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Mean-field approximation of stochastic population processes in games

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Abstract. We here establish an upper bound on the probability for deviations of a Markov population process from its mean-field approximation. The population consists of n distinct subpopulations of equal size N, where each subpopulation is associated with a player role in a finite n-player game. At discrete times $t = 0, 1/N, 2/N, ...$ one individual is drawn at random from the total population to review his or her pure strategy choice. We allow transition probabilities to depend smoothly on population size N and show that the probability bound converges exponenially to zero as $N \to \infty$. This generalizes a result in Benaïm and Weibull (2003).

Many population models in game theory hypothesize a continuum of interacting agents and describe the evolutionary selection process in terms of a system of ordinary differential equations. These equations usually concern changes in population shares associated with the different pure strategies in the game, and the changes are viewed as aggregates of large numbers of individual strategy switches. In this so-called *mass-action interpretation*, due to Nash (1950), individuals are randomly and recurrently drawn to review their own choice of pure strategy, and mixed strategies are population distributions over pure strategies, rather than randomizations implemented by individual players. This population model was not formalized by Nash (1950), but dynamic population models later emerged in evolutionary biology. Prime examples are different versions of the replicator dynamics, see Taylor and Jonker (1978), Taylor (1979) and Maynard Smith (1982). For wide classes of such population dynamics, results have been obtained that establish connections with non-cooperative concepts such as dominance, Nash equilibrium and strategic stability (see, e.g. Weibull (1995)).

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An important question for the relevance of these results is whether these dynamics are good approximations of the stochastic population processes that arise from individual strategy revision in finite but large populations. In an earlier study, Benaïm and Weibull (2003), we addressed this question and established approximation results under the hypothesis that switching rates do not depend on population size. However, robustness in this respect is desirable, since in applications switching probabilities may well, to some extent, depend on population size, see Fudenberg et al. (2006), Fudenberg and Imhof (2008) and Example 1 below.[1] In this note, we generalize our previous result to allow for such dependence on population size.[2] To the best of our knowledge, the present result is the most general and powerful approximation for this class of Markov chains, giving an exponential upper bound on deviation probabilities in bounded time intervals, a bound that permits asymptotic analysis by way of the Borel-Cantelli Lemma.[3] The present generalization is obtained by first establishing a result for the stochastic process' deviations from its finite-population mean-field flow, before going to its deviation from its limit mean-field flow.

1. A class of stochastic processes

Consider a finite n-player game with player roles $i \in I = \{1, ...n\}$, finite pure strategy sets $S_i = \{1, ..., m_i\}$, set of pure-strategy profiles $S = \times_i S_i$, mixed-strategy simplices

$$\Delta(S_i) = \left\{ x_i \in \mathbb{R}_+^{m_i} : \sum_{h \in S_i} x_{ih} = 1 \right\} , \tag{1}$$

and polyhedron $\square(S) = \times_i \Delta(S_i)$ of mixed-strategy profiles $x = (x_1, ..., x_n)$. The polyhedron $\square(S)$ is thus a subset of $\mathbb{R}^m$, for $m = \sum_i m_i$. For each player role i there is a subpopulation consisting of N individuals. Each individual is at every moment in time associated with a pure strategy in her strategy set. An individual in population i who is associated with pure strategy $h \in S_i$ is called an *h-strategist*. At times $t \in \mathbb{T} = \{0, \delta, 2\delta, ...\}$, where $\delta = 1/N$, and only then, exactly one individual has the opportunity to change his or her pure strategy. This individual is randomly drawn, with equal probability for all nN individuals, and with statistical independence between successive draws. With this fixed relationship between population size and period length, the expected time interval between two successive draws of one

[1] We are grateful to Drew Fudenberg for raising this issue.

[2] We also take the opportunity to correct a mistake in the statement of Lemma 1 in our previous paper. The statement should be "For every $T > 0$ there exists...", see Proposition 1 below. We are grateful to Sergiu Hart for spotting the mistake.

[3] For other stochastic approximation results, see Kurtz (1981), Benveniste et al. (1990), Duflo (1996), Kushner and Yin (1997), Benaïm (1999) and Benaïm and Le Boudec (2008).

and the same individual is n, independently of the population size N. We will call the times $t \in \mathbb{T}$ *transition times* - the only times when a transition *can* take place.

