Theorem of existence of ruptures for mean values on finite numerical segments. Discrete case

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The proof of the theorem of existence of the ruptures, namely the proof of maximality, is improved. The theorem may be used in economics and explain the well-known problems such as Allais' paradox. Illustrated examples of ruptures are presented.

Contents

Introduction

A man is the key subject of economic theory. Decisions of a man are fundamental operations of it. Utility theory, as a branch of the economic theory, is specially devoted to the research of decisions of a man.

Bernoulli (1738) had given rise to researches of problems of the utility theory. Von Neumann and Morgenstern (1947) had provided promises of feasibility of correct and, naturally, rational fundamentals of the economic theory. But Allais (1953) breached these promises. Other later works of various authors had shown that real man's decisions are undoubtedly inconsistent with rational models. As pointed by Kahneman and Thaler (2006), these inconsistencies are still not overcome by the economic theory.

These problems may be solved by taking into account ruptures.

1. Example of ruptures

1.1. Two points

Suppose a numerical segment $[A; B]$ (see figure 1). Suppose two points are determined on this segment: a left point x_{Left} and a right point x_{Right} : $x_{Left}<x_{Right}$. The coordinates of the middle, mean point may be calculated as $M=(x_{Left}+x_{Right})/2$.

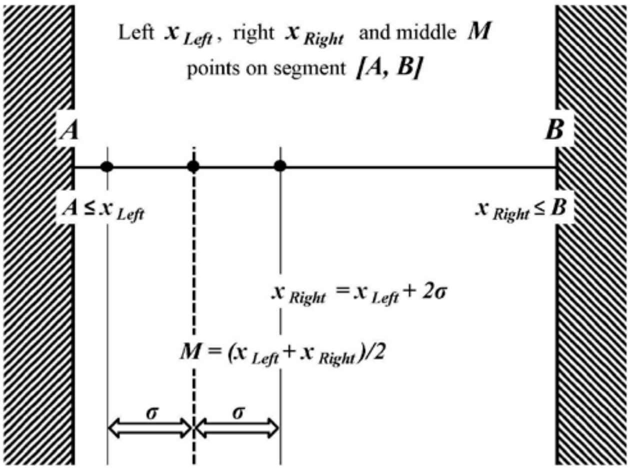

Figure 1. A segment $[A, B]$. Left x_{Left}, right x_{Right} and middle, mean M points on it

Suppose the points can not escape outside the boundaries of this segment. This means $A \leq x_{Left}$ and $x_{Right} \leq B$.

Suppose the points can not approach each other closer than a non-zero distance of two sigma $2\sigma>0$. This means $x_{Right} \geq x_{Left}+2\sigma$ or $x_{Left} \leq x_{Right}-2\sigma$. At that, $M-x_{Left}=x_{Right}-M \geq \sigma>0$.

For the sake of simplisity and obviousness, figures 1-3 represent a case: $x_{Right}=x_{Left}+2\sigma$ and $x_{Left}=x_{Right}-2\sigma$ and $M-x_{Left}=x_{Right}-M=\sigma$.

One can easily see two types of zones can exist on the segment:

The mean point M can be located only in the zone which may be named "allowed" (see figure 2).

The mean point M can not be located in the zones which may be named "forbidden" (see figure 3).

1.2. Allowed zone

Due to the conditions of the example, the left point x_{Left} may not be located more left than the left boundary of the segment $A \le x_{Left}$ and the right point x_{Right} may not be located more right than the right boundary of the segment $x_{Right} \le B$. For M we have $M - x_{Left} = x_{Right} - M = \sigma$.

