$, $\begin{pmatrix} m_{11} \\ m_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $R_1=R_2=\begin{pmatrix} \pi_{31} \\ \pi_{41} \end{pmatrix}$. Its parameter is identifiable under local identifiability conditions (2)

$$(\gamma_{31}\sigma_{44}-\gamma_{41}\sigma_{43})(-\gamma_{31}\sigma_{34}+\gamma_{41}\sigma_{33}) \neq 0.$$

Other examples can be constructed from a member of $F_a(S_1(4))$ by leaving its rows k_1 and k_3 unchanged and by replacing the exclusion restrictions on B_{k_2} and B_{k_4} by exclusion restrictions on Γ_{k_2} and Γ_{k_4}.

B. If a model in $F_b(S_1(4))$ satisfying $j_{k_1}=k_3$ is modified by replacing the exclusion restrictions in B_4 (or B_2) by exclusion restrictions in Γ_4 (or Γ_2), the new model has an identifiable parameter. For example the model

$$(B,\Gamma) = \begin{pmatrix} -1 & \beta_{12} & 0 & 0 & \gamma_{11} & \gamma_{12} \\ 0 & -1 & \beta_{23} & 0 & \gamma_{21} & \gamma_{22} \\ \beta_{31} & 0 & -1 & 0 & \gamma_{31} & \gamma_{32} \\ \beta_{41} & \beta_{42} & \beta_{43} & -1 & 0 & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & \sigma_{34} \\ 0 & 0 & \sigma_{43} & \sigma_{44} \end{pmatrix} \qquad (10)$$

is a modification of the model $\{[2312] \mid S_1(4), \beta_{23}\sigma_{34}\neq 0\} \in F_b(S_1(4))$. To see this when $G=4$, reduce the four equations (7) to two equations and then to the single equation

$$\left(\prod_{\ell=1}^{4}\bar{a}_\ell - \prod_{\ell=1}^{4}\bar{b}_\ell\right)\tau_{k_1} = -\bar{a}_1\bar{a}_2\,(\bar{a}_3\bar{d}_4 - \bar{b}_4\bar{d}_3) - \bar{b}_3\bar{b}_4\,(\bar{a}_1\bar{d}_2 - \bar{b}_2\bar{d}_1)(\tau_{k_1})^2 \quad (11)$$

With $j_{k_1}=k_3$, $B_{k_3}(B^{-1}\Sigma)^{k_i}=0$, $i=2,4$, $B_{k_3}(\Omega C')^{k_4}=\bar{b}_3$ imply $\bar{a}_3\bar{d}_4-\bar{b}_4\bar{d}_3 = -\bar{a}_3\bar{b}_3$, $\bar{a}_1\bar{d}_2 - \bar{b}_2\bar{d}_1 = \bar{b}_2\bar{a}_2$ and the coefficient of the quadratic term is

$$\overline{a_2}\overline{b_3}\,(\,-\overline{a_1}\overline{a_3} + \overline{b_2}\overline{b_4}\,) = \overline{a_2}\overline{b_3}\begin{pmatrix} (B^{-1}\Sigma)_{jk_1k_2} & (B^{-1}\Sigma)_{jk_1k_4} \\ (B^{-1}\Sigma)_{jk_3k_2} & (B^{-1}\Sigma)_{jk_3k_4} \end{pmatrix} = 0$$

and (10) has an identifiable parameter under the local identifiability conditions (2) i.e. when

$$\beta_{23}\sigma_{34}\,\left(\begin{vmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{31} & \gamma_{32} \end{vmatrix} + \begin{vmatrix} \gamma_{21} & \gamma_{22} \\ \gamma_{31} & \gamma_{32} \end{vmatrix}\right) \neq 0.$$

