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Document de Travail
n°2010-31

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July 2010

DT-GREQAM

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July 22, 2010

Abstract

We consider a model of interdependent efforts, with linear and possibly asymmetric interaction. We examine how a variation of the intensity of interaction affects aggregate effort. We show that the relevant information is given by the transposed system.

JEL: D85, C72.

Keywords: Asymmetric Interaction, Social Network, Aggregate Effort, Transposed System.

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Strategic interaction plays an important role in economics, and the study of interdependencies between individual efforts can be relevant for policy intervention. One central theme is how the sum of individual efforts varies with the intensity of interaction. This note considers systems of interacting efforts, where efforts are nonnegative and continuous, and interactions are linear. Under symmetric interaction, it is well-known that raising cross-effects generates an increase of the sum of individual efforts (Ballester, Calvó-Armengol and Zénou [2006]).[1] This intuitive result is based on the (also well-known) existence of a potential function associated with the game, the value of which is, at equilibrium, the sum of efforts. However, when interactions are asymmetric, there is in general no potential function. How does the introduction of asymmetry affect this result? This note addresses this issue. We show that, to assess the impact of an increase of cross-effects on the sum of efforts, one has to examine the solution of the transposed system. When the solution of the transposed system is nonnegative, increasing cross-effects always induces an increase of the sum of efforts. In contrast, when the solution of the transposed system admits a negative component, there always exists a perturbation of the interaction that increases cross-effects and that generates a decrease of the sum of efforts, and we build such a perturbation. We then present sufficient conditions to guarantee that the solution of the transposed system is positive. Finally, we distinguish between raising cross-effects and raising complementarities, the difference of which is economically meaningful when efforts are substitutes.

We consider a society $N = \{1, \cdots, n\}$. Let $X = (x_1, \cdots, x_n)$, with $x_i \in \mathbb{R}_+$ for all i, be a column-vector of efforts, and let $x = \sum_{i=1}^n x_i$ denote the sum of components of profile X, what we call *aggregate effort*. Consider a square matrix Γ, with $\gamma_{ii} \in \mathbb{R}_+^*$ and $\gamma_{ij} \in \mathbb{R}$ for all i, j, $j \neq i$. When $\gamma_{ij} < 0$ (resp. $\gamma_{ij} > 0$), agent j's effort is a strategic complement (resp. substitute) to agent i's effort. Note that our formulation allows for mixed effects. Let $A = (a_1, a_2, \cdots, a_n)$, with $a_i \in \mathbb{R}_+^*$. We consider systems of first order conditions (FOC) of the form:

$$\begin{cases} \gamma_{ii} x_i + \sum_{j \neq i} \gamma_{ij} x_j = a_i & \text{if } a_i - \sum_{j \neq i} \gamma_{ij} x_j > 0 \\ x_i = 0 & \text{if } a_i - \sum_{j \neq i} \gamma_{ij} x_j \leq 0 \end{cases} \tag{1}$$

[1]This result holds under low level of interaction. Bramoullé, Kranton and d'Amours (2010) recently complemented this result under large interaction.

This linear system of FOCs may arise in many contexts, like synergistic efforts with linear quadratic utilities (Ballester, Calvó-Armengol and Zénou [2006]), local public goods (Bramoulé and Kranton [2007]), Cournot oligopolies (Bramoullé, Kranton and D'Amours [2010]), or for describing equilibrium consumptions in pure exchange economies with positional goods (Ghiglino and Goyal [2010]), pricing with local network externalities (Bloch and Quérou [2009]), risk taking under informal risk sharing (Belhaj and Deroïan [2009]). Our setting allows an idiosyncratic component a_i.
Interior solutions of the above system of FOCs are such that $\frac{\partial x_i}{\partial x_j} = \frac{-\gamma_{ij}}{\gamma_{ii}}$, we call *intensity of interaction* the quantity $|\frac{\partial x_i}{\partial x_j}|$. Symmetric interaction thus arises when $\frac{\gamma_{ij}}{\gamma_{ii}} = \frac{\gamma_{ji}}{\gamma_{jj}}$ for all i, j. Conform to Ballester, Calvó-Armengol and Zénou (2006), raising cross-effects is formally defined as follows:

