# SEASONAL ADJUSTMENT AND MULTIPLE TIME SERIES ANALYSIS

Kenneth F. Wallis
University of Warwick

## INTRODUCTION

Despite the expenditure of enormous amounts of energy (and computer time) in the search for a solution, the seasonal adjustment problem flourishes. Shortcomings in existing methods of adjustment are continually perceived, and new methods proposed [2; 3]. The official statistician is concerned to provide a method of adjustment for series taken one at a time, paying little attention to the context in which the data are gathered, and often seems to feel that the more automatic and mechanical the adjustment procedure is, the better. The economist believes that variables are related, whether a formal model to represent the interrelationships is constructed or an informal examination of a number of series taken together is carried out when the current economic situation is under study. Never the twain shall meet! Thus, with the U.K. unemployment series, adjustment methods break down, and new methods are introduced by official statisticians at dates that often coincide with the occurrence of labour market phenomena that are, in turn, a subject of independent study by economists; when institutional arrangements are changed, possible impacts on the seasonal components of relevant variables and, hence, on the adjustment methods employed seem not to be considered. While statisticians seem unwilling to examine related series together, the linear constant-parameter models, constructed by economists, seem unable to adequately capture seasonal effects except by the use of proxy variables—usually seasonal dummy variables, again a somewhat automatic procedure. A different approach is presented by the methods popularly known as Box-Jenkins methods [1], in which a single time series is modeled as an (possibly seasonal) autoregressive-integrated-moving-average (ARIMA) process.

This paper compares and contrasts these various approaches to the analysis of seasonal series, particularly in a multiple time series context. The view taken is that "seasonality in one economic variable is not necessarily an isolated phenomenon but may be related to the seasonality in other economic variables with which that variable interacts" [13, p.19].

The second section is concerned with the consequences of different treatments of seasonality, in turn, considering univariate models, single-equation models, and systems of equations; the third section contains two illustrative examples.

## SEASONAL TIME SERIES: ANALYSIS AND ADJUSTMENT

### Univariate Analysis

We begin by briefly considering alternative approaches to the modeling of a single seasonal time series. It is assumed that the effect of a seasonal adjustment procedure can be studied by considering the effect of a linear filter on the series; thus, the adjusted series, $y^a$, is obtained from the original series $y$ as $y^a_t=\sum a_j y_{t-j}=a(L)y_t$, where $L$ is the lag operator. For the adjustment of current data the filter $a(L)$ is one sided, but a symmetric two-sided filter is usually used for the adjustment of historical series, given sufficient data.

A convenient starting point is the zero-mean stationary process, generated by the linear relation

$$\phi(L)y_t=\theta(L)\epsilon_t \qquad (1)$$

where $\phi(L)$ and $\theta(L)$ are polynomials in $L$ of degree $p$ and $q$, respectively, and $\{\epsilon_t\}$ is a white-noise error—the ARMA $(p, q)$ model. For nonstationary seasonal series, Box and Jenkins [1] propose the class of model

$$\phi(L)\Phi(L^s)(1-L)^d(1-L^s)^D\, y_t=\theta_0+\theta(L)\Theta(L^s)\epsilon_t \qquad (2)$$

of order $(p, d, q) \times (P, D, Q)_s$, where $\Phi(L^s)$ and $\theta(L^s)$ are polynomials of degree $P$ and $Q$, respectively, in $L^s$, $s$ being the number of seasons per year. Such models can certainly capture correlations between observations for the same season in successive years, and it has been assumed, as "would usually be reasonable" [1, p.304], that the polynomials $\Phi$ and $\theta$ are the same for each season. In practice, the choice between a seasonal difference $(1-L^s)$ and a seasonal AR operator $(1-\Phi_1 L^s)$ with coefficient less than 1 for modeling a particular phenomenon is not unambiguous when short series are analysed [9]. However, the implications clearly differ if the model is used for forecasting, because the use of $(1-L^s)$ implies that a seasonal pattern with constant amplitude will be maintained in the forecasts, whereas a seasonal AR operator

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with $0<\Phi_1<1$ implies that the seasonal pattern will gradually decay.

The effect of filtering is to replace the model (2) for the $y$ series by the following model for the $y^a$ series:

$$\phi(L)\Phi(L^s)(1-L)^d(1-L^s)^D y_t^a=a(L)\theta_0+a(L)\theta(L)\Theta(L^s)\epsilon_t$$

Since the effect of a typical seasonal adjustment filter is to reduce autocorrelation at lags $s$, $2s$, $3s$, ..., but to induce small autocorrelation coefficients at nonseasonal lags [13], $a(L)$ might be expected to contain factors that would approximately cancel with the seasonal factors in the model, but such cancellations would not completely account for the action of the filter. Thus, the adjusted series might be modelled in standard ARIMA fashion as

$$\phi(L)(1-L)^d y_t^a=\theta_0'+a'(L)\theta(L)\epsilon_t$$

where the effect of $a'(L)$ is to increase the order of the (nonseasonal) MA operator, although the associated coefficients may be small in absolute value; alternatively, these effects might be more effectively captured on the autoregressive side. Hence, in practice, it might be difficult to say much about the underlying model for $y_t$ on the basis of an analysis of $y_t^a$, although if both series have been analysed, one might expect to find common elements in the nonseasonal AR and MA operators for the two series.

The classical additive components model for an economic time series assumes that the series contains trend cycle, seasonal and irregular components, often taken to be independent of one another, and the seasonal component is variously modeled in a deterministic or stochastic fashion. The simplest deterministic model is that of a constant additive seasonal pattern, familiarly estimated in econometrics by the use of seasonal dummy variables, while a slowly changing seasonal pattern can be represented by dummy variables interactive with trend. The stochastic model of Hannan [6] represents an alternative approach to an evolving seasonal pattern, and a stochastic components model can be obtained by selecting appropriate members of the class (2) to represent each component, as in Grether and Nerlove [5]: The sum of such independent components is again a member of the class (2). However, this may not represent a fruitful approach in practice, for the optimality of Grether and Nerlove's optimal signal extraction approach to seasonal adjustment rests, in part, on the assumption that the ARMA model for each of the separate unobserved components is known.

A regular seasonal component can be removed from a series by use of the operator $(1-L^s)$, but, in practice, this presents further difficulties in estimating the model (2). For example, if the process is generated by $y_t=\eta_t+\mu_t$, where $\eta_t$ has the stochastic representation $\phi(L)(1-L)^d\eta_t=\theta(L)\epsilon_t$, and, in the extreme case, the seasonal component $\mu_t$ is deterministic with $\mu_t=\mu_{t-s}$, then

$$\phi(L)(1-L)^d y_t=\theta(L)\epsilon_t+\phi(L)(1-L)^d \mu_t$$

and use of the seasonal differencing operator to remove the seasonal component results in

$$\phi(L)(1-L)^d(1-L^s)\, y_t=\theta(L)(1-L^s)\,\epsilon_t=\theta'(L)\,\epsilon_t$$

The resulting MA operator has a root on the unit circle, which causes considerable difficulties for the usual statistical estimation and inference procedures. Such problems can be avoided by the use of seasonal variables within the time series model. In the present example, this is accomplished by including $s$ seasonal means in the ARIMA $(p,d,q)$ model, as follows:

$$\phi(L)(1-L)^d\, y_t=\sum_{j=1}^{s}\theta_{0,j}+\theta(L)\,\epsilon_t \qquad (3)$$

Some examples of this approach are given in the subsection on an aggregate demand model.

**Related Series**

The implications of the use of seasonally adjusted data in studying the relationship between a pair of series are considered in [13]. (See also [12].) The argument is presented in terms of linear filters, but a Monte Carlo study, using artificial data adjusted by the Census Bureau X–11 program, indicates that, at least for the cases studied, the linear filter analysis provides a good guide to the results of using the official nonlinear procedures. For the distributed lag model

$$y_t=\sum\alpha_j x_{t-j}+u_t$$

various possibilities arise depending on whether $y$ and/or $x$ exhibit seasonality.

If $y$ is a seasonal series, then the implication of the model is that this is caused either by $x$ or $u$. Whether seasonal filtering is required is determined by the behaviour of the disturbance term, since applying least squares methods to filtered data can be regarded as equivalent to an efficient generalized least squares procedure if the autocorrelation structure of $u_t$ is approximately removed by the filter. Whatever choice is made, the same procedure should be applied to both $y$ and $x$ in order to avoid distorting the distributed lag function. If the $u$ series is nonseasonal, the seasonality in $y$ being entirely caused by $x$, then what seems to be the obvious thing to do, namely to filter or adjust both $y$ and $x$, will, in fact, prove less efficient than using the raw data; the more seasonal is the $x$ series, the less efficient are the estimates based on adjusted data. When the $x$ series is nonseasonal, the same filter should, nevertheless, be used for both $y$ and $x$ to preserve the structure of $\alpha(L)$; in this case, one would expect some filtering to be applied, since the $u$ series will be seasonal, this being the cause of $y$'s seasonality. Note that these arguments differ from that advanced by Box and Jenkins [1] in the context of transfer function identification procedures. They argue that if a prewhitening filter for the $x$ series has been identified and estimated, the same filter should be applied to the $y$ series in order to facilitate identification of the structure of $\alpha(L)$. Our own argument falls more in the econometric tradition of seeking efficient estimators for a given structure; nevertheless, the requirement to use the same filter on both $y$ and $x$ is common to the two approaches.

