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Intergenerational equity and stationarity

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Abstract: We consider quasi-orderings of infinite utility streams satisfying the strong Pareto axiom (i.e., Paretian quasi-orderings) and study the question of how strong a notion of intergenerational equity one can impose on these quasi-orderings without generating an impossibility theorem. Building on a result by Mitra and Basu (2007), we first show that there exist many possible extensions of the finite anonymity axiom that are satisfied by some Paretian quasi-ordering. Then we study how the additional requirement of stationarity à la Koopmans (1960) affects this result. After proving a possibility theorem for this case, we demonstrate that stationarity imposes strong restrictions on the extendability of the finite anonymity axiom.

Journal of Economic Literature **classification codes:** D60, D70, D90

Key words: intergenerational equity, stationarity, quasi-orderings, strong Pareto axiom, infinite utility streams.

1 Introduction

Intergenerational equity is an important issue for policy decisions that have long lasting effects. The design and implementation of mechanisms to reduce greenhouse gas emissions or the reform of public pension systems in response to demographic changes are just two examples of policy problems of that sort that are currently high on the agenda in many countries around the world.

The formal definition of intergenerational equity and the analysis of its implications has therefore captured the attention of many good economists and philosophers at least since Sidgwick (1907) and Ramsey (1928). A broadly accepted minimal requirement for any sensible notion of intergenerational equity is that swapping the utility levels of any two generations should produce a utility allocation which is judged to be equally good by the society. If one further assumes that social preferences are transitive, this requirement is easily seen to be equivalent to the so-called finite anonymity axiom which postulates that society should be indifferent between any utility allocation and the one that arises if the original allocation is subjected to a finite permutation.

In a seminal contribution, Diamond (1965) demonstrated that there does not exist a continuous (with respect to the sup metric) social welfare function satisfying both finite anonymity and the strong Pareto axiom.[1] Later it has been shown by Basu and Mitra (2003) that this impossibility result remains true if one drops the continuity requirement. Svensson (1980), on the other hand, proved that there exist (complete) social welfare orderings (not representable by social welfare functions) that satisfy the finite anonymity axiom as well as the strong Pareto axiom. The problem with that result is that it necessarily requires non-constructive methods like the axiom of choice. This means that social welfare orderings satisfying the above mentioned two axioms, even if they are known to exist, cannot be explicitly described; see, e.g., Zame (2007) or Lauwers (2010).

Motivated by these impossibility results, researchers have started to study social welfare quasi-orderings (i.e., binary relations that are reflexive and transitive but not necessarily complete) satisfying the strong Pareto axiom as well as finite anonymity. Examples of such quasi-orderings are well known: e.g., the grading principle of Suppes (1966) and Svensson (1980), the overtaking criterion of von Weizsäcker (1965), or the catching-up criterion due to Gale (1967). Dropping the requirement of completeness of the social preferences of course allows one to strengthen the anonymity axiom. It has therefore been suggested to consider extended anonymity axioms, which require social indifference between any two utility allocations that are related to each other by a permutation from a set that properly includes all finite permutations; see, e.g., Lauwers (1997a) or Fleurbaey and Michel (2003). The set of permutations that has been suggested most often in this context is the set of fixed-step permutations; see, e.g., Lauwers (1997b), Fleurbaey and Michel (2003), Banerjee (2006), Kamaga and Kojima (2008, 2009), or Asheim and Banerjee (2010). How far one can go in that direction has been precisely characterized by Mitra and Basu (2007). These authors have shown that the set of all permutations that are compatible with indifference under a given Paretian quasi-ordering must satisfy two properties: this set must only contain cyclical permutations and it must form a subgroup of the group of all permutations (when endowed with the concatenation operation). In the present paper we

[1] Although Diamond (1965) is usually credited for this result, he himself attributes it to Yaari.

extend this analysis in two directions.

First, we explore the implications of the result from Mitra and Basu (2007) by constructing a rich family of sets of permutations that satisfy the two criteria mentioned above (cyclicality and group structure). Our construction uses the concept of a filter base and yields permutations within the set of so-called variable-step permutations; see Fleurbaey and Michel (2003). We also give an example of a set of permissible permutations that is not generated by a filter base and, hence, does not fall into the variable-step category. We show that permutations that are permissible for Paretian quasi-orderings can generate orbits of arbitrary length, and that these permutations can affect all generations. Thus, the Pareto axiom and intergenerational equity as formalized by the extended anonymity axiom are not only consistent with each other, but there exist many quasi-orderings that satisfy both of these requirements. As a by-product of our analysis, we address a question that has been posed by Mitra and Basu (2007) and that has recently been answered in the negative by Lauwers (2010), namely whether the set of fixed-step permutations forms a maximal group in the set of cyclic permutations. Our contribution to that question is that we confirm Lauwer's result by explicitly constructing simple groups of cyclic permutations that properly include all fixed-step permutations.

Our second contribution is that we study the consistency of the Pareto axiom, the extended anonymity axiom, and stationarity as defined by Koopmans (1960). In an intertemporal context like the one under consideration in the present paper, stationarity is a very natural requirement because it allows for a recursive representation of preferences. Surprisingly, the compatibility of stationarity and intergenerational equity has so far received only very little attention. Dutta (2007) has raised the issue and has demonstrated that stationarity of Paretian quasi-orderings implies time-preference, at least in comparisons of certain pairs of infinite utility streams that are related to each other by an infinite permutation. Mitra (2007) shows among other things that neither the overtaking criterion nor the catching-up criterion, probably the two most popular stationary Paretian quasi-orderings, satisfy any extended anonymity axiom. Our approach is more general than Dutta (2007) and Mitra (2007) and allows us to complement their findings with a possibility result. We begin by splitting stationarity in two separate properties: shift-invariance and truncation-invariance. Shift-invariance means that, whenever two utility streams are ranked in a certain way, this ranking is not changed by shifting these utility streams forward in time by one generation and assigning in both streams the same utility level to the (new) first generation. Truncation-invariance, on the other hand, reflects the idea that a comparison of two utility streams that assign the same utility level to generation 1 can be reduced to a comparison of the truncated utility streams from period 2 onwards. For both of these properties, we provide an analogue to the result of Mitra and Basu (2007), that is, we provide a characterization of the sets of permutations that can be permissible for a Paretian quasi-ordering satisfying either one of the two invariance properties. These characterizations add one property to cyclicality and group structure, namely the invariance of the set of permissible permutations with respect to a shift operator or a truncation operator, respectively.

As an application and illustration of the characterization theorems we show that both of the invariance properties, extended anonymity, and the strong Pareto axiom are compatible with each other, but that many of the Paretian quasi-orderings that satisfy the two criteria of the characterization theorem in Mitra and Basu (2007) fail to satisfy the invariance properties. This demonstrates that the two invariance properties (and, hence, also stationarity) severely limit the

ways in which the finite anonymity axiom can be extended in order to produce stronger notions of intergenerational equity. In particular, we show that any set of permissible permutations for a stationary Paretian quasi-ordering which is generated by a filter basis contains only permutations that leave infinitely many generations unaffected.

The rest of the paper is organized as follows. In section 2 we describe the framework of our analysis and introduce some terminology with regard to permutations. In section 3 we state and discuss the characterization theorem for Paretian quasi-orderings from Mitra and Basu (2007) and construct a large class of sets of permutations which satisfy the characterizing conditions of cyclicality and group structure. We begin section 4 by introducing the notions of stationarity, shift-invariance, and truncation-invariance. Then we prove characterization theorems for Paretian quasi-orderings satisfying either one of the two invariance properties. Finally, we demonstrate that these invariance properties are consistent with extended anonymity and the Pareto axiom, but that they present a severe restriction to extending the finite anonymity axiom. Section 5 summarizes our findings.

2 Terminology and notation

In this section we introduce the objects of study of the present paper, namely Paretian quasi-orderings on infinite utility streams. Furthermore, we collect some terminology regarding permutations that will be used throughout the paper.

2.1 Paretian quasi-orderings

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of all natural numbers, let $Y = [0, 1]$ be the unit interval on the real line, and let $X = Y^{\mathbb{N}}$. An element $\mathbf{x} = (x_1, x_2, \ldots) \in X$ is interpreted as an infinite utility stream, whereby $x_n \in Y$ denotes the utility of generation $n \in \mathbb{N}$.

A quasi-ordering $\succeq$ on X is a binary relation on X which is reflexive and transitive. We interpret $\mathbf{x} \succeq \mathbf{x}'$ in the sense that $\mathbf{x}$ is socially preferred to $\mathbf{x}'$. The symmetric and asymmetric components of $\succeq$ are defined in the usual way and will be denoted by $\sim$ and $\succ$, respectively. That is, $\mathbf{x} \sim \mathbf{x}'$ if and only if both $\mathbf{x} \succeq \mathbf{x}'$ and $\mathbf{x}' \succeq \mathbf{x}$ are true, and $\mathbf{x} \succ \mathbf{x}'$ if and only if both $\mathbf{x} \succeq \mathbf{x}'$ and $\mathbf{x} \not\sim \mathbf{x}'$ are true. If $\succeq$ and $\succeq'$ are two binary relations on X, then we say that $\succeq'$ is a subrelation of $\succeq$ if the following two implications hold for all pairs $(\mathbf{x}, \mathbf{x}') \in X^2$:

$$\mathbf{x} \succ' \mathbf{x}' \Rightarrow \mathbf{x} \succ \mathbf{x}' \quad \text{and} \quad \mathbf{x} \sim' \mathbf{x}' \Rightarrow \mathbf{x} \sim \mathbf{x}'.$$

For any two utility streams $\mathbf{x} = (x_1, x_2, \ldots) \in X$ and $\mathbf{x}' = (x'_1, x'_2, \ldots) \in X$ we write $\mathbf{x} \geq \mathbf{x}'$ if $x_n \geq x'_n$ holds for all $n \in \mathbb{N}$, and we write $\mathbf{x} > \mathbf{x}'$ if $\mathbf{x} \geq \mathbf{x}'$ and $\mathbf{x} \neq \mathbf{x}'$. A quasi-ordering $\succeq$ on X is said to satisfy the strong Pareto axiom if for all pairs $(\mathbf{x}, \mathbf{x}') \in X^2$ it holds that

$$\mathbf{x} > \mathbf{x}' \Rightarrow \mathbf{x} \succ \mathbf{x}'.$$

A quasi-ordering satisfying this axiom is called a Paretian quasi-ordering.

