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Abstract

This paper studies learning under multiple priors by characterizing the decision maker's attitude toward information. She is incredulous if she integrates new information with respect to only those measures that minimizes the likelihood of the new information and credulous if she uses the maximum likelihood procedure to update her priors. Both updating rules expose her to dynamic inconsistency. We explore different ways to resolve this problem. One way consists to assume that the decision maker's attitude toward information is not relevant to characterize conditional preferences. In this case, we show that a necessary and sufficient condition, introduced by [Epstein L. and Schneider M., 2003. Recursive multiple priors. *Journal of Economic Theory* 113, 1-31], is the rectangularity of the set of priors. Another way is to extend optimism or pessimism to a dynamic set-up. A pessimistic (max-min expected utility) decision maker will be credulous when learning bad news but incredulous when learning good news. Conversely, an optimistic (max-max expected utility) decision maker will be credulous when learning good news but incredulous when learning bad news. It allows max-min (or max-max) expected utility preferences to be dynamically consistent but it violates consequentialism because conditioning works with respect to counterfactual outcomes. The implications of our findings when the set of priors is the core of a non-additive measure are explored.

Key-words: Multiple priors; Learning; Dynamic consistency; Consequentialism; Attitude toward information

JEL classification: D81, D83

1 Introduction

In decision theory, a current way to model decision making under uncertainty is the Max-min Expected Utility (MEU) approach, firstly axiomatized by Gilboa and Schmeidler (1989). The MEU approach allows to describe Ellsberg-type preferences by assuming that the decision maker's beliefs are represented by multiple priors, instead of a single (additive) prior. Because various economic situations involve information arrivals, an important problem concerns the updating of MEU preferences when new information comes. Several axiomatically-based responses have been supplied in the literature.

Gilboa and Schmeidler (1993) axiomatized Bayesian update rules for the case where the set of prior is the core of a convex capacity. They interpreted the Dempster-Shafer updating as pessimistic and the Bayes updating as optimistic, depending on which event the worst or the best outcome would be drawn. Pires (2002) axiomatized the generalized Bayes updating rule for multiple priors. His key axioms are consequentialism, that is, only those consequences that can be reached are relevant when conditioning, and a weakening of dynamic consistency. Assuming dynamic consistency and consequentialism, Wang (2003) axiomatized conditional preferences and obtained the three rules mentioned above. However, contrary to previous works, he did not assume reduction

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of compound acts, that is, the equivalence between the static and the dynamic representation of an act.

Hanany and Klibanoff (2007) proposed an updating rule which depends on the unconditional preference relation. It consists to update such and such subset of the initial set of priors in order to obtain conditional preferences that are consistent with ex-ante preferences. This allows to simultaneously keep a weakened version of dynamic consistency and reduction of compound acts. Whereas conditioning does not depend on counterfactuals outcomes, consequentialism is violated in the sense that the updating is made with respect to ex-ante preferences.

In many real life situations involving information arrivals, our conditional decisions differ, depending on our attitude toward the information. For instance, on financial markets, investors and traders can react disproportionately to new information (e.g. annual operating results) about a given security. Indeed, they can exhibit overconfidence or lake of confidence toward the information. Hence we propose to approach the updating of MEU preferences by characterizing the decision maker's attitude toward information. When facing new information, she can be *incredulous* and only update those priors that give the lower likelihood to the observed event. On the opposite, she can be *credulous* and only consider the priors ascribing the higher value to the realized event. In this case, she will use the maximum likelihood procedure to update her priors. Consequently, credulity and incredulity can be used to characterize conditional preferences. However, both credulous and incredulous decision makers are dynamically inconsistent and so they are exposed to some money pump.

A way to avoid the problem of dynamic inconsistency of credulous and incredulous MEU decision makers is to assume that credulity or incredulity do not impose any constraint on conditional preferences. In this case, credulous and incredulous ways of updating give the same conditional set of probabilities and it is equivalent to apply the Bayes updating rule on each prior, i.e. the generalized Bayes rule. Our first contribution is to show that the decision maker's attitude toward information do not affect MEU conditional preferences if and only if the set of priors is rectangular[1], in the sense of Epstein and Schneider (2003). Our result allows to give a behavioral interpretation of the rectangularity condition. We investigate the implications of this result when the set of priors is the core of a convex capacity. In this case, according to Sarin and Wakker (1998)[2], the capacity is additive over the first stage of a two-stage filtration. The rectangularity of the core of the capacity can be seen as the condition under which MEU and Choquet Expected Utility (CEU) are equivalent in dynamic choice situations.

An other way to avoid the problem of dynamic inconsistency is to mix different characterizations of the decision maker's attitude toward information. It consists in assuming that the decision maker will be incredulous or credulous depending on the nature of the information with respect to counterfactuals outcomes. We distinguish pessimistic and optimistic decision makers. An uncertainty-averse decision maker will be credulous when learning bad news but incredulous when receiving good news. Therefore, such a way of updating can be interpreted as a dynamic extension of pessimism, understood as ambiguity-aversion. On the opposite, an uncertainty-lover decision maker will be credulous when learning good news but incredulous when receiving bad news. Then our second contribution is to propose an updating rule based on attitudes toward uncertainty that allows preferences to have a recursive structure. From an axiomatic point of view, dynamic consistency holds but consequentialism is violated, in the sense that the way of conditioning is depending on counterfactuals outcomes.

The paper is organized as follows. The next section presents the set-up and introduces main notions throughout a motivating example. Section 3 exposes the link between attitudes toward information and rectangularity. Section 4 presents our approach of the updating based on attitudes toward uncertainty and section 5 concludes.

[1] The rectangularity condition allows the MEU criterion to have a recursive structure. The recursive multiple priors utility model has been applied to real option (Nishimura and Ozaki 2007, Trojanowska and Kort 2007, Miao and Wang 2007), job search (Nishimura and Ozaki 2004) and portfolio selection (Chen and Epstein 2002, Epstein and Schneider 2007, Epstein and Schneider 2008).

