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MONOTONICITY IMPLIES STRATEGY-PROOFNESS FOR CORRESPONDENCES

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ABSTRACT

We show that Maskin monotone social choice correspondences on sufficiently rich domains satisfy a generalized strategy-proofness property, thus generalizing Muller and Satterthwaite's (1977) theorem to correspondences.

From the point of view of Nash implementation theory, the result yields a partial characterization of the restrictions entailed by Nash implementability. Alternatively, the result can be viewed as a possibility theorem on the dominant-strategy-implementability of monotone SCCs via set-valued mechanisms for agents who are completely ignorant about the finally selected outcome. It is shown by examples that stronger strategy-proofness properties fail easily.

1. INTRODUCTION

In a famous paper, Maskin (1977) has shown a condition called "monotonicity" to be necessary and not far from sufficient for the implementation of a social choice correspondence (SCC) in Nash equilibrium[1] For single-valued SCCs on sufficiently rich domains, this is a highly negative result, since it was also shown by Muller-Satterthwaite (1977) as well as Dasgupta-Hammond-Maskin (1979) that under these assumptions, monotonicity implies strategy-proofness, which in turn is well known to be highly restrictive. The main result of this paper, Theorem 1, shows that Muller and Satterthwaite's result can be generalized to correspondences: monotonicity on "comprehensive" domains implies "generalized strategy-proofness" (GSP).

Thus generalized to correspondences, the result looses its highly negative implications for Nash implementation, since the class of monotone SCCs is large; it includes, for example, the Pareto correspondence and, more generally, the class of core correspondences with respect to some effectivity function[2], as well as the constrained Walrasian correspondence on economic domains. Moreover, the class of monotone SCCs is closed under (pointwise) intersection and union; thus any SCC has a unique *minimal* monotone extension (Sen (1995); see also Thomson (1992)). Theorem 1 yields information about the restrictiveness of the conditions of monotonicity respectively Nash implementability. In particular, by showing a qualitative continuity of the properties of monotone SCCs with the single-valued case, it suggests that on comprehensive domains, monotone SCCs will be multi-valued to a significant extent (that goes beyond mere tie-breaking, for instance); this is borne out by a more detailed

[1] "Monotonicity" is also sometimes referred to as "strong" or "Maskin" monotonicity or as "Strong Positive Association". Maskin's result (see also Maskin (1985)) has been refined by for example Moore-Repullo (1990), Sjostrom (1991), and Danilov (1992).

[2] In the sense of Moulin-Peleg(1982).

analysis (see section 4).

On an alternative reading of the result, Theorem 1 can be viewed as a ***possibility*** theorem which says that there exists a rich and interesting class of non-empty-valued SCCs defined on comprehensive domains that can be implemented in dominant-strategy equilibrium as *correspondences*, i.e. via "indeterminate" mechanisms with sets of alternatives as outcomes ("quasi-mechanisms"): the class of monotone SCCs. Since this class contains the class of core correspondences with respect to some effectivity function, monotone SCCs allow to model property-rights based social choice quite generally; similarly, no-envy- and libertarian decisiveness-conditions give rise to monotone SCCs.

The dominant-strategy interpretation is substantiated in section 3, in which agents' behavior in quasi-mechanisms is formalized. "Generalized Strategy-Proofness" is shown to characterize dominant-strategy implementable SCCs in which agents are "completely ignorant" about the final selection from the outcome set; it can thus be viewed as the ***weakest*** meaningful strategy-proofness condition for correspondences.

Generalized Strategy-Proofness may seem very weak; however, stronger strategy-proofness properties are simply not in the cards in most cases, even for very well-behaved SCCs such as core correspondences; see examples 2 and 3 below, as well as Barbera (1977) and Kelly (1977) who obtained impossibility results for the somewhat stronger property of "weak strategy-proofness". As a result, Generalized Strategy-Proofness is arguably the most *informative* strategy-proofness property for correspondences. We note that for core correspondences derived from convex[3] effectivity functions, Demange (1987) has shown their coalitional strategy-proofness. Theorem 1

[3] In the sense of Peleg (1982).

can be viewed as a generalization of the non-cooperative aspect of her contribution.[4]

The remainder of the paper is organized as follows. Section 2 proves and discusses the main result. The strategic interpretation of the employed strategy-proofness condition GSP is given in section 3; it is argued that GSP has considerable appeal in an incomplete information context (dominant-strategy implementation), but will often be unsatisfactory under complete information (direct Nash implementation). Section 4 applies the main result to core correspondences based on effectivity-functions; in particular, their strategy-proofness and non-empty-valuedness on a comprehensive domain are shown to be intimately related. Section 5 concludes.