2.2 Permutations

A permutation of $\mathbb{N}$ is a bijective function $\pi : \mathbb{N} \mapsto \mathbb{N}$. We denote the set of all permutations of $\mathbb{N}$ by $\mathbf{P}$. The identity mapping $\mathrm{id}_{\mathbb{N}} : \mathbb{N} \mapsto \mathbb{N}$ is defined by $\mathrm{id}_{\mathbb{N}}(n) = n$ for all $n \in \mathbb{N}$. For any pair of permutations $(\pi, \pi') \in \mathbf{P}^2$, the concatenation of π and π' is another permutation, denoted by $\pi \circ \pi' \in \mathbf{P}$ and defined by $\pi \circ \pi'(n) = \pi(\pi'(n))$ for all $n \in \mathbb{N}$. Since any permutation is bijective, it has an inverse which we denote by π^{-1}. It holds that $\pi^{-1} \in \mathbf{P}$ and $\pi \circ \pi^{-1} = \pi^{-1} \circ \pi = \mathrm{id}_{\mathbb{N}}$. The above properties reflect the fact that $(\mathbf{P}, \circ)$ has the algebraic structure of a group.

It will be useful to define some important classes of permutations. The simplest non-trivial permutations are the transpositions. For any pair $(i, j) \in \mathbb{N}^2$, the transposition $\tau_{i,j} \in \mathbf{P}$ is defined by

$$\tau_{i,j}(n) = \begin{cases} j & \text{if } n = i, \\ i & \text{if } n = j, \\ n & \text{if } n \notin \{i, j\}. \end{cases} \tag{1}$$

Note that $\tau_{i,j} = \tau_{j,i}$ holds for all $(i, j) \in \mathbb{N}^2$ and that $\tau_{i,j} = \mathrm{id}_{\mathbb{N}}$ holds whenever $i = j$. A permutation $\pi \in \mathbf{P}$ is called finite if there exists $k \in \mathbb{N}$ such that $\pi(n) = n$ holds for all $k \geq n$. We denote the set of all finite permutations by $\mathbf{F}$. It is known that $(\mathbf{F}, \circ)$ is the smallest subgroup of $(\mathbf{P}, \circ)$ that contains all transpositions; see, e.g., Rotman (1995, theorem 1.3).

For given $\pi \in \mathbf{P}$ and $k \in \mathbb{N} \cup \{0\}$ we denote by π^k the k-th iterate of π, that is, $\pi^0 = \mathrm{id}_{\mathbb{N}}$ and $\pi^k = \pi \circ \pi^{k-1}$ for all $k \in \mathbb{N}$. Given $\pi \in \mathbf{P}$ and $n \in \mathbb{N}$, the orbit of π starting in n is defined as

$$\mathcal{O}_\pi(n) = \{\pi^k(n) \,|\, k \in \mathbb{N} \cup \{0\}\}.$$

A permutation $\pi \in \mathbf{P}$ is called cyclic, if $\mathcal{O}_\pi(n)$ is a finite set for all $n \in \mathbb{N}$. We denote by $\mathbf{C}$ the set of all cyclic permutations of $\mathbb{N}$.

It is worth emphasizing that the set of cyclic permutations $\mathbf{C}$ itself is not closed under concatenation such that $(\mathbf{C}, \circ)$ is not a group. This can be seen by means of the following simple example.

Example 1 Define $\pi \in \mathbf{P}$ by

$$\pi(n) = \begin{cases} n + 1 & \text{if } n \text{ is odd}, \\ n - 1 & \text{if } n \text{ is even}, \end{cases}$$

and let $\pi' \in \mathbf{P}$ be given by

$$\pi'(n) = \begin{cases} 1 & \text{if } n = 1, \\ n + 1 & \text{if } n \text{ is even}, \\ n - 1 & \text{if } n > 1 \text{ is odd}. \end{cases}$$

All orbits of π have length 2 and all orbits of π' have length 1 or 2. Hence, both π and π' are cyclic permutations. Their concatenation $\pi \circ \pi'$, however, is not cyclic. Indeed, we have

$$\pi \circ \pi'(n) = \begin{cases} 2 & \text{if } n = 1, \\ n+2 & \text{if } n \text{ is even}, \\ n-2 & \text{if } n > 1 \text{ is odd} \end{cases}$$

from which it is easy to see that the permutation $\pi \circ \pi'$ has a single orbit that consists of all $n \in \mathbb{N}$. Obviously, this orbit is not finite so that $\pi \circ \pi'$ is not cyclic.

3 Intergenerational equity

In this section we analyze how strong a notion of intergenerational equity one can impose on a quasi-ordering on X without violating the strong Pareto principle. This analysis is based on a result by Mitra and Basu (2007) which is stated as proposition 1 below.

3.1 Anonymity axioms and the Mitra-Basu result

Permutations of $\mathbb{N}$ arise quite naturally in the context of intergenerational equity. To see this, note that any element $\mathbf{x} \in X$ can be considered as a mapping from $\mathbb{N}$ to Y. Given $\mathbf{x} = (x_1, x_2, \ldots) \in X$ and $\pi \in \mathbf{P}$, we may also consider the concatenation of $\mathbf{x}$ and π, which is given by

$$\mathbf{x} \circ \pi = (x_{\pi(1)}, x_{\pi(2)}, \ldots).$$

The utility stream $\mathbf{x} \circ \pi$ describes the allocation of utility to different generations after the generation numbers have been permuted according to π. Since such a permutation simply reflects a re-labelling of generations, intergenerational equity is typically formulated by the requirement that

$$\mathbf{x} \circ \pi \sim \mathbf{x} \qquad \text{for all } \mathbf{x} \in X \tag{2}$$

holds for a certain class of permutations of $\mathbb{N}$. The larger this class is, the stronger is the notion of intergenerational equity. For any non-empty subset $\mathbf{G} \subseteq \mathbf{P}$, we say that $\succeq$ satisfies $\mathbf{G}$-anonymity if (2) holds for all $\pi \in \mathbf{G}$.

As a minimal requirement for a definition of intergenerational equity one should allow for the re-labelling of any two generations; see, e.g., Diamond (1965). Such a re-labelling of two generations is formally described by a transposition as defined in (1). It will follow from proposition 1 below that, whenever (2) holds for all transpositions, it must actually hold for all finite permutations $\pi \in \mathbf{F}$. Thus, the weakest sensible notion of intergenerational equity is $\mathbf{F}$-anonymity, which is also known as the finite anonymity axiom.

On the other hand, it has been proved that one cannot require (2) to hold for all permutations $\pi \in \mathbf{P}$ without violating the strong Pareto axiom; see Lauwers (1997a, lemma 1). The question therefore arises, of how big a set $\mathbf{G}$ of permutations can be such that the strong Pareto axiom

and **G**-anonymity are consistent with each other. Any notion of intergenerational equity that is stronger than finite anonymity is called extended anonymity. More formally, the Paretian quasi-ordering $\succeq$ satisfies extended anonymity if it satisfies **G**-anonymity for some **G** with $\mathbf{F} \subset \mathbf{G}$.

Before we can state the main result from Mitra and Basu (2007), we have to introduce two more concepts. First, we define the set of permissible permutations for a given Paretian quasi-ordering $\succeq$ on X by

$$\Pi(\succeq) = \{\pi \in \mathbf{P} \,|\, \mathbf{x} \circ \pi \sim \mathbf{x} \text{ holds for all } \mathbf{x} \in X\}.$$

Second, given a non-empty subset $\mathbf{G} \subseteq \mathbf{P}$ one can define a binary relation $\succeq_{\mathbf{G}}$ on X called the **G**-grading principle. According to that principle $\mathbf{x} \succeq_{\mathbf{G}} \mathbf{x}'$ holds if and only if there exists a permutation $\pi \in \mathbf{G}$ such that $\mathbf{x} \circ \pi \geq \mathbf{x}'$. The basic idea of the **G**-grading principle originates from Suppes (1966) and has later been studied and extended by several scholars including Sen (1970) and Svensson (1980).

Proposition 1 *(Mitra and Basu, 2007)*
(a) *If $\succeq$ is a Paretian quasi-ordering on X, then it follows that $\Pi(\succeq) \subseteq \mathbf{C}$ and that $(\Pi(\succeq), \circ)$ is a subgroup of $(\mathbf{P}, \circ)$.*
(b) *For every set $\mathbf{G} \subseteq \mathbf{C}$ such that $(\mathbf{G}, \circ)$ is a subgroup of $(\mathbf{P}, \circ)$, the **G**-grading principle $\succeq_{\mathbf{G}}$ defines a Paretian quasi-ordering on X such that $\Pi(\succeq_{\mathbf{G}}) = \mathbf{G}$.*

This result provides a complete characterization of those subsets of **P** that qualify as sets of permissible permutations for Paretian quasi-orderings on X. The fact that any permutation $\pi \in \Pi(\succeq)$ must be cyclic is a consequence of the strong Pareto axiom, whereas the group property of $(\Pi(\succeq), \circ)$ derives from reflexivity and transitivity of $\succeq$. Note that proposition 1 translates properties of the binary relation $\succeq$ into properties of its set of permissible permutations $\Pi(\succeq)$. It is also worth emphasizing that the **G**-grading principle is the least complete Paretian quasi-ordering on X satisfying **G**-anonymity, that is, any **G**-anonymous Paretian quasi-ordering $\succeq$ on X contains $\succeq_{\mathbf{G}}$ as a subrelation; see Banerjee (2006, proposition 2).