Of course, if the residual seasonality is deterministic, then its explicit modeling, through dummy variables, is to be preferred. A fixed pattern could be removed by seasonal differencing, but the difficulties described in the previous section could be expected to reappear.

The possibility that different components of the series are differently related can be considered by writing $x_t$ as the sum of nonseasonal and seasonal components

$$x_t = x_t^o + x_t^s$$

and the distributed lag relation as

$$y_t = \alpha_1(L)x_t^o + \alpha_2(L)x_t^s + u_t$$

This discussion is concerned with the special case $\alpha_1(L)=\alpha_2(L)$. In practice, the detection of different relations for the two components seems difficult; it is seldom that data are available in sufficient quantity to provide the required resolution for the relevant frequency-domain techniques. In the time domain, one might hope that the seasonally adjusted series $x^a$ would provide a close approximation to $x^o$, so that $x^a$ and $x$-$x^a$ could be used in place of $x^o$ and $x^s$. Note that even if $x^a$ is very close to $x^o$, an attempt to estimate $\alpha_1(L)$ by relating $y$ to $x^a$ will be subject to omitted variable bias, since, if $x^a$ is obtained by filtering, $x^a$ and $x$-$x^a$ are not orthogonal.

A further special case of interest occurs when $\alpha_2(L)\equiv 0$, so that the seasonal component of $x$ is genuinely noise. If $u$ is nonseasonal, this represents the opposite case to one previously considered, since the observed $x$ series exhibits seasonality, but $y$ does not. This case could surely be detected by examining the separate series. It is of the standard errors-in-variables form and again raises the question of using $x^a$ as an approximation to $x^o$, which may reduce the biases in estimation based on $x$ itself. Evidence of some success with this approach is contained in the simulation results in [13].

If a (possibly seasonal) ARIMA representation is postulated for the $x$ series, the nature of the implied ARIMA representation for $y$ can be readily deduced. Writing the rational distributed lag or transfer function relation as

$$y_t = \frac{\omega(L)}{\delta(L)} x_t + u_t$$

where $\omega(L)$ and $\delta(L)$ are of degree $h$ and $k$, respectively, and the models for $x_t$ and $u_t$ as

$$\phi_x(L)x_t = \theta_x(L)\epsilon_{1t}$$

$$\phi_u(L)u_t = \theta_u(L)\epsilon_{2t}$$

then on substituting and rearranging we obtain

$$\delta(L)\phi_x(L)\phi_u(L)y_t = \omega(L)\theta_x(L)\phi_u(L)\epsilon_{1t} + \delta(L)\phi_x(L)\theta_u(L)\epsilon_{2t}$$

Since $\epsilon_1$ and $\epsilon_2$ are independent, the right-hand side has a simple moving average representation [4], hence we have

$$\phi_y(L)y_t = \theta_y(L)\epsilon_{3t}$$

which is of order $(p_y, q_y)$, where $p_y = k + p_x + p_u$, assuming no cancellation of factors, and $q_y \leq \max(h + q_x + p_u, k + p_x + q_u)$. Note that if either $\phi_x(L)$ or $\phi_u(L)$ (or both) contains the factor $(1-L^s)$, then so does $\phi_y(L)$. Moreover, if $x$ possesses a deterministic component, so that the use of the seasonal differencing operator $(1-L^s)$ produces the difficulties referred to in the previous section when modelling the $x$ series, the same is true of $y$. Thus, if $x_t = \xi_t + \mu_t$, where $\mu_t = \mu_{t-s}$, then the deterministic component in $y_t$ is $\{\omega(L)/\delta(L)\}\mu_t$, and, on removing this by applying the operator $(1-L^s)$, a unit root is induced in the moving average operator. As argued in the previous section, explicit modeling of such components is preferable.

## Multiple Series and Final Equation Considerations

If the distributed lag relation of the previous section is but one of a set of such relations between endogenous and exogenous variables, then the final equation considerations become relevant. (See [9; 14; 15].) The general linear dynamic model, relating a vector of endogenous variables $\underset{\sim}{y}_t$ to a vector of exogenous variables $\underset{\sim}{x}_t$, is written in structural form as

$$\underset{\sim}{B}(L)\underset{\sim}{y}_t + \underset{\sim}{\Gamma}(L)\underset{\sim}{x}_t = \underset{\sim}{u}_t \tag{4}$$

where $\underset{\sim}{B}(L)$ and $\underset{\sim}{\Gamma}(L)$ are matrices of polynomials in the lag operator. The final equations are

$$|\underset{\sim}{B}(L)|\underset{\sim}{y}_t = -\underset{\sim}{b}(L)\underset{\sim}{\Gamma}(L)\underset{\sim}{x}_t + \underset{\sim}{b}(L)\underset{\sim}{u}_t \tag{5}$$

where $\underset{\sim}{b}(L)$ is the adjoint matrix and $|\underset{\sim}{B}(L)|$ the determinant of $\underset{\sim}{B}(L)$. This gives a set of multiple-input transfer function equations, each of which relates a given endogenous variable to its own past values and to the exogenous variables but to no other endogenous variable, current or past. Assuming that there is no cancellation of common factors across particular final equations, these have the special characteristic that the autoregressive operator $|\underset{\sim}{B}(L)|$ is the same for each endogenous variable, equivalently that the denominator polynomials in the rational distributed lag models are all the same, unless the model contains a recursive element.

To illustrate, consider the two-equation model

$$\begin{pmatrix} \beta_{11}(L) & \beta_{12}(L) \\ \beta_{21}(L) & \beta_{22}(L) \end{pmatrix} \begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} + \begin{pmatrix} \gamma_{11}(L) & 0 \\ 0 & \gamma_{22}(L) \end{pmatrix} \begin{pmatrix} x_{1t} \\ x_{2t} \end{pmatrix} = \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}$$

that is just-identified by Hatanaka's conditions [7]. The final equations are

$$\{\beta_{11}(L)\beta_{22}(L)-\beta_{12}(L)\beta_{21}(L)\}\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix}$$
$$=\begin{pmatrix} -\beta_{22}(L)\gamma_{11}(L) & \beta_{12}(L)\gamma_{22}(L) \\ \beta_{21}(L)\gamma_{11}(L) & -\beta_{11}(L)\gamma_{22}(L) \end{pmatrix}\begin{pmatrix} x_{1t} \\ x_{2t} \end{pmatrix}+\begin{pmatrix} w_{1t} \\ w_{2t} \end{pmatrix}$$

with error terms given by

$$w_{1t}=\beta_{22}(L)u_{1t}-\beta_{12}(L)u_{2t}$$
$$w_{2t}=-\beta_{21}(L)u_{1t}+\beta_{11}(L)u_{2t}$$

Seasonality in the endogenous variables may result from seasonality in the exogenous variables (which may include deterministic variables), in the disturbance terms, or from a particular pattern of lag coefficients. Whatever the nature of the generating mechanism and even if such a mechanism is present in only one structural equation, the final equations indicate that the question of whether an endogenous variable is seasonal or not is answered in the same way for each variable: whatever causes of seasonality are present in the model are present in both final equations. The detailed structure of seasonality may differ between $y_1$ and $y_2$, since the right-hand sides of the two final equations are not identical; in broad outline, the patterns can be expected to be similar, and, of course, the autoregressive coefficients are the same for each variable. Departures from this general conclusion can result from "coincidental situations" in Granger and Morris' term [4]; for example, if the only seasonal effects in the model are $u_{1t}=\Phi u_{1,t-s}+\epsilon_{1t}$ and $\beta_{22}(L)=1-\beta' L^s$ with, coincidentally, $\beta'=\Phi$, then the final equation errors differ in their seasonality, $w_{1t}$ having no autocorrelation at seasonal lags.

The model becomes recursive (in the more general dynamic sense) if $\beta_{12}(L)=0$. Then the AR operators differ between the two variables, for $\beta_{22}(L)$ cancels across the first final equation; equivalently, note that the first structural equation is already a final equation (which, of course, still holds if $x_2$ is introduced with coefficient $\gamma_{12}(L)$). The final equations are

$$\beta_{11}(L)y_{1t}=-\gamma_{11}(L)x_{1t}+u_{1t}$$
$$\beta_{11}(L)\beta_{22}(L)y_{2t}=\beta_{21}(L)\gamma_{11}(L)x_{1t}-\beta_{11}(L)\gamma_{22}(L)x_{2t}+w_{2t}$$

Seasonal effects generated in the first structural equation again carry through to both $y_1$ and $y_2$, but effects originating in the second equation do not feed back into $y_1$, so the seasonal implications for $y_1$ and $y_2$ may differ.

