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Abstract

We show in this paper that *instability* is an intrinsic cause of production variability in a dynamic inventory system. We first show that a unique stationary optimal policy exists for both full-backlog and lost-sales case and under the policy a firm replenishes its inventory to a constant target level. We then express the constant inventory target as the unique steady state of the Euler's equation governing the dynamics of target inventories. We finally show that the Euler's equation is locally instable at the steady state but a sufficiently large refund to unsold inventory in lost-sales case can stabilize the inventory system.

Keywords: stability, production variability, dynamic inventory system, full-backlog, lost-sales

JEL Codes: C61, C62, D92

1. Introduction

This paper sets out to demonstrate, using a simple inventory model, that instability can be a cause of production variability. The conclusion contributes to the literature that explain one of the most surprising empirical findings on macro business fluctuation summarized in Blinder and Maccini (1991) – production is more volatile than sales in most industries.

Inventory is seen as the buffer to smooth production in traditional microeconomic inventory theories such that production cannot be more variable than sales. Researchers have modified the traditional microeconomic inventory models in different ways in order to explain the larger production variability as compared to sales variability. We classify these studies into the following groups.

Demand factors to explain production variability. Kahn [10] shows that production can be more volatile than sales when demand shocks are serially correlated, and the resulted amplification in production variability is larger when the replenishment lead time[1] is longer. A conceptually different but clearly related case is the demand backlog; if unmet demand in a period can be backlogged into future periods, the backlogs impart some serial correlation to the total demand (backlogs plus new demand). Kahn [11] shows that production cannot be more volatile than sales when backlog is not allowed (the case of lost-sales) and demand shocks are *i.i.d.*. Another closely related case is the price speculation discussed by Hall and Rust [8]; serially correlated shocks on product price encourages the firm's speculative behavior to produce more if it expects a positive price change, and to produce less if it expects a negative price change.

Cost factors to explain production variability. Random shocks on production cost increases production variability as shown by Lee *et.al* [13]. On a different front, Ramsey [16] shows that firms tend to batch their productions when there are economies of scale in production and batching amplifies production variability. Batching also happens when there is a fixed component in production cost; (*s*, *S*)

[1] The time from production to storage.

type of inventory control[2] is optimal leading to larger production variability as shown by Blinder [4], Calpin [6], and Mosser [15].

Market structure to explain production variability. Lee *et. al* [13] show that strategic behaviors of firms in an oligopoly market can amplify production variability.

Behavioral factors to explain production variability. Sterman [18] demonstrates using lab experiments and field observations that when there is a lead time from production to storage, decision makers tend to give insufficient weight to the unfilled orders. Such bounded rationality causes production to be more volatile than sales.

We show in this paper that instability is an intrinsic cause of production variability in a dynamic inventory system. We obtain this result based on a simple inventory model which is consistent with the classical one in Arrow *et. al* [2] and excludes all aforementioned factors to explain production variability. A unique stationary optimal policy of the inventory model exists and is in the produce-up-to type[3] under which a firm replenishes its inventory to a constant target level at each period. Although production cannot be more volatile than sales under the stationary optimal policy, the constant inventory target, which is the steady state of the dynamics of optimal target inventory, is fragile to small changes in environment; a perturbation to the system leads to the divergent oscillation of the target inventory. The oscillation amplifies production variability such that production can still be more volatile than sales even after excluding aforementioned factors. When unmet demand in a period cannot be backlogged, the inventory system can be stabilized by a sufficiently large refund for unsold inventory.

2. The Inventory Model

[2] Arrow, Harris, and Marschak [1]. Under (*s*, *S*) policy, a firm orders a commodity up to *S* whenever the inventory position of the commodity drops below *s*.

[3] It is a special case of the (*s*, *S*) in which the two thresholds *s* and *S* are the same.

Consider a firm which produces and sells a product in a market and the market price of the product is p. Demand for the product at each period $\left(d_t \in [\underline{d}, \bar{d}]\right)$ is *i.i.d.* distributed. The firm employs inventory technology and the lead time of replenishing inventory is zero. Inventory position x_t follows a dynamic transition equation:

$$x_{t+1} = x_t + q_t - s_t \tag{1}$$

where q_t is the quantity produced; s_t is the quantity sold, i.e.,

$$s_t = \begin{cases} \min\{x_t + q_t, d_t\}, & \text{for lost-sales} \\ d_t, & \text{for full backlog} \end{cases} \tag{2}$$

The production cost is linear and has no the fixed component; the unit production cost is denoted by c. The costs of holding inventory and stock-out are also linear. We use h and γ to denote unit inventory and stock-out cost[4] such that the inventory and stock-out cost functions are $h \cdot [x_t + q_t - d_t]^+$ and $\gamma \cdot [d_t - x_t + q_t]^+$ respectively, where $[x]^+ \equiv \max\{0, x\}$. The one-period profit function under lost-sales is

$$\pi(x_t, d_t, q_t) = p \cdot s_t - h \cdot [x_t + q_t - d_t]^+ - \gamma \cdot [d_t - x_t - q_t]^+ - cq_t \tag{3}$$

