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Solving Models with Incomplete Markets and Aggregate Uncertainty Using the Krusell-Smith Algorithm: A Note on the Number and the Placement of Grid Points*

Michal Horvath†
University of Oxford and CDMA

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ABSTRACT

This paper shows that numerical solutions to models with incomplete markets and aggregate uncertainty obtained using the Krusell and Smith (1998) algorithm are sensitive to the parameterization of the grid in the aggregate asset holdings direction. Higher moments of the cross-sectional distribution of asset holdings can be particularly affected, which is important for welfare analysis. Using grids that are denser around the mean of the ergodic distribution of individual asset holdings can enhance the consistency of the results across parameterizations. The accuracy of the approximation to individual decision functions can be much improved this way.

JEL Classification: C6, C63, D52, E21
Keywords: Incomplete Markets, Aggregate Uncertainty, Heterogeneous agents, Simulations, Numerical Solutions.

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† Department of Economics and Nuffield College, University of Oxford, Manor Road Building, Manor Road, Oxford OX1 3UQ, United Kingdom, E-mail: michal.horvath@economics.ox.ac.uk

CASTLECLIFFE, SCHOOL OF ECONOMICS & FINANCE, UNIVERSITY OF ST ANDREWS, KY16 9AL
TEL: +44 (0)1334 462445 FAX: +44 (0)1334 462444 EMAIL: cdma@st-and.ac.uk

www.st-and.ac.uk/cdma

1. Introduction

When solving models with incomplete markets using grid-based numerical simulations, having the right grid is key to achieving a satisfactory accuracy of results, whilst keeping the time length of the simulation reasonable. Researchers have used equally-spaced and various unequally-spaced grids for individual asset holdings.[1] This paper highlights that when aggregate uncertainty is added as a source of risk in the model, and (moments of) the aggregate state variable(s) become an element in the state vector, the parameterization of the aggregate asset holdings dimension of the grid can affect the results in a significant way. Notably, the second and higher moments of the cross-sectional distribution of asset holdings are shown to be particularly affected by having an insufficient number of grid points and/or misplaced grid points. This has important implications for welfare analysis conducted in the context of such models.

The model in this paper is that of Krusell and Smith (1998), and we use a close relative of their stochastic simulation algorithm to solve the model. The variant we use involves a grid-based Euler-equation algorithm to solve for the individual decision functions as in Maliar et al. (2010).

Krusell and Smith (1997) state that since there is generally not much curvature in the value function in the direction of aggregate capital, it is sufficient to use a small number of grid points in this direction, and use polynomial interpolation to compute the value function for values of aggregate capital holdings not on the grid. In a similar vein, Krusell and Smith (1998) report that their results are not sensitive to increasing the number of grid points in the direction of aggregate capital. Whilst proposing an elaborate technique to parameterize the individual

[1] Logarithmic spacing or Chebyshev nodes are popular choices. Maliar et al. (2010) proposed a simple polynomial rule for the placement of the grid points in the individual capital holdings direction.

capital holdings direction, Maliar et al. (2010) also use only four equally spaced grid points for aggregate capital, distributed on a symmetrical interval around the mean of the ergodic distribution of capital.

This paper shows that there is non-negligible curvature in the individual decision functions with respect to aggregate capital, and that the approximation errors generated by polynomial interpolations over different grids can lead to decision functions that imply significantly different (and often implausible) second and higher moments of the distribution of individual capital holdings.

The proposed solution to this problem is the use of grids that are finer around the mean of the ergodic distribution of individual capital holdings. This way, one can substantially increase the accuracy of the solution without significantly increasing the computational cost.

The rest of the paper is organized as follows. Section 2 discusses alternative parameterizations of the grid. Section 3 presents and discusses the results from the stochastic simulation algorithm using alternative grids. Section 4 concludes.

2. Alternative parameterizations of the grid

Our model and its baseline parameterization is the same as in Den Haan et al. (2010). The baseline grid as well as other parameters of the simulation algorithm are the same as the ones used in the stochastic simulation algorithm of Maliar et al. (2010). More specifically, in the aggregate capital direction, we use four equally spaced points on the interval between 30 and 50.[2] The alternative parameterizations of the grid we consider are shown in Table 2.1. In these alternative simulations, we leave everything else unchanged, including the series for aggregate and individual shocks.