[2] Eichberger, Grant and Kelsey (2005) give a partial converse of this result by showing that if the capacity is convex, then a necessary and sufficient condition to dynamic consistency is that the capacity be additive over the last stage of the filtration. Dominiak and Lefort (2008) generalize this result to non-necessarily convex capacities.

2 The set-up and main notions

We consider a discrete and finite time represented by $T = \{0, 1, 2\}$ and a state space S containing a finite number of states noted s. An event is a subset of S and for all $B \subset S$ we note B^c the event $S \backslash B$. The information structure is given by a filtration $\{\mathcal{F}_t\}_{t \in T}$, where $\mathcal{F}_0$ is trivial and for all t in T we take $\mathcal{F}_t$ as an algebra. X is an outcome space, i.e. a subset of $\mathbb{R}$, and we denote by $\mathcal{A} \subseteq X^S = \{f : S \to X\}$ the set of acts, or measurable functions. We write $f_B \geq f_{B^c}$ if $\forall s \in B, \forall s' \in B^c, f(s) \geq f(s')$.

A decision maker is characterized by a class of binary relations $\{\succcurlyeq_B\}_{B \in \mathcal{F}_1}$ on $\mathcal{A}$. When $B = S$, we write $\succcurlyeq$ the unconditional preference relation. MEU over Savage-style acts with a finite state space is axiomatized in Alon and Schmeidler (2009)[3] and we will assume that $\succcurlyeq$ satisfies their axioms. Then for all f and g in $\mathcal{A}$, $f \succcurlyeq g$ if and only if $\mathbb{E}_{\min p \in \mathcal{P}}[u(f)] \geq \mathbb{E}_{\min p \in \mathcal{P}}[u(g)]$, where $\mathcal{P}$ is a (unique, non empty, closed, and convex) set of finitely probability measures on the filtered measurable space $(S, \{\mathcal{F}_t\}_{t \in T})$ and the utility function $u : X \to \mathbb{R}$ is unique up to a positive affine transformation. We denote by $\mathcal{P}_0^1 = \{m : \mathcal{F}_1 \to [0; 1] | \sum_{B \in \mathcal{F}_1} m(B) = 1\}$ the set of probabilities measuring first stage events.

Several rules can be adopted to update $\mathcal{P}$ when the right event in $\mathcal{F}_1$ is known. Each of them implies that the class of conditional preferences $\{\succcurlyeq_B\}_{B \in \mathcal{F}_1}$ is represented by a MEU functional. Given any first stage event B, the conditional probability obtained from any p in $\mathcal{P}$ is noted p_B. The most intuitive updating rule for MEU preferences is the following:

Definition 1 *The Generalized Bayes updating rule applied to $\mathcal{P}$ conditional on $B \in \mathcal{F}_1, \min\limits_{p \in \mathcal{P}} p(B) > 0$, gives a set $\mathcal{P}_B$ of conditional probabilities such that:*

$$\mathcal{P}_B = \{p_B | \forall A \in \mathcal{F}_2, p_B(A) = \frac{p(A \cap B)}{p(B)}, p \in \mathcal{P}\}$$

Given any event $B \in \mathcal{F}_1$, we distinguish the following subsets of $\mathcal{P}$:
$\mathcal{Q}^B = \{p \in \mathcal{P} : p \in \underset{p \in \mathcal{P}}{\text{argmax}} p(B)\}$ and $\mathcal{R}^B = \{p \in \mathcal{P} : p \in \underset{p \in \mathcal{P}}{\text{argmin}} p(B)\}$.

Definition 2 *The credulous updating rule applied to $\mathcal{P}$ conditional on $B \in \mathcal{F}_1, \min\limits_{p \in \mathcal{P}} p(B) > 0$, gives a set $\mathcal{Q}_B^B$ of conditional probabilities such that:*

$$\mathcal{Q}_B^B = \{p_B | \forall A \in \mathcal{F}_2, p_B(A) = \frac{p(A \cap B)}{p(B)}, p \in \mathcal{Q}^B\}$$

This is equivalent to the maximum likelihood procedure. The following example, closed to Kelsey (1995), shows that a credulous decision maker can be victim of money pumping.

Example 1 Consider a dynamic version of the Ellsberg paradox[4]. A MEU decision maker is facing an urn with 30 red balls and 60 blue or green balls. At time 1, a ball is drawn and the decision maker knows whether this ball is green or not. At time 2, the color of this ball is fully revealed to the decision maker. The state space is $S = \{R, B, G\}$ and the information at time 1 is delivered by $\mathcal{F}_1 = \{S, \varnothing, \{R, B\}, G\}$. A possible set of priors may be:

$$\mathcal{P}' = \{p' = (\frac{1}{3}, \beta, \frac{2}{3} - \beta) | \beta \in [\frac{1}{6}; \frac{1}{2}]\} \tag{1}$$

Assume that the decision maker initially owns the lottery $f \equiv (100, 900, 0)$ and that the utility $u(.)$ is linear. Assume that an entrepreneur proposes her to exchange the lottery f to the lottery $g \equiv (1200, 0, 0)$ for an amount of money, say $\varepsilon = 100$. Because $\mathbb{E}_{\min p' \in \mathcal{P}'}(g - \varepsilon) > \mathbb{E}_{\min p' \in \mathcal{P}'}(f)$, she will choose the lottery $g - \varepsilon$ rather than f. If event G occurs, the decision maker looses 100. Assume that event $(R \cup B)$ occurs. If the decision maker is credulous, she overweights the

[3]Other axiomatizations of MEU preferences in a purely subjective set-up are Casadesus-Masanell et al. (2000) and Ghirardato et al. (2003).