The multiple time series generalization of (1), considered by Quenouille [10], gives the vector ARMA model, and, if the exogenous variables of the model (4) possess ARMA representations, the extended model describing the generation of both endogenous and exogenous variables is a special case of the vector ARMA process, as noted by Zellner and Palm [15]:

$$\begin{pmatrix} \underline{F}(L) & \underline{O} \\ \underline{\Gamma}(L) & \underline{B}(L) \end{pmatrix}\begin{pmatrix} \underline{x}_t \\ \underline{y}_t \end{pmatrix}=\begin{pmatrix} \underline{G}(L)\underline{\xi}_t \\ \underline{H}(L)\underline{\epsilon}_t \end{pmatrix} \qquad (6)$$

The block-triangular nature of the autoregressive matrix, together with the requirement that the white-noise processes $\underline{\xi}_t$ and $\underline{\epsilon}_{t-k}$ are uncorrelated for all $t$ and $k$, ensures that the $x$ variables can be treated as exogenous in the structural form (4); for the present purpose, the structural disturbance term is assumed to have the MA representation $\underline{u}_t=\underline{H}(L)\underline{\epsilon}_t$. On diagonalizing (6), a set of final equations is obtained, comprising an ARMA equation for each variable of the model. In effect, this generalizes the analysis of the closing paragraph of the subsection on related series and gives a set of seemingly unrelated ARMA models, again with the AR operator, in general, common to all endogenous variables. This analysis implies that, when constructing models of the pure time-series kind for economic variables, the usual practice of considering variables one at a time, in separate univariate analyses, is inappropriate. If the variables being studied, for such purposes as statistical forecasting or seasonal adjustment, can be regarded as endogenous variables of some underlying structural model, then the ARMA models have cross correlated disturbances and AR coefficients subject to restrictions and to take account of either factor would improve the efficiency of the statistical methods. Procedures for specifying and jointly estimating such models, including a test of the common AR restriction, are described in [14]. In practice, conflicts may arise between the above algebraic derivation and the results of statistical analysis, especially as far as the degree of the common autoregressive polynomial is concerned. Thus, given that the highest order coefficients in $\beta_{ij}(L)$ are individually significant in some appropriate structural estimation procedure, the resulting $|\underline{B}(L)|$ will be of substantially greater degree, and the highest order coefficients may not be significant when these are estimated, free of restrictions, from short economic time series, even in the absence of cancellations or near cancellations.

With respect to seasonal effects, the considerations of the subsection on univariate analysis apply to the vector case, and, if seasonally adjusted data are employed, again the argument of the subsection on univariate analysis can be generalized. As far as deterministic elements are concerned, note the assumption in (6) that the exogenous variables of the econometric model have purely stochastic ARMA representations. Certain exogenous variables, in particular seasonal dummy variables, may not be well approximated in this manner and cannot be solved out in moving from a structural form to a set of seemingly unrelated ARMA models but must be retained explicitly in the solution, yielding models of the basic ARMA form but augmented by such deterministic effects. For example, if all the elements of the vector $\underline{x}_t$ are purely deterministic, then the final equations (5) give an example of the mixed deterministic-stochastic time series model under consideration; if $\underline{x}_t$ comprises the simple seasonal dummy variables, these final equations are a set of seemingly unrelated ARMA models, with varying seasonal means, of the form of (3). Examples of these various approaches are given in the next section.

## EMPIRICAL EXAMPLES

### Manufacturers' Shipments, Inventories, and Orders

The first example uses monthly data on shipments ($S_t$), inventories ($I_t$), new orders ($NO_t$), and unfilled orders ($UO_t$) for the U.S. durable goods manufacturing industry, January 1958—September 1975, in billions of dollars, available in both adjusted and unadjusted forms. Considerable attention has been given to the subject of inventory behaviour in applied econometrics (see, e.g., Rowley and Trivedi [11, chs. 2, 6], but, for our present purposes, we simply use the four series as an example of interrelated seasonal series. In fact, the series are also related by the identity

$$UO_t=UO_{t-1}+NO_t-S_t$$

hence, in joint estimation, one series must be dropped, and our multiple time series results, given in the following, are concerned with the series ($S_t$, $I_t$, $NO_t$). A further identity might also be used to define a production series, as follows:

$$P_t=S_t+I_t-I_{t-1}$$

Such a derived series has not been analysed, but the relation suggests that, in the relevant time series analyses, it might be necessary to difference the $I$-series once more than the other series in order to measure all variables in comparable flow terms; however, a simple acceleration principle view of inventory investment would postulate a relation between $\Delta I$ and $\Delta S$, suggesting that the series should be treated equally.

We first describe the results of univariate analyses of the individual series, broadly based on the Box-Jenkins guidelines, but mainly concentrating on autoregressive representations on grounds of computational convenience. Particular attention is paid to the choice of appropriate degrees of differencing, i.e., to the choice between a difference or quasi-difference operator, in both the seasonal and nonseasonal contexts. Estimation is by exact maximum likelihood methods, which, in particular, avoid the need to change the effective sample size as autoregressive orders are changed.

**Univariate analyses: unadjusted data**—For the shipments series we choose the following $(2, 1, 0) \times (2, 1, 0)_{12}$ model:

$$\underset{(0.07)}{(1-0.06L}+\underset{(0.07)}{0.14L^2})(1+\underset{(0.08)}{0.49L^{12}}+\underset{(0.08)}{0.24L^{24}})\Delta\Delta_{12}S_t=\epsilon_t,$$

$$\hat{\sigma}^2_\epsilon=0.635,\ Q(20)=13.00$$

(standard errors are given in parentheses, and $Q(f)$ denotes the portmanteau test statistic of model adequacy, based on residual autocorrelations and tested as a $\chi^2$ variate with $f$ degrees of freedom). One nonseasonal differencing operator is immediately suggested by an examination of the autocorrelation function, and the choice of a seasonal differencing operator is supported by a comparison with a $(2, 1, 0) \times (3, 0, 0)_{12}$ model: In such a model, $\Phi(L^{-12})$ has a root of 0.975, and a comparison of likelihoods favours the model previously reported. This has a pair of complex roots in both the nonseasonal and the seasonal AR operators, implying damped oscillations with periods of 4.23 (months) and 3.01 (years), respectively.

The inventories series displays a less regular seasonal pattern, and the question of the appropriate degree of differencing solely concerns the choice of the value of $d$. A $(4, 2, 0) \times (2, 0, 0)_{12}$ model is estimated as follows:

$$(1+\underset{(0.07)}{0.38L}+\underset{(0.07)}{0.11L^2}-\underset{(0.07)}{0.18L^3}-\underset{(0.07)}{0.14L^4})(1-\underset{(0.07)}{0.38L^{12}}-\underset{(0.08)}{0.31L^{24}})\Delta^2 I_t$$

$$=\epsilon_t,\ \hat{\sigma}^2_\epsilon=0.082,\ Q(18)=22.68$$

The restriction implicit in the use of two differencing operators can be tested by comparing with a $(5, 1, 0) \times (2, 0, 0)_{12}$ model, and, in fact, a likelihood ratio test rejects the restriction. Nevertheless, for reasons that will be discussed, we retain the specification of $d=2$. On adding a moving average component to this model, no evidence of overdifferencing is found. The nonseasonal AR operator contains a pair of complex roots with a period of 3.28 months, while the seasonal AR operator factors as $(1-0.78L^{12})(1+0.40L^{12})$.