Similar extensions of members in $F_b(S_1(4)$ stated above result in models with identifiable parameter. By special calculation, Mallela et. al. (1993) established the identifiability of the model

$$(B,\,\Gamma) = \begin{pmatrix} -1 & 0 & 0 & \beta_{14} & \gamma_{11} & \gamma_{12} \\ \beta_{21} & -1 & 0 & 0 & \gamma_{21} & \gamma_{22} \\ \beta_{31} & 0 & -1 & 0 & \gamma_{31} & \gamma_{32} \\ \beta_{41} & \beta_{42} & \beta_{43} & -1 & 0 & 0 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & \sigma_{34} \\ 0 & 0 & \sigma_{43} & \sigma_{44} \end{pmatrix} \qquad (11)$$

This is an extension of the model $\{[41121|S_1(4),\beta_{14}\sigma_{43}\neq 0\} \in F_b(S_1(4))\}$ with $j_{k_3} = k_1$ and therefore it has an identifiable parameter under its local identifiability conditions (2) that are

$$\beta_{14}(\sigma_{44}+\beta_{43}\sigma_{34})(\sigma_{43}+\beta_{43}\sigma_{33})\,\left(\begin{vmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{31} & \gamma_{32} \end{vmatrix} + \beta_{14}\beta_{42}\begin{vmatrix} \gamma_{21} & \gamma_{22} \\ \gamma_{31} & \gamma_{32} \end{vmatrix}\right) \neq 0.$$

10. Concluding Remarks.

Unique identifiability under- covariance restrictions is generally understood to be a property of recursive models or models where there is at least one equation identifiable without covariance restrictions. The parameters of the other equations can be further restricted by the zero correlations with residuals that are identifiable already. Without such first equation covariance restrictions in general imply local identifiability only and inequality restrictions must be relied upon for uniqueness.

Taking inspiration from the special model (11) analyzed by Mallela et.al. (1993), we have characterized four families of models with parameter identifiable under a subset of uncorrelated residuals and without having to invoke inequality restrictions. In these families, not counting the zero correlation, there is one coefficient restriction missing per equation so that riot one single equation is identifiable without the covariance restrictions. Further for all members of the family the string of zero correlations is a closed cycle of order at least four and the odd-numbered variables are siblings. For these families Theorem 1 states that the parameter is globally identifiable if it is locally identifiable.

We derived the complete membership list of four equation models with the above properties and we described by example how models with five and six equations are to be recognized as members of the above families. Since the analysis can be applied immediately to models that contain external variables, the results of this paper are important in rnodeling price and quantity variables under first and second moment reproducing inverse demand and supply equations with intercorrelated inverse demand shocks and intercorrelated supply shocks that are uncorrelated with the demand shocks. Models (9) and (10) provide an illustration.

Appendix.

1. Total Number of G_1-cyclical G-tuples *in* $G\times G$ Σ .

A. The number of G_1-cycles is $G!/2G_1(G-G_1)!$.

To see this, a G_1-cycle in $G\times G$ Σ has two elements in a row or column. The G_1-cycle with vertices $(\sigma_{1i},\sigma_{j1})$ contains the vertices

$$\sigma_{1i} = \sigma_{im_3} = .. = \sigma_{m_{k-1}j} = \sigma_{j1}, \quad m_1=m_{k+1}=1,\ m_2=i,\ m_k=j.$$

Given the subscripts 1,i,j, the remaining subscripts are G_1-3 integers out of G-3 possible, which implies the pair $(\sigma_{1i},\sigma_{1j})$, $1<i<j\leq G$, can be the side of $(G-3)(G-4)..[G-(G_1-1)]$ different G_1-cycles. $G_1>3$. When $i=2$ there are G-2 different values for j, when $i=3$ there are G-3 different values for j and for $i=G-1$, $j=G$ is the only value of j. Therefore the first row pairs $(\sigma_{1i},\sigma_{1j})$, $1<i<j$, are a side of

$$(G-3)(G-4)..[G-(G_1-1)][(G-2)+(G-3)+..+1]=\frac{1}{2}(G-1)(G-2)(G-3)..[G-(G_1-1)]$$

different G_1-cycles. When $G_1=3$, this is $(G-1)\#$.