2. THE MAIN RESULT

Let X denote a finite set of social alternatives, $\mathcal{L}$ the set of linear orders[5] on X with generic element P, $\mathcal{D} = \prod_{i \in I} \mathcal{D}_i \subseteq \mathcal{L}^I$ a domain of preference profiles $\mathbf{P} = (P_i)_{i \in I}$.[6] *A social choice correspondence (SCC)* C maps preference profiles to sets of social alternatives, $C : \mathcal{D} \to 2^X$.[7] P is an x-improvement over Q ("$P \trianglerighteq_x Q$") if yPx implies yQx for all $y \in X$.

Definition 1 C **is monotone** *if, for any* $i, \mathbf{P}, Q, x$ *such that* $Q \trianglerighteq_x P_i$, $x \in C(\mathbf{P})$ *implies* $x \in C(Q, \mathbf{P}_{-i})$.

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[4] Modulo a difference between generalized and "optimistic" strategy-proofness described in section 4.

[5] A linear order is an asymmetric, transitive and weakly connected ($x \neq y \Rightarrow xPy$ or yPx) relation.

[6] Throughout, preference profiles are distinguished notationally from preference relations through their bold face.

[7] The analysis can straightforwardly be generalized to allow for fixed indifference sub-relations for each agent (cf. remark 6 following Theorem 1 below).

To define an appropriate generalization of the notion of strategy-proofness, it is helpful to associate with an ordering P on X an extension to an order on the subsets of X (i.e., on 2^X), denoted by $\widehat{P}$.

Definition 2 (P) *For $S,T \in 2^X$: $S \ \widehat{P} \ T$ if $T \neq \emptyset$, and, for all $x \in S$ and all $y \in T$: $x \ P$ y.*

Definition 3 (GSP) C *is* (generalized) strategy-proof *if for no* i, P, Q :

$$C(Q, \mathbf{P}_{-i}) \ \widehat{P_i} \ C(\mathbf{P}).$$

Interpreted in strategic terms, generalized strategy-proofness asserts that misrepresentation of preferences is never *unambiguously advantageous;* alternatively put, for any $\mathbf{P} \in \mathcal{D}$ and any $i \in I$, there exists a selection g of $C(., \mathbf{P}_{-i})$ such that under g it is not in agent i's interests to misrepresent his preferences at the preference profile **P**; see section **3** for further discussion.

Definition 4 $\mathcal{D}$ *is* comprehensive *if for all* $i \in I$ *and all* $\mathbf{P}, Q \in \mathcal{D}_i, R \in \mathcal{L}$: $R \supseteq P \cap Q$ *implies* $R \in \mathcal{D}_i$.

To paraphrase: $\mathcal{D}$ is comprehensive if for any $i \in \mathbf{I}$ and any $\mathbf{P}, Q \in \mathcal{D}_i$, $\mathcal{D}_i$ contains all R *between* **P** and Q. In particular, any domain $\mathcal{D}$ such that each $\mathcal{D}_i$ consists of *all* linear orders extending a fixed strict partial order Q_i is comprehensive.

Theorem **1** *A monotone non-empty-valued SCC C with comprehensive domain is strategy-proof.*

Since both monotonicity and strategy-proofness are conjunctions of single-agent conditions, it is notationally simpler and conceptually cleaner to prove the result and conduct some of the following discussion in terms of single-person choice-correspondences

$f : \mathcal{D} \to 2$"; this is evidently without loss of generality since one can apply the result to f $= C(., P_{-i})$.

Proof. For k $\leq$ n $= \#X$ and $\mathbf{P} \in \mathcal{L}$, let $P(k)$ denote the k-th ranked alternative x (from the top). Also, for P, Q $\in \mathcal{L}$ and $l \leq$ n, with m defined implicitly by $P(m) = Q(l)$ given P,Q and l, let $\Phi_l(Q, \mathbf{P})$ be defined by

$$\Phi_l(Q,P)(k) = \begin{cases} P(m) = Q(l) & \text{if } k = l \\ P(k) & \text{if } k < \min(l,m) \text{ or } k > \max(l,m) \\ P(k+1) & \text{if } l < m \text{ and } l < k \leq m \\ P(k-1) & \text{if } l > m \text{ and } m \leq k < l \end{cases}$$

$\Phi_l(Q, P)$ results from $\mathbf{P}$ by moving the m-th alternative into l$-$th position, thus ensuring that the now $l-$th ranked alternative coincides with the alternative that is l$-$th ranked with respect to Q, $Q(l)$. To prove the theorem, fix any $P, Q \in \mathcal{D}$. It needs to be shown that there exist $x^* \in$ f (Q) and y* $\in$ f (P)such that not y^*Qx^*.

Define inductively the finite sequence $\{Q_l\}_{l=0,\cdots,n}$ in $\mathcal{D}$ such that $Q_0 = \mathbf{P}$, $Q_n =$ Q and $Q_l = \Phi_l(Q, Q_{l-1})$. It is straightforward from the construction that

$$\text{for k} \leq l : Q_l(k) = Q(k) \tag{1}$$

and

$$\text{for } k \neq l : Q_{l-1} \trianglerighteq_{Q_l(k)} Q_l \quad . \tag{2}$$

From (1), it follows that $Q_l \supseteq$ Q $\cap Q_{l-1}$, for all $l \leq$ n, hence that $Q_l \in \mathcal{D}$ for all 1 $\leq$ n by comprehensiveness.