The new orders series is modeled in similar form to the shipments series as the following $(3, 1, 0) \times (2, 1, 0)_{12}$ process:

$$(1+\underset{(0.07)}{0.08L}+\underset{(0.07)}{0.02L^2}-\underset{(0.07)}{0.19L^3})(1+\underset{(0.08)}{0.50L^{12}}+\underset{(0.08)}{0.24L^{24}})\Delta\Delta_{12}NO_t$$

$$=\epsilon_t,\ \hat{\sigma}^2_\epsilon=1.27,\ Q(19)=21.07$$

Both AR operators contain a pair of complex roots, implying damped oscillations with periods of 2.97 months and 2.98 years, respectively, which compare closely with the previous models.[1]

As noted, the unfilled orders series is not included in the joint estimations that will be reported, but, for the

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[1] Moving average representations are suggested as alternatives in some instances, particularly for the seasonal factor in the $S_t$ and $NO_t$ models, where the seasonal AR operator has the appearance of the first few terms of the expansion of $(1-\Theta L^{12})^{-1}$. The corresponding estimates are

$$(1-\underset{(0.07)}{0.07L}+\underset{(0.07)}{0.15L^2})\Delta\Delta_{12}S_t=(1-\underset{(0.07)}{0.45L^{12}})\epsilon_t,\ \hat{\sigma}^2_\epsilon=0.648,\ Q(21)=13.30$$

$$(1+\underset{(0.07)}{0.08L}+\underset{(0.07)}{0.02L^2}-\underset{(0.07)}{0.19L^3})\Delta\Delta_{12}NO_t=(1-\underset{(0.07)}{0.58L^{12}})\epsilon_t,$$

$$\hat{\sigma}^2_\epsilon=1.24,\ Q(20)=19.94$$

While the differences in goodness of fit are marginal, the principle of parsimony [1] would lead to the above models being preferred. However, at the time of writing, we have not developed a program for exact maximum likelihood estimation of vector seasonal mixed autoregressive-moving average models, and the joint estimates which will be reported are confined to vector autoregressive models.

sake of completeness, we describe our experience in univariate analysis. However, models of the form considered above do not pass the diagnostic checks quite so easily. For example, a $(2, 1, 0) \times (2, 0, 0)_{12}$ model is

$$(1-\underset{(0.07)}{0.64}L-\underset{(0.07)}{0.18}L^2)(1-\underset{(0.07)}{0.26}L^{12}-\underset{(0.08)}{0.21}L^{24})\Delta UO_t=\epsilon_t,$$

$$\hat{\sigma}_\epsilon^2=0.612,\ Q(20)=32.82$$

The $Q$-statistic for this model slightly exceeds the 5-percent critical value, but the main contributors are the residual autocorrelations at lags 6, 17 and 22, which have no ready interpretation. The seasonal AR operator factors as $(1-0.61L^{12})(1+0.35L^{12})$, which is of similar form to that of the inventories series. The choice of $d=1$ is supported by the roots of the nonseasonal AR operator, 0.85 and $-0.21$, the former being significantly different from 1.

**Univariate analyses: adjusted data**—The nonseasonal effects observed in the unadjusted shipments series were relatively small, though significant; in the adjusted series, these effects virtually disappear, thus,

$$(1-\underset{(0.07)}{0.08}L)\Delta S_t^a=\underset{(0.04)}{0.13}+\epsilon_t,\ \hat{\sigma}_\epsilon^2=0.39,\ Q(9)=8.01$$

The random-walk-plus-trend interpretation of the series is maintained when seasonal models are entertained, for no evidence of any effect (such as an over adjustment) calling for a seasonal factor is found.

For the adjusted inventories series we select the following (2, 2, 0) model:

$$(1+\underset{(0.07)}{0.45}L+\underset{(0.07)}{0.16}L^2)\ \Delta^2 I_t^a=\epsilon_t,\ \hat{\sigma}_\epsilon^2=0.067,\ Q(8)=11.73$$

In this case, the choice of $d=2$ is supported in a likelihood ratio test against a (3, 1, 0) alternative, the AR operator of the latter having a root of 0.95. The AR operator of the preferred model has a pair of complex roots with period 2.91 months, which compares with a period of 3.28 months in the model for the unadjusted series.

The adjusted new orders series exhibits the features discussed in the subsection on univariate analysis, namely the introduction of small autocorrelations at lags that, on the one hand, are nonseasonal but, on the other hand, are not readily interpretable. Thus, we have the (5, 1, 0) model

$$(1+\underset{(0.07)}{0.04}L-\underset{(0.07)}{0.14}L^2-\underset{(0.07)}{0.24}L^3+\underset{(0.07)}{0.10}L^4+\underset{(0.07)}{0.20}L^5)\Delta NO_t^a\ =\epsilon_t,$$

$$\hat{\sigma}_\epsilon^2=0.834,\ Q(15)=23.76$$

There are two pairs of complex roots in the AR operator: One has a period of 3.05 months, which compares closely to that obtained in modeling the unadjusted new orders series. The second pair of complex roots implies damped oscillations with a period of 12.23 months, which might suggest a residual seasonal effect or possible overadjustment, but, on estimating seasonal models, no significant effects are observed.

The adjusted unfilled orders series, likewise, displays small intermediate autocorrelations apparently introduced in spurious fashion by the adjustment filer. A (6, 1, 0) model is

$$(1-\underset{(0.07)}{0.64}L-\underset{(0.08)}{0.37}L^2-\underset{(0.08)}{0.04}L^3+\underset{(0.08)}{0.26}L^4+\underset{(0.08)}{0.21}L^5-\underset{(0.07)}{0.24}L^6)\Delta UO_t^a$$

$$=\epsilon_t,\ \hat{\sigma}_\epsilon^2=0.462,\ Q(14)=13.93$$

The roots of the AR operator are not very informative. There is a real root of 0.87, which compares closely with that estimated from unadjusted data, and a complex pair with period 11.26 months, suggesting residual seasonality. The remainder (real: $-0.759$; complex: Period 3.03) suggest noise induced by the adjustment when compared with the results based on unadjusted data for this variable, but damped oscillations with a period of approximately 3 months appear relatively frequently in our analyses of the other series.

It is clear that the seasonal adjustment procedure removes more from the unadjusted series than is modeled by the seasonal AR or differencing operators. A prior indication of this can·be obtained by examining the variances or standard deviations of the first differences of the variables in adjusted or unadjusted form (and noting the different behaviour of stock and flow variables). These are presented in table 1, with the additional aim of illuminating the residual variances of the models quoted above—the smaller residual variances of the models for the adjusted $S_t$ and $NO_t$ series clearly do not imply that these models have superior explanatory power, once the prefiltering of the data is borne in mind.

**Joint estimation: unadjusted data**—A modeling procedure for the multiple time series model with diagonal autoregressive matrix, which is interpreted as a multivariate representation of the individual time series models for the endogenous variables of an economic system, is presented in [14]. The procedure starts from joint estimation of models suggested by the univariate analyses and includes a test of the common autoregressive polynomial restriction. The argument of the previous section indicates that this procedure should be applied to series that have been treated equally, rather than to a mixture of adjusted and unadjusted series, and this is now considered. Estimation with the vector $(S_t, I_t, NO_t)$ runs into difficulties, the nature of which suggests that the inventory series should be differenced once more than the shipments and orders series. For the vector $\underset{\sim}{y}_t=(S_t, \Delta I_t, NO_t)'$, the following $(2, 1, 0) \times (4, 1, 0)_{12}$ model is preferred in likelihood ratio tests:

$$\begin{bmatrix} (1+\underset{(0.06)}{0.08}L+\underset{(0.06)}{0.02}L^2)(1+\underset{(0.06)}{0.55}L^{12}+\underset{(0.08)}{0.33}L^{24}+\underset{(0.08)}{0.08}L^{36}+\underset{(0.07)}{0.12}L^{48}) & & \\ & (1+\underset{(0.08)}{0.31}L+\underset{(0.08)}{0.11}L^2)(1+\underset{(0.08)}{0.70}L^{12}+\underset{(0.10)}{0.56}L^{24}+\underset{(0.10)}{0.48}L^{36}+\underset{(0.09)}{0.29}L^{48}) & \\ & & (1+\underset{(0.06)}{0.18}L+\underset{(0.06)}{0.14}L^2)(1+\underset{(0.06)}{0.57}L^{12}+\underset{(0.08)}{0.39}L^{24}+\underset{(0.08)}{0.23}L^{36}+\underset{(0.07)}{0.32}L^{48}) \end{bmatrix} \Delta\Delta_{12}\underset{\sim}{y}_t=\epsilon_t$$

$$\hat{\underset{\sim}{\Sigma}}=\begin{bmatrix} 0.652 & -0.033 & 0.644 \\ & 0.079 & -0.013 \\ & & 1.250 \end{bmatrix}$$

Table 1. **STANDARD DEVIATIONS OF ORIGINAL, DIFFERENCED, AND RESIDUAL SERIES**

| Variable | $S_t$ | $I_t$ | $NO_t$ | $UO_t$ |
|---|---|---|---|---|
| | Unadjusted data | | | |
| $X_t$ | 8.91 | 19.84 | 9.22 | 23.78 |
| $\Delta X_t$ | 1.96 | 0.52 | 1.89 | 1.22 |
| $\Delta_{12} X_t$ | 1.93 | 4.16 | 2.88 | 9.73 |
| $\Delta\Delta_{12} X_t$ | 0.89 | .58 | 1.28 | 1.54 |
| $\Delta^2 X_t$ | - | .38 | - | - |
| Residual of model reported | .80 | .29 | 1.13 | 0.78 |
| | Adjusted data | | | |
| $X_t^a$ | 8.89 | 20.02 | 9.23 | 24.00 |
| $\Delta X_t^a$ | 0.63 | 0.46 | 0.95 | 1.20 |
| $\Delta^2 X_t^a$ | - | .28 | - | - |
| Residual of model reported | .62 | .26 | .91 | 0.68 |

- Entry represents zero.