Next, the pair $(\sigma_{2i},\sigma_{2j})$, $2<i<j$, $j\leq G$, can be combined with G_1-3 integers out of G-4 to form $(G-4)(G-5)..(G-G_1)$ different polygons, also different from the polygons constructed before. When $i=3$ there are G-3 different values for j and when $i=G-1$ there is only one value for j. Therefore the second row pairs $(\sigma_{2i},\sigma_{2j})$, $2<i<j$, are a side of

$$(G-4)(G-5)..(G-G_1)\ [(G-3)+(G-4)+..+1] = \frac{1}{2}(G-2)(G-3)(G-4)..(G-G_1)$$

different G_1-cycles.

in this sequence the last row pairs are $(\sigma_{G-(G_1-1)\cdot i},\ \sigma_{G-(G_1-1)\cdot j})$, $G-(G_1-1) < i<j$, $j\leq G$, which are a side of $\frac{1}{2}(G_1-1)(G_1-2)..(1)$ different cycles. Therefore, adding over the sequence, the number of different G_1-cycles in $G\times G$ Σ is

$$\frac{1}{2}\sum_{m=G_1}^{G}(m-1)(m-2)..[m-(G_1-1)] = \frac{1}{2G_1}G(G-1)(G-2)..[G-(G_1-1)].$$

B. The number of connected G-tuples per G_1-cycle is $G_1 G^{G-(G_1+1)}$.

Each polygon of order $G_1<G$ can be combined with $\binom{G\#-G_1}{G-G_1}$ different choices of G-k restrictions to constitute assignable and not assignable sets of G restrictions.

Clearly the number of G_1-laterals per polygon of order G_1 is one. Renumbering the subscripts so that the vertices of the polygon stand in the first G_1 rows and columns, let the polygon of order G_1 be denoted

$$p_{G_1} \equiv \sigma_{12} = \sigma_{23} = .. = \sigma_{G_1 1} .$$

Connected G-tuples linked to the G_1-cycle number

i) G_1 in the form

$p_{G_1} = \sigma_{G_1+1 \cdot n_{G_1+1}} = 0$, $n_{G_1+1} \, \varepsilon \, \{1,2,..,G_1\}$, when $G=G_1+1$,

ii) $G_1(G_1+1)$ and G_1 repectively in the form

$p_{G_1} = \sigma_{G_1+1 \cdot n_{G_1+1}} = \sigma_{G_1+2 \cdot n_{G_1+2}} = 0$, $n_{G_1+2} \, \varepsilon \, \{1,2,..,G_1,G_1+1\}$, or

$p_{G_1} = \sigma_{G_1+2 \cdot n_{G_1+1}} = \sigma_{G_1+1 \cdot G_1+2} = 0$, when $G=G_1+2$,

iii) $G_1(G_1+1)(G_1+2)$, $G_1(G_1+1)$, $G_1(G_1+2)$, $G_1(G_1+1)$, G_1, G_1, G_1 respectively in the form

$p_{G_1} = \sigma_{G_1+1 \cdot n_{G_1+1}} = \sigma_{G_1+2 \cdot n_{G_1+2}} = \sigma_{G_1+3 \cdot n_{G_1+3}} = 0$,

$n_{G_1+3} \varepsilon \{1,2,..,G_1,G_1+1,G_1+2\}$,

or

$p_{G_1} = \sigma_{G_1+1 \cdot n_{G_1+1}} = \sigma_{G_1+3 \cdot n_{G_1+2}} = \sigma_{G_1+2 \cdot G_1+3} = 0$, or