3.2 Extended anonymity

In this subsection we present a large class of sets $\mathbf{G} \subseteq \mathbf{C}$ that allow the construction (via the **G**-grading principle) of Paretian quasi-orderings satisfying extended anonymity. To this end, we first have to introduce some notation.

We denote by $\mathcal{M}$ the set of all strictly increasing (infinite) sequences of natural numbers, that is,

$$\mathcal{M} = \{(m_n)_{n\in\mathbb{N}} \,|\, m_n \in \mathbb{N} \text{ and } m_n < m_{n+1} \text{ for all } n \in \mathbb{N}\}.$$

A non-empty subset $\mathcal{F}$ of $\mathcal{M}$ is called a filter base on $\mathcal{M}$ if[2]
(a) for all $(m_n)_{n\in\mathbb{N}} \in \mathcal{F}$ and $(m'_n)_{n\in\mathbb{N}} \in \mathcal{F}$ there exists $(k_n)_{n\in\mathbb{N}} \in \mathcal{F}$ such that

$$\{k_n \,|\, n \in \mathbb{N}\} \subseteq \{m_n \,|\, n \in \mathbb{N}\} \cap \{m'_n \,|\, n \in \mathbb{N}\}; \tag{3}$$

[2]Condition (b) is normally not part of the definition of a filter base but turns out to be helpful in the present context. For more on filter bases see, e.g., Bourbaki (1989).

(b) for all $k \in \mathbb{N}$ there exists $(m_n)_{n\in\mathbb{N}} \in \mathcal{F}$ such that $m_1 \geq k$.

A filter base is therefore a set of sequences, $\mathcal{F}$, such that any two sequences in $\mathcal{F}$ have at least one common subsequence that is contained in $\mathcal{F}$, and such that $\mathcal{F}$ contains sequences with arbitrarily large first elements.

Given a permutation $\pi \in \mathbb{N}$ and a set $A \subseteq \mathbb{N}$ we say that A is invariant under π if $\pi(A) := \{\pi(n) \,|\, n \in A\} = A$. Moreover, for every $n \in \mathbb{N}$ we define $A_n = \{1, 2, \ldots, n\}$. Finally, given a filter base $\mathcal{F}$, the set $\mathbf{G}(\mathcal{F}) \subseteq \mathbf{P}$ is defined as the set of all permutation $\pi \in \mathbf{P}$ with the following property: there exists $(m_n)_{n\in\mathbb{N}} \in \mathcal{F}$ such that A_{m_n} is invariant under π for all $n \in \mathbb{N}$. We have the following result.

Lemma 1 *For every filter base $\mathcal{F}$ in $\mathcal{M}$ it holds that $\mathbf{F} \subseteq \mathbf{G}(\mathcal{F}) \subseteq \mathbf{C}$ and that $(\mathbf{G}(\mathcal{F}), \circ)$ is a group.*

Proof: We first show that $\mathbf{F} \subseteq \mathbf{G}(\mathcal{F})$. Consider any $\pi \in \mathbf{F}$. Then there exists $k \in \mathbb{N}$ such that A_m is invariant under π for all $m \geq k$. From the definition of a filter base we know that there exists $(m_n)_{n\in\mathbb{N}} \in \mathcal{F}$ such that $m_1 \geq k$. Since $(m_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence, it follows that $m_n \geq k$ for all $n \in \mathbb{N}$. Taking these observations together we see that A_{m_n} is invariant under π for all $n \in \mathbb{N}$, which implies $\pi \in \mathbf{G}(\mathcal{F})$.

Next we show that every $\pi \in \mathbf{G}(\mathcal{F})$ is cyclic. Let $k \in \mathbb{N}$ be arbitrary and let $(m_n)_{n\in\mathbb{N}} \in \mathcal{F}$ be such that A_{m_n} is invariant under π for all $n \in \mathbb{N}$. Pick any $n \in \mathbb{N}$ such that $k \leq m_n$. Since A_{m_n} is invariant under π it follows that the orbit $\mathcal{O}_\pi(k)$ is contained in A_{m_n}. Therefore, we get $|\mathcal{O}_\pi(k)| \leq |A_{m_n}| = m_n$. This shows that $\mathcal{O}_\pi(k)$ is a finite set. Since k was arbitrary, it follows that all orbits of π are finite sets and, hence, that $\pi \in \mathbf{C}$.

Finally, we show that $(\mathbf{G}(\mathcal{F}), \circ)$ is a group. Since every set is invariant under the identity $\mathrm{id}_\mathbb{N}$, it follows that $\mathrm{id}_\mathbb{N} \in \mathbf{G}(\mathcal{F})$. Since any set $A \subseteq \mathbb{N}$ that is invariant under a permutation $\pi \in \mathbf{P}$ is also invariant under its inverse, it follows that $\mathbf{G}(\mathcal{F})$ contains with every π also the inverse π^{-1}. Now consider an arbitrary pair (π, π') of permutations in $\mathbf{G}(\mathcal{F})$. There exist sequences $(m_n)_{n\in\mathbb{N}}$ and $(m'_n)_{n\in\mathbb{N}}$ such that A_{m_n} and $A_{m'_n}$ are invariant under π and π', respectively, for all $n \in \mathbb{N}$. Choose a sequence $(k_n)_{n\in\mathbb{N}} \in \mathcal{F}$ such that (3) holds and consider an arbitrary element k_ℓ of this sequence. Since $k_\ell \in \{m_n \,|\, n \in \mathbb{N}\} \cap \{m'_n \,|\, n \in \mathbb{N}\}$ it follows that A_{k_ℓ} is invariant both under π and under π'. This implies that it is also invariant under $\pi \circ \pi'$. Since k_ℓ was an arbitrary element of the sequence $(k_n)_{n\in\mathbb{N}}$, we conclude that $\pi \circ \pi' \in \mathbf{G}(\mathcal{F})$ and it follows that $(\mathbf{G}(\mathcal{F}), \circ)$ must be a group. □

The lemma shows that, for every filter base $\mathcal{F}$, there exists a group of cyclic permutations $(\mathbf{G}(\mathcal{F}), \circ)$ that contains $(\mathbf{F}, \circ)$ as a subgroup. From proposition 1 it follows that for each of these groups one can use the $\mathbf{G}(\mathcal{F})$-grading principle to construct a Paretian quasi-ordering $\succeq_{\mathbf{G}(\mathcal{F})}$ that satisfies $\mathbf{G}(\mathcal{F})$-anonymity. In order to obtain extended anonymity, however, we have to ensure that the inclusion $\mathbf{F} \subset \mathbf{G}(\mathcal{F})$ is strict. This will be done in the next lemma. To be able to state it, we call a filter base $\mathcal{F}$ a filter base of tails if and only if for all sequences $(m_n)_{n\in\mathbb{N}} \in \mathcal{F}$ there exists $k \in \mathbb{N}$ such that $\{k, k+1, \ldots\} \subseteq \{m_n \,|\, n \in \mathbb{N}\}$. In words, a filter base is a filter base of tails if all of its sequences contain all but finitely many natural numbers.

Lemma 2 *Let $\mathcal{F}$ be a filter base on $\mathcal{M}$. It holds that $\mathbf{F} = \mathbf{G}(\mathcal{F})$ if and only if $\mathcal{F}$ is a filter base of tails.*

PROOF: Suppose that $\mathcal{F}$ is a filter base of tails and that $\pi \in \mathbf{G}(\mathcal{F})$. There exists a sequence $(m_n)_{n\in\mathbb{N}} \in \mathcal{F}$ such that A_{m_n} is invariant under π for all $n \in \mathbb{N}$. Since $\mathcal{F}$ is a filter base of tails, this implies that A_ℓ is invariant under π for all sufficiently large ℓ, say for all $\ell \geq k$. But this implies obviously that $\pi(\ell) = \ell$ for all $\ell > k$. This shows that π is finite. Since π was chosen arbitrarily from $\mathbf{G}(\mathcal{F})$ it follows that $\mathbf{G}(\mathcal{F}) \subseteq \mathbf{F}$. From lemma 1 we already know that $\mathbf{F} \subseteq \mathbf{G}(\mathcal{F})$ such that the proof of $\mathbf{F} = \mathbf{G}(\mathcal{F})$ is complete.