The null hypothesis of a diagonal $\sum$ matrix, or independent series, is rejected in a likelihood ratio test, the dominant feature being the correlation of 0.71 between the shipments and new orders residuals. On imposing the restriction of a common AR operator, we obtain the estimates

$$\underset{(0.04)}{(1+0.20L}\underset{(0.04)}{+0.08L^2)}\underset{(0.05)}{(1+0.61L^{12}}\underset{(0.06)}{+0.43L^{24}}\underset{(0.06)}{+0.27L^{36}}\underset{(0.05)}{+0.25L^{48})}$$

The likelihood ratio test statistic for the hypothesis of a common autoregression has a value of 25.78, and, on testing this as a $\chi^2$ variate with 12 degrees of freedom, the hypothesis is rejected. Despite the similarities in the implicit dynamics of the separate AR operators, which all have complex roots implying damped oscillations with similar periods (nonseasonal: Respectively, 3.36, 3.05, and 3.46 months; seasonal: Respectively, 2.66, 2.46, and 2.59 years and 5.36, 4.95, and 5.62 years) an inspection of the coefficients suggests the treatment of seasonality in the inventories series as the main source of difficulty. The univariate model for $I_t$ has $D=0$, and the application of $\Delta_{12}$ in joint estimation must then be compensated in the AR coefficients. Starting with the univariate seasonal AR operator, we calculate

$$(1-0.38L^{12}-0.31L^{24})/(1-L^{12})$$
$$=(1-0.38L^{12}-0.31L^{24})(1+L^{12}+L^{24}+L^{36}+\ldots)$$
$$=(1+0.62L^{12}+0.31L^{24}+0.31L^{36}+0.31L^{48}+\ldots)$$

which is reflected in the joint estimation for this series when seasonally differenced. Clearly, it is necessary to consider different treatments of seasonality within the multiple time series model, since the underlying generating mechanism is not transmitted in the same way to all variables of this system.

**Joint estimation: adjusted data**—The tendency of the adjustment procedure to induce autocorrelations at intermediate lags seems to be emphasized in joint estimation, because the following (7, 1, 0) model is preferred in likelihood ratio tests as a representation for the vector $\underset{\sim}{y}_t^a=(S_t^a, \Delta I_t^a, NO_t^a)'$:

$$\begin{bmatrix} \underset{(0.06)}{1+0.08L}\underset{(0.06)}{-0.09L^2}\underset{(0.06)}{-0.07L^3}\underset{(0.06)}{+0.06L^4}\underset{(0.06)}{+0.17L^5}\underset{(0.10)}{+0.00L^6}\underset{(0.06)}{+0.11L^7} \\ \underset{(0.07)}{1+0.38L}\underset{(0.08)}{+0.09L^2}\underset{(0.08)}{-0.08L^3}\underset{(0.08)}{-0.04L^4}\underset{(0.08)}{+0.03L^5}\underset{(0.08)}{-0.06L^6}\underset{(0.07)}{-0.15L^7} \\ \underset{(0.06)}{1+0.18L}\underset{(0.06)}{+0.03L^2}\underset{(0.06)}{-0.12L^3}\underset{(0.06)}{+0.10L^4}\underset{(0.06)}{+0.24L^5}\underset{(0.03)}{+0.07L^6}\underset{(0.06)}{+0.19L^7} \end{bmatrix} \Delta y_t=\epsilon_t$$

$$\hat{\underset{\sim}{\Sigma}}=\begin{bmatrix} 0.404 & -0.021 & 0.386 \\ & 0.066 & 0.010 \\ & & 0.851 \end{bmatrix}$$

Although the separate variances of the shipments and orders residuals are slightly increased from their single-series values, the generalized variance shows an overall reduction, a substantial correlation (0.66) between the residuals of these two series being again observed. On imposing the restriction of a common AR operator, we estimate

$$\underset{(0.04)}{(1+0.19L}\underset{(0.04)}{-0.04L^2}\underset{(0.04)}{-0.11^3}\underset{(0.04)}{+0.06L^4}\underset{(0.04)}{+0.17L^5}\underset{(0.04)}{+0.03L^6}\underset{(0.04)}{+0.07L^7)}$$

The likelihood ratio statistic for testing the hypothesis of a common AR operator has the value of 36.48; as a $\chi^2$ variate with 14 degrees of freedom, this is significant at the 1-percent level, and the final form restriction must be rejected. The use of seasonally adjusted data is clearly not a panacea for the problems described in the previous paragraph.

## An Aggregate Demand Model

Our second example is concerned with the endogenous variables of a dynamic model of aggregate demand in the United Kingdom, constructed by Hendry [8] as a vehicle for an investigation of the performance of alternative econometric estimation and specification methods. The data are quarterly constant-price series, in millions of pounds, for 1957 III–1967 IV (42 observations). The seven variables considered are consumers' expenditure on durable goods ($Cd$) and on all other goods and services ($Cn$), gross domestic fixed capital formation ($I$), inventory investment ($Iv$), imports ($M$), personal disposable income ($Yd$), and gross domestic product ($Y$). Hendry uses seasonally unadjusted data, and these have also been used by Prothero and Wallis [9] to investigate the performance of the Box-Jenkins methodology when applied to short seasonal macroeconomic series. They consider the univariate model selection problem and, after taking final equation considerations into account, choose representations of order $(p, 0, 0) \times (P, 1, 0)_4$ for comparison with Hendry's model—the values of $p$ range from 0 to 2, five of the series having $p=1$, while the values of $P$ range from 1 to 4. When models based on seasonally differenced data containing a seasonal $MA$ element are considered, a moving average coefficient tending to 1 often arises, hence, the alternative of quarterly seasonal means to represent a fixed seasonal pattern within the time series models should also be entertained. Since the data are also available in seasonally adjusted form, a further comparison is possible. The amount of variation in the series removed by the various operations, and that remaining to be explained by a time series model, is indicated in table 2, the first five rows being based on results reported in [9, tables 1-7].

**Unadjusted data, quarterly means**—The sixth row of the table indicates that the use of four quarterly means in place of a seasonal differencing operator achieves greater "explanatory power", moreover for three series the residual variance is already smaller than that of the previous autoregressive representation. On considering various ARMA models, it is immediately clear that the problems of unit roots in the MA operators previously found when working with seasonally differenced data now disappear,

Table 2. STANDARD DEVIATIONS OF VARIABLES AND RESIDUALS

| Variable | Cd | Cn | I | Iv | M | Yd | Y |
|---|---|---|---|---|---|---|---|
| | Unadjusted data | | | | | | |
| $X_t$ | 93 | 374 | 214 | 74 | 179 | 517 | 606 |
| $\Delta X_t$ | 58 | 267 | 56 | 90 | 60 | 178 | 246 |
| $\Delta_4 X_t$ | 53 | 40 | 70 | 79 | 65 | 127 | 118 |
| $\Delta\Delta_4 X_t$ | 45 | 38 | 55 | 80 | 69 | 117 | 101 |
| Residual of $(p,0,0)\times(P,1,0)_4$ | 37 | 28 | 38 | 47 | 40 | 96 | 83 |
| $\Delta X_t$, quarterly means | 41 | 30 | 37 | 59 | 48 | 92 | 79 |
| | Adjusted data | | | | | | |
| $X^a_t$ | 89 | 334 | 212 | 56 | 178 | 505 | 589 |
| $\Delta X^a_t$ | 32 | 24 | 36 | 50 | 43 | 73 | 70 |

in accordance with the discussion in the subsection on univariate analysis. In fact relatively few additional effects are observed, and AR representations are generally preferred, as follows:

$$\underset{(0.15)}{(1-0.40}L^4)\Delta Cd_t=\sum\hat{\theta}_{oj}+\epsilon_t,\ \hat{\sigma}^2_\epsilon=1399,\ Q(15)=6.18$$

$$\underset{(0.15)}{(1+0.30}L)\Delta Cn_t=\sum\hat{\theta}_{oj}+\epsilon_t,\ \hat{\sigma}^2_\epsilon=801,\ Q(15)=7.51$$

$$\underset{(0.15)}{(1+0.25}L^8)\Delta I_t=\sum\hat{\theta}_{oj}+\epsilon_t,\ \hat{\sigma}^2_\epsilon=1263,\ Q(15)=7.94$$

$$\underset{(0.13)}{(1+0.56}L)\Delta Iv_t=\sum\hat{\theta}_{oj}+\epsilon_t,\ \hat{\sigma}^2_\epsilon=2358,\ Q(15)=10.23$$

$$\underset{(0.14)}{(1+0.39}L)\Delta M_t=\sum\hat{\theta}_{oj}+\epsilon_t,\ \hat{\sigma}^2_\epsilon=1930,\ Q(15)=8.36$$

$$(1+\underset{(0.15)}{0.25}L+\underset{(0.14)}{0.31}L^2+\underset{(0.14)}{0.33}L^3)\Delta Yd_t=\sum\hat{\theta}_{oj}+\epsilon_t,\ \hat{\sigma}^2_\epsilon=6886,\ Q(13)=5.10$$

$$\underset{(0.14)}{(1+0.38}L)\Delta Y_t=\sum\hat{\theta}_{oj}+\epsilon_t,\ \hat{\sigma}^2_\epsilon=5344,\ Q(15)=6.50$$

Six of the reported models contain only a single AR coefficient, the exception being $Yd$; the $I$-series has no significant autocorrelation, but the strongest effect, at lag 8, is retained in this model for comparative purposes.