$p_{G_1} = \sigma_{G_1+2 \cdot n_{G_1+1}} = \sigma_{G_1+1 \cdot G_1+2} = \sigma_{G_1+3 \cdot n_{G_1+3}} = 0$, or

$$p_{G_1}=\sigma_{G_1+2\cdot n_{G_1+1}}=\sigma_{G_1+3\cdot n_{G_1+2}}=\sigma_{G_1+1\cdot G_1+3}=0, \text{ or}$$

$$p_{G_1}=\sigma_{G_1+3\cdot n_{G_1+1}}=\sigma_{G_1+1\cdot G_1+3}=\sigma_{G_1+2\cdot G_1+1}=0, \text{ or}$$

$$p_{G_1}=\sigma_{G_1+3\cdot n_{G_1+1}}=\sigma_{G_1+1\cdot G_1+3}=\sigma_{G_1+2\cdot G_1+3}=0, \text{ or}$$

$$p_{G_1}=\sigma_{G_1+3\cdot n_{G_1+1}}=\sigma_{G_1+2\cdot G_1+3}=\sigma_{G_1+1\cdot G_1+2}=0, \qquad \text{for } G=G_1+3.$$

Adding all forms shows the result.

2. *Proof of Lemma 1* .

We proceed in steps. With a single covariance restriction $(T\Sigma T')_{k_1k_2}=0$ we have $\tau_{k_1}\neq 0$, $\tau_{k_2}=0$ only if $(j_{k_1},k_2)=0$. Also $\tau_{k_1}=0$, $\tau_{k_2}\neq 0$ only if $(j_{k_2},k_1)=0$, or the parent of the k_2th variable is exogenous in the k_1th equation. Therefore, we have

$$(T\Sigma T')_{k_1k_2}=0,\ (j_{k_1},k_2)\neq 0,\ \Rightarrow\ \text{" not } (\tau_{k_1}\neq 0,\ \tau_{k_2}=0)\text{ "}$$
$$(T\Sigma T')_{k_1k_2}=0,\ (j_{k_2},k_1)\neq 0,\ \Rightarrow\ \text{" not } (\tau_{k_1}=0,\ \tau_{k_2}\neq 0)\text{ "}$$

and the combined conclusion that:

$$(j_{k_1},k_2)(j_{k_2},k_1)\neq 0,\ \ (T\Sigma T')_{k_1k_2}=0 \text{ and } (\tau_{k_1},\tau_{k_2})\neq 0 \Rightarrow \tau_{k_1}\tau_{k_2}\neq 0. \quad (2.1)$$

If two connected covariance restrictions $\sigma_{k_1k_2}=\sigma_{k_2k_3}=0$ are imposed. the two equations $(T\Sigma T')_{k_1k_2}=(T\Sigma T')_{k_2k_3}=0$ are

$$0=(j_{k_1},k_2)\ \tau_{k_1}+(j_{k_2},k_1)\ \tau_{k_2}+\tau_{k_1}\tau_{k_2}\ \omega_{j_{k_1}j_{k_2}},$$

$$0=(j_{k_3},k_2)\ \tau_{k_3}+(j_{k_2},k_3)\ \tau_{k_2}+\tau_{k_3}\tau_{k_2}\ \omega_{j_{k_3}j_{k_2}}.$$

By the argument applied to $(T\Sigma T')_{k_1k_2}=0$ leading to (2.1), conclude

$$(j_{k_1},k_2)(j_{k_2},k_3)(j_{k_3},k_2)(j_{k_2},k_1)\neq 0,\ (T\Sigma T')_{k_1k_2}=(T\Sigma T')_{k_2k_3}=0 \text{ and}$$
$$(\tau_{k_1},\tau_{k_2},\tau_{k_3})\neq 0\ \Rightarrow\ \tau_{k_1}\tau_{k_2}\tau_{k_3}\neq 0.$$

With three cyclical covariance restrictions $\sigma_{k_1k_2}=\sigma_{k_2k_3}=\sigma_{k_3k_1}=0$ we have the implications

$(j_{k_1},k_2)\neq0 \Rightarrow$ "not $(\tau_{k_1}\neq0,\tau_{k_2}=0)$", $(j_{k_2},k_1)\neq0 \Rightarrow$ "not $(\tau_{k_1}=0,\tau_{k_2}\neq0)$",
$(j_{k_2},k_3)\neq0 \Rightarrow$ "not $(\tau_{k_2}\neq0,\tau_{k_3}=0)$", $(j_{k_3},k_2)\neq0 \Rightarrow$ "not $(\tau_{k_2}=0,\tau_{k_3}\neq0)$",
$(j_{k_3},k_1)\neq0 \Rightarrow$ "not $(\tau_{k_3}\neq0,\tau_{k_1}=0)$", $(j_{k_1},k_3)\neq0 \Rightarrow$ "not $(\tau_{k_3}=0,\tau_{k_1}\neq0)$".