Definition 1 *A perturbation $\Theta = [\theta_{ij}]$ raises cross-effects if $\theta_{ij} \leq 0$ for all i, j.*

When $\theta_{ij} < 0$, the complementarity that agent j exerts on agent i's effort is increased (or the substitutability is decreased). When $\theta_{ii} < 0$, agent i's sensitiveness to others' efforts is increased.[2] Ballester, Calvó-Armengol and Zénou (2006) consider the impact of an increase of cross-effects under symmetric interaction. They focus on low interaction, which guarantees a unique and interior equilibrium. Examining the case where $a_i = a$ for all i, they show that an increase of cross-effects enhances aggregate effort (Ballester et al. [2006, Theorem 2 pp. 1409]). Their result can be formulated as follows (we normalize A to J, where J is the column-vector of ones, without loss of generality):

Theorem 1 (derived from Ballester, Calvó-Armengol and Zénou 2006) *Consider two invertible and symmetric matrices Γ and Γ', such that $\Gamma' = \Gamma + \Theta$ with $\Theta \leq 0$. If both Γ^{-1} and Γ'^{-1} are well defined and nonnegative, then the interior equilibria $X = \Gamma^{-1}J$ and $X' = \Gamma'^{-1}J$ are such that $x' \geq x$.*

Proof of theorem 1. Since matrices are invertible, and inverse matrices are nonnegative, the interior equilibria to both systems, X and X', exist. Then the following sequence of equalities holds:

$$x' = X'^T J = X'^T \Gamma X = X'^T (\Gamma' - \Theta) X = X'^T \Gamma' X - X'^T \Theta X$$

[2]For instance, if $U_i(X) = x_i - c_i x_i^2 + \sum_j g_{ij} x_i x_j$, with $g_{ij} \in \mathbb{R}$, a perturbation raising cross-effects can both lower c_i and increase g_{ij}.

By symmetry of Γ', $X'^T\Gamma' X = (\Gamma' X')^T X = x$. In total, $x' \geq x$. ■

Remark. As stated here the theorem is a little bit more general than as stated in Ballester, Calvó-Armengol and Zénou (2006), in two respects. First, the original theorem holds under additional condition entailing uniqueness of the equilibrium. Second, the original theorem supposes that γ_{ii} is constant across agents.

The following theorem addresses the same issue under asymmetric interaction. We consider now system (1) with differentiated levels of a_i:

Theorem 2 *Consider two matrices Γ and Γ', such that $\Gamma' = \Gamma + \Theta$, and such that both Γ^{-1} and Γ'^{-1} are well defined and nonnegative. Every perturbation Θ that raises cross-effects (meaning $\Theta \leq 0$) induces $x' \geq x$ if and only if the solution to $\Gamma^T Y = J$ is nonnegative.*

Proof of theorem 2. The following lemma is adapted from Farkas's lemma.

Lemma 1 *Let M be an $n \times n$ matrix. The equation $M^T Y = J$ admits a nonnegative solution if and only if, for all $Z \in \mathbb{R}^n$ such that $MZ \geq 0$, we have $z \geq 0$.*

Only if. Since inverse matrices are nonnegative, both systems admit an interior solution, X and X'. Basically, $\Gamma(X' - X)$ $(= A - A - \Theta X') = -\Theta X'$. As $X' > 0$, we have $-\Theta X' \geq 0$. If the solution to $\Gamma^T Y = J$ is nonnegative, lemma 1 applies (setting $M = \Gamma$ and $Z = X' - X$) and thus $x' \geq x$.