Conversely, assume that $\mathbf{F} = \mathbf{G}(\mathcal{F})$. We proceed by contradiction and therefore suppose that the filter base $\mathcal{F}$ is not a filter base of tails. This means that there exists $(m_n)_{n\in\mathbb{N}} \in \mathcal{F}$ and $(n_\ell)_{\ell\in\mathbb{N}} \in \mathcal{M}$ such that $m_{n_\ell} + 1 < m_{n_\ell+1}$ holds for all $\ell \in \mathbb{N}$. Now let us define the permutation

$$\pi(k) = \begin{cases} m_{n_\ell+1} & \text{if } k = m_{n_\ell} + 1 \text{ for some } \ell \in \mathbb{N}, \\ m_{n_\ell} + 1 & \text{if } k = m_{n_\ell+1} \text{ for some } \ell \in \mathbb{N}, \\ k & \text{otherwise.} \end{cases}$$

Obviously, this is not a finite permutation. On the other hand, it holds that $\pi \in \mathbf{G}(\mathcal{F})$. Since this is a contradiction to our assumption $\mathbf{F} = \mathbf{G}(\mathcal{F})$ it must be the case that $\mathcal{F}$ is a filter base of tails. □

We continue by giving some examples of filter bases $\mathcal{F}$ and their corresponding sets $\mathbf{G}(\mathcal{F})$. Let us start with the filter base $\mathcal{F}_1$ consisting of all (homogeneous) arithmetic progressions. That is, a sequence $(m_n)_{n\in\mathbb{N}}$ is in $\mathcal{F}_1$ if and only if there exists $k \in \mathbb{N}$ such that $m_n = kn$ for all $n \in \mathbb{N}$. The corresponding set of permutations $\mathbf{G}_1 = \mathbf{G}(\mathcal{F}_1)$ consists of all permutations $\pi \in \mathbf{P}$ for which there exists $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ the set A_{kn} is invariant under π. The set of permutations $\mathbf{G}_1$ is known as the set of fixed-step permutations. It was introduced by Lauwers (1997b) and is the set of permutations that has been most often used to formulate extended anonymity axioms; see Fleurbaey and Michel (2003), Banerjee (2006), Kamaga and Kojima (2008, 2009), or Asheim and Banerjee (2010). A generalization of fixed-step permutations is the set $\mathbf{G}_\ell = \mathbf{G}(\mathcal{F}_\ell)$ where, for any $\ell \in \mathbb{N}$, the filter base $\mathcal{F}_\ell$ is defined as follows: a sequence $(m_n)_{n\in\mathbb{N}}$ is in $\mathcal{F}_\ell$ if and only if there exists $k \in \mathbb{N}$ such that $m_n = (kn)^\ell$ for all $n \in \mathbb{N}$. The following result proves that $\mathcal{F}_\ell$ is indeed a filter base on $\mathcal{M}$.

Lemma 3 *For any $\ell \in \mathbb{N}$ it holds that $\mathcal{F}_\ell$ is a filter base. Moreover, it holds for all $(\ell, m) \in \mathbb{N}^2$ with $m > 1$ that $\mathbf{G}_\ell \subset \mathbf{G}_{\ell m}$.*

PROOF: Suppose that $(m_n)_{n\in\mathbb{N}}$ and $(m'_n)_{n\in\mathbb{N}}$ are two sequences in $\mathcal{F}_\ell$. Then there exist natural numbers k and k' such that $m_n = (kn)^\ell$ and $m'_n = (k'n)^\ell$ hold for all $n \in \mathbb{N}$. Now define $K = kk'$ and $k_n = (Kn)^\ell$ for all $n \in \mathbb{N}$. This implies that $k_n = (kn_1)^\ell$ with $n_1 = k'n$ and it follows that $(k_n)_{n\in\mathbb{N}}$ is a subsequence of $(m_n)_{n\in\mathbb{N}}$. Analogously, one can show that $(k_n)_{n\in\mathbb{N}}$ is a subsequence of $(m'_n)_{n\in\mathbb{N}}$. Moreover, since $(k_n)_{n\in\mathbb{N}} \in \mathcal{F}_\ell$ by construction, it follows that condition

(3) is satisfied. Obviously, it holds that $\mathcal{F}_\ell$ contains sequences with arbitrary large first element so that $\mathcal{F}_\ell$ is indeed a filter base on $\mathcal{M}$.

Now suppose that $\ell \in \mathbb{N}$ and $m \in \mathbb{N}$ are given with $m > 1$ and consider any permutation $\pi \in \mathbf{G}_\ell$. Then it follows that there exists $k \in \mathbb{N}$ such that $A_{(kn)^\ell}$ is invariant under π for all $n \in \mathbb{N}$. For $n = k^{m-1}n_1^m$, where $n_1 \in \mathbb{N}$ is arbitrary, we have $(kn)^\ell = (kn_1)^{\ell m}$ and it follows therefore that $A_{(kn_1)^{\ell m}}$ is invariant under π. Since $n_1 \in \mathbb{N}$ was arbitrary, this proves $\pi \in \mathbf{G}_{\ell m}$ and we obtain $\mathbf{G}_\ell \subseteq \mathbf{G}_{\ell m}$. The proof of $\mathbf{G}_\ell \neq \mathbf{G}_{\ell m}$ is quite obvious and therefore omitted. □

Lemmas 1-3 clearly demonstrate that extended anonymity is not only consistent with the strong Pareto axiom, but that there are actually many different subsets of $\mathbf{C}$ that can form the set of permissible permutations for an appropriately chosen Paretian quasi-ordering. It follows in particular that a permissible permutation can interchange generations that are arbitrarily far apart, and that there are permissible permutation that do not let any generation unaffected. Lemmas 1 and 3 together show in particular that the set of fixed-step permutations, $\mathbf{G}_1$, is not a maximal group of cyclic permutations.

We would also like to point out that every permutation that belongs to some set $\mathbf{G}(\mathcal{F})$ is a so-called variable-step permutation; see Fleurbaey and Michel (2003). A variable-step permutation π is characterized by the property that for every $k \in \mathbb{N}$ there exists $n \geq k$ such that A_n is invariant under π. The set of variable-step permutations is a subset of $\mathbf{C}$ but it is not a group when endowed with the operation $\circ$. An example of a cyclic permutation $\bar{\pi}$ that is not a variable-step permutation and, hence, not contained in $\mathbf{G}(\mathcal{F})$ for any filter base $\mathcal{F}$ is

$$\bar{\pi}(n) = \begin{cases} 3 & \text{if } n = 1, \\ 1 & \text{if } n = 3, \\ n+3 & \text{if } n \text{ is even}, \\ n-3 & \text{if } n > 3 \text{ is odd}. \end{cases} \tag{4}$$

It is straightforward to see that every orbit of $\bar{\pi}$ has length 2 such that $\bar{\pi} \in \mathbf{C}$. The fact that every orbit has length 2 also implies that $\bar{\pi} = \bar{\pi}^{-1}$ and that $(\{\mathrm{id}_{\mathbb{N}}, \bar{\pi}\}, \circ)$ is a group. On the other hand, there does not exist any $n \in \mathbb{N}$ such that A_n is invariant under $\bar{\pi}$ which proves that $\bar{\pi}$ is not a variable-step permutation. In order to check whether $\bar{\pi}$ can be a permissible permutation for any Paretian quasi-ordering $\succeq$ that satisfies extended anonymity, we have to find out whether there exists a set $\bar{\mathbf{G}}$ such that $\bar{\pi} \in \bar{\mathbf{G}}$, $\mathbf{F} \subset \bar{\mathbf{G}} \subseteq \mathbf{C}$, and such that $(\bar{\mathbf{G}}, \circ)$ forms a group. That this is indeed the case can be seen as follows.

Let $\bar{\mathbf{G}}_0 = \mathbf{F} \cup \{\bar{\pi}\}$ and let $(\bar{\mathbf{G}}, \circ)$ be the subgroup of $(\mathbf{P}, \circ)$ generated by $\bar{\mathbf{G}}_0$.[3] It follows from Rotman (1995, theorem 2.7) and the fact that both $(\mathbf{F}, \circ)$ and $(\{\mathrm{id}_{\mathbb{N}}, \bar{\pi}\}, \circ)$ are groups that this construction yields

$$\bar{\mathbf{G}} = \mathbf{F} \cup \{\pi_1 \circ \bar{\pi} \circ \pi_2 \,|\, (\pi_1, \pi_2) \in \mathbf{F}^2\}.$$

Note that any permutation $\pi \in \bar{\mathbf{G}}$ is either finite or there exists $k \in \mathbb{N}$ such that $\pi(n) = \bar{\pi}(n)$ holds for all $n \geq k$. Since $\bar{\pi}$ is cyclic, this shows in particular that $\bar{\mathbf{G}} \subseteq \mathbf{C}$.

[3] See page 22 in Rotman (1995) for the definition of what it means that a subgroup is generated by a given set.

4 Stationarity

Let us now turn to stationarity as introduced by Koopmans (1960).[4] To define this concept we use the following notation. For each $\mathbf{x} = (x_1, x_2, \ldots) \in X$ and each $y \in Y$ we denote by $(y, \mathbf{x})$ the utility stream $(y, x_1, x_2, \ldots) \in X$. The quasi-ordering $\succeq$ on X is said to be stationary if for all $\mathbf{x} \in X$, all $\mathbf{x}' \in X$, and all $y \in Y$ it holds that

$$\mathbf{x} \succeq \mathbf{x}' \Leftrightarrow (y, \mathbf{x}) \succeq (y, \mathbf{x}').$$

For reasons that will become apparent shortly, it is convenient to break Koopmans' original definition into two parts. More specifically, we say that the quasi-ordering $\succeq$ on X is shift-invariant if for all $\mathbf{x} \in X$, all $\mathbf{x}' \in X$, and all $y \in Y$ it holds that

$$\mathbf{x} \succeq \mathbf{x}' \Rightarrow (y, \mathbf{x}) \succeq (y, \mathbf{x}'),$$

and we say that $\succeq$ is truncation-invariant if for all $\mathbf{x} \in X$, all $\mathbf{x}' \in X$, and all $y \in Y$ it holds that

$$(y, \mathbf{x}) \succeq (y, \mathbf{x}') \Rightarrow \mathbf{x} \succeq \mathbf{x}'.$$

Using this terminology it follows that $\succeq$ is stationary if and only if it is both shift-invariant and truncation-invariant.