Therefore,

$(T\Sigma T')_{k_1k_2}=(T\Sigma T')_{k_2k_3}=(T\Sigma T')_{k_3k_1}=0$, $(\tau_{k_1},\tau_{k_2},\tau_{k_3})\neq0$ and *either* $(j_{k_1},k_2)(j_{k_2},k_3)(j_{k_3},k_1)\neq0$ or $(j_{k_2},k_1)(j_{k_3},k_2)(j_{k_1},k_3)\neq0 \Rightarrow \tau_{k_1}\tau_{k_2}\tau_{k_3}\neq0$.

To see this, **if** $(\tau_{k_1},\tau_{k_2},\tau_{k_3})\neq0$ with $\tau_{k_1}\neq0$, then $(j_{k_1},k_2)\neq0$ implies $\tau_{k_2}\neq0$ and then $(j_{k_2},k_3)\neq0$ implies $\tau_{k_3}\neq0$. If $(\tau_{k_1},\tau_{k_2},\tau_{k_3})\neq0$ with $\tau_{k_2}\neq0$, then $(j_{k_2},k_3)\neq0$ implies $\tau_{k_3}\neq0$ and then $(j_{k_3},k_1)\neq0$ implies $\tau_{k_1}\neq0$. Finally, if $(\tau_{k_1},\tau_{k_2},\tau_{k_3})\neq0$ because $\tau_{k_3}\neq0$, then $(j_{k_3},k_1)\neq0$ implies $\tau_{k_1}\neq0$ and then $(j_{k_1},k_2)\neq0$ implies $\tau_{k_2}\neq0$. We could also use the implications from the second column above to prove the or part of the statement.

The same argument goes through for general G_1-cycles.

3. *Relations between elements of* B, Σ, $B^{-1}\Sigma$ and Ω.

The equations (1) to be solved contain elements of $B^{-1}\Sigma$ and of Ω. Observe the following properties :

(P1) $B_m\Omega^{j_n} = (j_n,m) = -\omega_{mj_n} + \beta_{mj_m}\,\omega_{j_mj_n}$.

(P2) $\sigma_{mn} = B_m(B^{-1}\Sigma)^n = -(m,n) + \beta_{mj_m}(j_m,n)$

$\sigma_{nm} = B_n(B^{-1}\Sigma)^m = -(n,m) + \beta_{nj_n}(j_n,m)$

$\sigma_{mn}=0$ implies $(m,n)\,\omega_{j_mj_n} - (j_m,n)\,\omega_{mj_n} = (j_m,n)\,(j_n,m)$.

(P3) $\sigma_{mn}=0$, $(m,n)\neq0$, $\beta_{mj_m}\neq0$ imply

$(j_n,m)\,(-1/\beta_{mj_m}) + (j_m,n)\,(-\omega_{mj_n}/(m,n)) = -\,\omega_{j_mj_n}$

(P4) $\sigma_{mn_1}=\sigma_{mn_2}=0$, imply $\beta_{mj_m}((j_m,n_1)\ \ (j_m,n_2)) = ((m,n_1)\ \ (m,n_2))$.

$\sigma_{nm_1}=\sigma_{nm_2}=0$, imply $\beta_{nj_n}((j_n,m_1)\ \ (j_n,m_2)) = ((n,m_1)\ \ (n,m_2))$.