[4]Similar extensions of this experience have been proposed by Epstein and Schneider (2003) and Hanany and Klibanoff (2007).

information. In this case, she will use the maximum likelihood procedure to update her priors and only consider those priors that give the higher value to the realized event. Then:

$$q_{R\cup B}(R) = \frac{\frac{1}{3}}{\max\limits_{p\in\mathcal{P}} p(R\cup B)} = \frac{2}{5} \tag{2}$$

$$q_{R\cup B}(B) = \frac{\max\limits_{p\in\mathcal{P}} p(B)}{\max\limits_{p\in\mathcal{P}} p(R\cup B)} = \frac{3}{5} \tag{3}$$

and, obviously, $q_{R\cup B}(G) = 0$. Hence the conditional expectations calculated w.r.t. $q_{R\cup B}$ are $\mathbb{E}_{r_{R\cup B}}(f-\varepsilon) > \mathbb{E}_{r_{R\cup B}}(g-\varepsilon)$. Then the decision maker is willing to exchange $g-\varepsilon$ for $f-\varepsilon$, and, given event $R\cup B$, she will end up with the lottery $(0, 800, -100)$ instead of $(100, 900, 0)$.

The example shows that a credulous decision maker can be victim of money pumping. This is due to the fact that she is dynamically inconsistent, in the sense that new information arrivals may generate a reversal between ex-ante and ex-post preferences. This conclusion applies as well as to incredulous decision makers, i.e. decision makers who are suspicious about the information.

Definition 3 *The incredulous updating rule applied to $\mathcal{P}$ conditional on $B \in \mathcal{F}_1, \min\limits_{p\in\mathcal{P}} p(B) > 0$, gives a set $\mathcal{R}_B^B$ of conditional probabilities such that:*

$$\mathcal{R}_B^B = \{p_B | \forall A \in \mathcal{F}_2, p_B(A) = \frac{p(A\cap B)}{p(B)}, p \in \mathcal{R}^B\}$$

Because such decision makers will not found event B very credible, they will uniquely update their priors with regards to the lower envelope of probability priors measuring the realized event.

As noted above, when we characterize the decision maker's attitude toward information, she exhibits dynamic inconsistency. Hence a way to avoid dynamic inconsistency is to assume that the decision maker's attitudes toward information do not affect conditional preferences: it is the recursive multiple priors approach.

3 The recursive multiple priors approach

As mentioned in the introduction, Epstein and Schneider (2003) have axiomatized the recursive multiple priors utility model. "Recursive" means that the following relation holds:

$$\mathbb{E}_{\min p\in\mathcal{P}}[u(f)] = \mathbb{E}_{\min m\in\mathcal{P}_0^1}[u(f)|\mathcal{F}_1] \tag{4}$$

The left member of this equation represents the static evaluation of the act f, whereas the right member represents its dynamic evaluation. Such a relation is allowed by the following condition on the structure of $\mathcal{P}$:

Definition 4 *$\mathcal{P}$ is $\mathcal{F}_t$-rectangular if for all t in T and B in $\mathcal{F}_1$,*

$$\mathcal{P} = \{\int_S p_B(.)\mathrm{d}m | B \in \mathcal{F}_1, p_B \in \mathcal{P}_B, m \in \mathcal{P}_0^1\} \tag{5}$$

Each p in $\mathcal{P}$ is a dynamic coherent risk measure in the sense of Riedel (2004). Given any first stage event B in $\mathcal{F}_1$ and any second stage event A in $\mathcal{F}_2$ such that $A \subset B$, the rectangularity condition can be simply expressed as:

$$\mathcal{P}(A) = \{p(A) = p_B(A) \otimes m(B) | p_B \in \mathcal{P}_B, m \in \mathcal{P}_0^1\} \tag{6}$$

Example 2 Let us reconsider the previous example. The rectangular set of priors $\mathcal{P}$ is such that, for any $q, q' \in \mathcal{P}'$ and m in $\{m = (m(R\cup B) = \frac{1}{3}+\beta, m(G) = \frac{2}{3}-\beta) | \beta \in [\frac{1}{6}; \frac{1}{2}]\}$, a generic element

p in $\mathcal{P}$ is defined by $\forall s \in S, p(s) = q_G(s) \otimes m(G) + q'_{R\cup B}(s) \otimes m(R\cup B)$ and then definition 4 yields:

$$\mathcal{P} = \{p = (\frac{1}{3}\frac{m(R\cup B)}{\frac{1}{3}+\beta}, \beta\frac{m(R\cup B)}{\frac{1}{3}+\beta}, \frac{2}{3} - m(R\cup B))|\beta \in [\frac{1}{6}; \frac{1}{2}], m(R\cup B) \in [\frac{1}{2}; \frac{5}{6}]\} \tag{7}$$

The set of conditional probabilities $\mathcal{P}_{R\cup B}$ obtained by the GB rule on $\mathcal{P}'$ and $\mathcal{P}$ are identical:

$$\mathcal{P}_{(R\cup B)} = \{p_{R\cup B} = (\frac{\frac{1}{3}}{\frac{1}{3}+\beta}, \frac{\beta}{\frac{1}{3}+\beta})|\beta \in [\frac{1}{6}, \frac{1}{2}]\} \tag{8}$$

We distinguish the following subsets of $\mathcal{P}$: $\mathcal{Q}^{R\cup B} = \{p|p \in \mathcal{P}, p(R) + p(B) = \frac{5}{6}\}$ and $\mathcal{R}^{R\cup B} = \{p|p \in \mathcal{P}, p(R) + p(B) = \frac{1}{2}\}$. If the decision maker is credulous, she will update $\mathcal{P}$ with the maximum likelihood procedure and obtain:

$$\mathcal{Q}^{R\cup B}_{R\cup B} = \left\{q_{R\cup B}|q_{R\cup B}(.) = \frac{q(.)}{\frac{5}{6}}, q(R) = \frac{1}{3}\frac{\frac{5}{6}}{\frac{1}{3}+\beta}, q(B) = \beta\frac{\frac{5}{6}}{\frac{1}{3}+\beta}, \beta \in [\frac{1}{6}; \frac{1}{2}]\right\} \tag{9}$$