For the case of full backlog, the profit of the backlogged sales, $[d_t - x_t - q_t]^+$, is assumed to be realized in the next period. With a discount rate of β, the profit on backlogged sales can be then written as $\beta(p-c) \cdot [d_t - x_t - q_t]^+$. Thus, the one-period profit function under backlogging has the exact form as equation (3), except for γ is replaced as $\tilde{\gamma} \equiv \gamma - \beta(p-c)$, denoting the smaller stock-out

[4] The unit stock-out penalty γ, captures both quantifiable shortage costs (e.g., additional customer service and management cost) and subjective components in terms of customers' brand loyalty and preference.

cost in full-backlog case. Taking expectation with respect to the random demand for equation (3), we get

$$E\pi(x_t, q_t) \equiv E_d\{\pi(x_t, d_t, q_t)\} = ES(x_t + q_t) - EG(x_t + q_t) - cq_t \tag{4}$$

where $ES(x_t + q_t)$ denotes the expected sale revenue; $EG(x_t + q_t)$ denotes the sum of expected stock-out and holding costs. Specifically, these terms are

$$ES(x_t + q_t) = p \cdot \left[\int_{\underline{d}}^{x_t + q_t} z_t dF_d(z_t) + (1 - F_d(x_t + q_t)) \cdot (x_t + q_t) \right] \tag{5}$$

$$EG(x_t + q_t) = h \int_{\underline{d}}^{x_t + q_t} (x_t + q_t - z_t) dF_d(z_t) + \gamma \int_{x_t + q_t}^{\bar{d}} (z_t - x_t - q_t) dF_d(z_t) \tag{6}$$

where F_d is the probability distribution function of the demand d_t. In equation (6), γ is replaced as $\tilde{\gamma}$ for full-backlog case.

The firm's objective is to maximize the present value of future profits discounted by the discount rate $\beta \in (0,1)$. Let $\bar{x}$ denote the upper bound of inventory holding and $\Omega(x_t) \equiv [0, \bar{x} - x_t]$ denote the constraint correspondence called admissible policy space, the firm's profit-maximization problem can be stylized in the following dynamic programming formulation over infinite planning horizon.

$$V(x_t) = \sup_{q_t \in \Omega(x_t)} [E\pi(x_t, q_t) + \beta EV(x_{t+1})] \tag{7}$$

s.t. $\quad x_{t+1} = x_t + q_t - s_t$

Given $x_t \in X$

where for lost-sales $X = [0, \bar{x}]$ and

$$EV(x_{t+1}) = E_d\{\mathbf{1}(x_t + q_t \le d_t)V(0) + \mathbf{1}(x_t + q_t > d_t)V(x_t + q_t - d_t)\}$$

$$= \left(1 - F_d\left(x_t + q_t\right)\right) \cdot V(0) + \int_{\underline{d}}^{x_t + q} V\left(x_t + q_t - z\right) dF_d(z) \tag{8}$$

with $\mathbf{1}(\cdot)$ denoting the indicator function. For full-backlog $X = \left[-\bar{d}, \bar{x}\right]$ and

$$EV\left(x_{t+1}\right) = EV(x_t + q_t - d_t) = \int_{\underline{d}}^{\bar{d}} V(x_t + q_t - z) dF_d(z) \tag{9}$$

3. Produce-up-to Policy, Instability, and Production Variability

Since the planning horizon is infinite and the demand distribution is i.i.d., the optimal production policy is stationary[5] and is expressed as the solution of

$$q(x) = \underset{q \in \Omega(x)}{\arg\max} \left\{ E\pi(x,q) + \beta EV(x,q) \right\} \tag{10}$$

According to Iglehart [9], the optimal stationary policy for the full-backlog case exists uniquely and has the form of a produce-up-to type, which is expressed as

$$q(x) = \left(K^* - x\right)^+ = \begin{cases} K^* - x, & \text{for } x < K^* \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

where K^* is a constant produce-up-to or target inventory level. We confirm also the optimality of the produce-up-to policy for the lost-sales case, and the results are summarized in proposition 1.

Proposition 1: *There exists uniquely optimal stationary policy for the considered inventory problem in (7), including both lost-sales and full-backlog case. The optimal stationary policy is in the produce-up-to type defined in (11).*

Proof: See Appendix 1.

Under the stationary produce-up-to policy, the production variability, which is usually defined as the variance of the time series $\{q_t\}$, can be expressed as

[5] It is only a function of state (inventory position), not an explicit function of time.

$$Var(q_t) = Var\left(K^* - x_t\right) = Var(x_t) \tag{12}$$

where $x_t = K^* - d_{t-1}$ in full-backlog case and $x_t = K^* - s_{t-1}$ in lost-sales case. Therefore production variability equals to demand variability in full-backlog case and is less than demand variability in lost-sales case, since $s_{t-1} = \max\{0, K^* - d_{t-1}\}$ in lost-sales case. For the considered inventory problem, we can then conclude that production cannot be more volatile than demand under the stationary optimal policy.