(P5) $\sigma_{n_1m_1}=\sigma_{n_1m_2}=\sigma_{n_2m_1}=\sigma_{n_2m_2}=0$, $\beta_{n_1j_{n_1}}\neq 0$, $\beta_{n_2j_{n_2}}\neq 0$ imply

$$\rho\begin{pmatrix}(j_{n_1},m_1) & (j_{n_1},m_2)\\ (j_{n_2},m_1) & (j_{n_2},m_2)\end{pmatrix} = \rho\begin{pmatrix}(n_1,m_1) & (n_1,m_2)\\ (n_2,m_1) & (n_2,m_2)\end{pmatrix}.$$

4. *Proof of Theorem 1* .

We have to show that the solution $\tau(G_1)\equiv(\tau_{k_1},\tau_{k_2},\ldots,\tau_{k_{G_1}})=0$ of the system $(T\Sigma T')_{k_ik_{i+1}}=0$, $i=1,\ldots,G_1$, is unique under the stated conditions. This is the system of G_1 equations

$$a_i\,\tau_{k_i} + b_i\,\tau_{k_{i+1}} = -\tau_{k_i}\tau_{k_{i+1}}\omega_{j_{k_i}j_{k_{i+1}}}, \qquad i=1,\ldots,G_1, \tag{4.1}$$

Observe that the Jacobian matrix of these equations at $\tau(G_1)=0$ has determinant $\Delta(G_1)=z_1(G_1)-z_2(G_1)$. By assumption, either $a_i\neq 0$, $i=1,\ldots,G_1$, or $b_i\neq 0$, 1 $,\ldots,G_1$. Therefore from Lemma 1, $\tau(G_1)\neq 0$ implies each component of $\tau(G_1)$ is different from zero.

Eliminating $\tau_{k_i}\neq 0$ from the equations corresponding to $(T\Sigma T')_{k_{i-1}k_i} = (T\Sigma T')_{k_ik_{i+1}} = 0$, we have the $G_1/2$ equations

$$[\,a_{i-1}\,\omega_{j_{k_i}j_{k_{i+1}}} - b_i\,\omega_{j_{k_{i-1}}j_{k_i}}\,]\,\tau_{k_{i-1}}\tau_{k_{i+1}} + a_{i-1}a_i\,\tau_{k_{i-1}} - b_{i-1}b_i\tau_{k_{i+1}}=0, \quad i=2,4,\ldots,G_1. \tag{4.2,i}$$

From (P2), Appendix 3. we have

$$\sigma_{k_n k_i} = 0 \rightarrow (k_n,k_i)\ \omega_{j_{k_i} j_{k_n}} - (j_{k_n},k_i)\ \omega_{k_n j_{k_i}} - (j_{k_n},k_i)\ (j_{k_i},k_n) = 0. \quad (4.3)$$

Hence, when $\sigma_{k_i k_n}=0$ adding (4.3) to (4.2,i), the latter is equivalent to

$$\{ [a_{i-1}\omega_{j_{k_i} j_{k_{i+1}}} - (j_{k_n},k_i)\omega_{k_n j_{k_i}}] - [b_i\omega_{j_{k_{i-1}} j_{k_i}} - (k_n,k_i)\omega_{j_{k_i} j_{k_n}}]$$

$$-(j_{k_i},k_n)\ (j_{k_n},k_i)\}\ \tau_{k_{i-1}}\tau_{k_{i+1}} + a_{i-1}a_i\ \tau_{k_{i-1}} - b_{i-1}b_i\ \tau_{k_{i+1}} = 0, \quad (4.4,i,n)$$

or subtracting (4.3) from (4.2,i), the latter is also equivalent to

$$\{ [a_{i-1}\ \omega_{j_{k_i} j_{k_{i+1}}} - (k_n,k_i)\omega_{j_{k_i} j_{k_n}}] - [b_i\ \omega_{j_{k_{i-1}} j_{k_i}} - (j_{k_n},k_i)\omega_{k_n j_{k_i}}]$$

$$+ (j_{k_i},k_n)(j_{k_n},k_i)\ \}\ \tau_{k_{i-1}}\tau_{k_{i+1}} + a_{i-1}a_i\ \tau_{k_{i-1}} - b_{i-1}b_i\ \tau_{k_{i+1}} = 0. \quad (4.5,i,n)$$

The first quadratic term does not depend on elements of Ω if

1. k_{i-1} and k_{i+1} are siblings i.e.