Let k* $= \max\{k \mid Q(k) \in$ f (P)),set y^* equal to $Q(k^*)$, the Q$-$worst alternative in f (P),and fix any x* $\in$ f (Q_{k^*}) . We will show that not y^*Qx^* as well as x* $\in f(Q)$, as desired.

From **(2)** and the monotonicity of f (modus tollens), it follows that

$$\text{for } 1 \leq l \leq n : f(Q_l) \subseteq f(Q_{l-1}) \cup \{Q(l)\}.$$

From this one obtains by induction

$$f(Q_{k^*}) \subseteq f(P) \cup \{Q(l)\}_{l \leq k^*} \quad . \tag{3}$$

Since by (1) and the definition of y* one has in particular

$$Q_{k^*}(k^*) = y^*,$$

it follows from (3) and the definition of k^* that

$$\text{for no } z \in f(Q_{k^*}) : y^* Q z. \tag{4}$$

In particular,

$$\text{not } y^* Q x^*. \tag{5}$$

Moreover, it follows from (4), the definition of k*, (1) and the monotonicity of f that

$$f(Q) \supseteq f(Q_{k^*}),$$

which implies by the definition of x*

$$x^* \in f(Q). \tag{6}$$

(5) and (6) demonstrate the claim. ■

Discussion:

We begin the discussion by an overall assessment of the result, to be followed by remarks on technical aspects of the theorem and on the possibility of generalizations.

1. Viewed as a possibility result, Theorem 1 ensures the existence of a large and natural class of strategy-proof correspondences. From this perspective, its importance hinges on the intrinsic appeal of the strategy-proofness condition, as well as on the degree to which the existence of attractive strategy-proof correspondences is not evident without the result. The former aspect is discussed in section **3.** As to the latter, the existence of some minimally attractive strategy-proof is very easily established; for example, the correspondence of all Pareto efficient alternatives ("Pareto correspondence") is GSP[8]. Moreover, any super-correspondence of a strategy-proof SCC is strategy-proof as well. Thus, for example, any SCC that contains at each profile the top alternative of each agent is strategy-proof and anonymous; such SCCs will be referred to as "indeterminately dictatorial".

So there are lots of strategy-proof SCCs; however, it is not so obvious that there exist strategy-proof SCCs that are sufficiently selective, "small" in a relevant sense. Note first that "smallness" is not adequately measured by the cardinality of the selected sets. The smallest indeterminately dictatorial SCC, for example, selects at any profile at most as many alternatives as there are agents; however, measured in terms of the range of agents' utilities generated by the chosen alternatives at each profile, this correspondence is very large. Indeed, in many cases it is hardly more selective than the Pareto correspondence or even than the constant correspondence selecting the all feasible[g] alternatives. In particular, indeterminately dictatorial SCCs will generally fail to ensure satisfaction of lower bounds on agent's utility (i.e. violate non-trivial individual rationality constraints).

Theorem 1 ensures the ekistence of a wide range of selective strategy-proof SCCs. Take any monotone SCC G as an "upper bound" on "admissible" $C \subsetneq$ G; for example,

[8] Indeed, it satisfies much stronger strategy-proofness properties than GSP.

'Which may be a strict subset of the domain X.

G can incorporate efficiency and individual rationality conditions on C. In view of Theorem 1, for every monotone G, there exist admissible strategy-proof SCCs: simply put $C = G$.

One may want to go further and ask how closely it is possible to approximate single-valuedness; specifically, one may want to find, for some given single-valued $H \subseteq G$, its minimal strategy-proof extension. Strategy-proofness being not an intersection-closed property, there typically will be more than one such correspondence. Theorem 1 yields useful information about these: since the pointwise intersection of monotone correspondences is monotone, any correspondence B has a unique minimal monotone extension B^*. Thus, for any minimal strategy-proof extension C of H, H^* is an upper bound for C by Theorem 1: $C \subseteq H^* \subseteq G$. In particular, H can be approximated well by a strategy-proof SCC whenever it can be approximated well by a monotone SCC (for a study of minimal monotone extensions, see Sen (1995) and Thomson (1992)).

2. One might have expected to obtain a characterization of monotone SCCs in terms of a strategy-proofness property (as in Muller-Satterthwaite's (1977) theorem for the single-valued case), rather than a uni-directional implication. However, monotonicity implies the following IIA-type condition:

Condition 1 *For any* i, P, Q, x *such that* $Q \trianglerighteq_x P_i$ *and* $P_i \trianglerighteq_x Q$: $x \in C(\mathbf{P}) \Leftrightarrow x \in C(Q, \mathbf{P}_{-i})$.

This condition seems to be devoid of any strategy-proofness content.