For each $N \in N$, let $X^N = \langle X^N(t) \rangle_{t \in \mathbb{T}}$ be a Markov chain, with finite state space $\square^N(S)$, defined as follows. First, $\square^N(S)$ is the subset ("grid") of points $x \in \square(S)$ such that Nx_{ih} is a nonnegative integer for each $i \in I$ and $h \in S_i$. Secondly, for every player role i and pair $(h,k) \in S_i^2$ of pure strategies for that role, there exists a continuous function $p_{ik}^{hN} : \square(S) \to [0,1]$ such that $p_{ik}^{hN}(x) = 0$ if $x_{ik} = 0$ and

$$\Pr\left[X_i^N(t + \frac{1}{N}) = x_i + \frac{1}{N}\left(e_i^h - e_i^k\right) \mid X^N(t) = x\right] = p_{ik}^{hN}(x) \tag{2}$$

for all $i \in I$, $h,k \in S_i$, $N \in \mathbb{N}$ and $x \in \square^N(S)$. Here $p_{ik}^{hN}(x)$ is the conditional probability, given the current population state x, that a k-strategist in player population i will be drawn for strategy revision and switch to pure strategy h. Thus, for any $v = (v_1, ..., v_n) \in \mathbb{R}^m$:

$$\Pr\left[X^N(t + \frac{1}{N}) = x + \frac{1}{N}v \mid X^N(t) = x\right] = \begin{cases} p_{ik}^{hN}(x) & \text{if } v_i = e_i^h - e_i^k \text{ and } v_j = 0 \ \forall j \neq i \\ 0 & \text{otherwise} \end{cases}$$

For any player role $i \in I$ and pure strategy $h \in S_i$, and any population size N, the *expected net increase* in the subpopulation of h-strategists, from one transition time to the next, and conditional upon the current state x, is

$$F_{ih}^N(x) = \sum_{k \neq h} p_{ik}^{hN}(x) - \sum_{k \neq h} p_{ih}^{kN}(x) \ . \tag{3}$$

It follows from the probability specification above that $F_{ih}^N : \square(S) \to \mathbb{R}$ is bounded, with $\sum_h F_{ih}^N(x) \equiv 0$ and $F_{ih}^N(x) \geq 0$ if $x_{ih} = 0$. We view each function F^N as a mapping from E to E, where $E = \mathbb{R}^M$, for $M = m - n$, is the tangent space of the polyhedron $\square(S)$.[4] We assume that every function F^N, for $N \in \mathbb{N}$, is bounded and locally Lipschitz continuous, that there for every compact set $C \subset E$ exists a common Lipschitz constant λ for all functions F^N, and, moreover, that $F^N \to F$ uniformly. Then also F is bounded and locally Lipschitz continuous.

We are interested in deterministic continuous-time approximation of Markov chains X^N in the class defined above, when the population size N is large, and thus the time interval $\delta = 1/N$ between successive transition times is short. The key element for such approximation is the vector field $F^N : E \to E$ defined above, which, for short

[4] Recall that $\square(S)$ is the Cartesian product of n unit simplices, $\Delta(S_i)$, and that the latter is a subset of a hyperplan in $\mathbb{R}^{m_i}$. Hence, the dimension of E is $M = m - n$.

time intervals, gives the expected net increase in each population share during the time interval, per time unit.[5] The associated system of *mean-field equations,*

$$\dot{x}_{ih} = F^N_{ih}(x) \qquad \forall i \in I, h \in S_i, x \in E \tag{4}$$

define this limiting dynamic. In force of the Picard-Lindelöf Theorem, the system (4) of first-order ordinary differential equations has a unique solution through every point x in E (see, e.g., Hale (1969)). Moreover, as noted above, the sum of all population shares in each population remains constant over time, and no population share can turn negative. Hence, the system of equations (4) defines a solution mapping ξ^N : $\mathbb{R} \times \square(S) \to E$ that leaves each mixed-strategy simplex $\Delta(S_i)$, and hence also the polyhedron $\square(S)$ of mixed-strategy profiles, invariant. In other words, the system of differential equations determines a solution for all times $t \in \mathbb{R}$, and if the initial state is in $\square(S)$, then also all future states are in $\square(S)$.[6] We will call ξ^N the *flow* induced by F^N. Similarly, let ξ be the flow induced by the limit vector field F.