If. Consider $X \geq 0$ solution of $\Gamma X = A$, and suppose that the system $\Gamma^T Y = J$ admits a solution containing a negative component. By lemma 1, there exists a profile $Z = (z_1, \cdots, z_n)$ such that $\Gamma Z > 0$ while $z < 0$. Denote $\beta = \Gamma Z$ for convenience. We will show that there exists a matrix $\Theta \leq 0$, and vector $X' = (\Gamma + \Theta)^{-1} A \geq 0$, such that $x' < x$.

Consider $\epsilon \in \mathbb{R}^+_*$. Define $\Theta = [\theta_{ij}]$, with $\theta_{ij} = \theta_i$ for all i, j, with

$$\theta_i = \frac{-\epsilon \beta_i}{\sum_j x_j + \epsilon \sum_j z_j} \tag{2}$$

Then $\Theta \leq 0$ for ϵ small enough (the denominator is positive for small values of ϵ). By construction, we have $\Theta(X + \epsilon \cdot Z) = -\epsilon \cdot \Gamma Z$. Define $X' = X + \epsilon \cdot Z$, the latter condition writes $\Theta X' = \Gamma(X - X')$. Hence, X' is the solution of the system $(\Gamma + \Theta)X' = A$, with ϵ small enough to ensure $X' \geq 0$. Since $X' - X = \epsilon \cdot Z$, $z < 0$ implies $x' < x$. $\blacksquare$

Theorem 2 indicates that raising cross-effects can lead to a decrease in the sum of efforts. Moreover, to guarantee that an increase of cross-effects fosters aggregate effort, the intensity of strategic interaction of the transposed system should be low enough. In contrast, no condition is imposed on the intensity of interaction of the original system (although solutions of the original system and the transposed system are related).
Note that the linear part of system (1) is written $\Gamma X = A$, while the comparative statics is based on the system $\Gamma^T Y = J$. That is, idiosyncratic constants a_i play no role in the comparative statics. It is also worth noting that the perturbation Θ can be of arbitrary magnitude, provided that solutions are positive.[3] The above analysis extends straightforwardly to local perturbations around equilibria containing corner solutions, even in case of multiple equilibria. Indeed, theorem 2 holds when the perturbation keeps unchanged the set of corner agents. Moreover, we obtain:[4]

Corollary 1 *Suppose that system (1) admits one interior solution X and one solution with corners X'. If $\Gamma^T Y = J$ admits a nonnegative solution, $x < x'$.*

Proof of corollary 1. Since X' admits a corner, suppose without loss of generality that $x'_1, x'_2, \cdots, x'_p > 0$, and $x'_{p+1}, x'_{p+2}, \cdots, x'_n = 0$. Consider the profile $\Gamma(X' - X)$. Basically, $[\Gamma(X' - X)]_i = a_i - a_i = 0$ for all $i \leq p$, while for all $i > p$, $[\Gamma(X' - X)]_i = [\Gamma X']_i - [\Gamma X]_i$; recalling that $[\Gamma X']_i > a_i$ and $[\Gamma X]_i = a_i$, we obtain in total that $\Gamma(X' - X) \geq 0$ with at least one positive component. Then we can apply lemma 1 with $M = \Gamma$ and $Z = X' - X$, and we are done. $\square$

To illustrate, consider the matrices $\mathbf{\Gamma} = \begin{pmatrix} 1 & .3 & .3 \\ .7 & 1 & .3 \\ .7 & .3 & 1 \end{pmatrix}$, $\mathbf{\Theta} = \begin{pmatrix} -.02 & -.02 & -.02 \\ -.001 & -.001 & -.001 \\ -.001 & -.001 & -.001 \end{pmatrix}$, and assume $A = J$. We have $X_\Gamma \simeq (.79, .34, .34)$ and $X_{\Gamma^T} \simeq (-.11, .79, .79)$. Since

[3]This is irrespective of multiplicity of equilibria. Indeed by linearity there is a unique interior equilibrium.