If she is incredulous, then:

$$\mathcal{R}^{R\cup B}_{R\cup B} = \left\{r_{R\cup B}|r_{R\cup B}(.) = \frac{r(.)}{\frac{1}{2}}, r(R) = \frac{1}{3}\frac{\frac{1}{2}}{\frac{1}{3}+\beta}, r(B) = \beta\frac{\frac{1}{2}}{\frac{1}{3}+\beta}, \beta \in [\frac{1}{6}; \frac{1}{2}]\right\} \tag{10}$$

It can be readily seen that $\mathcal{P}_{R\cup B} = \mathcal{Q}^{R\cup B}_{R\cup B} = \mathcal{R}^{R\cup B}_{R\cup B}$, hence the decision maker's attitude toward information does not impose any constraint on conditional preferences.

The result is generally stated in the following:

Theorem 1 *Let $\mathcal{P}$ be a convex, compact and non-empty set of priors on $(S, \{\mathcal{F}_t\}_{t\in T})$. Then the following two statements are equivalent:*

i. $\mathcal{P}$ is $\mathcal{F}_t$-rectangular as in definition 4;

ii. $\mathcal{P}_B = \mathcal{Q}^B_B = \mathcal{R}^B_B$.

Proof See Appendix A.1. □

The main implication of this result is that the rectangularity condition does not allow to take into account different attitudes toward information.

In various uncertain situations dealing with Ellsberg-type preferences, the decision maker's beliefs can be represented by a Choquet capacity, i.e. a set function $\nu : 2^S \to [0; 1]$ such that $\nu(\varnothing) = 0$, $\nu(S) = 1$ and $\forall A, B \in 2^S, A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$. It is convex if $\forall A, B \in 2^S, \nu(A) + \nu(B) \leq \nu(A \cup B) + \nu(A \cap B)$.

In general, $\mathcal{P}$ is not characterizable by its lower envelope. However, if there exists a convex capacity, then it is a lower envelope and it defines $\mathcal{P}$:

$$\mathcal{P} = \{p : 2^S \to [0; 1]|p\,\text{additive}, \nu \leq p \leq \bar{\nu}\}$$

where $\bar{\nu}$ denotes the conjugate capacity such that $\forall A \in 2^S, \bar{\nu}(A) = 1 - \nu(A^c)$. In this case, $\mathcal{P}$ is the core of ν, noted core(ν), and ν is a lower probability from $\mathcal{P}$. Moreover, the Choquet integral of utility w.r.t. ν is equivalent to the MEU approach w.r.t. $\mathcal{P}$ (Schmeidler, 1986) and the updating rules defined for $\mathcal{P}$ can be used for ν.

Definition 5 *The Dempster-Shafer updating rule for ν conditional on $B \in \mathcal{F}_1, \bar{\nu}(B) > 0$, is given by:*

$$\forall A \in \mathcal{F}_2, \nu_B(A) = \frac{\nu((A \cap B) \cup B^c) - \nu(B^c)}{1 - \nu(B^c)}$$

This is equivalent to the maximum likelihood procedure applied to $\mathcal{P}$ (Gilboa and Schmeidler 1993). Symmetrically, the second updating rule for ν corresponds to the incredulous updating rule applied to $\mathcal{P}$ (see Chateauneuf et al. 2001).

Definition 6 *The Bayes updating rule for ν conditional on $B \in \mathcal{F}_1, \nu(B) > 0$, is given by:*

$$\forall A \in \mathcal{F}_2, \nu_B(A) = \frac{\nu(A \cap B)}{\nu(B)}$$

Therefore, it is obvious that these rules can be used to model the CEU decision maker's attitude toward information. If she is incredulous, she will use the Bayes updating rule whereas if she is credulous, she will use the Demspter-Shafer updating rule. The third rule has been developed, for instance, by Jaffray (1992) and Denneberg (1994).

Definition 7 *The Generalized Bayes updating rule for ν conditional on $B \in \mathcal{F}_1, \bar{\nu}(B) > 0$[5], is given by:*

$$\forall A \in \mathcal{F}_2, \nu_B(A) = \frac{\nu(A \cap B)}{1 + \nu(A \cap B) - \nu(A \cup B^c)}$$

Jaffray (1992) shows that the set of conditional probabilities $\mathcal{P}_B$ is in general a subset of core(ν_B), and further characterizations[6]of the capacity are needed in order to prove that $\mathcal{P}_B =$ core(ν_B). Note that a sufficient (but not necessary) condition is that core(ν) be rectangular, this will made clear in proposition 1.

Symmetrically to MEU preferences, a way to avoid the problem of dynamic inconsistency of CEU decision makers is to assume that core(ν) is rectangular.

Proposition 1 *Let ν be a convex capacity on $(S, \{\mathcal{F}_t\}_{t \in T})$. Then the following statements are equivalent:*

i. ν defines a $\mathcal{F}_t$-rectangular set of priors as in definition 4;

ii. For all B in $\mathcal{F}_1$ s.t. $\nu(B) > 0$, $\bar{\nu}(B) > 0$, and for all A in $\mathcal{F}_2$,

$$\nu_B(A) = \frac{\nu(A \cap B) \cup B^c) - \nu(B^c)}{1 - \nu(B^c)} = \frac{\nu(A \cap B)}{\nu(B)} = \frac{\nu(A \cap B)}{1 + \nu(A \cap B) - \nu(A \cup B^c)}$$

iii. ν is additive on $\mathcal{F}_1$.