4.1 Shift-invariance

We continue to use the strategy from the previous section and discuss properties of a quasi-ordering $\succeq$ on X by means of its set of permissible permutations $\Pi(\succeq)$. To this end, let us now define the shift operator $S : \mathbf{P} \mapsto \mathbf{P}$ by

$$S\pi(n) = \begin{cases} 1 & \text{if } n = 1, \\ \pi(n-1) + 1 & \text{otherwise.} \end{cases}$$

Notice that the property $S\pi(1) = 1$ implies that $S\pi \in \mathbf{P}_1 := \{\pi \in \mathbf{P} \,|\, \pi(1) = 1\}$. The set $\mathbf{P}_1$ is the set of all permutations which do not affect the first generation. This set is a strict subset of $\mathbf{P}$ and it is easy to see that $(\mathbf{P}_1, \circ)$ is a subgroup of $(\mathbf{P}, \circ)$. Because $S\pi \in \mathbf{P}_1$ for all $\pi \in \mathbf{P}$ we can consider the shift operator S also as a mapping from $\mathbf{P}$ to $\mathbf{P}_1$. Note furthermore that

$$(y, \mathbf{x}) \circ S\pi = (y, \mathbf{x} \circ \pi)$$

holds for all $y \in Y$ and all $\mathbf{x} \in X$. For instance, in example 1 from subsection 2.2 we have $\pi' = S\pi$. The relevance of the shift operator for our analysis derives from the following two lemmas.

Lemma 4 *Let $\succeq$ be a shift-invariant quasi-ordering on X and let $\Pi(\succeq)$ be the set of permissible permutations for $\succeq$. Then it holds that $S\Pi(\succeq) \subseteq \Pi(\succeq)$.*

[4] Dutta (2007) refers to stationarity as separability.

PROOF: Consider any $\pi \in \Pi(\succeq)$. This means that $\mathbf{x} \circ \pi \sim \mathbf{x}$ holds for all $\mathbf{x} \in X$. Because of shift-invariance of $\succeq$ this property implies that $(y, \mathbf{x} \circ \pi) \sim (y, \mathbf{x})$ holds for all $\mathbf{x} \in X$ and all $y \in Y$. Since $(y, \mathbf{x} \circ \pi)$ can be written as $(y, \mathbf{x}) \circ S\pi$, we have shown that $(y, \mathbf{x}) \circ S\pi \sim (y, \mathbf{x})$ holds for all $y \in Y$ and all $\mathbf{x} \in X$. This means that $S\pi \in \Pi(\succeq)$. Since $\pi \in \Pi(\succeq)$ was chosen arbitrarily, the proof of the lemma is complete. □

For the next lemma recall the **G**-grading principle introduced in subsection 3.1.

Lemma 5 *Let* $\mathbf{G} \subseteq \mathbf{C}$ *be given such that* $(\mathbf{G}, \circ)$ *is a subgroup of* $(\mathbf{P}, \circ)$ *and let* $\succeq_{\mathbf{G}}$ *be defined by the* **G***-grading principle. If* $S\mathbf{G} \subseteq \mathbf{G}$*, then it follows that* $\succeq_{\mathbf{G}}$ *is shift-invariant.*

PROOF: Consider an arbitrary pair $(\mathbf{x}, \mathbf{x}') \in X^2$ and suppose that $\mathbf{x} \succeq_{\mathbf{G}} \mathbf{x}'$. This means that there exists $\pi \in \mathbf{G}$ such that $\mathbf{x} \circ \pi \geq \mathbf{x}'$. For any $y \in Y$ we therefore have

$$(y, \mathbf{x}) \circ S\pi = (y, \mathbf{x} \circ \pi) \geq (y, \mathbf{x}').$$

Since $S\pi \in S\mathbf{G} \subseteq \mathbf{G}$, it follows therefore from the above inequality that $(y, \mathbf{x}) \succeq_{\mathbf{G}} (y, \mathbf{x}')$. This completes the proof. □

The following theorem is an immediate consequence of proposition 1 and lemmas 4-5 and summarizes these results in a way that is directly comparable to proposition 1. Whereas proposition 1 provides a characterization of those sets of permutations that are permissible for Paretian quasi-orderings, theorem 1 provides a characterization of those sets of permutations that are permissible for shift-invariant Paretian quasi-orderings.

Theorem 1 (a) *If* $\succeq$ *is a shift-invariant Paretian quasi-ordering on* X*, then it follows that* $\Pi(\succeq) \subseteq \mathbf{C}$*, that* $(\Pi(\succeq), \circ)$ *is a subgroup of* $(\mathbf{P}, \circ)$*, and that* $S\Pi(\succeq) \subseteq \Pi(\succeq)$*.*
(b)*For every set* $\mathbf{G} \subseteq \mathbf{C}$ *such that* $(\mathbf{G}, \circ)$ *is a subgroup of* $(\mathbf{P}, \circ)$ *and such that* $S\mathbf{G} \subseteq \mathbf{G}$*, the* **G***-grading principle defines a shift-invariant Paretian quasi-ordering* $\succeq_{\mathbf{G}}$ *on* X *such that* $\Pi(\succeq_{\mathbf{G}}) = \mathbf{G}$*.*

4.2 Truncation-invariance

In this subsection we want to derive a characterization for truncation-invariant Paretian quasi-orderings that is analogous to theorem 1. We begin by defining the backshift operator $B : \mathbf{P}_1 \mapsto \mathbf{P}$ via

$$B\pi(n) = \pi(n+1) - 1$$

for all $n \in \mathbb{N}$. Note that the domain of B is the set $\mathbf{P}_1$ and not the set of all permutations $\mathbf{P}$. Note furthermore that for every $\pi \in \mathbf{P}_1$ and every $\pi' \in \mathbf{P}$ it holds that $S \circ B\pi = \pi$ and $B \circ S\pi' = \pi'$. Thus, $B : \mathbf{P}_1 \mapsto \mathbf{P}$ and $S : \mathbf{P} \mapsto \mathbf{P}_1$ are inverse to each other.

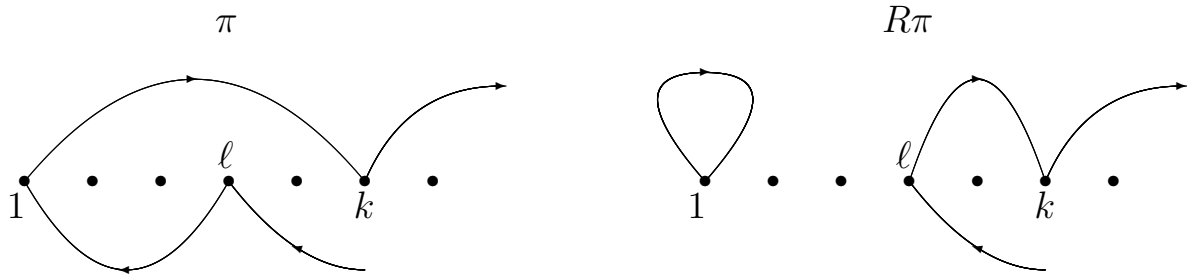

Figure 1: Illustration of the replacement operator R with $k = \pi(1)$ and $\ell = \pi^{-1}(1)$.

The backshift operator B can be extended to all of $\mathbf{P}$ in many different ways, but only the following one is useful for our purpose. We define the replacement operator $R : \mathbf{P} \mapsto \mathbf{P}$ by

$$R\pi(n) = \begin{cases} 1 & \text{if } n = 1, \\ \pi(1) & \text{if } n = \pi^{-1}(1), \\ \pi(n) & \text{if } n \notin \{1, \pi^{-1}(1)\}. \end{cases}$$

It is easy to see that $R\pi = \pi$ holds whenever $\pi \in \mathbf{P}_1$; see lemma 6(b) below. Now suppose that $\pi \notin \mathbf{P}_1$. Then $R\pi$ and π differ from each other only on the orbit $\mathcal{O}_\pi(1)$. More specifically, the operator R 'cuts out' the first generation from this orbit, turns it into the singleton orbit $\mathcal{O}_{R\pi}(1) = \{1\}$, and 'closes' the resulting hole in $\mathcal{O}_\pi(1)$ by connecting generation $\pi^{-1}(1)$ with generation $\pi(1)$; see figure 1. The following lemma summarizes a few simple properties of the operator R.

Lemma 6 *Let the operator $R : \mathbf{P} \mapsto \mathbf{P}$ be defined as above and let $\tau_{i,j}$ be the transposition of generations i and j as defined in (1). For any permutation $\pi \in \mathbf{P}$ the following statements are true.*
(a) *It holds that $R\pi \in \mathbf{P}_1$.*
(b) *If $\pi \in \mathbf{P}_1$, then it holds that $R\pi = \pi$.*
(c) *It holds that $\pi = (R\pi) \circ \tau_{1,\pi^{-1}(1)}$.*

Proof: Statements (a) and (b) follow immediately from the definitions of R and $\mathbf{P}_1$. To verify statement (c) just consider the following three cases. If $n = 1$ then

$$(R\pi) \circ \tau_{1,\pi^{-1}(1)}(n) = (R\pi) \circ \tau_{1,\pi^{-1}(1)}(1) = R\pi(\pi^{-1}(1)) = \pi(1) = \pi(n).$$

If $n = \pi^{-1}(1)$ then

$$(R\pi) \circ \tau_{1,\pi^{-1}(1)}(n) = (R\pi) \circ \tau_{1,\pi^{-1}(1)}(\pi^{-1}(1)) = R\pi(1) = 1 = \pi(n).$$

And if $n \notin \{1, \pi^{-1}(1)\}$ then

$$(R\pi) \circ \tau_{1,\pi^{-1}(1)}(n) = R\pi(n) = \pi(n).$$

Since these three cases are exhaustive, the proof of part (c) is complete. □

We are now ready to define the truncation operator $T : \mathbf{P} \mapsto \mathbf{P}$ by $T = B \circ R$. Note that this definition is feasible as R maps $\mathbf{P}$ to $\mathbf{P}_1$ according to lemma 6(a) and B maps $\mathbf{P}_1$ to $\mathbf{P}$ by definition. Note furthermore that lemma 6(b) implies that T and B coincide on $\mathbf{P}_1$ so that T is indeed an extension of B to the set of all permutations of $\mathbb{N}$. Finally, it is easy to verify that

$$(y, \mathbf{x}) \circ R\pi = (y, \mathbf{x} \circ T\pi) \tag{5}$$

holds for all $y \in Y$, all $\mathbf{x} \in X$, and all $\pi \in \Pi$. We can now state and prove results for the truncation operator T that correspond to the results for the shift operator stated in lemmas 4-5 and in theorem 1.