[4]Corollary 1 complements some recent results about encapsulated corners found in Bramoullé, Kranton and D'Amours (2010) in the context of symmetric interaction.

X_{Γ^T} contains a negative component, there exist some perturbation that both raises cross-effects and that lowers aggregate effort. The perturbation Θ is one: indeed, we find $x_\Gamma \simeq 1.477$ and $x_{\Gamma+\Theta} \simeq 1.476$.

Positive solutions in games with strategic substitutes. Theorem 2 suggests the interest of obtaining conditions guaranteeing positive solution to $\Gamma^T Y = J$. Consider a nonnegative matrix M with positive diagonal. Define matrix $G_M = [g_{ij}]$, with $g_{ii} = 0$ and $g_{ij} = \frac{m_{ij}}{m_{ii}}$ for all $i, j \neq i$, and define profile $B_M = (b_1, \cdots, b_n)$ such that $b_i = \frac{1}{m_{ii}}$. Let $\rho(.)$ denote the greatest modulus of eigenvalues of any square matrix. We consider the following properties:

Property 1 *If Q is a nonnegative square matrix with null diagonal, $\rho(Q) < 1$.*

Property 1, applied to the matrix G_{Γ^T}, guarantees that the system $\Gamma^T Y = J$ admits a unique and interior solution (and that this solution can be developed as a series of powers of the matrix G_{Γ^T}).

Property 2 *If M is a nonnegative square matrix with positive diagonal, $\sum_{j\neq i} \frac{m_{ij}}{m_{jj}} \leq 1$ for all i.*

Next lemma elaborates upon these two properties to guarantee a positive solution:

Lemma 2 *Consider a nonnegative square matrix M with positive diagonal. If G_M satisfies property 1 and M satisfies property 2, the solution to $MY = J$ is postitive, and belongs to $(0, B)$.*

Proof of lemma 2. The system $MY = J$ can be written $(I + G_M)Y = B_M$. As $\rho(G_M) < 1$, the solution exists and can be written $Y = \sum_{k=0}^{\infty} (-G_M)^k B_M$ (see Herstein and Debreu [1953], and Ballester, Calvó-Armengol and Zénou [2006]). Rearranging, we find

$$Y = \left(\sum_{k=0}^{\infty} G_M^{2k} \right) \cdot (I - G_M) B_M \tag{3}$$

The series $\sum_{k=0}^{\infty} G_M^{2k}$ converges since Y is finite. Since M is nonnegative, $[G_M^{2k}]_{ij} \geq 0$ for all k, i, j. The solution is positive if and only if $(I - GM)B_M > 0$, which happens to be property 2 applied to matrix M. Moreover, a solution of $(I + G_M)Y = B_M$ is also written $Y = B_M - G_M Y$. Since $M \geq 0$, we have $G_M \geq 0$. Thus, $Y > 0$ implies $Y < B_M$. $\square$

From lemma 2 we deduce immediately:

Corollary 2 *When Γ is nonnegative, if G_{Γ^T} satisfies property 1 and Γ^T satisfies property 2, the solution to $\Gamma^T Y = J$ is positive.*

Note that when $\gamma_{ii} = \gamma$ for all i, which is the case of Ballester, Calvó-Armengol and Zénou (2006), property 2 applied to the matrix Γ is a condition of diagonal dominance, and this implies $\rho(G_{\Gamma^T}) < 1$.[5] Hence, the diagonal dominance of a nonnegative matrix Γ implies an interior solution to system (1).[6] Note also that if $\gamma = 1$ for all i, the solution to system (1) belongs to $(0, 1)$.