3. GSP may be substantially weaker than monotonicity, as illustrated by the following example.

Example 1 *Let* $X = \{a, b, c\}$, $I = \{1, 2, 3\}$, *and* $\mathcal{D} = \mathcal{L}^I$.

Define an SCC C^* by setting $C^*(\mathbf{P})$ equal to the unique Condorcet winner if it exists, and equal to X otherwise. It is easily seen that C^* is GSP but not monotone. Indeed, no Condorcet consistent non-empty-valued SCC is monotone. This can be seen as follows.

The following matrix labels the set of six preference orderings in $\mathcal{L}$ (listing alternatives from top to bottom here and throughout):

P_1	P_2	P_3	Q_1	Q_2	Q_3
a	b	c	b	c	a
b	c	a	a	b	c
c	a	b	c	a	b

By Condorcet consistency, $C(Q_1,P_2,P_3) = \{b\}$. Hence by monotonicity (modus tollens), $c \notin C(P_1,P_2,P_3)$. By analogous arguments, $C(P_1,Q_2,P_3) = \{c\}$, hence $a \notin C(P_1,P_2,P_3)$, as well as $C(P_1,P_2,Q_3) = \{a\}$, hence $b \notin C(P_1,P_2,P_3)$. These implications jointly contradict the assumed non-empty-valuedness of C.

4. For single-valued choice-functions, the domain assumption can be significantly weakened, in particular to *connected* domains defined as follows.

Definition 5 *$\mathcal{D}$ is* connected *if for all $i \in I$ and all $P,Q \in \mathcal{D}_i$ the following holds: if there exists $R \in \mathcal{L}$ such that $R \supseteq P \cap Q$ and $R \notin \{P,Q\}$, then there exists $R \in \mathcal{D}_i$ such that $R \supseteq P \cap Q$ and $R \notin \{P,Q\}$.*[10]

For example, the class of preferences that are single-peaked with respect to some linear order Q^* on X is connected but not comprehensive. Such weakening is not possible in the set-valued case, as the following example shows.

[10]Connectedness can be paraphrased thus: for any i and any non-neighboring $P,Q \in \mathcal{D}_i$, $\mathcal{D}_i$ must contain a preference relation R strictly between P and Q.

Example *2* $X = \{a,b,c,d\}$, $\mathcal{D} = \{P, P', P''\}$, *with* P, P', P'' *given by the following matrix:*

P	P'	P''
a	a	b
b	b	a
c	d	d
d	c	c

Define f *by* $f(P) = \{a\}$, $f(P') = \{a,d\}$ *and* $f(P'') = \{d\}$. f *is monotone, but violates GSP.* $\mathcal{D}$ *is connected; the smallest comprehensive domain containing* $\mathcal{D}$ *is* $\mathcal{D} \cup \{Q\}$, *with* $b\ Q\ a\ Q\ c\ Q\ d$. *It is easily verified that* f *cannot be extended to a monotone and non-empty-valued correspondence* f' *on* $\mathcal{D} \cup \{Q\}$.

5. GSP cannot be strengthened in Theorem 1 to Kelly's (1977) "weak strategy-proofness" (WSP) which is obtained by extending the induced partial set-order $\widehat{P}$ to $\tilde{P}$, reflecting an attribution of strictly positive weight (lower probability) to any alternative in the outcome set.

Definition 6 *i)* *For* $S, T \in 2^X$: $S\ \tilde{P}\ T$ *if for some* $x \in S$ *and* $y \in T$, $x\ P\ y$, *and for no* $x' \in S$ *and* $y' \in T$, $y'\ P\ x'$.

ii) (WSP) C *is* weakly strategy-proof *if for no* i, P, Q : $C(Q, \mathbf{P}_{-i}) \widetilde{P_i}\ C(\mathbf{P})$.

We will see below in section 4 that violations are quite common; for instance, the restriction of f in example 2 to the comprehensive domain $\{P, P'\}$ fails to be WSP. While desirable, WSP is simply not in the cards in many cases.

6. A straightforward but important generalization of Theorem 1 is to situations in which agents care only about (possibly different) aspects of the social state, as for instance in discrete private-goods economies or matching problems.

Technically, the assumption that all preference relations are linear orderings on X can be weakened to the assumption that agent i's preference relations are linear on X_i (with $X = X_i \times X_{-i}$), i.e. that they are asymmetric, transitive and satisfy the condition:

$$(xPy \text{ or } yPx) \quad \Leftrightarrow \quad x_i \neq y_i \quad \text{for all } x, y \in X.$$

7. Dasgupta-Hammond-Maskin (1979) have shown for single-valued SCCs the validity of an analogue to Theorem 1 for economic domains in which preferences are assumed to be convex and continuous. Such an analogue does not exist for choice correspondences; for example, the (constrained) Walrasian correspondence is monotone but not generalized strategy-proof since it is not even strategy-proof at single-valued points.[11] This shows that the conclusion of the result for correspondences is substantially stronger, and thus confirms the need for substantially stronger domain assumptions for its validity.[12]

3. ON THE STRATEGIC INTERPRETATION OF GENERALIZED STRATEGY-PROOFNESS

The intrinsic strategic meaning of CSP is best elucidated in the context of mechanisms $F : \prod_{i \in I} S_i \rightarrow 2^X \backslash \emptyset$ that map strategy profiles $s = (s_i)_{i \in I}$ to non-empty *sets*

[11] Note that the Walras correspondence itself is monotone if X is taken to be the set of all (not necessarily feasible) allocations.