Let $||\cdot||_\infty$ denote the L^∞-norm on $E = \mathbb{R}^M$. Then $||\hat{X}^N(t) - \xi^N(t,x)||_\infty$ represents the deviation of the interpolated Markov chain from the deterministic approximation solution ξ^N at time t, measured as the largest deviation in any population share at time t:

$$||\hat{X}^N(t) - \xi^N(t,x)||_\infty = \max_{i \in I, h \in S_i} \left| \hat{X}^N_{ih}(t) - \xi^N_{ih}(t,x) \right| . \tag{5}$$

The random variable

$$D^N_N(T,x) = \max_{0 \le t \le T} ||\hat{X}^N(t) - \xi^N(t,x)||_\infty \tag{6}$$

is thus the *maximal deviation* in any population share, from the flow induced by F^N through x, during a bounded time interval $[0,T]$.

Likewise, the random variable

$$D^N(T,x) = \max_{0 \le t \le T} ||\hat{X}^N(t) - \xi(t,x)||_\infty \tag{7}$$

is the maximal deviation in any population share, from the flow ξ induced by F through x, during the same time interval.

Example 1. *Suppose that (a) every individual has the same probability of being drawn for strategy revision, (b) the revising individual draws another individual in his*

[5] There are N transition times per time unit and N individuals in each player population.

[6] More exactly: $\xi(0,x) = x$ for all x, $\frac{\partial}{\partial t}\xi_{ih}(t,x) = F_{ih}[\xi(t,x)]$ for all i, h, x and t, and $\xi_i(t,x) \in \Delta(S_i)$ for all $i \in I$, $x \in \square(S)$, and $t \in \mathbb{R}$. The time domain of the solution mapping ξ can be taken to be the whole real line in force of the compactness of $\square(S)$.

or her own player subpopulation, and (c) depending on the information then available about her own and the other individual's payoffs, imitates the other individual. Then (3) holds for all $N \geq 2$, with

$$p_{ik}^{hN}(x) = \frac{x_{ik}}{n} \cdot \frac{Nx_{ih}}{N-1} \cdot q_{ik}^{h}(x),$$

where $q_{ik}^{h}(x)$ is the conditional imitation probability from pure strategy k to pure strategy h. If all functions $q_{ik}^{h} : E \to [0,1]$ are Lipschitz continuous, then

$$F_{ih}^{N}(x) = \frac{1}{n(1-1/N)} \left[\sum_{k \neq h} x_{ik} q_{ik}^{h}(x) - \sum_{k \neq h} x_{ik} q_{ih}^{k}(x) \right] x_{ih}$$

so all functions F^N are bounded, have a common Lipschitz constant, and converge uniformly to the function F defined by setting $1/N = 0$ in this formula.

2. Mean-field approximation of $D_N^N(T,x)$

Let U_k, for $k \in \mathbb{N}$, be the difference between the step taken by the Markov chain X^N between periods k and $k+1$, per time unit, and the vector field F^N at the state:

$$U_k = \frac{1}{\delta} \left[X^N((k+1)\delta) - X^N(k\delta) \right] - F^N(X^N(k\delta)) , \tag{8}$$

where $\delta = 1/N$ is the length of a period. Let $\mathcal{F}_k, k \in \mathbb{N}$ denote the sigma-field generated by $\left\{X^N(0), \ldots, X^N(k\delta)\right\}$. The following result provides a useful upper bound on the difference U_k. Let $\langle ., . \rangle$ denote the inner product in the tangent space of the polyhedron of mixed-strategy profiles: $\langle x, y \rangle = \sum_{i=1}^{M} x_i y_i$ for any vectors $x, y \in E$.