   $j_{k_{i-1}} = j_{k_{i+1}}$ and then from (4.2,i)

$$a_{i-1}a_i\ \tau_{k_{i-1}} - b_{i-1}b_i\ \tau_{k_{i+1}} = 0, \quad (4.6,i)$$

2. k_{i-1} or a sibling of k_{i-1} (for $n \neq i-1$) is the parent of k_{i+1}:

   $j_{k_{i-1}} = j_{k_n}$, $k_n = j_{k_{i+1}}$ and then from (4.4,i,n) provided $\sigma_{k_i k_n}=0$,

$$a_{i-1}a_i\tau_{k_{i-1}} - b_{i-1}b_i\tau_{k_{i+1}} = c_{in}\tau_{k_{i-1}}\tau_{k_{i+1}}, \quad c_{in} \equiv a_{i-1}(j_{k_i},k_n), \quad (4.7,i,n)$$

3. k_{i+1} or a sibling of k_{i+1} (for $n \neq i+1$) is the parent of k_{i-1}:

   $j_{k_{i+1}} = j_{k_n}$, $k_n = j_{k_{i-1}}$ and then from (4.5,i,n) provided $\sigma_{k_i k_n} = 0$,

$$a_{i-1}a_i\tau_{k_{i-1}} - b_{i-1}b_i\tau_{k_{i+1}} = -d_{in}\tau_{k_{i-1}}\tau_{k_{i+1}}, \quad d_{in} \equiv b_i(j_{k_i},k_n). \tag{4.8,i,n}$$

The models with possible unique solution $\tau(G_1)=0$ must be those for which the bilinear system consisting of $G_1/2$ equations and unknowns from the three equations above can be reduced further to a linear system of fewer equations and unknowns. These models are:

a) $[m,j_{k_2},m,j_{k_4},...,m,j_{k_{G_1}}]$ which has $j_{k_{i-1}}=j_{k_{i+1}}=m$, with the parent m any one of the variables $(k_2,k_4,...,k_{G_1})$. The coefficient of $\tau k_{i-1}\tau k_{i+1}$ is zero and (4.6,i)) is a linear equation in $\tau_{k_{i-1}}$ and $\tau_{k_{i+1}}$. When the variables $(k_1,k_3,...,k_{G_1-1})$ are all siblings, the equations form a linear homogeneous system in $\tau_{k_1},\tau_{k_3},...,\tau_{k_{G-1}}$, with matrix having $z_1(G_1)$ - $z_2(G_1)$ as determinant. The solution $\tau(G_1)=0$ is unique if and only if this is different from zero.

b) The adjacent equations (4.8,i,n) and (4.7,i+2,n) correspond to the model assumption that k_{i+1} and k_n are siblings with k_n the parent of k_{i-1} and k_{i+3}, not excluding the possibility that $k_n=k_{i+1}$, and provided $\sigma_{k_ik_n}=0$. Eliminating $\tau_{k_{i+1}}$ from these two equations, we get the equation

$$a_{i-1}a_ia_{i+1}a_{i+2}\tau_{k_{i-1}} - b_{i-1}b_ib_{i+1}b_{i+2}\tau_{k_{i+3}} = d_1\ \tau_{k_{i-1}}\tau_{k_{i+3}}, \tag{4.9,i}$$

with

$$d_1 \equiv a_{i-1}a_ic_{i+2,n} - b_{i+1}b_{i+2}d_{in} = a_{i-1}a_ia_{i+1}(j_{k_{i+2}},k_n) - b_{i+1}b_{i+2}b_i(j_{k_i},k_n),$$