[12] In view of the connectedness but non-comprehensiveness of the class of single-peaked preferences (cf. #4 above) and the fact that single-peakedness with respect to a a given linear order can be viewed as a convexity restriction, we conjecture that the culprit is the convexity assumption on preferences necessary to ensure the existence of Walrasian equilibria, rather than the infinite cardinality of the domain or the continuity of preferences.

of alternatives ("quasi-mechanisms"), the set-valuedness reflecting an indeterminacy of the final outcome. The agents' (possibly partial) orderings over sets Π_i are determined by their rankings of the alternatives and their beliefs about the final selection (as well as possibly their attitudes toward ignorance).

One interesting level of analysis derives from assuming that the agents rely only on the information about the final outcome given by the mechanism, in other words: that they act as-if they were completely ignorant about the final selection. Such ignorance is captured by endowing the agents with the *partial* orderings $\Pi_i = \widehat{P_i}$ defined in section 2; agents with such preferences will be called *"agnostic"*. The notions of Nash and dominant-strategy equilibrium are defined naturally; one just needs to accommodate the possible incompleteness of the set rankings.

Definition 7 *i) s is a* **best response** *to* $\mathbf{s}_{-i}$ *in the (quasi-)game* (F,Π) *if there does not exist* $s' \in S_i$ *such that* $F(s',\mathbf{s}_{-i})\Pi_i F(s,\mathbf{s}_{-i})$.

ii) s is a **Nash equilibrium** *in* (F,Π) *if, for all* $i \in I$, s_i *is a best response to* $\mathbf{s}_{-i}$.

iii) s is a **dominant-strategy equilibrium** *(DSE) in* (F,Π) *if, for all* $i \in I$ *and all* $\mathbf{s}'_{-i} \in \prod_{j:j\neq i} S_j$, s_i *is a best response to* $\mathbf{s}'_{-i}$.

A prominent class of mechanisms are the *direct* or revelation mechanisms with $S_i = \mathcal{D}_i$ and $F = C$; "truth-telling" in a revelation mechanism is described by the strategy-profile $\mathbf{P}$. Standard arguments yield the following result for agnostic agents.

Proposition 1 *The following four statements are equivalent:*

i) C satisfies GSP.

ii) P is a Nash equilibrium in $(C,\mathbf{P})$ *for all* $\mathbf{P} \in \mathcal{D}$.

iii) $\mathbf{P}$ *is a DSE in* (C,P) *for all* $\mathbf{P} \in \mathcal{D}$.

iv) C can be implemented in DSE[13] *via some quasi-mechanism F when agents are*

[13] In *the* obvious sense.

agnostic.

Remark 1: If one replaces GSP with WSP and $\widehat{\mathbf{P}}$ by $\mathbf{P}$, one obviously obtains an analogous result.

Remark 2: GSP is consistent with subjective expected-utility maximization if no restrictions at all on the subjective probabilities of the final outcome selected from $C(\widehat{\mathbf{P}})$ are imposed[14]. If however one requires dl agents to have identical probability distributions over the selection, one obtains effectively a single-valued mechanism with lotteries as outcomes; in a social choice-setting, such mechanisms do not significantly enlarge the class of strategy-proof SCCs (see Gibbard (1977)).

In view of proposition 1, Theorem 1 can be viewed as a general possibility theorem establishing the existence of a large class of SCCs that are DSE implementable for agnostic agents. The plausibility (at least as an approximation) and relevance in application of the assumption of agnosticism critically depends on the context, in particular on whether agents' information about each others preferences is complete or incomplete. On the one hand, in a Nash context in which agents' preferences are commonly known, the outcome set $C(\mathbf{P})$ resulting from truth-telling is also commonly known; it is then natural for agent to consult his beliefs about, and even form more definite beliefs about, the final selection from $C(\mathbf{P})$ and from the sets $C(P, \mathbf{P}_{-i})$ resulting from hypothetical deviations; this may well lead to perceived opportunities of advantageous manipulation. In particular, failures of weak strategy-proofness are detrimental to any claim that truth-telling constitutes a satisfactory Nash equilibrium.