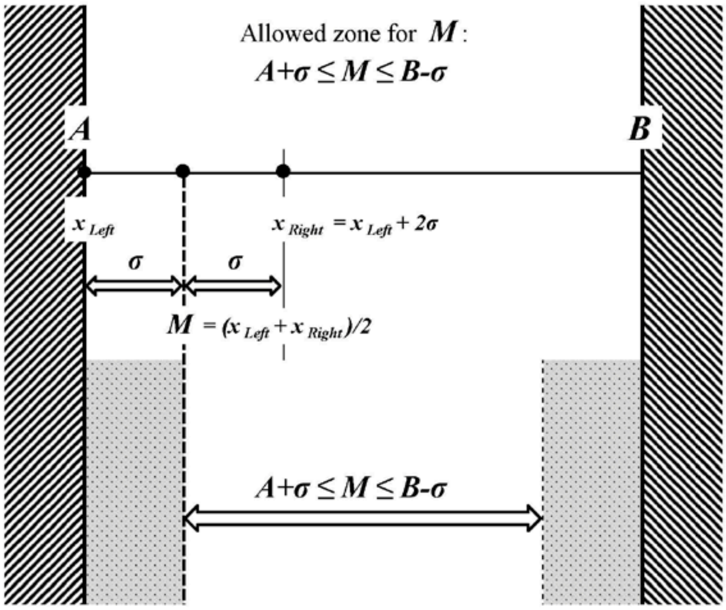

Figure 2. Allowed zone for M

The allowed zone for M is equal to $(B-A)-2\sigma$. It is less than the segment on 2σ. If the distance 2σ between the left x_{Left} and right x_{Right} points is non-zero then the difference between the allowed zone and the segment is non-zero also.

So, the mean point M can not be located in any position of the segment.

1.3. Forbidden zones, ruptures

If $A \le x_{Left}$, $x_{Right} \le B$ and x_{Right}-$x_{Left} \ge 2\sigma$, then there are the ruptures of one sigma between the mean point and the boundaries of the segment. So, the mean point M can not be located in two zones located near the boundaries of the segment. These zones may be named forbidden zones or ruptures.

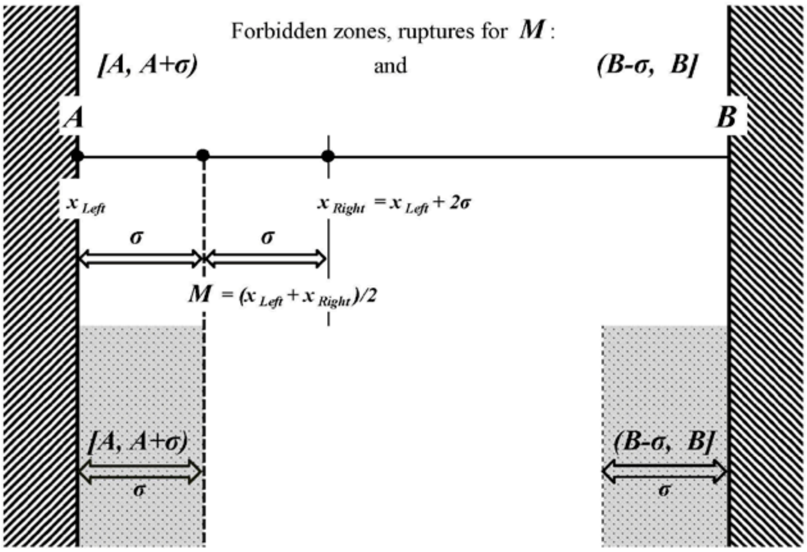

Figure 3. Forbidden zones, ruptures for M

As we can easily see, the non-zero ruptures exist between the allowed zone of the mean M of the quantity and the boundaries of the segment. The width of every rupture is equal to σ. If the distance 2σ between the left x_{Left} and right x_{Right} points is non-zero then the forbidden zones, ruptures for M are non-zero also.

1.4. From possibility to inevitability of existence of ruptures

It is enough to adduce only one example of existence of a phenomenon to prove the statement this phenomenon can exist. In other words, it is enough to adduce only one counterexample of existence of a phenomenon to disprove a statement this phenomenon can not exist.

So, by means of the above examples we have proved a statement: At the condition of a non-zero scattering of data of a quantity, the non-zero ruptures can exist for mean value of the quantity near the boundaries of the numerical segment.

Following theorem should specify conditions, namely the conditions of the scattering of data, at which the rupture must exist.

2. General theorem of existence of ruptures

2.1. Preliminary notes

2.1.1. General conditions, assumptions and notations

Suppose a numerical segment $X=[A, B] : 0<(B-A)<\infty$, and a quantity $f_K(x_k) : k=1, 2, \dots K : 2\le K\le\infty$; for $x_k<A$ and $x_k>B$, the statement $f_K(x_k)\equiv 0$ is true; for $A\le x_k\le B$ the statement $f_K(x_k)\ge 0$ is true, and

$$\sum_{k=1}^{K} f_K(x_k) = W_K ,$$

where W_K (the total weight of $f_K(x_k)$) is such that

$$0 < W_K < \infty .$$

Keeping generality, $f_K(x_k)$ may be normalized so that $W_K=1$.