Proof See Appendix A.2. □

It is clear that if core(ν) is $\mathcal{F}_t$-rectangular, then MEU preferences and CEU preferences are equivalent in dynamic choice situations. Indeed, when both criterion admit a recursive representation, we should have:

$$\mathbb{E}_{\min p \in \mathcal{P}}[u(f)] = \mathbb{E}_\nu[u(f)]$$

if and only if:

$$\mathbb{E}_{\min m \in \mathcal{P}_0^1}[u(f)\,|\mathcal{F}_1] = \mathbb{E}_\nu[u(f)|\mathcal{F}_1]$$

4 A Pessimistic approach

Pessimism is usually defined as ambiguity aversion. That is why the question arises of how a decision maker who always considers the worst case should update her preferences. Our approach suggests that a pessimistic decision maker will update her priors with regards to the nature of the information with respect to counterfactuals outcomes. Given any act f in $\mathcal{A}$ and any event B in $\mathcal{F}_1$, if the information constitutes good news, i.e. if $f_B \geq f_{B^c}$, she will only update priors from the subset $\mathcal{R}^B$. Indeed, in this case, she minimizes the weight of the information because she doesn't find it very credible. In other words, she is incredulous. However, if the event B^c occurs, then she will use the maximum likelihood procedure to update her priors. Indeed, the information constitutes a bad new. Therefore, conditional probabilities are given by the updating of $\mathcal{Q}^B$ and

[5] Such a condition is proved to ensure that the conditional capacity be defined in Denneberg (1994, proposition 2.1)

[6] Eichberger et al (2009) show that a necessary and sufficient condition to $core(\nu_B) = \mathcal{P}_B$ is that the capacity be a generalized version of a convex neo-additive capacity (see Chateauneuf et al. (2007)

she maximizes the weight of the information. Hence a pessimistic decision maker is credulous when learning bad news. In the context of non-additive measures, such a definition of pessimism can be found in Chateauneuf et al. (2001) and is linked to Gilboa and Schmeidler (1993).

Definition 8 *The pessimistic updating rule applied to $\mathcal{P}$ conditional on $B \in \mathcal{F}_1$ is given by: $\forall f \in \mathcal{A}$,*

- *If $f_B \geq f_{B^c}$, then the updating is incredulous;*
- *If $f_B \leq f_{B^c}$, then the updating is credulous.*

Whereas credulity and incredulity of MEU decision makers expose them to dynamic inconsistency, pessimism is normatively appealing. Indeed, this allows the criterion to have a recursive representation.

Theorem 2 *Assume that the decision maker is pessimistic. Then, for all f in $\mathcal{A}$ and B in $\mathcal{F}_1$, $f_B \leq f_{B^c}$ or $f_B \geq f_{B^c}$ imply the recursive relation*

$$\mathbb{E}_{\min p \in \mathcal{P}}[u(f)] = \mathbb{E}_{\min m \in \mathcal{P}_0^1}[u(f)|\mathcal{F}_1] \tag{11}$$

holds true.

Proof See Appendix B.1. □

On the opposite, we can define the optimistic way of updating corresponding to the max-max expected utility $\mathbb{E}_{\max p \in \mathcal{P}}[u(.)]$:

Definition 9 *The optimistic updating rule applied to $\mathcal{P}$ conditional on $B \in \mathcal{F}_1$ is given by: $\forall f \in \mathcal{A}$,*

- *If $f_B \geq f_{B^c}$, then the updating is credulous;*
- *If $f_B \leq f_{B^c}$, then the updating is incredulous.*

Obviously, if the criterion used is the max-max expected utility instead of MEU, the optimistic rule also allows the recursive relation 11 to be hold. Therefore, in the α-MEU framework[7], where the value of any f in $\mathcal{A}$ is given by:

$$\alpha \mathbb{E}_{\min p \in \mathcal{P}}[u(f)] + (1 - \alpha)\mathbb{E}_{\max p \in \mathcal{P}}[u(f)]$$

with $\alpha \in [0; 1]$, the pessimistic and the optimistic updating rules correspond to the case where $\alpha = 1$ and $\alpha = 0$, respectively. Then both rules can be seen as dynamic extensions of ambiguity-averse or ambiguity-lover preferences.

It should be noted that, whereas the rectangularity condition allows the recursive relation 11 to hold on all acts, and not only when information is good or bad news, the pessimistic updating rule does not. However, contrary to rectangularity, the recursive relation implied by the pessimistic updating rule is not tied to the filtration. Indeed, our rule can be applied for all B in 2^S, and it is not needed to restrict the domain of events on a given and fixed filtration. It allows to compare ambiguous situations where uncertainty is differently resolved while preserving a recursive structure.

Now we define dynamic consistency:

Property 1 *(Dynamic consistency) For all B in $\mathcal{F}_1$ and f, g in $\mathcal{A}$, $f \succcurlyeq_B g$ and $f \succcurlyeq_{B^c} g$ implies $f \succcurlyeq g$.*

Dynamic consistency avoids money pumps arguments. Because the pessimistic updating rule allows MEU preferences to have a recursive structure, dynamic consistency is satisfied by a pessimistic decision maker.

Corollary 1 *Assume that the decision maker is pessimistic. Then $\{\succcurlyeq_B\}_{B \in \mathcal{F}_1}$ satisfies dynamic consistency.*

[7] α-MEU preferences have been axiomatized by Ghirardato et al. (2004).

This result can be illustrated in the example 1 if the decision maker uses the incredulous updating rule instead of the credulous one. Another dynamic property is the following:

Property 2 *(Consequentialism) For all B in $\mathcal{F}_1$ and f, g in $\mathcal{A}$, $f_B = g_B$ implies $f \sim_B g$.*

Consequentialism is not satisfied, in general, by a pessimistic decision maker, except, obviously, in the case where $\mathcal{P}$ is $\mathcal{F}_t$-rectangular.