On the other hand, in an incomplete information context in which agents' preferences are mutually uncertain (as is generally presupposed in the search for DSE-

[14] Across agents as well as any agents' types.

implementable choice-functions), these considerations have much less force, since a deviation that is advantageous at some preference profile of the other agents may well be disadvantageous at others; this indeed seems highly probable if the mechanism is known to satisfy GSP. GSP may thus be viewed as a criterion of *prima-facie* incentive-compatibility. In addition, to ascertain that a contemplated deviation is advantageous overall requires much greater computational effort due to the need of checking for countervailing trade-offs, and may thus not even be attempted by a boundedly rational agent in view of the prima-facie optimality of an "agnostically dominant" strategy.

4. APPLICATION TO CORE CORRESPONDENCES

An *effectivity* function 9 maps non-empty subsets of agents ("coalitions") to sets of subsets of alternatives, $\Psi : 2^I \backslash \emptyset \to 2^{(2^X \backslash \emptyset)}$; $\Psi(S)$ describes those restrictions on the social outcome of the form "$C(\mathbf{P}) \subseteq Y$", for $Y \in 2^X \backslash \emptyset$, that coalition S is entitled to enforce. Given a preference profile $\mathbf{P}$, the core of Ψ $C_\Psi(\mathbf{P})$ is given as the (possibly empty) set of alternatives that no coalition can block.

Definition 8 (Core) $C_\Psi(\mathbf{P}) = \{x \in X \mid \textit{For}$ no $S \subseteq \mathbf{I}$ and $Y \in \Psi(S) : \forall i \in S, y \in Y : yP_i x\}$.

In a social choice context with universal domain $\mathcal{L}^I$, anonymous and neutral effectivity functions can be parametrically described by a function $\gamma : (1,..,\#\mathbf{I}) \to \{1,..,\#X)$ according to

$$\Psi_\gamma(S) = \{Y \subseteq X \mid \#Y \geq \gamma(\#S)\}.$$

Moulin (1981) has shown the following result[15].

[15] See Moulin-Peleg (1982) for a generalization of the analysis to non-anononymous and non-neutral effectivity functions.

Proposition 2 (Moulin) *C_{Ψ_γ} is non-empty-valued on $\mathcal{L}^I$ if and only if, for all $h \leq \#I$: $\gamma(h) > \#X \cdot (1 - \frac{h}{\#I})$.*

Corollary 1 *There exists a minimal anonymous and neutral non-empty-valued core correspondence $C_{\Psi_{\gamma^*}}$ on $\mathcal{L}^I$ (the "proportional veto correspondence"), with γ^* given by*

$$\gamma^*(h) = \textit{smallest integer strictly exceeding } \#X \cdot (1 - \frac{h}{\#I}).$$

From this and Theorem 1 one obtains immediately:

Proposition 3 *For any $\gamma \geq \gamma^*$, C_{Ψ_γ} is strategy-proof.*

In "heuristic continuity" with the Gibbard-Satterthwaite type impossibility results for single-valued SCCs, one surmises that C_{Ψ_γ} should be multi-valued to a non-negligible extent. This is borne out by inspection.

For instance, consider the proportional veto correspondence in situations in which $\#X = 3$ and $\#I \geq 2$. Let $\mu_{\mathbf{P}}$ denote the distribution of preference orderings associated with the profile $\mathbf{P}$ given by $\mu_{\mathbf{P}}(Q) = \frac{\#\{i \in I | P_i = Q\}}{\#I}$. It is easily verified that $\#C_{\Psi_{\gamma^*}}(\mathbf{P}) > 1$ if and only if i) for all $x \in X$, $\mu_{\mathbf{P}}(\{Q \mid Q(1) = x\}) \leq \frac{2}{3}$, and ii) for at most one $x \in X$, $\mu_{\mathbf{P}}(\{Q \mid Q(3) = x\}) > \frac{1}{3}$. If $\mu_{\mathbf{P}}$ is viewed as element of the $\mathcal{L}$-unit simplex $\Delta^{\mathcal{L}}$, the set of distributions μ in $\Delta^{\mathcal{L}}$ satisfying i) and ii) has non-empty interior and does not shrink as the number of agents increases. Sen (1995) shows this phenomenon to be entirely general for the class of monotone SCCs that are non-dictatorial and have a range of at least three alternatives.[16]

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[16]Sen himself expresses the point in terms of an asymptotic statement saying that monotone SCCs are multi-valued for a non-negligible fraction of preference profiles even in the limit as the number of agents becomes infinitely large.

In a related vein, Barbera (1977) has shown that ***weakly*** strategy-proof[17] choice-correspondences that respect conditions of unanimity and "positive responsiveness" (which formalizes the notion that multiplicity may arise exclusively to accommodate ties) must be dictatorial; this result seems however of limited significance to the study of monotone SCCs since the gap between generalized and weak strategy-proofness is significant (cf. example **3** below).

One may wonder whether the additional structure of core-correspondences can be exploited to strengthen Theorem 1. The following example proves this expectation to be wrong: in general, neither can the domain assumptions be weakened nor the strategy-proofness implications be strengthened.