However, the stability of the stationary produce-up-to policy has been ignored. As we will see later, the firm's profit-maximization problem in (7) can be formulated as a optimal sequence problem to decide the optimal target inventory levels for each period given the realized the inventory position x_t; we denote the solution as $\{K_t^*\}$ and decision today K_t^* affects decision tomorrow K_{t+1}^* through the inventory transition equation $x_{t+1} = K_t^* - s_t$. Given the solutions of target inventory, production plan at each period is $q_t = \left[K_t^* - x_t\right]^+$; therefore the stationary optimal policy in (11) is the production plan under the steady-state of the dynamic process of $K_{t+1}^* = f\left(K_t^*\right)$. From proposition 1 we know that the steady state exists uniquely. However, the stability problem investigated by this paper is concerned about the question: what if the target inventory once deviates from the steady-state, e.g., $K_t \neq K^*$, because of a perturbation to the system? In order to answer this question, we need the formal definition of stability and here is the one in Stokey, Lucas, and Prescott [19].

Definition 1: *Let $Z \subseteq \Re$ and $g : Z \to Z$ be a continuous function, and for any $z_0 \in Z$, consider the sequence $\{z_t\}$ defined by $z_{t+1} = g(z_t), t = 0,1,....$ Let $\bar{z} \in Z$ be a steady-state or fixed point of g,*

1. *the point $\bar{z}$ is globally asymptotically stable if for all $z_0 \in Z$ and $\{z_t\}$ satisfies g, we have $\lim_{t\to\infty} z_t = \bar{z}$;*

2. *the point $\bar{z}$ is locally asymptotically stable if there exists a compact set $\tilde{Z} \subset Z$ that contains $\bar{z}$, such that for all $z_0 \in \tilde{Z}$ and $\{z_t\}$ satisfies g, we have $\lim_{t \to \infty} z_t = \overline{Z}$. The point $\overline{Z}$ is said to be instable if it is not locally asymptotically stable.*

For the considered inventory problem, we will show that the target inventory is instable at the steady state; a perturbation leads to divergent oscillation of the target inventory; when the target inventory oscillates, production variability is amplified because it comes from two sources: demand variability and the oscillation of target inventory.

4. Solving the Target Inventory

With proposition 1, we now proceed to solve the stationary target inventory K^*. Differing from the approach directly from the recursive equation of value function, our solution strategy is based on the classical variational approach[6] in which the profit maximization problem in (7) is equivalently reformulated as the optimal sequence problem. Let $K_t = x_t + q_t$, the one period expected profit function in (4) can be expressed as the function of K_t as

$$E\pi(K_t) = ES(K_t) - EG(K_t) - c \cdot [K_t - x_t] \tag{13}$$

Given current inventory position x_t, the firm solves the following problem

$$V(x_t) \equiv \underset{\{K_j\}_{j=t}^{\infty}}{Max} \sum_{j=t}^{\infty} \beta^{j-t} E\pi(K_j) \tag{14}$$

Proposition 2 summarizes interior solutions of problem (14) for both the full-backlog and lost-sales case.

Proposition 2.1 (Full-backlog case):

[6] Stokey, Lucas, and Prescott [19, page 95]

1. *For a full-backlog inventory system, the optimal target inventories satisfy the following difference equation*

$$F_d\left(K_{t+1}^*\right)=\frac{(1+\beta)\cdot(p+\tilde{\gamma})-c}{\beta\cdot(p+h+\tilde{\gamma})}-\frac{1}{\beta}F_d\left(K_t^*\right),\quad K_t^*\in\left(0,\bar{x}\right)\text{ for each } t \tag{15}$$

2. *The stationary optimal produce-up-to level in (11) is the unique steady-state of (15) which is expressed as*

$$K^*=F_d^{-1}\left(\frac{p+\tilde{\gamma}}{p+h+\tilde{\gamma}}-\frac{c}{\left(1+\beta\right)\cdot\left(p+h+\tilde{\gamma}\right)}\right) \tag{16}$$

where $\tilde{\gamma}=\gamma-\beta\cdot\left(p-c\right)$

Proof. See Appendix 2.

Proposition 2.2 (Lost-sales case):

1. *For a lost-sales inventory system, the optimal target inventories satisfy the following difference equation*

$$F_d\left(K_{t+1}^*\right)=\frac{w}{F_d\left(K_t^*\right)}\left(\tilde{\pi}-\tilde{c}\right)+\tilde{\pi}-w\,,\quad K_t^*\in\left(0,\bar{x}\right)\text{ for each } t \tag{17}$$

2. *The stationary optimal produce-up-to level in (11) is the unique steady-state of (17) which is expressed as*

$$K^*=F_d^{-1}\left(-\frac{1}{2}\left(w-\tilde{\pi}\right)+\frac{1}{2}\sqrt{\left(\tilde{\pi}-w\right)^2+4w\left(\tilde{\pi}-\tilde{c}\right)}\right) \tag{18}$$

where $w\equiv\frac{1}{\beta}>1,\ \tilde{\pi}\equiv\frac{p+\gamma}{p+h+\gamma}\in\left(0,1\right),\ \tilde{c}\equiv\frac{c}{p+h+\gamma}\in\left(0,1\right)$

Proof. See Appendix 2.