Proposition 2 *Assume that the decision maker is pessimistic. Then $\{\succsim_B\}_{B\in\mathcal{F}_1}$ does not satisfy consequentialism.*

Proof See Appendix B.2. □

Similarly, if the decision maker uses the max-max expected utility criterion, then the optimistic updating rule implies dynamic consistency but violates consequentialism. In our knowledge, such a way of updating is the only one to drop consequentialism, understood as the assumption that counterfactuals outcomes are not relevant to the decision maker. In the CEU framework, an axiomatically-based pessimistic updating rule has been developed by Chateauneuf et al. (2001). It consists to use the Bayes updating rule when information brings good news and the Dempster-Shafer updating rule when information brings bad news. If the capacity is convex, then it is equivalent to our approach. On the opposite, if the capacity is concave, then it is equivalent to the optimistic updating. In both cases, the recursive relation

$$\mathbb{E}_\nu[u(f)] = \mathbb{E}_\nu[u(f)|\,\mathcal{F}_1]$$

holds true when $f_B \geq f_{B^c}$ or $f_B \leq f_{B^c}$, but consequentialism is violated.

5 Conclusion

We have studied learning under multiple priors by characterizing the decision maker's attitude toward information. We have distinguished credulous and incredulous decision makers. Both of them exhibit dynamically inconsistent preferences and can be victim of money pumps. Then two approaches can be adopted to preserve dynamic consistency. First, we assume that attitude toward information, i.e. the way of updating, does not impose any constraint on conditional preferences. In this case, the set of priors must be rectangular. Second, when information is good news or bad news, we suppose that the way of updating is relevant but can vary depending on the counterfactuals outcomes. Then we propose a dynamic extension to pessimism/optimism allowing dynamic consistency but relaxing consequentialism.

APPENDIX

A Proofs of results of section 3

A.1 Proof of theorem 1

$(i) \Rightarrow (ii)$. Let $\mathcal{P} = \{\int_S p_B(.)\mathrm{d}m | B \in \mathcal{F}_1, p_B \in \mathcal{P}_B, m \in \mathcal{P}_0^1\}$ be the rectangular set of priors obtained from an arbitrary (closed, convex and non-empty) set $\mathcal{P}' = \{p' | p' \,\text{is additive on}\, 2^S\}$ of probability measures. Given any events $A \in \mathcal{F}_2$ and $B \in \mathcal{F}_1$ s.t. $A \subset B$, $\mathcal{P}$ can be rewritten as:

$$\mathcal{P}(A) = \{p(A) = p_B(A) \otimes m(B) | p_B(A) = \frac{p'(A)}{p'(B)}, p' \in \mathcal{P}', m \in \mathcal{P}_0^1\} \tag{12}$$

Because

$$p_B(A) = \frac{\frac{p'(A)}{p'(B)} \otimes m(B)}{\sum_{A\subset B} \frac{p'(A)}{p'(B)} \otimes m(B)} = \frac{\frac{p'(A)}{p'(B)} \otimes m(B)}{m(B)} = \frac{p'(A)}{p'(B)} \tag{13}$$

the GB updating rule applied to $\mathcal{P}'$ or $\mathcal{P}$ generates the same set of conditional probabilities

$$\mathcal{P}'_B = \{p'_B = \frac{p'(.)}{p'(B)} | p' \in \mathcal{P}'\} = \{p_B = \frac{p'(.)\frac{m(B))}{p'(B)}}{m(B)} | m \in \mathcal{P}_0^1, p' \in \mathcal{P}'\} = \mathcal{P}_B \tag{14}$$

Then the update from p is given by $p_B = \frac{p'(.)}{p'(B)}$ for all m in $\mathcal{P}_0^1$. Hence for all $B \in \mathcal{F}_1$ and $A \in \mathcal{F}_2$ s.t. $A \subset B$, the application of the maximum likelihood procedure to $\mathcal{P}$ implies that each conditional probability $p_B(.)$ in $\mathcal{Q}_B^B$ is given by:

$$p_B(A) = \frac{p'(A)\frac{\max_{m \in \mathcal{P}_0^1} m(B)}{p'(B)}}{\max_{m \in \mathcal{P}_0^1} m(B)} = \frac{p'(A)}{p'(B)} \tag{15}$$

Similarly, $\forall p_B \in \mathcal{R}_B^B$,

$$p_B(A) = \frac{p'(A)\frac{\min_{m \in \mathcal{P}_0^1} m(B)}{p'(B)}}{\min_{m \in \mathcal{P}_0^1} m(B)} = \frac{p'(A)}{p'(B)} \tag{16}$$

Therefore,

$$\mathcal{Q}_B^B = \mathcal{R}_B^B = \{p_B | p_B(A) = \frac{p(A \cap B)}{p(B)}, A \in \mathcal{F}_2, A \subset B, p \in \mathcal{P}\} = \mathcal{P}_B \tag{17}$$

$(ii) \Rightarrow (i)$. We have: $\mathcal{P}_B = \mathcal{R}_B^B = \mathcal{Q}_B^B$. Then, for all p in $\mathcal{P}$, B in $\mathcal{F}_1$ and A in $\mathcal{F}_2$ such that $A \subset B$, there exists r in $\mathcal{R}^B$ and q in $\mathcal{Q}^B$ s.t.:

$$p_B(A) = r_B(A) = q_B(A) \tag{18}$$

with $p_B \in \mathcal{P}_B$, $r_B \in \mathcal{R}_B^B$ and $q_B \in \mathcal{Q}_B^B$. As $r(B) \leq p(B) \leq q(B)$, it follows that $r(A) \leq p(A) \leq q(A)$ and then:

$$\mathcal{P}(A) = \{p(A) : r(A) \leq p(A) \leq q(A)\} \tag{19}$$

Moreover, $r_B(A) = p_B(A)$ if and only if $r(A) = \frac{p(A)}{p(B)} \otimes r(B)$ and $q_B(A) = p_B(A)$ if and only if $q(A) = \frac{p(A)}{p(B)} \otimes q(B)$. Therefore:

$$\mathcal{P}(A) = \{p(A) | \frac{p(A)}{p(B)} \otimes r(B) \leq p(A) \leq \frac{p(A)}{p(B)} \otimes q(B)\} \tag{20}$$