Lemma 7 *Let $\succeq$ be a truncation-invariant Paretian quasi-ordering on X and let $\Pi(\succeq)$ be the set of permissible permutations for $\succeq$. Furthermore, assume that $\mathbf{F} \subseteq \Pi(\succeq)$. Then it holds that $T\Pi(\succeq) \subseteq \Pi(\succeq)$.*

Proof: Consider an arbitrary permutation $\pi \in \Pi(\succeq)$. This implies that $(y, \mathbf{x}) \sim (y, \mathbf{x}) \circ \pi$ holds for all $\mathbf{x} \in X$ and all $y \in Y$. Using lemma 6(c), one therefore obtains

$$(y, \mathbf{x}) \sim (y, \mathbf{x}) \circ \pi \sim (y, \mathbf{x}) \circ [(R\pi) \circ \tau_{1,\pi^{-1}(1)}] = [(y, \mathbf{x}) \circ R\pi] \circ \tau_{1,\pi^{-1}(1)}.$$

Moreover, because of $\tau_{1,\pi^{-1}(1)} \in \mathbf{F} \subseteq \Pi(\succeq)$ and (5) it must hold that

$$[(y, \mathbf{x}) \circ R\pi] \circ \tau_{1,\pi^{-1}(1)} \sim (y, \mathbf{x}) \circ R\pi = (y, \mathbf{x} \circ T\pi).$$

Using the transitivity of the quasi-ordering $\succeq$ it follows from the last two displayed formulas that $(y, \mathbf{x}) \sim (y, \mathbf{x} \circ T\pi)$. Together with truncation-invariance of $\succeq$ this implies $\mathbf{x} \sim \mathbf{x} \circ T\pi$. Because $\mathbf{x} \in X$ was chosen arbitrarily, it follows that $T\pi \in \Pi(\succeq)$. Finally, because $\pi \in \Pi(\succeq)$ was chosen arbitrarily, it follows that $T\Pi(\succeq) \subseteq \Pi(\succeq)$. □

Lemma 8 *Let $\mathbf{G} \subseteq \mathbf{C}$ be given such that $(\mathbf{G}, \circ)$ is a subgroup of $(\mathbf{P}, \circ)$ and let $\succeq_{\mathbf{G}}$ be defined by the $\mathbf{G}$-grading principle. If $T\mathbf{G} \subseteq \mathbf{G}$, then it follows that $\succeq_{\mathbf{G}}$ is truncation-invariant.*

Proof: Since $(\mathbf{G}, \circ)$ is a group of cyclic permutations it follows from proposition 1(b) that $\succeq_{\mathbf{G}}$ is a Paretian quasi-ordering and that $\mathbf{G} = \Pi(\succeq_{\mathbf{G}})$. To prove the truncation-invariance of $\succeq_{\mathbf{G}}$, consider two utility streams with common first element, $\mathbf{z} = (z_1, z_2, \ldots) = (y, \mathbf{x})$ and $\mathbf{z}' = (z'_1, z'_2, \ldots) = (y, \mathbf{x}')$, and suppose that $\mathbf{z} \succeq_{\mathbf{G}} \mathbf{z}'$. This means that there exists $\pi \in \mathbf{G}$ such that $\mathbf{z} \circ \pi \geq \mathbf{z}'$. We can rewrite this inequality componentwise as $z_{\pi(n)} \geq z'_n$ for all $n \in \mathbb{N}$. This implies in particular that $z_{\pi(1)} \geq z'_1 = y$ and $y = z_1 = z_{\pi(\pi^{-1}(1))} \geq z'_{\pi^{-1}(1)}$. It follows therefore that $z_{\pi(1)} \geq z'_{\pi^{-1}(1)}$. Finally, we have $z_1 = y = z'_1$ such that $z_1 \geq z'_1$. Recalling the definition of the operator R it follows that $z_{R\pi(n)} \geq z'_n$ holds for all $n \in \mathbb{N}$ or, equivalently, $(y, \mathbf{x}) \circ R\pi \geq (y, \mathbf{x}')$. Because of (5) this can be written as $(y, \mathbf{x} \circ T\pi) \geq (y, \mathbf{x}')$ which obviously implies $\mathbf{x} \circ T\pi \geq \mathbf{x}'$. Finally, because $\mathbf{x} \in X$ was chosen arbitrarily and because of $T\pi \in T\mathbf{G} \subseteq \mathbf{G} = \Pi(\succeq_{\mathbf{G}})$ it must therefore hold that $\mathbf{x} \succeq_{\mathbf{G}} \mathbf{x}'$. This completes the proof of truncation-invariance of $\succeq_{\mathbf{G}}$. □

As in the previous subsection, we summarize our characterization of the permissible sets for truncation-invariant Paretian quasi-orderings in a theorem that is directly comparable to proposition 1. This result is an immediate consequence of proposition 1 and lemmas 7-8.

Theorem 2 **(a)** *If $\succeq$ is a truncation-invariant Paretian quasi-ordering on X such that $\mathbf{F} \subseteq \Pi(\succeq)$, then it follows that $\Pi(\succeq) \subseteq \mathbf{C}$, that $(\Pi(\succeq), \circ)$ is a subgroup of $(\mathbf{P}, \circ)$, and that $T\Pi(\succeq) \subseteq \Pi(\succeq)$.*
(b) *For every set $\mathbf{G} \subseteq \mathbf{C}$ such that $(\mathbf{G}, \circ)$ is a subgroup of $(\mathbf{P}, \circ)$ and such that $T\mathbf{G} \subseteq \mathbf{G}$, the $\mathbf{G}$-grading principle defines a truncation-invariant Paretian quasi-ordering $\succeq_{\mathbf{G}}$ on X such that $\Pi(\succeq_{\mathbf{G}}) = \mathbf{G}$.*

For the sake of completeness, let us add one more lemma.

Lemma 9 *Let $\succeq$ be a stationary Paretian quasi-ordering on X and let $\Pi(\succeq)$ be the set of permissible permutations of $\succeq$. Furthermore, assume that $\mathbf{F} \subseteq \Pi(\succeq)$. Then it holds that $T\Pi(\succeq) = \Pi(\succeq)$.*

Proof: We know from lemma 7 that $T\Pi(\succeq) \subseteq \Pi(\succeq)$. It is therefore sufficient to show that $\Pi(\succeq) \subseteq T\Pi(\succeq)$. Consider an arbitrary $\pi \in \Pi(\succeq)$. From lemma 4 we know that $S\Pi(\succeq) \subseteq \Pi(\succeq)$ such that $T(S\pi) \in T(S\Pi(\succeq)) \subseteq T\Pi(\succeq)$. Finally, we note that $T(S\pi) = \pi$ such that we obtain $\pi \in T\Pi(\succeq)$. This completes the proof. □

4.3 A possibility theorem

In the rest of the paper we explore to what extent the additional requirements of shift-invariance or truncation-invariance restrict the set of Paretian quasi-orderings that satisfy an extended anonymity axiom. We start by proving that stationarity (i.e., the conjunction of shift-invariance and truncation-invarince) is consistent with extended anonymity by exhibiting a group $(\mathbf{G}^*, \circ)$ of cyclic permutations that strictly contains all finite permutations and that satisfies $S\mathbf{G}^* \subseteq \mathbf{G}^*$ and $T\mathbf{G}^* \subseteq \mathbf{G}^*$. To this end we need the following auxiliary lemma.[5]

Lemma 10 *Let $(m_n)_{n\in\mathbb{N}} \in \mathcal{M}$ be given and define $M_n = \{m_\ell \,|\, \ell \in \mathbb{N}\} \cap A_n$ and $a_n = |M_n|$. Then it holds that*

$$\lim_{n\to+\infty} m_n/n = 1 \;\Leftrightarrow\; \lim_{n\to+\infty} n/a_n = 1.$$

Proof: Note that a_n is the number of elements of the sequence $(m_\ell)_{\ell\in\mathbb{N}}$ which are smaller than or equal to n. Since $(m_\ell)_{\ell\in\mathbb{N}}$ is a sequence of strictly increasing natural numbers, it follows that the inequalities

$$a_n \leq n \leq m_n \tag{6}$$

[5]Recall that $\mathcal{M}$ is the set of all infinite and strictly increasing sequences of natural numbers and that $A_n = \{1, 2, \ldots, n\}$.

and

$$m_{a_n} \leq n < m_{a_n+1} \tag{7}$$

hold for all $n \in \mathbb{N}$, where we have set $m_0 = 1$ for convenience. For all $n \in \mathbb{N}$ which are large enough such that $a_n > 0$, the above two inequalities imply that

$$m_{a_n}/a_n = (m_{a_n}/n)(n/a_n) \leq n/a_n \tag{8}$$

and

$$m_{a_n+1}/(a_n+1) = (m_{a_n+1}/n)[n/(a_n+1)] > (n/a_n)[a_n/(a_n+1)]. \tag{9}$$

Now suppose that $\lim_{n\to+\infty} m_n/n = 1$. This implies that the left-hand sides of both (8) and (9) converge to 1 as n approaches $+\infty$. From (8) it follows therefore that $\liminf_{n\to+\infty} n/a_n \geq 1$. Moreover, because $\lim_{n\to+\infty} a_n/(a_n+1) = 1$, condition (9) yields $\limsup_{n\to+\infty} n/a_n \leq 1$. Taking these results together we have $\lim_{n\to+\infty} n/a_n = 1$.