The first-order difference equations in (15) and (17) are the first-order conditions (Euler's equations) of optimal solutions to the optimal sequence problem in (14); in the two equations,

$F_d\left(K_t^*\right) \equiv \Pr\left(d_t \le K_t^*\right)$ is the probability of non stock-out under the optimal target inventory level. A unique steady state exists for both the two difference equations and the steady states determined by (15) and (17) are the constant produce-up-to points of the stationary optimal policy in (11) for full-backlog and lost-sales case respectively.

5. Stability Analysis

Based on the Euler's equations, we can investigate the stability of the stationary optimal policy of the considered inventory systems. The stability analysis follows directly the results from Scheinkman [17].

Lemma 1. *Let Let $Z \subseteq \Re$ and $g : Z \to Z$ be a continuous function, and for any $z_0 \in Z$, consider the sequence $\{z_t\}$ defined by a difference equation (or system) $z_{t+1} = g(z_t)$, $t = 0,1,....$ Let $\bar{z} \in Z$ be a steady-state or fixed point of g, and g is differentiable in a neighborhood N of $\bar{z}$. Then, the difference equation is locally asymptotically stable at $\bar{z}$ if and only if the first-order derivative of g evaluated at $\bar{z}$ is less than 1 in absolute value, i.e., $\left|g'(\bar{z})\right| < 1$.*

Applying Lemma 1 to the Euler's equations in (15) and (17), we have:

Proposition 3 (Stability Conditions). *The Euler's equations in (15) and (17) are locally asymptotically stable at K^* if and only if,*

1) *For full-backlog:* $\left|\frac{1}{\beta}\right| < 1$

2) *For lost-sales:* $0 < (\beta\tilde{\pi} - 1)F_d(K^*)$ *when* $\left(\tilde{\pi} - \tilde{c}\right) \ge 0$; *or*

   $2(\tilde{c} - \tilde{\pi}) < (\beta\tilde{\pi} - 1)F_d(K^*)$ *when* $\left(\tilde{\pi} - \tilde{c}\right) < 0$

Proof: See Appendix 3.

It can then be verified that the stability conditions given in Proposition 3 cannot be met by the inventory systems considered in this paper so far. Theorem 1 summarizes the instability of the steady-state K^*.

Theorem 1 (Instability). *The dynamic inventory system defined in (7) is instable under either full-backlog or lost-sales case. Furthermore, the trajectories of inventory target starting from* K_0^**, with* $\|K_0^* - K^*\| \geq \delta, \forall \delta > 0$*, are of oscillatory divergence.*

Proof. For full-backlog case, the system is instable by Proposition 3 since $\beta \leq 1$. For lost-sales case, since probability distribution function $F_d(\cdot)$ must be nonnegative, it is immediate by Proposition 3 that the steady-state is locally asymptotically stable if and only if $\tilde{\pi} > 1$. However since $\tilde{\pi} = \dfrac{p+\gamma}{p+h+\gamma} \leq 1$, K^* cannot be a stable steady-state. Furthermore, the trajectories starting from K_0^*, with $\|K_0^* - K^*\| \geq \delta, \forall \delta > 0$, are oscillatory because the Euler's equations in (15) and (17) are strictly decreasing; the trajectories are divergent, i.e., moving farther and farther from K^*, because of the linearity of Euler's equation for the full-backlog case, and strict convexity of the Euler's equation for the lost-sales case. Q.E.D.

Theorem 1 concludes that: 1) a full-backlog inventory system is unconditionally instable, unless $\beta > 1$ which is infeasible under the concerned circumstance; 2) a lost-sales inventory system is conditionally unstable, for which we will elaborate in the next Section.

6. Achieving Stability by Inventory Refund Policies under Lost-Sales

Firms can outsource their production parts to suppliers. After outsourcing, a firm behaves as a retailer to order the product from the suppliers and sell the product into a market. We show in this section that the inventory refund from suppliers to retailers, which is common in retailing industry and is mainly seen as marketing incentive for promoting new and innovative products as discussed in Kandel [12], can in fact stabilize a retail inventory system with lost-sales.

We consider a retailer who receives inventory credit ξ for each unit of unsold inventory; total inventory credits received at the end of each period t is then $\xi \cdot [x_t + q_t - d_t]^+$. The one-period profit with inventory refunds can be derived as:

$$\hat{\pi}_t(x_t, d_t, q_t) = ps_t - \hat{h} \cdot [x_t + q_t - d_t]^+ - \gamma \cdot [d_t - x_t - q_t]^+ - c_t q_t \tag{19}$$

where $\hat{h} = h - \xi$. Noting that for the lost-sales case, the retailer's inventory problem with inventory refunds differs only in $\hat{h}$ instead of h. Let $\hat{\pi} = \dfrac{p+\gamma}{p+\hat{h}+\gamma}$ and $\hat{c} = \dfrac{c}{p+\hat{h}+\gamma}$, and the unique stead-state with inventory refunds is then (from proposition 2.2)

$$K^* = F_d^{-1}\left(-\frac{1}{2}(w-\hat{\pi}) + \frac{1}{2}\sqrt{(\hat{\pi}-w)^2 + 4w(\hat{\pi}-\hat{c})}\right) \tag{20}$$

The stability condition of the steady-state is summarized in proposition 4.