Lemma 1. *Let $\Gamma_N = (\sqrt{2} + \left\|F^N\right\|_2)^2$. For any $\theta \in \mathbb{R}^M$:*

$$\mathbb{E}(e^{\langle \theta, U_k \rangle} | \mathcal{F}_k) \leq e^{\Gamma_N ||\theta||_2^2/2}$$

Proof. By definition of U_k it is easy to verify that

$$||U_k||_2 \leq \max_{i,h,k} ||e_i^h - e_i^k||_2 + \left\|F^N\right\|_2 = \sqrt{\Gamma_N} \tag{9}$$

Let $g(t) = \log \mathbb{E}(e^{t\langle \theta, U_k \rangle} | \mathcal{F}_k)$. The function $g : \mathbb{R} \to \mathbb{R}$ is convex and satisfies $g(0) = g'(0) = 0$, $g''(t) \leq ||\theta||_2^2 \Gamma_N$. Therefore $g(1) \leq ||\theta||_2^2 \Gamma_N / 2$. ■

We are now in a position to state and prove our result for $D_N^N(T,x)$:

Lemma 2. *For every $T > 0$ there exists a scalar $c > 0$ such that, for any $\varepsilon > 0$, and any N large enough:*

$$\Pr\left[D_N^N(T,x) \geq \varepsilon \mid X^N(0) = x\right] \leq 2Me^{-\varepsilon^2 cN} \qquad \forall x \in \square^N(S)$$

Proof. In order to prove this, let λ_N be the Lipschitz constant of F^N on the compact set $\square(S) \subset E$, with respect to the L^∞-norm, let $\|\cdot\|_2$ denote the L^2-norm, and let $\|F^N\|_2$ be the maximum of $\|F^N(x)\|_2$ on $\square(S)$. Let U_k, for $k \in \mathbb{N}$, be as defined above, and let $U : \mathbb{R}_+ \to E$ be the map defined by $U(t) = U_k$ for $k\delta \leq t < (k+1)\delta$. Likewise, let $\bar{X}^N$ be the continuous-time (right-continuous) *step process* generated by the Markov chain X^N: $\bar{X}^N(t)$ is defined for all $t \in \mathbb{R}_+$ by $\bar{X}^N(t) = X^N(k\delta)$ for $k\delta \leq t < (k+1)\delta$. Suppose that $X^N(0) = x \in \square(S)$. Then

$$\begin{aligned} \hat{X}^N(t) - x &= \int_0^t \left[F^N(\bar{X}^N(s)) + U(s)\right] ds \qquad (10) \\ &= \int_0^t \left[F^N(\hat{X}^N(s)) + F^N(\bar{X}^N(s)) - F^N(\hat{X}^N(s)) + U(s)\right] ds \ . \end{aligned}$$

Since $\xi^N(t,x) - x = \int_0^t F^N(\xi^N(s,x))ds$, we obtain

$$||\hat{X}^N(t) - \xi^N(t,x)||_\infty \leq \lambda_N \left[\int_0^t (||\hat{X}^N(s) - \xi^N(s,x)||_\infty)ds + 2\delta T\right] + \Psi(T) \ , \qquad (11)$$

where

$$\Psi(T) = \max_{0\leq t\leq T} || \int_0^t U(s)ds||_\infty \ . \qquad (12)$$

Grönwall's inequality implies

$$D_N^N(T,x) = \max_{0\leq t\leq T} ||\hat{X}^N(t) - \xi^N(t,x)||_\infty \leq \left[\Psi(T) + 2\delta\lambda_N T\right] e^{\lambda_N T} \ . \qquad (13)$$

Thus, for $\delta \leq \frac{\varepsilon}{4\lambda_N T} e^{-\lambda_N T}$,

$$\Pr\left[D_N^N(T,x) \geq \varepsilon\right] \leq \Pr\left[\Psi(T) \geq \frac{\varepsilon}{2} e^{-\lambda_N T}\right] \ . \qquad (14)$$

Our next goal is to estimate the probability on the right-hand side. For $k \in \mathbb{N}$, let

$$Z_k(\theta) = \exp\left(\sum_{i=0}^{k-1} \langle \theta, \delta U_i \rangle - \frac{\Gamma}{2} k\delta^2 ||\theta||_2^2\right) \ . \qquad (15)$$