Conversely, suppose that $\lim_{n\to+\infty} n/a_n = 1$. From (8) it follows that $\limsup_{n\to+\infty} m_{a_n}/a_n \leq 1$ holds, and from (9) we obtain $\liminf_{n\to+\infty} m_{a_n+1}/(a_n+1) \geq 1$. The definition of a_n implies furthermore that $a_{n+1} \in \{a_n, a_n+1\}$ which shows that $\mathbb{N} \subseteq \{a_n \,|\, n \in \mathbb{N}\}$ and $\mathbb{N}\setminus\{1\} \subseteq \{a_n+1 \,|\, n \in \mathbb{N}\}$. Hence, we have $\limsup_{n\to+\infty} m_n/n = \limsup_{n\to+\infty} m_{a_n}/a_n \leq 1$ and $\liminf_{n\to+\infty} m_n/n = \liminf_{n\to+\infty} m_{a_n+1}/(a_n+1) \geq 1$. These two inequalities yield $\lim_{n\to+\infty} m_n/n = 1$. □

We are now ready to prove the following result.

Theorem 3 *Define the set* $\mathcal{F}^* = \{(m_n)_{n\in\mathbb{N}} \in \mathcal{M} \,|\, \lim_{n\to+\infty} m_n/n = 1\}$.
(a) *It holds that* $\mathcal{F}^*$ *is a filter base on* $\mathcal{M}$.
(b) *It holds that* $\mathbf{F} \subset \mathbf{G}(\mathcal{F}^*)$.
(c) *It holds that* $S\mathbf{G}(\mathcal{F}^*) \subseteq \mathbf{G}(\mathcal{F}^*)$.
(d) *It holds that* $T\mathbf{G}(\mathcal{F}^*) \subseteq \mathbf{G}(\mathcal{F}^*)$.

Proof: (a) We start by verifying that for any pair of sequences $(m_n)_{n\in\mathbb{N}} \in \mathcal{F}^*$ and $(m'_n)_{n\in\mathbb{N}} \in \mathcal{F}^*$ there exists another sequence $(k_n)_{n\in\mathbb{N}} \in \mathcal{F}^*$ such that condition (3) holds. As in the proof of lemma 10, let us define a_n and a'_n as the numbers of elements of the sequences $(m_\ell)_{\ell\in\mathbb{N}}$ and $(m'_\ell)_{\ell\in\mathbb{N}}$, respectively, that are smaller than or equal to n. It follows from lemma 10 that $\lim_{n\to+\infty} n/a_n = \lim_{n\to+\infty} n/a'_n = 1$. Consequently, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\min\{a_n, a'_n\} \geq n/(1+\varepsilon)$ holds for all $n \geq N$. In words, for every $n \geq N$, both $(m_\ell)_{\ell\in\mathbb{N}}$ and $(m'_\ell)_{\ell\in\mathbb{N}}$ contain at least $n/(1+\varepsilon)$ elements that are smaller than or equal to n. If $\varepsilon < 1$ this implies that the set

$$\{m_\ell \,|\, \ell \in \mathbb{N}\} \cap \{m'_\ell \,|\, \ell \in \mathbb{N}\} \cap A_n$$

must be nonempty. More specifically, denoting by b_n the number of elements in this set, it must hold that

$$b_n \geq n - 2[n - n/(1+\varepsilon)] = n(1-\varepsilon)/(1+\varepsilon).$$

Because $\varepsilon \in (0,1)$ was arbitrarily chosen and $n/b_n \geq 1$ for all n, we obtain $\lim_{n\to+\infty} n/b_n = 1$. Now let us define k_n as the n-th smallest element in the set

$$\{m_\ell \,|\, \ell \in \mathbb{N}\} \cap \{m'_\ell \,|\, \ell \in \mathbb{N}\}.$$

By construction, it holds that b_n is the number of elements of the sequence $(k_\ell)_{\ell\in\mathbb{N}}$ that are smaller than or equal to n. Recalling that $\lim_{n\to+\infty} n/b_n = 1$ and appealing to lemma 10 again, it follows that $\lim_{n\to+\infty} k_n/n = 1$. Thus, the sequence $(k_n)_{n\in\mathbb{N}}$ is an element of $\mathcal{F}^*$ and the verification of (3) is complete.

To verify the second condition in the definition of a filter base, consider any fixed $k \in \mathbb{N}$ and define $m_n = k+n$ for all $n \in \mathbb{N}$. Then it holds that $\lim_{n\to+\infty} m_n/n = 1$ such that $(m_n)_{n\in\mathbb{N}} \in \mathcal{F}^*$. Furthermore, we have $m_1 \geq k$ which proves that condition (b) in the definition of a filter base is also satisfied.

(b) It is easy to see that $\mathcal{F}^*$ is not a filter base of tails, that is, it contains sequences other than those which contain all but finitely many natural numbers. For example, the sequence $(m_n)_{n\in\mathbb{N}}$ that consists of all natural numbers except the square numbers is contained in $\mathcal{F}^*$. From lemma 2 it follows therefore that $\mathbf{F} \subset \mathbf{G}(\mathcal{F}^*)$.

(c) A permutation π is an element of $\mathbf{G}(\mathcal{F}^*)$ if and only if there exists a sequence $(m_n)_{n\in\mathbb{N}} \in \mathcal{F}^*$ such that A_{m_n} is invariant under π for all $n \in \mathbb{N}$. This implies obviously that A_{m_n+1} is invariant under $S\pi$ for all $n \in \mathbb{N}$. Because of $\lim_{n\to+\infty}(m_n + 1)/n = \lim_{n\to+\infty} m_n/n = 1$ it follows that $S\pi \in \mathbf{G}(\mathcal{F}^*)$ holds as well. We conclude that $S\mathbf{G}(\mathcal{F}^*) \subseteq \mathbf{G}(\mathcal{F}^*)$.

(d) The proof of this part is similar to that of part (c). This time, however, one uses the observation that A_{m_n-1} is invariant under $T\pi$ for all sufficiently large $n \in \mathbb{N}$. Because of $\lim_{n\to+\infty}(m_n - 1)/n = \lim_{n\to+\infty} m_n/n = 1$ it follows therefore that $T\pi \in \mathbf{G}(\mathcal{F}^*)$ and one obtains $T\mathbf{G}(\mathcal{F}^*) \subseteq \mathbf{G}(\mathcal{F}^*)$. □

Combining lemma 1 and theorems 1-3, it follows that the Paretian quasi-ordering $\succeq_{\mathbf{G}(\mathcal{F}^*)}$ satisfies extended anonymity, shift-invariance, and truncation-invariance. This shows that the strong Pareto axiom, extended anonymity, and stationarity are consistent with each other. Hence, we have a possibility theorem analogous to that by Svensson (1980). However, whereas Svensson (1980) considers complete orderings satisfying finite anonymity, our result deals with quasi-orderings that satisfy an extended anonymity axiom. Furthermore, our result is not based on the axiom of choice or other non-constructive methods but defines the stationary Paretian quasi-ordering by means of the well-know grading principle and a specific set of permutations $\mathbf{G}(\mathcal{F}^*)$.

4.4 Restrictions imposed by stationarity

In this subsection we demonstrate that the requirement of stationarity severely restricts the extendability of the finite anonymity axiom. We start by translating the defining property of $\mathcal{F}^*$, namely $\lim_{n\to+\infty} m_n/n = 1$ for all $(m_n)_{n\in\mathbb{N}}$, into properties of the permutations in $\mathbf{G}(\mathcal{F}^*)$.

Theorem 4 *Let $\pi \in \mathbf{G}(\mathcal{F}^*)$ be given.*
(a) *It holds that $\lim_{n\to+\infty} \pi(n)/n = 1$.*
(b) *There are infinitely many generations $n \in \mathbb{N}$ such that $\pi(n) = n$.*

Proof: Since $\pi \in \mathbf{G}(\mathcal{F}^*)$ there exists a sequence $(m_\ell)_{\ell\in\mathbb{N}} \in \mathcal{F}^*$ such that A_{m_ℓ} is invariant under π for all $\ell \in \mathbb{N}$.

(a) As in the proof of lemma 10 we let a_n be the number of elements of the sequence $(m_\ell)_{\ell\in\mathbb{N}}$ that are smaller than or equal to n. Hence, inequalities (6)-(7) must hold. For sufficiently large n (such that $a_n \geq 2$ holds) this implies that

$$m_{a_n-1} < n \leq m_n. \tag{10}$$

Because both $A_{m_{a_n-1}}$ and A_{m_n} are invariant under π it follows that the set $\{m_{a_n-1}+1, m_{a_n-1}+2, \ldots, m_n\}$ is also invariant under π. Together with (10) this implies that

$$m_{a_n-1} < \pi(n) \leq m_n.$$

Dividing this inequality by n and considering the limit as n approaches $+\infty$, we get

$$1 = \lim_{n\to+\infty}[m_{a_n-1}/(a_n-1)][(a_n-1)/n] = \lim_{n\to+\infty} m_{a_n-1}/n \leq \lim \pi(n)/n \leq \lim_{n\to+\infty} m_n/n = 1,$$

where we have made use of lemma 10 and the assumption $\lim_{\ell\to+\infty} m_\ell/\ell = 1$. Obviously, this proves statement (a).