Proposition 4: *The Euler's equation under lost-sales case with inventory refund is locally asymptotically stable at* K^* *if and only if*

$$\xi > h - (1-\beta)(p+\gamma) \tag{21}$$

Proof. From Proposition 3, Euler's equation under lost-sales is stable at K^* if and only if $0 < \beta\hat{\pi} - 1$, or equivalently, $\xi > h - (1-\beta)(p+\gamma)$. Q.E.D.

By Theorem 1, a lost-sales inventory system without inventory refund is intrinsically prone to instability. While according to Proposition 4 on the other hand, when inventory refunds are allowable, the lost-sales inventory system can be stabilized locally under a sufficient refund credited back to the retailer for unsold inventory. The right hand side of the inequality in (21) is intuitive: the marginal cost of carrying one more inventory is *h* but this additional inventory reduces the marginal opportunity cost of $(p+\gamma)$ in the event of stock-out; $(1-\beta)(p+\gamma)$ measures then the marginal benefit of ordering one more product today relative to ordering it tomorrow. Inequality (21) states that in order to achieve local stability for the inventory system with lost-sales, the inventory refund per unit unsold (ξ) must be greater than the *net* marginal cost of ordering today. The more patient the retailer is (or the larger β is), the larger inventory refund is necessary to stabilize the retailer inventory locally.

Appendix 1

Proof for Proposition 1

For any $t < \infty$ and both the full-backlog and lost-sales inventory system, the value function $V(x_t) \equiv \sup_{q_k \in \Omega(x_t)} E_t \left\{ \sum_{k=t}^{\infty} \beta^{k-t} \pi(x_k, d_k, q_k) \right\}$ a well-defined function of inventory position x_t. Since our dynamic programming problem is stationary, without loss of correctness for any $t < \infty$, we drop the index t in the rest of the proof. The Bellman's equation can then be written as

$$V(x) = \sup_{q \in \boldsymbol{\Omega}(x)} \left[W(x+q) - cq \right] \tag{A1.1}$$

where

$$W(x+q) = ES(x+q) - EG(x+q) + \beta EV(x+q) \tag{A1.2}$$

The retailer's optimal order plan can then be expressed as

$$q^*(x) = \arg\max_{q \in \boldsymbol{\Omega}(x)} \left[W(x+q) - cq \right] \tag{A1.3}$$

Define a space of continuous and bounded functions $\mathbf{B}$, such that $V(x) \in \mathbf{B}$; and a mapping $\mathbf{T} : \mathbf{B} \to \mathbf{B}$, with the form

$$\mathbf{T}(V)(x) = \sup_{q \in [0, \bar{x} - x]} E\pi(x, q) + \beta EV(x+q) \tag{A1.4}$$

The Bellman's equation can then be written as $V(x) = \mathbf{T}(V)(x)$. Without loss of correctness, it is assumed that the state space, $X \equiv \left[-\bar{d}, \bar{x} \right]$ in full-backlog case and $X \equiv [0, \bar{x}]$ in lost-sales case, is compact and convex, and the constraint correspondence $\Omega(x)$ is compact for each $x \in X$. Then, the image of $\boldsymbol{\Gamma} \equiv X \times \boldsymbol{\Omega}$ under $E\pi$ where,

$$E\pi(\Gamma) = \{ z \in \mathbf{R} : z = E\pi(w) \; for\; some\; w \in \Gamma \},$$

is also compact. Thus $E\pi(x,q)$ is bounded. Given $\beta \in (0,1)$, it can be verified that there exists a unique V that solves the Bellman's equation by applying the contraction mapping theorem in Fuente [7, Theorem 1.5].

Following the existence and uniqueness of $V(x)$, we proceed to show the concavity of $V(x)$, and thus the optimality of the order-up-to policy.

Lemma A1: *The one-period expected profit function, $E\pi(x,q)$, is a strictly concave function of x in both full-backlog and lost-sales case.*

Proof: By direct differentiation and for lost-sales case,

$$\nabla_{x_t} E\pi(x_t, q_t) = p + \gamma - (p + h + \gamma) \cdot F_d(x_t + q_t) \tag{A1.5}$$

$$\nabla^2_{x_t} E\pi(x_t, q_t) = -(p + h + \gamma) \cdot f_d(x_t + q_t) < 0 \tag{A1.6}$$

γ is replaced as $\tilde{\gamma} \equiv \gamma - \beta(p - c)$ in full-backlog case. Still,

$$\nabla^2_{x_t} E\pi(x_t, q_t) = -(p + h + \tilde{\gamma}) \cdot f_d(x_t + q_t) < 0 \tag{A1.7}$$

Q.E.D.