According to lemma 1, $(Z_k(\theta))_{k\in\mathbb{N}}$ is a supermartingale. Thus, for any $\beta > 0$

$$\begin{aligned}\Pr\left[\max_{0\le k\le n}\langle\theta,\sum_{i=0}^{n-1}\delta U_i\rangle\ge\beta\right] &\le \Pr\left[\max_{0\le k\le n} Z_k(\theta)\ge\exp\left(\beta-\frac{\Gamma_N}{2}||\theta||_2^2 n\delta^2\right)\right]\\ &\le \exp\left(\frac{\Gamma_N}{2}||\theta||_2^2 n\delta^2-\beta\right)\ . \qquad (16)\end{aligned}$$

Let $u_1,\ldots,u_M$ be the canonical basis of $E=\mathbb{R}^M$, $\varepsilon>0$, and $u=\pm u_i$ for some i. Set $\beta=\varepsilon^2/(\Gamma_N n\delta^2)$ and $\theta=(\beta/\varepsilon)u$. Then

$$\begin{aligned}\Pr\left[\max_{0\le k\le n}\langle u,\sum_{i=0}^{k-1}\delta U_i\rangle\ge\varepsilon\right] &= \Pr\left[\max_{0\le k\le n}\langle\theta,\sum_{i=0}^{k-1}\delta U_i\rangle\ge\beta\right] \qquad (17)\\ &\le \exp(\frac{-\varepsilon^2}{2\Gamma_N n\delta^2})\ .\end{aligned}$$

It follows that

$$\Pr\left[\Psi(T)\ge\varepsilon\right]\le 2M\exp\left(\frac{-\varepsilon^2}{2\delta\Gamma_N T}\right)\ . \qquad (18)$$

Therefore,

$$\begin{aligned}\Pr\left[\Psi(T)\ge\frac{\varepsilon}{2}e^{-\lambda_N T}\right] &\le 2M\exp\left(-\varepsilon^2\frac{e^{-2\lambda_N T}}{8\delta\Gamma_N T}\right) \qquad (19)\\ &= 2M\exp\left[-\varepsilon^2 c_N N\right]\ , \qquad (20)\end{aligned}$$

where

$$c_N=\frac{e^{-2\lambda_N T}}{8\Gamma_N T}=\frac{e^{-2\lambda_N T}}{8T(\sqrt{2}+\|F^N\|_2)^2}$$

Hence, the claim in the lemma holds for any $c\in(0,\gamma)$, where

$$\gamma=\lim\inf_{N\to\infty}\frac{e^{-2\lambda_N T}}{8T(\sqrt{2}+\|F^N\|_2)^2} \qquad (21)$$

is positive by our hypotheses about the sequence (F^N). ■

3. Mean-field approximation of $D^N(T,x)$

We now turn to the stochastic variable $D^N(T,x)$. By Grönwall's inequality,

$$\lim_{N\to\infty}\max_{0\le t\le T}||\xi(t,x)-\xi^N(t,x)||_\infty=0$$

Hence, for any $\varepsilon > 0$ there exists a N_ε such that

$$\max_{0 \le t \le T} ||\xi(t,x) - \xi^N(t,x)||_\infty < \varepsilon/2$$

for all $N \ge N_\varepsilon$. Hence, by the Triangle Inequality, for such N, the event

$$A_N = \left\{\omega \in \Omega : D^N(T,x) \ge \varepsilon \mid X^N(0) = x\right\}$$

implies the event

$$B_N = \left\{\omega \in \Omega : D_N^N(T,x) \ge \varepsilon/2 \mid X^N(0) = x\right\}$$

that is, $A_N \subset B_N$, and hence $\Pr(A_N) \le \Pr(B_N)$. Our main result follows immediately from Lemma 2 above:

Proposition 1. *For every $T > 0$ there exists a scalar $c > 0$ such that, for any $\varepsilon > 0$, and any N large enough:*

$$\Pr\left[D^N(T,x) \ge \varepsilon \mid X^N(0) = x\right] \le 2Me^{-\varepsilon^2 cN} \qquad \forall x \in \square^N(S).$$

Remark 1. *The claim holds for any $c \in (0, \gamma/4)$, where $\gamma > 0$ is defined in equation (21).*

Remark 2. *It is easily verified that propositions 1-5 in Benaïm and Weibull (2003) hold also under the new, weaker hypothesis.*

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