Because $r(B) = \min_{m \in \mathcal{P}_0^1} m(B)$ and $q(B) = \max_{m \in \mathcal{P}_0^1} m(B)$, we have:

$$\mathcal{P}(A) = \{p(A) | p(A) = m(B) \otimes p_B(A), m \in \mathcal{P}_0^1\} \tag{21}$$

which can be rewritten as:

$$\mathcal{P} = \{\int_S p_B(.) \mathrm{d}m | B \in \mathcal{F}_1, p_B \in \mathcal{P}_B, m \in \mathcal{P}_0^1\} \tag{22}$$

A.2 Proof of proposition 1

The implications $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$ are obvious. To see that $(i) \Rightarrow (iii)$, note that if core(ν) is $\mathcal{F}_t$-rectangular, then equation 4 becomes:

$$\mathbb{E}_\nu[u(f)] = \mathbb{E}_\nu[u(f)|\mathcal{F}_1] \tag{23}$$

which is equivalent to (iii) (see Sarin and Wakker 1998, theorem 3.1).

Finally, note that (ii) gives:

$$\frac{\nu(A)}{\nu(B)} = \frac{\nu(A)}{1 + \nu(A) - \nu(A \cup B^c)} \tag{24}$$

when $A \subset B$. It implies:

$$\nu(B) = 1 + \nu(A) - \nu(A \cup B^c) \tag{25}$$

hence $\nu(A \cup B^c) - \nu(B^c) = 1 + \nu(A) - \nu(B) - \nu(B^c)$ and then (ii) implies:

$$\frac{1 + \nu(A) - \nu(B) - \nu(B^c)}{1 - \nu(B^c)} = \frac{\nu(A)}{\nu(B)} \tag{26}$$

and if ν is not additive, then there exists $\varepsilon \in \mathbb{R}$ such that:

$$\frac{\nu(A) - \varepsilon}{\nu(B) - \varepsilon} = \frac{\nu(A)}{\nu(B)} \tag{27}$$

If $\nu(A) \neq \nu(B)$, then $\varepsilon = 0$ and $\nu(.)$ is additive on $\mathcal{F}_1$.

B Proofs of results of section 4

B.1 Proof of theorem 2

We successively consider $f_B \leq f_{B^c}$ and $f_B \geq f_{B^c}$.

Case 1. $f_B \leq f_{B^c}$. In this case, the decision maker considers $\mathcal{Q}^B$ if B occurs and $\mathcal{R}^{B^c}$ if B^c occurs to update her priors. Moreover, we have:

$$\mathbb{E}_{\min p_B \in \mathcal{Q}_B^B}[u(f)] \leq \mathbb{E}_{\min p_{B^c} \in \mathcal{R}_{B^c}^{B^c}}[u(f)] \tag{28}$$

Let us define conditional probabilities q_B and r_{B^c} as $q_B \in \underset{p_B \in \mathcal{Q}_B^B}{\operatorname{argmin}} \int_S u(f) dp_B$ and $r_{B^c} \in \underset{p_{B^c} \in \mathcal{R}_{B^c}^{B^c}}{\operatorname{argmin}} \int_S u(f) dp_{B^c}$.

Then inequation 28 implies:

$$\mathbb{E}_{\min m \in \mathcal{P}_0^1}[u(f)|\mathcal{F}_1] = \max_{m \in \mathcal{P}_0^1} m(B) \mathbb{E}_{q_B}[u(f)] + \min_{m \in \mathcal{P}_0^1} m(B) \mathbb{E}_{r_{B^c}}[u(f)] \tag{29}$$

Therefore, we can define a measure π such that $\forall A \subset B, \pi(A) = q(A)$, $\forall A \subset B^c, \pi(A) = r(A)$ and $\pi(B) = \max_{m \in \mathcal{P}_0^1} m(B)$. We have:

$$\mathbb{E}_\pi[u(f)] = \mathbb{E}_{\min m \in \mathcal{P}_0^1}[u(f)|\mathcal{F}_1] \tag{30}$$

Let $p^* \in \underset{p \in \mathcal{P}}{\operatorname{argmin}} \int_S u(f) dp$. Because $f_B \leq f_{B^c}$, the pessimistic updating rule gives $p_B^*(.) = \frac{p^*(.)}{\max_{m \in \mathcal{P}_0^1} m(B)}$ and $p_{B^c}^*(.) = \frac{p^*(.)}{\min_{m \in \mathcal{P}_0^1} m(B^c)}$ and, moreover, $\int_S u(f) dp_B^* \leq \int_S u(f) dp_{B^c}^*$. It implies:

$$\mathbb{E}_{p^*}[u(f)] = \max_{m \in \mathcal{P}_0^1} m(B) \mathbb{E}_{p_B^*}[u(f)] + \min_{m \in \mathcal{P}_0^1} m(B) \mathbb{E}_{p_{B^c}^*}[u(f)] \tag{31}$$

If $p_B^* \notin \underset{p_B \in \mathcal{Q}_B^B}{\operatorname{argmin}} \int_S u(f) dp_B$ and/ or $p_{B^c}^* \notin \underset{p_{B^c} \in \mathcal{R}_{B^c}^{B^c}}{\operatorname{argmin}} \int_S u(f) \mathrm{d}p_{B^c}$, then $\int_S u(f) \mathrm{d}p_B^* > \int_S u(f) \mathrm{d}q_B$ and/or $\int_S u(f) \mathrm{d}p_{B^c}^* > \int_S u(f) \mathrm{d}r_{B^c}$, hence:

$$\max_{m \in \mathcal{P}_0^1} m(B) \mathbb{E}_{p_B^*}[u(f)] + \min_{m \in \mathcal{P}_0^1} m(B) \mathbb{E}_{p_{B^c}^*}[u(f)] > \mathbb{E}_{\min m \in \mathcal{P}_0^1}[u(f)|\mathcal{F}_1] \tag{32}$$