To facilitate the proof, we need to prepare some preliminaries as presented in Lemma A2 below. Since the state space X is convex, we can obtain

Lemma A2:

1. *The set of weakly concave functions in* B *is a closed subset of* B.
2. *For both full-backlog and lost-sales case, the constraint correspondence* $\mathbf{\Omega}(x)$ *is convex in the sense that* $\forall x_0, x_1 \in X$, $\lambda \in [0,1]$, $y_0 \in \mathbf{\Omega}(x_0)$, *and* $y_1 \in \mathbf{\Omega}(x_1)$, *we can have* $(1-\lambda)y_0 + \lambda y_1 \in \mathbf{\Omega}((1-\lambda)x_0 + \lambda x_1)$.

Proof: The first claim is the direct result from Lemma 1.13 of Fuente [7, page 565]. For the second one, we just need to show that the graph of $\mathbf{\Omega}$ is convex. The graph of $\mathbf{\Omega}$ is the set

$\{(x,y)\in X\times X: y\in\mathbf{\Omega}(x)\}$, which is a triangular in both full-backlog and lost-sales case and is thus convex.   Q.E.D.

With the above necessary preliminaries, we can prove in the order of the following two claims, which hold for both full-backlog and lost-sales case:

1. *The value function* $V(x)$ *is strictly concave in* x.
2. $q^*(x)$ *is a continuous function in the form of*

$$q(x)=\left(K^*-x\right)^+=\begin{cases}K^*-x, & \text{for } x\le K^*\\ 0, & \text{otherwise}\end{cases} \tag{A1.8}$$

where K^* *represents the firm's desired inventory, and it is defined as*

$$K^*=\arg\max_{q\in\mathbf{\Omega}(0)}\left[W(q)-cq\right] \tag{A1.9}$$

Let $x_0,x_1\in X$, $\lambda\in[0,1]$, $\hat{x}=\lambda x_0+(1-\lambda)x_1$, we now want to show that $V(\hat{x})>\lambda V(x_0)+(1-\lambda)V(x_1)$. Let $q_0^*=q^*(x_0)$, $q_1^*=q^*(x_1)$, and $\hat{q}^*=q^*(\hat{x})$, from Lemma 1, we can get

$$\begin{aligned}\lambda V(x_0)+(1-\lambda)V(x_1)&=\lambda\left[E\pi\left(x_0,q_0^*\right)+\beta EV\left(x_0+q_0^*\right)\right]+(1-\lambda)\left[E\pi\left(x_1,q_1^*\right)+\beta EV\left(x_1+q_1^*\right)\right]\\ &<E\pi\left(\hat{x},\hat{q}^*\right)+\lambda\beta EV\left(x_0+q_0^*\right)+(1-\lambda)\beta EV\left(x_1+q_1^*\right)\end{aligned} \tag{A1.10}$$

Given the contraction mapping theorem, point 1 of Lemma A2 implies that the unit fixed point of $\mathbf{T}$ is in the space of weakly concave functions. Then following (A1.10), we can have

$$E\pi\left(\hat{x},\hat{q}^*\right)+\beta\left\{\lambda EV\left(x_0+q_0^*\right)+(1-\lambda)EV\left(x_1+q_1^*\right)\right\}\le E\pi\left(\hat{x},\hat{q}^*\right)+\beta EV\left(\hat{x}+\hat{q}^*\right)=V(\hat{x})$$

The equality in the last step is the result from point 2 of Lemma A2. **This finishes claim 1**.

For claim 2, first from the Theorem of the Maximum, we know that $q^*(x)$ is non-empty and upper-hemicontinuous (uhc). Concavity of $V(x)$ implies then that $q^*(x)$ exits uniquely and is thus a continuous function of the state. The inventory target K, which is expressed as the solution of

$K^* = \arg\max_{q\in[0,\bar{x}]}[W(q) - cq]$, exists also uniquely. Now we want to show equation (A1.4) is the optimal strategy. By definition, for each $x \in X$, we know

$$\begin{aligned} &W(x) - cx < W(K^*) - cK^* \Rightarrow \\ &\frac{W(K^*) - W(x)}{K^* - x} > c, \quad if \quad x < K^* \\ &\frac{W(x) - W(K^*)}{x - K^*} < c, \quad if \quad x > K^* \end{aligned}$$

that is, for any $x < K^*$, ordering $K^* - x$ units is optimal. Also, from the strict concavity of $W(x)$, we can get

$$\frac{W(x') - W(x)}{x' - x} < \frac{W(x) - W(K^*)}{x - K^*} < c, \quad \forall x' > x > K^*$$

which implies that the optimal strategy is not to order when $x > K^*$. Finally, the optimal strategy when $x = K^*$ is obtained by the continuity of $q^*(x)$. This finishes claim 2 and thus the proof of proposition 1. Q.E.D.