Let us denote a moment of n-th order $E(X-X_0)^n$ of the quantity $f_K(x_k)$ as

$$E(X - X_0)^n = \frac{1}{W_K}\sum_{k=1}^{K}(x_k - x_0)^n f_K(x_k) .$$

Suppose the mean value of the quantity $f_K(x_k)$ exists

$$EX = \frac{1}{W_K}\sum_{k=1}^{K} x_k f_K(x_k) \equiv M .$$

Suppose at least one central moment of the quantity $f_K(x_k)$ exists such as $1<n<\infty$

$$E(X - M)^n = \frac{1}{W_K}\sum_{k=1}^{K}(x_k - M)^n f_K(x_k) ,$$

2.1.2. Maximal possible value of central moment for finite segment

The maximal possible absolute value of a central moment of the quantity $f_K(x_k)$ may be estimated from its definition

$$| E(X - M)^n | \equiv | \frac{1}{W_K}\sum_{k=1}^{K}(x_k - M)^n f_K(x_k) | \le \frac{1}{W_K}\sum_{k=1}^{K} | (x_k - M)^n f_K(x_k) | \le$$
$$\le \frac{1}{W_K}(B - A)^n \sum_{k=1}^{K} f_K(x_k) = (B - A)^n .$$

More precise estimation (see Appendix A1) of this value is provided by the sum of modules of the central moments of the functions that are concentrated at the borders of the segment $(A-M)^n(B-M)/(B-A)$ and $(B-M)^n(M-A)/(B-A)$

$$Max(E(X - M)^n) \le | (A - M)^n \frac{B - M}{B - A} | + | (B - M)^n \frac{M - A}{B - A} | .$$

It leads to the well-known maximum for $n=2$ and $M_{max}=(B-A)/2$

$$Max(E(X - M)^2) = (\frac{B - A}{2})^2 ,$$

and, for $n=2k>>1$, - to maximums at $M_{max}\approx A+(B-A)/2n$ and $M_{max}\approx B-(B-A)/2n$

$$Max(E(X - M)^n) \approx \frac{1}{\sqrt{e}} \frac{(B - A)^n}{2n} .$$

2.2. General lemma about tendency to zero for central moments

If, for the quantity $f_K(x_k)$, defined in the section 2.1.1, $M\equiv E(X)$ tends to A or to B, then, for $1<n<\infty$, $E(X-M)^n$ tends to zero.

The proof (in detail see, e.g., Harin 2010-2): For $M \to A$,

$$|E(X-M)^n| \le |(A-M)^n \frac{B-M}{B-A}| + |(B-M)^n \frac{M-A}{B-A}| \le$$
$$\le [(B-A)^{n-1} + (B-A)^{n-1}] \frac{(M-A)(B-M)}{B-A} \le .$$
$$\le 2(B-A)^{n-1}(M-A) \xrightarrow[M\to A]{} 0$$

So, if $(B\text{-}A)$ and n are finite and $M \to A$ (that is $(M\text{-}A)\to 0$), then $E(X\text{-}M)^n \to 0$. For $M \to B$, the proof is similar.

The lemma has been proved.

Note. More precise (see, e.g., Harin 2010-2) estimation may be obtained for central moments' tendency to zero, e.g. for $M \to A$

$$|E(X-M)^n| \le (B-A)^{n-1}(M-A) \xrightarrow[M\to A]{} 0$$

2.3. General theorem of existence of ruptures for mean values

If there are: the quantity $f_K(x_k)$ defined in the section 2.1.1, $n : 1<n<\infty$, and $r_{dispers}$: $|E(X\text{-}M)^n| \ge r_{dispers} > 0$, then $r_{mean}>0$ exists such as the statement $A<(A+r_{mean})\le E(X)\le(B\text{-}r_{mean})<B$ is true.

The proof (in detail see, e.g., Harin 2010-2): From the lemma, for $M \to A$,

$$0 < r_{dispers} \le |E(X-M)^n| \le 2(B-A)^{n-1}(M-A)$$

$$0 < \frac{r_{dispers}}{2(B-A)^{n-1}} \le (M-A)$$

$$r_{mean} \equiv \frac{r_{dispers}}{2(B-A)^{n-1}}$$

For $M \to B$, the proof is similar.

As long as $(B\text{-}A)$, n and $r_{dispers}$ are finite and $r_{dispers}>0$, then r_{mean} is finite, $r_{mean}>0$, both $(M\text{-}A)\ge r_{mean}>0$ and $(B\text{-}M)\ge r_{mean}>0$.

The theorem has been proved.