**Adjusted data**—An examination of the correlograms of the adjusted series, and of the last two rows of the above table, immediately points to the need for a differencing operator. Once this has been applied, however, little remains to be modeled, and, for five of the series, no improvement can be achieved by either AR or MA elements.

$$\Delta Cd^a_t=\underset{(4.95)}{8.17}+\epsilon_t,\ \hat{\sigma}^2_\epsilon=1006,\ Q(10)=5.95$$

$$\Delta Cn^a_t=\underset{(3.71)}{26.63}+\epsilon_t,\ \hat{\sigma}^2_\epsilon=566,\ Q(10)=6.99$$

$$\Delta I^a_t=\underset{(5.67)}{15.61}+\epsilon_t,\ \hat{\sigma}^2_\epsilon=1318,\ Q(10)=6.54$$

$$(1+\underset{(.14)}{0.45}L)\Delta Iv^a_t=\epsilon_t,\ \hat{\sigma}^2_\epsilon=2017,\ Q(9)=6.80$$

$$(1+\underset{(.15)}{0.32}L)\Delta M^a_t=\underset{(6.70)}{18.36}+\epsilon_t,\ \hat{\sigma}^2_\epsilon=1645,\ Q(9)=6.46$$

$$\Delta Yd^a_t=\underset{(11.41)}{39.48}+\epsilon_t,\ \hat{\sigma}^2_\epsilon=5335,\ Q(10)=5.23$$

$$\Delta Y^a_t=\underset{(10.87)}{41.61}+\epsilon_t,\ \hat{\sigma}^2_\epsilon=4843,\ Q(10)=10.46$$

In no case is any seasonal AR or MA operator called for, either to supplement the adjustment or to correct for overadjustment.

**Joint estimation**—The final analyses are concerned with joint estimation of models suggested by the univariate analyses. For the unadjusted data, we first consider the joint estimation of the $(p, 0, 0) \times (P, 1, 0)_4$ models selected in [9]. The results are as follows:

$$\begin{bmatrix} (1-\underset{(0.14)}{0.63}L)(1+\underset{(0.22)}{0.45}L^4+\underset{(0.19)}{0.38}L^8) & & & & & & \\ & (1-\underset{(0.13)}{0.42}L)(1+\underset{(0.16)}{0.54}L^4+\underset{(0.16)}{0.17}L^8) & & & & & \\ & & (1-\underset{(0.11)}{0.53}L)(1+\underset{(0.13)}{0.61}L^4+\underset{(0.12)}{0.52}L^8+\underset{(0.13)}{0.18}L^{12}) & & & & \\ & & & (1-\underset{(0.11)}{0.25}L-\underset{(0.10)}{0.15}L^2)(1+\underset{(0.11)}{0.81}L^4+\underset{(0.12)}{0.56}L^8+\underset{(0.11)}{0.44}L^{12}) & & & \\ & & & & (1+\underset{(0.17)}{0.66}L^4+\underset{(0.24)}{0.53}L^8+\underset{(0.23)}{0.29}L^{12}+\underset{(0.17)}{0.18}L^{16}) & & \\ & & & & & (1-\underset{(0.13)}{0.45}L)(1+\underset{(0.18)}{0.34}L^4) & \\ & & & & & & (1-\underset{(0.11)}{0.52}L)(1+\underset{(0.14)}{0.57}L^4+\underset{(0.13)}{0.31}L^8) \end{bmatrix} \Delta_4 \underset{\sim}{y}_t = \underset{\sim}{\hat{\theta}}_0 + \underset{\sim}{\epsilon}_t$$

Compared with the earlier results, the nonseasonal AR coefficients are generally reduced in magnitude, while a number of the higher order coefficients become nonsignificant. The joint model has not been revised to take this into account. The residual standard deviations are, respectively, 37, 30, 40, 52, 51, 99, and 88, which are all greater than the corresponding values in separate estimation. Of course, the generalized variance is decreased—there are substantial residual cross-correlations, and the hypothesis that the residual covariance matrix is diagonal is decisively rejected. The residual correlation matrix is

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 0.274 | 0.074 | −0.134 | 0.061 | 0.396 | 0.084 |
| | 1 | 0.442 | 0.478 | 0.554 | 0.590 | 0.529 |
| | | 1 | 0.308 | 0.448 | 0.344 | 0.608 |
| | | | 1 | 0.684 | 0.313 | 0.697 |
| | | | | 1 | 0.448 | 0.376 |
| | | | | | 1 | 0.439 |
| | | | | | | 1 |

Joint estimation of the models based on first differences and including seasonal means, with an amendment of the specification for $\Delta I_t$, gives the following results:

$$\begin{bmatrix} 1-\underset{(0.19)}{0.20}L^4 & & & & & & \\ & 1+\underset{(0.15)}{0.39}L & & & & & \\ & & 1+\underset{(0.13)}{0.33}L & & & & \\ & & & 1+\underset{(0.10)}{0.49}L & & & \\ & & & & 1+\underset{(0.11)}{0.49}L & & \\ & & & & & (1+\underset{(0.13)}{0.33}L+\underset{(0.12)}{0.28}L^2+\underset{(0.14)}{0.40}L^3) & \\ & & & & & & 1+\underset{(0.12)}{0.50}L \end{bmatrix} \Delta \underset{\sim}{y}_t = \sum \underset{\sim}{\hat{\theta}}_{oj} + \underset{\sim}{\epsilon}_t$$

The coefficient in the $Cd$ equation becomes nonsignificant. As in the previous case, the residual variances (respectively, 1481, 807, 1445, 2383, 1949, 6957, and 5426) show slight increases, but the hypothesis of uncorrelated residuals is rejected in a likelihood ratio test. The residual correlation matrix is

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | −0.006 | 0.189 | −0.245 | 0.150 | 0.258 | 0.098 |
| | 1 | 0.203 | 0.419 | 0.334 | 0.328 | 0.375 |
| | | 1 | 0.228 | 0.339 | 0.451 | 0.632 |
| | | | 1 | 0.569 | 0.135 | 0.515 |
| | | | | 1 | 0.110 | 0.280 |
| | | | | | 1 | 0.426 |
| | | | | | | 1 |

Estimating a model for the adjusted series with a simple first-order specification for all variables gives

$$\begin{bmatrix} 1-\underset{(0.13)}{0.08}L & & & & & & \\ & 1+\underset{(0.11)}{0.27}L & & & & & \\ & & 1+\underset{(0.14)}{0.25}L & & & & \\ & & & 1+\underset{(0.11)}{0.34}L & & & \\ & & & & 1+\underset{(0.10)}{0.46}L & & \\ & & & & & 1+\underset{(0.14)}{0.20}L & \\ & & & & & & 1+\underset{(0.10)}{0.27}L \end{bmatrix} \Delta \underset{\sim}{y}^a_t = \underset{\sim}{\hat{\theta}}_0 + \underset{\sim}{\epsilon}_t$$

In this case, joint estimation produces additional significant coefficients, and, with the exception of the first coefficient that is nonsignificant, these are remarkably similar. The residual variances are, respectively, 1013, 574, 1710, 2051, 1674, 5327, and 4517, which show slight increases except for the last two variables. Again, the generalized variance is significantly reduced, and the residual correlation matrix is

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 0.619 | −0.040 | −0.020 | 0.086 | 0.542 | 0.284 |
| | 1 | 0.282 | 0.201 | 0.387 | 0.588 | 0.554 |
| | | 1 | 0.185 | 0.363 | 0.124 | 0.545 |
| | | | 1 | 0.678 | 0.066 | 0.506 |
| | | | | 1 | 0.264 | 0.255 |
| | | | | | 1 | 0.243 |
| | | | | | | 1 |

Comparing with the first residual correlation matrix, we see that with the exception of five coefficients mainly concerned with the $Cd$ variable the correlations are reduced when adjusted data are employed—not only is the scope for time series model building reduced by the adjustment procedure, but the interrelationships are apparently weakened. Also, in moving from the first to the second case, a reduction in correlation coefficients is, in general, observed.

**Conclusions**

Tentative conclusions based on these examples can be stated briefly:

1. The use of deterministic seasonal variables in place of seasonal differencing operators in time series models can be recommended.

2. However, such variables cast no light on the interrelatedness of seasonal variations in related series, and explicit modeling of seasonal effects to capture this is still required.