$$= (j_{k_{i-1}},k_i)(j_{k_i},k_{i+1})(j_{k_{i+1}},k_{i+2})(j_{k_{i+2}},k_n)$$

$$- (j_{k_{i+2}},k_{i+1})(j_{k_{i+3}},k_{i+2})(j_{k_{i+1}},k_i)(j_{k_i},k_n)$$

$$= (j_{k_i},k_{i+1})(j_{k_{i+2}},k_n)\begin{vmatrix}(j_{k_{i-1}},k_i) & (j_{k_{i+3}},k_{i+2})\\ (j_{k_{i+1}},k_i) & (j_{k_{i+1}},k_{i+2})\end{vmatrix}, \text{ if } k_n = k_{i+1} \text{ or } j_{k_i}=j_{k_{i+2}},$$

$$= (j_{k_i},k_{i+1})(j_{k_{i+2}},k_n)\begin{vmatrix}(k_n,k_i) & (k_n,k_{i+2})\\ (j_{k_n},k_i) & (j_{k_n},k_{i+2})\end{vmatrix}=0,$$

from (P4) of Appendix 3. provided $\sigma_{k_nk_i}=\sigma k_nk_{i+2}=0$.

When $k_n=k_{i+1}$, both covariances are zero and the equation (4.9,i) is linear. For instance, when i=2, n-3 and we have the model $[k_3,j_{k_2},j_{k_3},j_{k_4},k_3,j_{k_6},..,k_3,j_{k_{G_1}}]$ with the vector of unknowns $(\tau_{k_1},\tau_{k_5},\tau_{k_7},..,\tau_{k_{G_1-1}})$ satisfying the homogeneous linear system of equations (4.9,2) and (4.6,i), $i=6,..,G_1$. Again $z_1(G_1)-z_2(G_1)\neq 0$ implies the null solution is unique.

c) When $j_{k_i}=j_{k_{i+2}}$ the equation (4.9,i) is linear if $\sigma_{k_nk_i}=\sigma_{k_nk_{i+2}}=0$. For example **if** i=2, n=6 we have the model $[k_6,j_{k_2},j_{k_6},j_k\ ,k_6,j_{k_6},...,k_6,j_{k_{G_1}}$**I.**

d) The adjacent equations (4.7,i,n) and (4.8,i+2,n) correspond to the model assumption that k_{i-1}, k_n and k_{i+3} are siblings with k_n the parent of k_{i+1}, not excluding the possibility that k_n is either k_{i-1} or k_{i+3}, and provided $\sigma_{k_ik_n}=\sigma_{k_{i+2}k_n}=0$ in the population. Eliminating $\tau_{k_{i+1}}$ from these two equations, we get the equation

$$a_{i-1}a_ia_{i+1}a_{i+2}\,\tau_{k_{i-1}} - b_{i-1}b_ib_{i+1}b_{i+2}\,\tau_{k_{i+3}} = -d_3\,\tau_{k_{i-1}}\tau_{k_{i+3}}, \qquad (4.10,i)$$

with

$$d_3 \equiv a_{i-1}a_id_{i+2,n}-c_{in}b_{i+1}b_{i+2} = a_{i-1}b_{i+2}\begin{vmatrix}(j_{k_i},k_{i+1}) & (j_{k_i},k_n)\\ (j_{k_{i+2}},k_{i+1}) & (j_{k_{i+2}},k_n)\end{vmatrix}.$$

The equation (4.10,i) is linear if $j_{k_i}=j_{k_{i+2}}$, or $k_i=j_{k_{i+2}}$, or $k_{i+2}=j_{k_i}$ with $\sigma_{k_i k_n}=\sigma_{k_{i+2} k_n}=0$. Corresponding to the given model, the equations (4.10,2) and (4.6,i), $i=6,8,\ldots,G_1$, determine values of $(\tau_{k_1},\tau_{k_5},\ldots,\tau_{k_{G_1-1}})$. If $z_1(G_1)-z_2(G_1)\neq 0$, the zero solution is unique.

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