So, if a finite ($n<\infty$) central moment of a quantity, which is defined for a finite segment, cannot approach 0 closer, than by a non-zero value $r_{dispers}>0$, then the mean value of the quantity also cannot approach a border of this segment closer, than by the non-zero value $r_{mean}>0$.

More general: If a quantity is defined for a finite segment and a non-zero rupture $r_{dispers}>0$ exists between zero and the zone of possible values of a finite ($n<\infty$) central moment of the quantity, then the non-zero ruptures $r_{mean}>0$ also exist between a border of the segment and the zone of possible values of the mean value of this quantity.

3. Applications of theorem in economics

The theorem of existence of the ruptures may be used in economics and explain the well-known problems and paradoxes.

In the presence of a data scattering, the forbidden zones, the ruptures can exist for the mean values near the boundaries of finite numerical segments and of the probability scale. These ruptures can bias the results of experiments. This bias can explain (at least partially) the well-known problems and paradoxes of utility theory and economics.

The simple example of the problem:

Suppose Mr. Somebody offers you a choice of two prizes

A) a guaranteed gain, prize of *$99* (with the probability *100%*), or

B) a probable gain of *$100* with the probability *99%*.

(For the experiment accuracy, both *$99* and *$100* should be in *$1* banknotes. So *99* and *100* banknotes of *$1*)

The mean values to get the probable gain and to get the guaranteed gain are precisely equal to each other, but the well-determined experimental fact is: in similar experiments, the overwhelming majority of people choose the guaranteed gain instead of the probable one (see, e.g., Tversky and Wakker 1995).

In the ideal standard explanations we have the mean values for the guaranteed and probable cases

$$\$99 \times 100\% = \$99 \,,$$
$$\$100 \times 99\% = \$99 \,,$$
$$\$99 = \$99 \,.$$

So, in the ideal standard explanations the mean of the guaranteed gain and the mean of the probable gain are both equal to *$99* and are precisely equal to each other.

If scattering of real data leads to the rupture (near *100%*) which is more than *1%* and is equal to, say, *3%*, then the probability of the probable gain cannot be equal to *99%* and is not more than *97%*

$$\$99 \times 100\% = \$99 \,,$$
$$\$100 \times 97\% = \$97 \,,$$
$$\$99 > \$97 \,.$$

So, the mean value of the probable gain is less than the mean value of the guaranteed gain. The paradox can be explained by taking into account the rupture and the bias of mean result.

Conclusions

In the paper, the general possibility of existence of ruptures in the scale of possible values of mean values of quantities, which are defined for the finite segments, has been proved. The possibility of existence of the ruptures in the probability scale both for the probability estimation and for the probability has been proved in, e.g., (Harin 2010-1).

The theorem allows to obtain and to support solutions of a number of the well known paradoxes of the utility theory (see, e.g., Harin 2007 and 2009).

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Appendix A1. Detailed estimation of the maximal possible value of central moment for finite segment

A1.1. The proof of maximality

A1.1.1. Couple of elements

Let us find a function that reaches maximal possible value of central moments on a segment *[A, B]*

Consider a couple of elements $f_K(x_A)$ and $f_K(x_B)$

$$A \le x_A < M < x_B \le B .$$

such as they are tied together by the conditions of constant total weight f and constant mean point M

$$f_K(x_A) + f_K(x_B) \equiv f = Const .$$

$$(M - x_A) f_K(x_A) = (x_B - M) f_K(x_B) .$$

Consider the central moment of this couple of elements

$$E(X - M)^n \equiv (x_A - M)^n f_K(x_A) + (x_B - M)^n f_K(x_B) .$$

Its absolute value do not exceed the sum of absolute values of its parts

$$\begin{aligned} &| E(X - M)^n | \le | (x_A - M)^n | f_K(x_A) + | (x_B - M)^n | f_K(x_B) = \\ &= (M - x_A)^n f_K(x_A) + (x_B - M)^n f_K(x_B) \end{aligned} .$$

A1.1.2. Modification of the basic expression

By substituting $f_K(x_B)$ to f and $f_K(x_A)$

$$f_K(x_B) = f_K(x_A) \frac{M - x_A}{x_B - M} = f - f_K(x_A) .$$

and replacing $f_K(x_A)$ and $f_K(x_B)$ by functions of x_A and x_B

$$f_K(x_A) + f_K(x_B) = f_K(x_A) \frac{x_B - M + M - x_A}{x_B - M} = f_K(x_A) \frac{x_B - x_A}{x_B - M} = f ,$$

$$f_K(x_A) = f \frac{x_B - M}{x_B - x_A} ,$$

$$f_K(x_B) = f \frac{M - x_A}{x_B - x_A} ,$$

this expression may be reorganized to the expression which depends only on x_A and x_B

$$| E(X - M)^n | \le (M - x_A)^n \frac{x_B - M}{x_B - x_A} f + (x_B - M)^n \frac{M - x_A}{x_B - x_A} f .$$

A1.1.3. Determination of maximum by analysis of derivatives

Let us find a maximum of the absolute value of central moments of this pair of elements by the analysis of derivatives.