Together with 30 and 31, 32 gives:

$$\mathbb{E}_{p^*}[u(f)] > \mathbb{E}_\pi[u(f)] \tag{33}$$

which is a contradiction. Therefore, $\mathbb{E}_{p^*}[u(f)] = \mathbb{E}_\pi[u(f)]$ hence $\pi \in \underset{p \in \mathcal{P}}{\operatorname{argmin}} \int_S u(f) dp$. It implies:

$$\min_{p \in \mathcal{P}} \int_S u(f) \mathrm{d}p = \min_{m \in \mathcal{P}_0^1} \int_S \left(\min_{p_B \in \mathcal{Q}_B^B} \int_S u(f) \mathrm{d}p_B, \min_{p_{B^c} \in \mathcal{R}_{B^c}^{B^c}} \int_S u(f) \mathrm{d}p_{B^c} \right) \mathrm{d}m \tag{34}$$

which is equivalent to equation 11 when $f_B \leq f_{B^c}$.

Case 2. $f_B \geq f_{B^c}$. In this case, the decision maker considers $\mathcal{R}^B$ if B occurs and $\mathcal{Q}^{B^c}$ if B^c occurs to update her priors. Then the method is similar to case 1. We have:

$$\mathbb{E}_{\min p_B \in \mathcal{R}_B^B}[u(f)] \geq \mathbb{E}_{\min p_{B^c} \in \mathcal{Q}_{B^c}^{B^c}}[u(f)] \tag{35}$$

We define conditional probabilities r_B and q_{B^c} as $r_B \in \underset{p_B \in \mathcal{R}_B^B}{\operatorname{argmin}} \int_S u(f) dp_B$ and $q_{B^c} \in \underset{p_{B^c} \in \mathcal{Q}_{B^c}^{B^c}}{\operatorname{argmin}} \int_S u(f) dp_{B^c}$. Therefore:

$$\mathbb{E}_{\min m \in \mathcal{P}_0^1}[u(f)|\mathcal{F}_1] = \min_{m \in \mathcal{P}_0^1} m(B) \mathbb{E}_{r_B}[u(f)] + \max_{m \in \mathcal{P}_0^1} m(B) \mathbb{E}_{q_{B^c}}[u(f)] \tag{36}$$

Therefore, we can define a measure π such that $\forall A \subset B, \pi(A) = r(A)$, $\forall A \subset B^c, \pi(A) = q(A)$ and $\pi(B) = \min_{m \in \mathcal{P}_0^1} m(B)$. We have:

$$\mathbb{E}_\pi[u(f)] = \mathbb{E}_{\min m \in \mathcal{P}_0^1}[u(f)|\mathcal{F}_1] \tag{37}$$

Let $p^* \in \underset{p \in \mathcal{P}}{\operatorname{argmin}} \int_S u(f) \mathrm{d}p$. Because $f_B \geq f_c$, the pessimistic updating rule gives $p_B^*(.) = \frac{p^*(.)}{\min_{m \in \mathcal{P}_0^1} m(B)}$ and $p_{B^c}^*(.) = \frac{p^*(.)}{\max_{m \in \mathcal{P}_0^1} m(B^c)}$ and, moreover, $\int_S u(f) \mathrm{d}p_B^* \geq \int_S u(f) \mathrm{d}p_{B^c}^*$ It implies:

$$\mathbb{E}_{p^*}[u(f)] = \min_{m \in \mathcal{P}_0^1} m(B) \mathbb{E}_{p_B^*}[u(f)] + \max_{m \in \mathcal{P}_0^1} m(B) \mathbb{E}_{p_{B^c}^*}[u(f)] \tag{38}$$

If $p_B^* \notin \underset{p_B \in \mathcal{R}_B^B}{\operatorname{argmin}} \int_S u(f) \mathrm{d}p_B$ and/or $p_{B^c}^* \notin \underset{p_{B^c} \in \mathcal{Q}_{B^c}^{B^c}}{\operatorname{argmin}} \int_S u(f) \mathrm{d}p_{B^c}$, then $\int_S u(f) \mathrm{d}p_B^* > \int_S u(f) \mathrm{d}r_B$ and/or $\int_S u(f) \mathrm{d}p_{B^c}^* > \int_S u(f) \mathrm{d}q_{B^c}$, hence equations 37 and 38 imply:

$$\mathbb{E}_{p^*}[u(f)] > \mathbb{E}_\pi[u(f)] \tag{39}$$

which is a contradiction. Therefore, $\mathbb{E}_{p^*}[u(f)] = \mathbb{E}_\pi[u(f)]$ hence $\pi \in \underset{p \in \mathcal{P}}{\operatorname{argmin}} \int_S u(f) dp$. It implies:

$$\min_{p \in \mathcal{P}} \int_S u(f) \mathrm{d}p = \min_{m \in \mathcal{P}_0^1} \int_S \left(\min_{p_B \in \mathcal{R}_B^B} \int u(f) \mathrm{d}p_B, \min_{p_{B^c} \in \mathcal{Q}_{B^c}^{B^c}} \int_S u(f) \mathrm{d}p_{B^c} \right) \mathrm{d}m \tag{40}$$

which is equivalent to equation 11 when $f_B \geq f_{B^c}$.

B.2 Proof of proposition 2

Consider acts f and g in $\mathcal{A}$ and an event B in $\mathcal{F}_1$ such that $f_B = g_B$, $f_B \geq f_{B^c}$ and $g_B \leq g_{B^c}$. The conditional MEU of f and g are given, respectively, by $\mathbb{E}_{\min p_B \in \mathcal{R}_B^B}[u(f)]$ and $\mathbb{E}_{\min p_B \in \mathcal{Q}_B^B}[u(g)]$, which differ in general.

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