Appendix 2

<u>*Proof for Proposition 2*</u>

We first get the differentiability of the value function in the interior region, as presented in Lemma A3, which holds for both full-backlog and lost-sales case.

Lemma A3: *$V(x)$ is differentiable at x' for each $x' \in \operatorname{int} X$ and $q^*(x') \in \operatorname{int} \mathbf{\Omega}(x')$. Furthermore,*

$\nabla_x V(x)\big|_{x=x'} = \nabla_x E\pi(x, q^*(x'))\big|_{x=x'}$ for each $x' \in \operatorname{int} X$ and $q^*(x') \in \operatorname{int} \mathbf{\Omega}(x')$.

Proof: For any $x' \in \operatorname{int} X$ and $q' \equiv q^*(x') > 0$, let $N(x')$ be the neighborhood of x'. For any $x \in N(x')$, $q' \in \operatorname{int} \mathbf{\Omega}(x)$ because $q' \in \operatorname{int} \mathbf{\Omega}(x')$ and $\mathbf{\Omega}$ is continuous. Define

$$H(x) = E\pi(x, q') + \beta EV(x' + q') \quad \text{(A2.1)}$$

Clearly, $H(x)$ is differentiable with respect to x because $E\pi(\cdot)$ is differentiable with respect to x and the second term in the *r.h.s.* of (A1.4) is not the function of x. Thus, $\nabla_x H(x) = \nabla_x E\pi(x, q')$. By definition, $H(x') = V(x')$ and $\forall x \in N(x')$, $H(x) \le V(x)$. Because of the concavity of $V(x)$, we can apply the Envelop Theorem of Benveniste and Scheinkman [3] to get that $V(x)$ is differentiable at x', and

$$\nabla_x V(x)\big|_{x=x'} = \nabla_x H(x)\big|_{x=x'} = \nabla_x E\pi(x, q')\big|_{x=x'} \quad \text{(A2.2)}$$

Q.E.D.

With lemma A3, we now proceed to prove proposition 2. The first order condition of the profit maximizing problem in the interior region, $\nabla_{q_t} V_t = 0$, leads to the following equality:

$$\nabla_{q_t} ES(x_t + q_t^*) - \nabla_{q_t} EG(x_t + q_t^*) + \beta \nabla_{q_t} EV_{t+1}(x_{t+1}) = c_t \quad \text{(A2.3)}$$

where $q_t^* = q^*(x_t)$ is the optimal order as given in Proposition 1, and

$$\nabla_{q_t} EV_{t+1}(x_{t+1}) = \nabla_{x_{t+1}} EV_{t+1}(x_{t+1}) \cdot \nabla_{q_t} x_{t+1} = \nabla_{x_{t+1}} EV_{t+1}(x_{t+1}) \quad \text{(A2.4)}$$

Let $K_t^* = x_t + q_t^*$, we write (A2.3) as

$$\nabla_{q_t} ES(K_t^*) - \nabla_{q_t} EG(K_t^*) + \beta \nabla_{x_{t+1}} V_{t+1}(x_{t+1}) = c_t \quad \text{(A2.5)}$$

Applying Lemma A3, we can then obtain

For full-backlog:

$$F_d(K_{t+1}^*) = \frac{(1+\beta)\cdot(p+\tilde{\gamma}) - c}{\beta \cdot (p + h + \tilde{\gamma})} - \frac{1}{\beta} F_d(K_t^*) \quad \text{(A2.6)}$$

For lost-sales:

$$F_d\left(K_{t+1}^*\right)=\frac{p+\gamma-c}{\beta F_d\left(K_t^*\right)\cdot(p+h+\gamma)}+\frac{p+\gamma}{p+h+\gamma}-\frac{1}{\beta} \tag{A2.7}$$

Reorganizing (A2.7) we can get equation (16).

From proposition1, we know that the order policies of the considered inventory systems have the stationary order-up-to type, in which the retailer orders up to K^* for each period. Since the order-up-to level must satisfy the Euler's equation in the interior region, it is then the stationary point or the steady-state of the Euler's equation. The Euler's equations in (A2.6) and (A2.7) define a mapping from $[0,1]$ to itself, and by the monotonic property of the mapping, it is straightforward to verify by the Brouwer's fixed point theorem that there exists a unique fixed point or steady-state $K^* = K_{t+1}^* = K_t^*$, which solves the following equation:

Full-backlog: $$F_d\left(K^*\right)=\frac{(1+\beta)\cdot(p+\tilde{\gamma})-c}{\beta\cdot(p+h+\tilde{\gamma})}-\frac{1}{\beta}F_d\left(K^*\right) \tag{A2.8}$$

Lost-sales: $$F_d\left(K^*\right)=\frac{p+\gamma-c}{\beta F_d\left(K^*\right)\cdot(p+h+\gamma)}+\frac{p+\gamma}{p+h+\gamma}-\frac{1}{\beta} \tag{A2.9}$$