3. The use of seasonally adjusted data as a substitute for a solution to this problem has nothing to commend it.

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# COMMENTS ON "SEASONAL ADJUSTMENT AND MULTIPLE TIME SERIES ANALYSIS" BY KENNETH F. WALLIS

Charles R. Nelson
University of Washington

It's a pleasure to discuss Ken Wallis' paper on seasonal adjustment and analysis of multiple time series, since my own thoughts on the subject have, to a great extent, been stimulated by his earlier work [4] on the subject and by many enjoyable discussions with him while I was a visitor at the London School of Economics. This paper successively takes up the implications of seasonal adjustment for univariate analysis, distributed lag relations, and the final equations for the endogenous variables of a linear dynamic simultaneous equation system. Illustrative examples draw on adjusted and unadjusted data for the U.S. durable goods manufacturing industry and U.K. aggregate income and expenditures.

The discussion of univariate analysis emphasizes the necessary cancellation that occurs between the seasonal parts of a multiplicative seasonal ARIMA process and a filter that successfully deseasonalizes the series. This would suggest that the design of seasonal adjustment procedures should focus on effecting this cancellation rather than on discovering one all-purpose filter. The obvious filter, as I have suggested elsewhere [2, pp. 174–175], is the one that just cancels the seasonal factors of the process, namely their inverse. The advantage of these filters is that they do not introduce new autocorrelation structure into the adjusted series, nor do they induce the negative seasonality, so characteristic of series adjusted by methods, such as X–11. The main drawback of my procedure is that the traditional notion of additive seasonal and nonseasonal components no longer applies. This objection is probably more conceptual than practical, since, at least for aggregate economic indicators, the actual numerical magnitudes have real meaning; economy-watchers are interested in the relative magnitude of the nonseasonal change in the series. For example, consider the process $z_t=(1-\theta L)\,(1-\Delta L^S)u_t$. Associated with this process are the nonseasonal process $y_t=(1-\theta L)u_t$ and the seasonal process $s_t=(1-\Delta L^S)u_t$. Note that $z$ is not the sum of $y$ and $s$, the variance of $z$ is larger than the sum of the variances of $y$ and $s$, and $(z-y)$ does not display seasonality. In many situations, it may suffice to report $u_t$, the unanticipated change in the series, since many consumers of economic data are concerned essentially with whether economic conditions have improved or deteriorated. Reports in the *Wall Street Journal* of economic statistics that have no specific meaning to most readers are often accompanied by the helpful comment that the reported number was better or worse than what had been anticipated by informed observers.

I do not mean to imply, by this discussion, that I regard the multiplicative model, posited by Box and Jenkins [1], to be entirely satisfactory as a representation of seasonal time series. Cases in which the multiplicative specification does not seem to be borne out by the data are not uncommon, and, in these cases, an additive decomposition will be more appropriate, e.g., in the case $z_t=(1-\theta L-\Delta L^S)u_t$. Other constraints in the Box-Jenkins model should also come in for further scrutiny. It is certainly plausible that coefficient parameters and disturbance variance are a function of seasonal period, and models that are flexible in these dimensions should be investigated.

The ramifications of seasonal adjustment for distributed lag models are well summarized by Wallis, drawing in part on his earlier work [4] on the problem. Wallis mentions models that assume different responses to seasonal and nonseasonal variation in an independent variable and the difficulty in practice of isolating these different effects. The hypothesis that the effects should be different often stems, I think, from the notion that seasonal variation is predictable and, therefore, will not involve as strong a reaction on the part of economic agents, while nonseasonal variation is largely unanticipated and, thus, will cause plans to be altered. A more meaningful distinction could be drawn between anticipated and unanticipated variation, since there will be unanticipated changes in seasonal pattern, as well as in nonseasonal movements, if seasonality is statistically evolutionary.

Wallis points out that the final equation analysis of linear dynamic systems implies that the autoregressive aspect of seasonality would be common to all endogenous variables and suggests that this may explain the success of X–11 when applied to a wide range of series. I would agree that this is quite suggestive but would only point out that the same restriction does not hold on the moving average side. Seasonality of a moving average nature need not be shared by all the endogenous variables, and, therefore, the same seasonal adjustment procedure may not be appropriate to all the endogenous variables.

The implication of final equation analysis that all endogenous variables will, in general, share the same AR polynomial, both seasonal and nonseasonal, is one that receives considerable attention in Wallis' empirical work. The rejection of the hypothesis that this restriction holds for U.S. durable goods manufacturing shipments, inventories, and new orders adds to the accumulating evidence (see [5; 6]) that the common AR property is difficult to confirm in practice. Some of the reasons for this are illustrated by the present example. One reason is that it is often difficult to distinguish, in practice, between AR and MA structure. The seasonal AR polynomial estimated for the shipments data, for example, would be difficult to distinguish from the inverse of $(1\text{–}0.5L^{12})$ and similarly for the new orders model. The nonseasonal AR polynomial estimated for inventories, both adjusted and unadjusted, is roughly the inverse of $(1\text{–}0.4L)$. It may be too much to expect that identification procedures will separate AR and MA structure sufficiently to permit testing of the AR restriction. A second reason for not confirming the AR restriction would be that it does not, in fact, hold in the underlying system because of lack of simultaneity, a possibility noted by Wallis. In the present context, it would not be implausible that feedbacks from shipments and inventories to new orders would be virtually nil. The AR restriction may also break down if some variables in the system are strongly dependent on expectations. It is not difficult to show that the long-horizon expectation of an ARIMA process will be a random walk [3]. Many economic variables are, of course, strongly dependent on expectations, e.g., inventories and new orders where supplies and purchaser attempt to anticipate future requirements. One would expect that changes in these variables would display little autocorrelation of a nonseasonal nature, and this would indeed seem to be the case for durable goods new orders and inventories (taking into account that the model presented is for the second differences of inventories), adjusted and unadjusted, as well as for the aggregate U.K. inventory series. Regardless of the reasons for the rejection of the AR restriction, it is clear, as Wallis emphasizes, that seasonal adjustment is not the solution to this problem, nor should it be expected to be.

Analysis of the U.K. aggregate data focuses on the choice between the Box-Jenkins multiplicative model and a model with shifting seasonal means as the preferred representation for unadjusted data. Wallis concludes, tentatively, that the latter is to be preferred to the former, presumably on the basis of a better fit. The standard deviations of residuals are substantially lower for two of the seven variables but substantially higher for another when several dummies replace seasonal differencing and seasonal AR parameters. If seasonality is, in fact, evolutionary, as I suspect it is, then the comparison should shift in favor of the stochastic representation if the sample period were lengthened beyond 1957–67. It would also be interesting to see whether stochastic models that are not strictly multiplicative would offer any significant improvement over either the multiplicative or dummy variable models.

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# COMMENTS ON "SEASONAL ADJUSTMENT AND MULTIPLE TIME SERIES ANALYSIS" BY KENNETH F. WALLIS

Thomas Sargent
University of Minnesota

Kenneth Wallis's paper is quite good. Since, in most cases, I see no reason to disagree with the positions that he has taken in the paper on various technical statistical matters, I have decided to try to expand a little on a theme addressed in Wallis's introduction: the relation of the official statistician's views on seasonal adjustment to those of the economist interested in estimating structures. In particular, what rationale can there be for recommending that economists use seasonally adjusted data?[1] Equivalently, what reasons are there for supposing that different time-domain models relate variables at different frequencies? The economist's ancient hunch that the response of $y$ to $x$ is somehow different depending on whether seasonal, cyclical, or trend variations in $x$ are involved is not sufficient reason. For a single time-domain, distributed lag linking $y$ and $x$ is sufficiently flexible to imply very different responses of $y$ (gain and phase) with respect to different frequency components of $x$.

Wallis and Sims have set down a neat statistical model that can be used to rationalize the use of seasonally adjusted data. Their model, essentially an error in variables model, is

$$y_t=b_1(L)x_{1t}+b_2(L)x_{2t}+u_t$$

$$x_t=x_{1t}+x_{2t} \qquad (1)$$

where $Ex_{1t}x_{2s}=Ex_{1t}u_s=Ex_{2t}u_s=0$ for all $t$, $s$; $y_t$, $x_{1t}$, $x_{2t}$, and $u_t$ are jointly stationary, purely linearly indeterministic processes, and $b_1(L)$ and $b_2(L)$ are one-sided polynomials in the lag operator $L$. Here $x_{2t}$ is an indeterministic seasonal process, i.e., a process with a spectrum having most of its power at the seasonals. In this model, if one is really interested in estimating $b_1(L)$, it makes sense to use seasonally adjusted data.