(b) First note that there must exist infinitely many $\ell \in \mathbb{N}$ such that $m_{\ell+1} = m_\ell + 1$. If this property were not true, we would have $m_{\ell+1} \geq m_\ell + 2$ for all but finitely many $\ell \in \mathbb{N}$ which, in turn, is not consistent with $\lim_{\ell\to+\infty} m_\ell/\ell = 1$. Writing n instead of $m_\ell + 1$ it follows therefore that there are infinitely many $n \in \mathbb{N}$ such that both $A_{n-1} = A_{m_\ell}$ and $A_n = A_{m_{\ell+1}}$ are invariant under π. Obviously, this implies that $\pi(n) = n$ and the proof of part (b) is complete. □

Both parts of theorem 4 demonstrate that the set of permutations $\mathbf{G}(\mathcal{F}^*)$ is "close" to the set of finite permutations $\mathbf{F}$. Permutations satisfying $\lim_{n\to+\infty} \pi(n)/n = 1$ are called bounded permutations; see Lauwers (1995). Theorem 4(a) proves therefore that $\mathbf{G}(\mathcal{F}^*)$ contains only bounded permutations. We would like to point out, however, that not every bounded permutation is an element of $\mathbf{G}(\mathcal{F}^*)$. For example, the permutation $\pi \circ \pi'$ from example 1 in subsection 2.2 is bounded. On the other hand, since this permutation is not cyclic, it follows from lemma 1 that it cannot be an element of $\mathbf{G}(\mathcal{F}^*)$.

Statement (b) of theorem 4 shows a related property: the condition $\pi(n)/n = 1$ does not only hold asymptotically, but it must hold for infinitely many generations n. We will now prove that this property is actually necessary for any shift-invariant or truncation-invariant Paretian quasi-ordering which has a set of permissible permutations that is generated by a filter base.

A filter base $\mathcal{F}$ is said to be coarse if there exists a sequence $(m_n)_{n\in\mathbb{N}} \in \mathcal{F}$ such that the equation $m_n + 1 = m_{n+1}$ holds at most for finitely many $n \in \mathbb{N}$. Note that, whenever $\mathcal{F}$ is a coarse filter base, the corresponding set $\mathbf{G}(\mathcal{F})$ contains a permutation for which $\pi(n) = n$ holds for at most finitely many $n \in \mathbb{N}$.

Theorem 5 *Let $\succeq$ be a Paretian quasi-ordering on X that is shift-invariant or truncation-invariant. Furthermore, let $\mathcal{F}$ be a coarse filter base on $\mathcal{M}$. Then it holds that $\Pi(\succeq) \neq \mathbf{G}(\mathcal{F})$.*

PROOF: Suppose to the contrary that $\Pi(\succeq) = \mathbf{G}(\mathcal{F})$ for some coarse filter base $\mathcal{F}$. There exists a sequence $(m_n)_{n\in\mathbb{N}} \in \mathcal{F}$ such that $m_n + 1 = m_{n+1}$ holds for at most finitely many $n \in \mathbb{N}$ and

such that $m_1 > 1$. For notational convenience, define $m_0 = 0$ and consider the permutation

$$\pi(k) = \begin{cases} k+1 & \text{if } k \neq m_n \text{ for any } n \in \mathbb{N} \\ m_{n-1}+1 & \text{if } k = m_n. \end{cases}$$

Note that the only sets of integers which are invariant under π are the sets $\{1, 2, \ldots, m_1\}$, $\{m_1+1, m_1+2, \ldots, m_2\}$, $\ldots$ as well as all unions of these sets. This shows that $\pi \in \mathbf{G}(\mathcal{F}) = \Pi(\succeq)$.

Now assume that $\succeq$ is shift-invariant such that $S\pi \in \Pi(\succeq)$ according to lemma 4. The permutation $S\pi$ is given by

$$S\pi(k) = \begin{cases} 1 & \text{if } k = 1, \\ k+1 & \text{if } k \neq m_n + 1 \text{ for any } n \in \mathbb{N} \\ m_{n-1}+2 & \text{if } k = m_n + 1. \end{cases}$$

The only sets of integers which are invariant under $S\pi$ are the sets $\{1\}$, $\{2, 3, \ldots, m_1+1\}$, $\{m_1+2, m_1+3, \ldots, m_2+1\}$, $\ldots$ as well as all unions of these sets. Since $S\pi \in \Pi(\succeq) = \mathbf{G}(\mathcal{F})$, there must exist a sequence $(m'_n)_{n\in\mathbb{N}} \in \mathcal{F}$ which is a subsequence of $(1, m_1+1, m_2+1, \ldots)$. Since $\mathcal{F}$ is a filter base, there must exist another sequence $(k_n)_{n\in\mathbb{N}} \in \mathcal{F}$ such that

$$\{k_n \,|\, n \in \mathbb{N}\} \subseteq \{m_n \,|\, n \in \mathbb{N}\} \cap \{m'_n \,|\, n \in \mathbb{N}\} \subseteq \{m_1, m_2, \ldots\} \cap \{1, m_1+1, m_2+1, \ldots\}.$$

However, because the equation $m_n + 1 = m_{n+1}$ holds for at most finitely many $n \in \mathbb{N}$, the set on the right-hand side of this inclusion is a finite set, whereas the set on the left-hand side must be infinite. This contradiction proves the theorem in the case where $\succeq$ is shift-invariant. The case where $\succeq$ is truncation-invariant can be dealt with analogously. □

The above result shows that the set of permissible permutations for a shift- or truncation-invariant Paretian quasi-ordering cannot be a group that is generated by a coarse filter base. Note that the filter bases used to generate the groups $\mathbf{G}_\ell$ are coarse. Hence, we know that $\Pi(\succeq) \neq \mathbf{G}_\ell$ holds for all $\ell \in \mathbb{N}$ provided that $\succeq$ is shift- or truncation-invariant and Paretian. We interpret the result from theorems 5 in the sense that either of these two invariance properties (and, hence, also the joint requirement of stationarity) provides a severe restriction to extending the finite anonymity axiom for Paretian quasi-orderings.

We corroborate our findings by returning to the example discussed at the end of section 3. In that example we constructed a group $(\bar{\mathbf{G}}, \circ)$ that is not generated by any filter base and that satisfies $\mathbf{F} \subset \bar{\mathbf{G}} \subseteq \mathbf{C}$. According to proposition 1, the set $\bar{\mathbf{G}}$ is the set of permissible permutations for the Paretian quasi-ordering $\succeq_{\bar{\mathbf{G}}}$, and this Paretian quasi-ordering satisfies extended anonymity. Does there also exist a shift- or truncation-invariant Paretian quasi-ordering $\succeq$ with $\Pi(\succeq) = \bar{\mathbf{G}}$? The answer is 'no' as can be seen from the following argument.

Suppose first that $\Pi(\succeq) = \bar{\mathbf{G}}$ for some shift-invariant Paretian quasi-ordering. This implies in particular that $\bar{\pi} \in \Pi(\succeq)$, where $\bar{\pi}$ has been defined in (4). From theorem 1 we know that

$(\Pi(\succeq), \circ)$ must be a group of cyclic permutations and that $S\bar{\pi} \in \Pi(\succeq)$ must hold. We have

$$S\bar{\pi}(n) = \begin{cases} 1 & \text{if } n = 1, \\ 4 & \text{if } n = 2, \\ 2 & \text{if } n = 4, \\ n+3 & \text{if } n > 1 \text{ is odd}, \\ n-3 & \text{if } n > 4 \text{ is even}. \end{cases}$$

Now consider $\tilde{\pi} := S\bar{\pi} \circ \bar{\pi}$. Since $\Pi(\succeq)$ is a group, it must hold that $\tilde{\pi} \in \Pi(\succeq) \subseteq \mathbf{C}$. On the other hand, we have $\tilde{\pi}(n) = n + 6$ for all even numbers n such that $\mathcal{O}_{\tilde{\pi}}(n)$ is an infinite set for all even n. This shows that $\tilde{\pi}$ is not cyclic which is a contradiction to what we have shown above. This proves that there does not exist a shift-invariant Paretian quasi-ordering $\succeq$ such that $\Pi(\succeq) = \bar{\mathbf{G}}$.

The existence of a truncation-invariant Paretian quasi-ordering $\succeq$ with $\Pi(\succeq) = \bar{\mathbf{G}}$ can be ruled out in a similar way. From theorem 2 we know that $(\Pi(\succeq), \circ)$ must be a group of cyclic permutations and that $T\bar{\pi} \in \Pi(\succeq)$ must hold. We have

$$T\bar{\pi}(n) = \begin{cases} 2 & \text{if } n = 2, \\ n+3 & \text{if } n \text{ is odd}, \\ n-3 & \text{if } n > 2 \text{ is even}. \end{cases}$$

Now consider $\tilde{\pi}' := T\bar{\pi} \circ \bar{\pi}$ which must be a cyclic permutation by the same argument as above. But again we find that $\tilde{\pi}'(n) = n + 6$ for all even numbers n which shows that $\tilde{\pi}' \notin \mathbf{C}$. This proves that there does not exist a truncation-invariant Paretian quasi-ordering $\succeq$ such that $\Pi(\succeq) = \bar{\mathbf{G}}$.

5 Concluding remarks

In this paper we have

- explored by how much the finite anonymity axiom can be strengthened for Paretian quasi-orderings on infinite utility streams and
- provided a complete characterization of those sets of permutations that can be permissible for Paretian quasi-orderings which are stationary in the sense of Koopmans (1960).

Whereas our results regarding the first issue indicate that there are many different ways how intergenerational equity of Paretian quasi-orderings can be formalized by means of an extended anonymity axiom, the results regarding the second issue clearly show that stationarity (as a matter of fact, even less stringent invariance conditions than stationarity) impose severe restrictions on the extendability of the finite anonymity axiom.

In future work we would like to further explore the question of which kind of permutations are definitely not permissible for stationary Paretian quasi-orderings. Given our results and examples from the present paper it is tempting to conjecture that permissible permutations for stationary Paretian quasi-orderings necessarily leave infinitely many generations unaffected.

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