Let us differentiate the expression for the absolute value of a central moment by x_A

$$[(M-x_A)^n \frac{x_B-M}{x_B-x_A} f + (x_B-M)^n \frac{M-x_A}{x_B-x_A} f]'_{x_A} =$$

$$= (x_B-M)\frac{-n(M-x_A)^{n-1}(x_B-x_A)+(M-x_A)^n}{(x_B-x_A)^2} f +$$

$$+ (x_B-M)^n \frac{-(x_B-x_A)+(M-x_A)}{(x_B-x_A)^2}$$

and, continuing,

$$[(M-x_A)^n \frac{x_B-M}{x_B-x_A} f + (x_B-M)^n \frac{M-x_A}{x_B-x_A} f]'_{x_A} =$$

$$= (x_B-M)\times$$

$$\times[(M-x_A)^{n-1}\frac{-n(x_B-x_A)+(M-x_A)}{(x_B-x_A)^2} f + (x_B-M)^{n-1}\frac{-x_B+M}{(x_B-x_A)^2} f] = .$$

$$= -\frac{(x_B-M)}{(x_B-x_A)^2} f \times$$

$$\times\{(M-x_A)^{n-1}[(n-1)(x_B-x_A)+(x_B-M)]+(x_B-M)(x_B-M)^{n-1}\} < 0$$

The first derivative by x_A is strictly less than zero for any $A \le x_A < M$ and its sign is independent of x_B for any $M < x_B \le B$. Hence, for any $x_B : M < x_B \le B$, the maximum of the absolute value of a central moment $E(X\text{-}M)^n$ is attained at the minimal x_A, that is at $x_A = A$.

Let us differentiate the expression for the absolute value of a central moment by x_B

$$[(M-x_A)^n \frac{x_B-M}{x_B-x_A} f + (x_B-M)^n \frac{M-x_A}{x_B-x_A} f]'_{x_B} =$$
$$= (M-x_A)^n \frac{(x_B-x_A)-(x_B-M)}{(x_B-x_A)^2} f +$$
$$+ (M-x_A)\frac{(x_B-x_A)n(x_B-M)^{n-1}-(x_B-M)^n}{(x_B-x_A)^2} f$$
.

and, continuing,

$$[(M-x_A)^n \frac{x_B-M}{x_B-x_A} f + (x_B-M)^n \frac{M-x_A}{x_B-x_A} f]'_{x_B} =$$
$$= (M-x_A)^n \frac{M-x_A}{(x_B-x_A)^2} f + (M-x_A)(x_B-M)^{n-1}\frac{n(x_B-x_A)-(x_B-M)}{(x_B-x_A)^2} f =$$
$$= (M-x_A)^n \frac{M-x_A}{(x_B-x_A)^2} f + (M-x_A)(x_B-M)^{n-1}\frac{M+x_B(n-1)-nx_A}{(x_B-x_A)^2} f =$$
$$= \frac{(M-x_A)}{(x_B-x_A)^2} f\{(M-x_A)^n + (x_B-M)^{n-1}[(M-x_A)+(x_B-x_A)(n-1)]\} > 0$$
.

The first derivative by x_B is strictly more than zero for any $M<x_B\leq B$ and its sign is independent of x_A for any $A\leq x_A<M$. Hence, for any $x_A : A\leq x_A<M$, the maximum of the absolute value of the central moment $E(X\text{-}M)^n$ is attained at the maximal x_B, that is at $x_B=B$.

So, the maximum of the absolute value of the central moment $E(X\text{-}M)^n$ is attained at $x_A=A$ and $x_B=B$. That is, it is attained for the functions that are concentrated at the boundaries of the finite numerical segment $[A, B]$

$$Max(E(X-M)^n) \leq |(A-M)^n \frac{B-M}{B-A}| + |(B-M)^n \frac{M-A}{B-A}|.$$