However, we seem to lack a plausible economic model that is capable of generating a statistical model like (1). To indicate the problem, consider the following structure:

$$p_t=aE_tp_{t+1}+bm_t+\epsilon_t+s_t, \quad |a|<1 \qquad (2)$$

where $E_t[x]$ is the linear least squares projection of $x$ on an information set available at time $t$, $\epsilon_t$ is a disturbance process orthogonal to $m$ at all lags, and $s_t$ is a seasonal disturbance process, orthogonal to $m$ and $\epsilon$ at all lags. Equation (2) can be interpreted as a rational expectations version of a portfolio balance schedule like Cagan's, with $p$ being the price level and $m$ being the money supply. I use equation (2), because it is a simple example of a decision rule in which agents' expectations about the future appear, a standard feature that, in a sense, can give rise to different behavior at the seasonal and nonseasonal frequencies. Assume that $m$, $\epsilon$, and $s$ are jointly covariance stationary and purely indeterministic processes (which implies that their spectral densities vanish, at most, at a set of frequencies of Lebesque measure zero, so that $s$ must have some power at nonseasonal frequencies, while $\epsilon$ has some power at seasonal frequencies). The process $m$ can be assumed to possess an indeterministic seasonal in the sense of having spectral peaks at seasonal frequencies.

A solution of the difference equation (2) is

$$p_t=b\sum_{j=0}^{\infty}a^jE_tm_{t+j}+\sum_{j=0}^{\infty}a^j(E_t\epsilon_{t+j}+E_ts_{t+j}) \qquad (3)$$

This equation determines the projection of $p_t$ on current, past, and future $m$'s, as well as the projection of $p_t$ on current and past $m$'s and past $p$'s. Thus, letting agents' information set at $t$ consist (at least) of past $m$'s and past $p$'s, we have, e.g., $E_tm_{t+j}=v_j(L)m_t$, so that

$$p_t=\left[b\sum_{j=0}^{\infty}a^jv_j(L)\right]m_t+\sum_{j=0}^{\infty}a^j(E_t\epsilon_{t+j}+E_ts_{t+j}) \qquad (4)$$

the first term of which is the projection of $p_t$ on current, past, and future $m$'s, with future $m$'s bearing zero coefficients. This is a model in which agents, in a sense, respond differently to seasonal and nonseasonal variation in the money supply. Thus, to the extent that seasonal variations in the money supply are more predictable than nonseasonal variations, these seasonal variations will be

[1] As Wallis and Sims have pointed out, if in the model $y_t=b(L)x_t+u_t$ ($u$ process orthogonal to $x$ process), $u$ has a seasonal, then use of symmetrically seasonally adjusted data can be viewed as a way of approximating generalized least squares, thus, providing some rationale for seasonal adjustment in one case. Of course, by using the Hannan efficient estimator with the unadjusted data, one could always do at least as well asymptotically as with the adjusted data.

more fully discounted in advance according to equation (3). This will be reflected in the distributed lag weights in (4) and in interesting behavior at the seasonal frequencies of the Fourier transform of the lag weights. But, despite the possibility that it is easier for agents to forecast seasonal than nonseasonal movements in money, it still remains true that $p$ and $m$ will be linked by the single one-sided distributed lag (4). Our setup does not lead to a statistical model of the Wallis-Sims form (1). And our setup is quite a standard one that seems to capture the key features of an important class of the decision rules, e.g., investment, consumption, and portfolio balance schedules, that are the subjects of econometric studies: Those decision rules are usually viewed as the solution of dynamic programing problems in which agents face exogenous stochastic processes that they have to forecast. Their optimal forecasting rules are impounded into their decision rule, as illustrated in (4). While there is a sense in which agents respond to different frequency components differently, it is not a sense that can be used to rationalize using seasonally adjusted data.

Furthermore, this setup generates restrictions across the parameters of the stochastic process for $m$ (the $v_j(L)$'s) and the projection of $p_t$ on the $m$ process, the usual kind of cross-equation restrictions characteristic of rational expectations models. If the model is true, these restrictions are predicted to hold for the raw unadjusted data; however, those restrictions are predicted not to hold for seasonally adjusted and even symmetrically adjusted data. Therefore, the model should be tested by using unadjusted data.

One setup that can generate separate time-domain models holding at seasonal and nonseasonal frequencies results if it is assumed that $m_t$ contains a deterministic component. Let

$$m_t=d_t+i_t,\ Ei_td_s=0$$

for all $t$, $s$ where $d_t$ is a deterministic component, i.e., one that can be forecast arbitrarily well far into the future, given only the past values of $m_t$; $i_t$ is a purely indeterministic process. The linear decomposition above is Wold's and characterizes any covariance stationary process. We now have

$$E_tm_{t+j}=d_{t+j}+h_j(L)i_t$$

where

$$E_ti_{t+j}=h_j(L)i_t$$

The expression for the linear least squares forecast of $i_{t+j}$ is unique; but $E_td_{t+j}=d_{t+j}$ has an infinite number of representations in terms of lagged values of $d_t$ or in terms of lagged values of $m_t$. Substituting any of these representations in this equation together with (3) leads to a version of (4) in which separate linear models do connect the deterministic and indeterministic parts of $p$ and $m$. Thus, this signal extraction setup can rationalize the separate treatment of a purely deterministic seasonal. But, we have still not arrived at a rationalization for the Sims-Wallis statistical model that describes purely indeterministic processes.

To indicate the kind of behavior that seems needed to rationalize seasonal adjustment in purely indeterministic models, let us drop the assumption that $m$ is exogenous in (2) and instead assume

$$m_t=w(L)s_{t-1}+\eta_t \tag{5}$$

where $w(L)$ is one sided on the present and past, and where $n_t$ is now a nonseasonal exogenous process. I assume that the authority somehow knows past values of the seasonal noise $s_t$ and that it is able to choose $w(L)$ so as to offset expected movements in $s_t$. In particular, the authority sets $w(L)$ so that

$$bw(L)s_{t-1}=-E_{t-1}s_t \tag{6}$$

Under (5) and (6), equation (2) implies

$$p_t=aE_tp_{t+1}+b\eta_t+\epsilon_t+(s_t-E_{t-1}s_t)$$

A solution of this difference equation is

$$p_t=b\sum_{j=0}^{\infty}a^jE_t\eta_{t+j}+E_t\sum_{j=0}^{\infty}a^j(\epsilon_{t+j}+(s_{t+j}-E_{t+j-1}s_{t+j})) \tag{7}$$

The disturbance process in (7) is orthogonal to $\eta_t$ at all lags, so that if people's information at time $t$ includes (at least) current and past values of $\eta_t$, then (7) determines the projection of $p_t$ on the entire $\eta$ process as a distributed lag on $\eta$ that is one sided on the present and past. Letting $E_t\eta_{t+j}=w_j(L)\eta_t$, we have

$$p_t=b\left[\sum_{j=0}^{\infty}a^jw_j(L)\right]\eta_t+E_t\sum_{j=0}^{\infty}a^j(\epsilon_{t+j}+(s_{t+j}-E_{t+j-1}s_{t+j})) \tag{8}$$

which is a projection equation and is, therefore, consistently estimated by least squares. On the other hand, equation (4), a version of which still holds, is not a projection equation because of the dependence of the $m$ process on the $s$ process via (5) and, thus, cannot, in general, be consistently estimated by least squares.[2] To the extent that seasonally adjusting $m$ permits the re-

[2] In particular, project $m_{t+j}$ against current and past $m$'s results in $\hat{m}_{t+j}=h_j(L)m_t$. Projecting both sides of (3), which still holds, against current and past $m$'s gives

$$E[p_t|m_t,\ m_{t-1},\ \ldots]=b\sum_{j=0}^{\infty}a^jh_j(L)m_t$$

$$+\sum_{j=0}^{\infty}a^jE[s_{t+j}|m_t,\ m_{t-1},\ \ldots]$$

In the current setup in which (5) holds, the second summation on the right does not, in general, vanish, so that the first term does not equal the projection of $p_t$ on the $m$ process (or even the projection on current and past $m$'s. In general, in this setup, future $m$'s will enter the projection of $p$ on the $m$ process.

searcher to recover $\eta$, it would be preferable in this setup to use the seasonally adjusted data and estimate (8).

This example is admittedly a strained one and works by requiring the authority to feedback only on the seasonal part of the disturbance in portfolio balance. The example is set up to lead to a breakdown of the exogeneity of $m$ mainly at the seasonal frequencies, a situation that can approximately be corrected by using seasonally adjusted data. But why would the authority ever choose to try to offset only the seasonal part of disturbances to portfolio balance? According to reasonable objective functions, it would not.

The preceding matters reveal a close connection between attitudes about strict econometric exogeneity and the proper handling of seasonality. If one were not concerned with estimating relationships with strictly exogenous variables on the right side, there would have been no reason for the models (1) to have been proposed. A two-sided projection of $y$ on the entire $x$ process, together with a moving average or autoregressive representation for $x$, could then completely characterize the second-order properties of the $(y, x)$ process. But, economists are appropriately concerned with estimating relations with strictly exogenous right-hand processes; one reason the model (1) seems to have been proposed was to rationalize an apparently greater tendency for seasonally adjusted data than unadjusted data to survive time series tests of the null hypothesis of exogeneity. Such findings would not be predicted by models like our (4) (unless the data were asymmetrically seasonally adjusted, as Sims and Wallis pointed out). If such findings represent a regularity, there should be an economic model that can explain them.

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