Example 3 Let $X = \{a, b, c, d\}$ *and* $I = \{1, 2, 3, 4\}$.

Consider the "75%-majority rule" $G = C_{\Psi_{\gamma^{**}}}$ *, with* $\gamma^{**}(h) = \begin{cases} 1 & \text{if } h \geq 3 \\ 4 & \text{if } h \leq 2 \end{cases}$

The orders P_k *of agent* k*, for* $k > 1$ *, are given by the following table:*

P_2	P_3	P_4
a	b	c
b	c	d
c	d	a
d	a	b

Consider the following orderings for agent 1:

[17] Barbera's condition of "uniform non-manipulability" effectively amounts to Kelly's WSP defined above with $\tilde{P}$ restricted to sets S,T such that, $\#S + \#T \leq 3$.

P	P'	P''	P'''
d	a	a	d
a	d	d	a
c	c	b	b
b	b	c	c

G fails to be strategy-proof at single-valued points and thus violates GSP since $G(P'', P_2, P_3, P_4) = \{a\}\widehat{P}\{c\} = G(P, P_2, P_3, P_4)$. *By Theorem 1, G cannot be non-empty-valued on* $\{P, P', P'', P'''\} \times \{P_2\} \times \{P_3\} \times \{P_4\}$, *the smallest comprehensive domain containing the two preference profiles involved in the violation. Indeed, one easily verifies* $G(P''', P_2, P_3, P_4) = \emptyset$.

Thus, as a result about core-correspondences (for which monotonicity is trivial), Theorem 1 can be read as deriving GSP from the non-empty-valuedness of the correspondence on a comprehensive set.

Note also that G violates WSP[18] *on* $\{P, P'\} \times \{P_2\} \times \{P_3\} \times \{P_4\}$. *It is easy to give examples of such violations for core correspondences* C_{Ψ_γ} *that are non-empty-valued on all of* $\mathcal{L}^I$. *For instance, modify example 3 by setting* $X^+ = X \cup \{e\}$, *and*

$$G^+ = C_{\Psi_{\gamma^+}}, \text{ with } \gamma^+(h) = \begin{cases} 1 & \text{if } h = 4 \\ 2 & \text{if } h = 3 \\ 5 & \text{if } h \leq 2 \end{cases}, \text{ and define } P^+, P^{+\prime}, P_k^+ \text{ from } P, P', P_k$$

above by inserting e as second-ranked element into each preference ordering.

The continued necessity of strong domain assumptions suggests that a direct verification of the strategy-proofness of a core-correspondence will be non-trivial. This is confirmed by the work of Demange *(1987)* that is directly devoted to this question. She has shown that in fact somewhat stronger strategy-proofness properties can be obtained when the effectivity-functions are *convex* in the sense of Peleg *(1982);* for

[18] As well as OSP just below.

example, the effectivity-function defining the proportional veto correspondence is convex, while weaker effectivity-functions typically are not. Specifically, Demange proved that under convexity, core correspondences are "coalitionally non-manipulable".[19] This strenghtens GSP by considering coalitional deviations, and by assuming that agents evaluate SCCs optimistically in terms of an ordering $\overline{P}$ containing $\widehat{P}$ and defined follows.

Definition 9 *i) For* $S, T \in 2^X$: $S \;\overline{P}\; T$ *if* $T \neq \emptyset$, *and there exists* $x \in S$ *such that for all* $y \in T$: $x \; P \; y$.

ii) **(OSP)** *C is* optimistically strategy-proof *if for no* $i, \mathbf{P}, Q$: $C(Q, \mathbf{P}_{-i})\overline{P_i}\, C(\mathbf{P})$.

To the extent to which the differences between optimistic and general strategy-proofness can be neglected,[20] Theorem 1 can be viewed as generalization of the non-cooperative dimension of Demange's result.

At least at first glance, the mathematics underlying the two results is very different. While hers relies on the "holistic" property of convexity, ours seems essentially "individualistic" in that both the monotonicity and the comprehensive-domain condition are agent-by-agent assumptions. However, a closer connection can be established if one views the (holistic) non-empty-valuedness assumption of Theorem 1 as a substantive consistency assumption on the social choice correspondence. In the context of a universal domain, this connection is indeed remarkably precise in view of Peleg's (1982) result who showed that a core correspondence C_Ψ is non-empty-valued on $\mathcal{L}^I$ if and only if it contains the core correspondence $C_{\Psi'}$ associated with some convex effectivity function Ψ'.

[19] Demange also suggested a complete-ignorance interpretation of the set-valuedness of the correspondences, and noted the weakness of the strategy-proofness properties of core-correspondences in general. She did not consider the weaker property GSP.

[20] It is easily verified that OSP does not imply WSP, which seems to be the more interesting notion from the present non-cooperative point of view.