The solutions from (A2.8) and (A2.9) are the optimal order-up-to points in the interior region for full-backlog and lost-sales case respectively. By the continuity of order-up-to decision rule, it is also applied to the cases when $x \geq K^*$; $x = -\bar{d}$ (for full-backlog), and $x = 0$ (for lost-sales). **This finishes the proof of proposition 2.**

Appendix 3

Proof of Proposition 3

Denoting $\xi_t = F_d(K_t)$ and $\xi^* = F_d(K^*)$, we rewrite the Euler's equations, respectively for backlog and lost-sales as:

- Full-backlog: $\xi_{t+1} = g_{FB}(\xi_t) = \frac{1}{\beta}\xi_t + \frac{(1+\beta)\cdot(p+\tilde{\gamma})-c}{\beta\cdot(p+h+\tilde{\gamma})}$

- Lost-sales: $\xi_{t+1} = g_{LS}(\xi_t) = \frac{w}{\xi_t}(\tilde{\pi} - \tilde{c}) + \tilde{\pi} - w$

It is then easy to obtain that:

$$g'_{FB}(\xi_t) = \frac{1}{\beta}, \quad \text{and} \quad g'_{LS}(\xi_t) = -\frac{w}{\xi'_t}(\tilde{\pi} - \tilde{c})$$

Then, according to the stability condition, $|g'(K^*)| < 1$, as given in Lemma 1, we confirm that the Euler's equation under full-backlog is stable if and only if

$$|g'_{FB}(K^*)| = \frac{1}{\beta} < 1, \text{ where } \xi^* = F_d(K^*)$$

For the lost-sales case, the stability condition $|g'_{LS}(K_t)| = \left|\frac{w(\tilde{\pi} - \tilde{c})}{\xi_t^{\,2}}\right| < 1$ at $\xi^* = F_d(K^*)$ can be equivalently expressed as a set of two mutually exclusive inequalities:

1) For $(\tilde{\pi} - \tilde{c}) \geq 0$, $0 < (\beta\tilde{\pi} - 1)F_d(K^*)$, or otherwise
2) For $(\tilde{\pi} - \tilde{c}) < 0$, $2(\tilde{c} - \tilde{\pi}) < (\beta\tilde{\pi} - 1)F_d(K^*)$

This concludes the proof for Proposition 3. Q.E.D.

References

[1] K. J. Arrow, T. Harris, J. Marschak. Optimal inventory policy. Econometrica 19 (1951) 250-272.

[2] K. J. Arrow, S. Karlin, H. Scarf. Studies in the mathematical theory of inventory and production. Stanford University Press, Stanford, CA, 1958.

[3] L. M. Benveniste, J. A. Scheinkman. On the differentiability of the value function in dynamic models of economics. Econometrica 47 (1979) 727 – 732.

[4] A. S. Blinder. Inventories and Sticky Prices: More on the microfoundations of macroeconomics. Amer. Econ. Rev. 72 (1981) 334-348.

[5] A. S. Blinder, L. J. Maccini. Taking Stock: A Critical Assessment of Recent Research on Inventories. J. Econ. Perspect. 5 (1991) 73-96.

[6] A. S. Caplin. The variability of aggregate demand with (s, S) inventory policies. Econometrica 53 (1985) 1395-1409.

[7] A. Fuente. Mathematical Methods and Models for Economists. Cambridge University Press, Cambridge, U.K., 2000.

[8] G. J. Hall, J. Rust. An empirical model of inventory investment by durable commodity intermediaries. Carnegie-Rochester Conf. Ser. Public Pol., 52 (2000) 171-214.

[9] D. S., Iglehart. Optimality of (s,S) Policies in the Infinite Horizon Dynamic Inventory Problem. Manage. Sci. 9 (1963) 257-267.

[10] J. A. Kahn. Inventories and the volatility of production. Amer. Econ. Rev. 77 (1987) 667-679.

[11] J. A. Kahn. Why is production more volatile than sales? Theory and evidence on the stockout-avoidance motive for inventory holding. Quart. J. Econ. 107 (1992) 481-510.

[12] E. Kandel. The right to return. J. Law Econ. 39 (1996) 329-356.

[13] H. L. Lee, V. Padmanabhan, S. Whang. Information distortion in a supply chain: The bullwhip effect. Manage. Sci. 43 (1997) 546-558.

[15] P. Mosser. Trade inventories and (S,s). Quart. J. Econ. 106 (1991). 1267-86.

[16] Ramsey, V. A. Nonconvex costs and the behavior of inventories. J. Polit. Economy 99 (1991) 306-334.

[17] J. A. Scheinkman. On optimal steady states of n-sector growth models when utility is discounted. J. Econ. Theory 12 (1976) 11-30.

[18] J. Sterman. Operational and behavioral causes of supply chain instability. The bullwhip effect in supply chains, in: O. Carranza, F. Villegas (Eds.), Palgrave McMillan, New York, NY, 2005.

[19] N. L. Stokey, R. E. Lucas, E. C. Prescott. Recursive methods in economic dynamics. Harvard University Press, Cambridge, MA, 1989.