Raising complementarities. The intensity of interaction in the perturbed system is $\frac{-(\gamma_{ij}+\theta_{ij})}{\gamma_{ii}+\theta_{ii}}$. When all Γ's off-diagonal elements are negative, i.e. the game contains only strategic complements, raising cross-effects increases the intensity of interaction, and thus raises complementarities. In opposite, when Γ is nonnegative, raising cross-effects no longer implies a decrease of substitutability in the sense that the intensity of interaction is not necessarily reduced. This motivates the following definition:

Definition 2 *In a game with strategic substitutes, a perturbation $\Theta = [\theta_{ij}]$ raises complementarities if $\theta_{ij} \leq 0$ for all $i, j \neq i$ and $\theta_{ii} \geq 0$.*

However, the perturbations adapted to this context generate no clear-cut prediction about aggregate effort. The problem is simply seen when considering perturbations that only affect the diagonal of the matrix Γ. Basically, raising complementarities (i.e. decreasing the intensity of interaction) by only increasing the diagonal of the matrix Γ entails less aggregate effort if the solution of the transposed system $\Gamma^T Y = J$ is nonnegative.[7] This result is easily explained. The linear system $\Gamma^T Y = J$ is also

[5] Note that $\rho(G_{\Gamma^T}) = \rho(G_\Gamma)$.

[6] What is usually known is that diagonal dominance implies uniqueness of the solution, not that it is interior.

[7] $\Gamma(X' - X) = -\Theta X'$, and since $\Theta \geq 0$, $-\Theta X' \leq 0$. Then, if the solution to $\Gamma^T Y = J$ is nonnegative, lemma 1 (setting $M = \Gamma$ and $Z = X' - X$) induces that $x' \leq x$.

written

$$y_i + \sum_{j \neq i} \frac{\gamma_{ji}}{\gamma_{ii}} y_j = \frac{1}{\gamma_{ii}} \text{ , for all } i \tag{4}$$

Increasing γ_{ii} has two opposite effects: it decreases the ratio $\frac{\gamma_{ji}}{\gamma_{ii}}$ but it also increases the quantity $\frac{1}{\gamma_{ii}}$. The latter effect dominates. Hence, imposing $\theta_{ii} > 0$ contributes to decrease aggregate effort, while $\theta_{ij} > 0$ contributes to enhance aggregate effort. Therefore, in a game with strategic substitutes, a perturbation that increases the diagonal of Γ and decrease its off-diagonal elements has in general an ambiguous impact on aggregate effort.

However, under some circumstances, it is possible to sign the variation of aggregate effort. The next lemma gives conditions under which the diagonal effect (θ_{ii}) dominates (resp. is dominated by) the off-diagonal effect (θ_{ij}). The conditions make the link between the perturbation and the minimum and maximum effort levels at equilibrium:

Lemma 3 *Consider two real numbers q_l, q_h such that $0 < q_l \leq q_h$. Consider a system (1) that admits an interior solution X such that $x_i \in (q_l, q_h)$ for all i, and a perturbation Θ that raises complementarities. Last, suppose that $\Gamma^T Y = J$ admits a nonnegative solution. If $\theta_{ii} \geq \frac{q_h}{q_l} \sum_{j \neq i} |\theta_{ij}|$ for all i, the perturbation induces a decrease of aggregate effort. If $\theta_{ii} \leq \frac{q_l}{q_h} \sum_{j \neq i} |\theta_{ij}|$ for all i, the perturbation induces an increase of aggregate effort.*

Proof of lemma 3. Consider profile $\Gamma(X' - X) = -\Theta X'$. Line i writes:

$$-[\Theta X']_i = -\theta_{ii} x'_i + \sum |\theta_{ij}| x'_j \tag{5}$$

As $q_l \leq x'_j \leq q_h$ for all j,

$$-\theta_{ii} q_h + q_l \sum_{j \neq i} |\theta_{ij}| \leq -[\Theta X']_i \leq -\theta_{ii} q_l + q_h \sum_{j \neq i} |\theta_{ij}| \tag{6}$$