5. CONCLUDING REMARKS

Theorem 1 is situated at the "edge of possibility": weaken GSP, and the strategy-proofness interpretation is lost; strengthen GSP, and few SCCs will satisfy the strengthened condition on comprehensive domains. The result helps explaining the celebrated impossibility theorem of Gibbard (1973) and Satterthwaite (1975) when viewed as a result about *single-valued* GSP correspondences: the impossibility is caused not so much by strategy-proofness per se, as by the additional requirement of single-valuedness. We note that while single-valuedness is attractive from a mechanism-*design* perspective, it seems too strong to make this requirement absolute: in particular, it seems misguided to rule out as fundamental a mechanism as the Walrasian merely on the ground of its set-valuedness.[21]

In response to Gibbard-Satterthwaite's impossibility theorem, a spate of recent work has concentrated on obtaining more positive results by imposing (typically strong) restrictions on the domain of preferences. It would be an interesting question for future research to try to analogously improve on Theorem 1 by such domain restrictions. In principle, improvements of two kinds might be obtained: the strategy-proofness properties of the SCC may be strengthened, and/or SCCs may emerge that are especially "small" in a relevant sense. In view of the ease of obtaining violations of even weak strategy-proofness, the prospects for the former seem to be slim. As a likely example of the latter, consider ε-cores of finite private goods economies with non-convex preferences in the sense of Wooders (1983). For given I, these are non-empty for sufficiently large, ε, but not very selective at many preference profiles. In view of the absence of substantive domain-restrictions on preferences, it seems highly

[21] For a spirited recent argument for the game-theoretic interest in set-valued mechanisms, see Brandenburger-Stuart (1996).

likely that non-empty-valued ε-core correspondences are strategy-proof.[22] Ranade (1995) has shown that for any given $\varepsilon > 0$, non-emptiness of the ε-core is assured for sufficiently large I. Thus, as the number of economic agents becomes large, "small" (approximately Walrasian) strategy-proof correspondences emerge.

[22] Cf. remarks 6 and 7 of section 2.

REFERENCES

[1] Barbera, S.. (1977): "The Manipulation of Social Choice Mechanisms that Do Not Leave Too Much to Chance," *Econometrica* 45, 1573-1588.

[2] Brandenburger, A. and H. Stuart (1996): "Biform Games," mimeo.

[3] Danilov, V. (1992): "Implementation via Nash Equilibria," *Econometrica* 60, 43-56.

[4] Dasgupta, P. , P. Hammond and E. Maskin (1979): "The Implementation of Social Choice Rules," *Review of Economic Studies 46*, 185-216.

[5] Demange, G. (1987): "Non-Manipulable Cores," *Econometrica* 55, 1057-1074.

[6] Gibbard, A. (1973): "Manipulation of Voting Schemes: A General Result," *Econometrica* 41, 587-601.

[7] Gibbard, A. (1977): "Manipulation of Schemes that Mix Voting with Chance," *Econometrica* 45, 587-601.

[8] Kelly, J. (1977): "Strategy-Proofness and Social Choice Functions without Single-Valuedness," *Econometrica* 45, 439-446.

[9] Maskin, E. (1977): "Nash Equilibrium and Welfare Optimality," rnimeo.

[10] Maskin, E. (1985): "The Theory of Implementation in Nash Equilibrium: A Survey", in L. Hurwicz et al.: *Social Goals and Social Organization. Volume in the Honor of Elisha Pazner*. Cambridge, Cambridge *U.P*.

[11] Moore, J. and R. Repullo (1990): "Nash Implementation: A Full Characterization," *Econometn'ca* 58, 1083-1099.

[12] Moulin, H. (1981): "The Proportional Veto Principle," *Review of Economic Studies* 48, 407-416.

[13] Moulin, H. and B. Peleg (1982): "Cores of Effectivity Functions and Implementation Theory," *Journal of Mathematical Economics* 10, 115-145.

[14] Muller, E. and M. Satterthwaite (1977): "The Equivalence of Strong Positive Association and Strategy-Proofness," *Journal of Economic Theory* 14, 412-418.

[15] Peleg, B. (1982): "Convex Effectivity Functions", mimeo.

[16] Peleg, B. (1984): *Game-Thwretic Analysis of Voting in Committees.* Cambridge, Cambridge UP.

[17] Ranade, A. (1995): "Large Finite Economies are Almost Balanced", mimeo.

[18] Satterthwaite, M.A. (1975): "Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions," *Journal of Economic Theory* 10, 198-217.

[19] Sen, A. (1995): "The Implementation of Social Choice Functions via Social Choice Correspondences: A General Formulation and a Limit Result," *Social Choice and Welfare* 12, 277-292.

[20] Sjostrom, T. (1991): "On the Necessary and Sufficient Conditions for Nash Implementation," *Social Choice and Welfare* 8, 333-340.

[21] Thomson, W. (1992): "Mohotonic Extensions," mimeo.

[22] Wooders, M. (1983): "Epsilon Cores of Replica Games," *Journal of Mathematical Economics* 11, 277-300.