If $\theta_{ii} \geq \frac{q_h}{q_l} \sum_{j \neq i} |\theta_{ij}|$, we have $-[\Theta X']_i \leq 0$, while if $\theta_{ii} \leq \frac{q_l}{q_h} \sum_{j \neq i} |\theta_{ij}|$, we find $-[\Theta X']_i \geq 0$. Then lemma 1 applies in both cases (with $M = \Gamma$ and $Z = X' - X$) and we are done. $\square$

Using lemma 3, we finally provide an economic example in which it is possible to sign the variation of aggregate effort. Consider $\delta \in (0, 1)$, and a *bi-stochastic* matrix

$\Lambda = [\lambda_{ij}]$. Suppose that the matrix $\Gamma = \Gamma^0$, with $\gamma^0_{ii} = \lambda_{ii}$, $\gamma^0_{ij} = \delta\lambda_{ij}$. Also, set $a_i = 1$ for all i. One possible corresponding economic situation is risk taking under informal risk sharing (Belhaj and Deroïan [2009]). System (1) becomes:

$$\begin{cases} \lambda_{ii}x_i + \delta\sum_{j\neq i}\lambda_{ij}x_j = 1 & \text{if } 1-\delta\sum_{j\neq i}\lambda_{ij}x_j > 0 \\ x_i = 0 & \text{if } 1-\delta\sum_{j\neq i}\lambda_{ij}x_j \leq 0 \end{cases} \tag{7}$$

Recall that G_Λ denotes the null diagonal matrix such that $g_{ij} = \frac{\lambda_{ij}}{\lambda_{ii}}$.

Corollary 3 *Suppose that Γ^0 is diagonal dominant, i.e. $\lambda_{ii} > \frac{\delta}{1+\delta}$ for all i, and consider a perturbation Θ that raises complementarities. If $\theta_{ii} \geq \frac{1}{\delta}\sum_{j\neq i}|\theta_{ij}|$ for all i, the perturbation induces a decrease of aggregate effort. If $\theta_{ii} \leq \delta\sum_{j\neq i}|\theta_{ij}|$ for all i, the perturbation induces an increase of aggregate effort.*

Proof of corollary 3. We show that diagonal dominance of Γ^0, combined with row-stochasticity of Λ, implies $x_i \in (1, \frac{1}{\delta})$. Indeed, the first equation of system (7) is written

$$x_i + \delta\sum_{j\neq i}\frac{\lambda_{ij}}{\lambda_{ii}}x_j = \frac{1}{\lambda_{ii}} \tag{8}$$

Define $w_i = x_i - \frac{1}{\delta}$. Given that $\sum_{j\neq i}\lambda_{ij} = 1-\lambda_{ii}$, we obtain $(I+\delta G_\Lambda)W = -\frac{1-\delta}{\delta}J$. Since $\lambda_{ii} > \frac{\delta}{1+\delta}$ for all i, and recalling that Λ is row-stochastic, the matrix $I + \delta G_\Lambda$ basically satisfies property 2. Moreover, note that $\delta\rho(G_\Lambda) < 1$ (the sum over every row in matrix G_Λ is smaller than 1). We can then apply lemma 2 to the system $(I+\delta G_\Lambda)\hat{W} = J$. This means that $\hat{w}_i \in (0,1)$ for all i. But we have $W = -\frac{1-\delta}{\delta}\hat{W}$, which entails that $w_i \in (-\frac{1-\delta}{\delta}, 0)$ for all i. And thus $x_i \in (1, \frac{1}{\delta})$.

Second, since Λ is bi-stochastic, Λ^T is row-stochastic. Then, we deduce that diagonal dominance of Γ^0, combined with row-stochasticity of Λ^T, implies that the solution to $(\Gamma^0)^T Y = J$ is positive. Applying therefore lemma 3 with $\Gamma = \Gamma^0$, $q_l = 1$ and $q_h = \frac{1}{\delta}$, the corollary follows directly. ■

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