[2] Interestingly, they almost coincide with the Chebyshev collocation nodes. On the mentioned interval, these would be located at $\{30.8, 36.2, 43.8, 49.2\}$.

Scenario (1) is there to illustrate the case of a sparse grid in which the mean of the ergodic distribution falls between grid points approximately in the middle of the interval used for aggregate capital. The grid is therefore similar to the baseline case but with looser boundaries.[3]

Scenario (2) is the case when the mean of the distribution falls nearer to the lower boundary of a grid spread out asymmetrically around the mean.[4]

Scenarios (3) and (4) use a finer grid on the intervals used in scenarios (1) and (2) respectively.

Scenarios (5) and (6) use an uneven grid of five points on the intervals used in scenarios (1) and (2) with the grid points concentrated in the vicinity of the mean of the ergodic distribution.[5]

Scenario (7) uses a slightly denser grid on the baseline interval. Finally, scenario (8) uses only 4 grid points on the baseline interval but the points are unevenly distributed.

Cubic interpolation is used throughout to find solutions for values not on the grid.

3. Results

Our analysis reveals that the numerical solution to the Krusell and Smith (1998) model is highly sensitive to the parameterization of the grid in the aggregate capital direction. Table 3.1 summarizes the results for different moments of the

[3] When solving models of this kind, the initial iterations might lead to grossly inaccurate results, depending on the initial guess. Hence, a priori, it is often useful to start with a wider interval for aggregate capital to prevent hitting of the boundary value.

[4] The mentioned problem of obtaining inaccurate results in the first few iterations may also lead the researcher to consider an asymmetric grid.

[5] A (however inaccurate) simulation based on an initial guess or even the non-stochastic steady state solution to the model can give the researcher an idea about where the mean of the ergodic distribution is going to be located.

Scenario	Grid points for aggregate capital
baseline	$\{30.0, 36.7, 43.3, 50.0\}$
(1)	$\{20.0, 33.3, 46.7, 60.0\}$
(2)	$\{35.0, 43.3, 51.7, 60.0\}$
(3)	$\{20.0, 25.7, 31.4, 37.1, 42.9, 48.6, 54.3, 60.0\}$
(4)	$\{35.0, 38.6, 42.1, 45.7, 49.3, 52.9, 56.4, 60.0\}$
(5)	$\{20, 37, 39, 41, 60\}$
(6)	$\{35, 37, 39, 41, 60\}$
(7)	$\{30, 34, 38, 42, 46, 50\}$
(8)	$\{30, 37, 39, 50\}$

Table 2.1: Alternative parameterizations of the grid in the aggregate capital direction

cross-sectional distribution of asset holdings. We also include the Gini coefficient to indicate that the differences might have significant implications for the welfare assessment conducted based on the results. Although we do not report the R^2 from the aggregate law of motion regressions for bad and good aggregate states separately, it is important to note that these had consistently very high values across all simulations (0.99996 and 0.99998 respectively).

It is obvious from the comparison of the baseline scenario and scenarios (1) and (2) that the parameterization of the grid can have a significant effect on the results, in particular on the higher moments of the distribution of capital holdings. A simple change in the grid can deliver anything between very high equality and high inequality in capital holdings across agents. Scenario (2) indicates an extreme result in which almost all agents have very small capital holdings and a lucky few end up being very rich.[6] Researchers might then be misled to believe

[6] The maximum level of individual capital holdings obtained from the simulation under scenario (2) is almost 800, approaching the maximum value on the grid of 1,000, which indicates that a wider grid for individual holdings would be justified. Note, however that, should we increase the outer boundary for individual holdings, the maximum holding would rise further accordingly. An examination of individual decision functions is an easy way of confirming this.

Scenario	Mean	Variance (x10^3)	Skewness	Kurtosis	Gini
baseline	38.723	1.344	3.271	16.210	0.389
(1)	38.103	0.217	0.312	2.829	0.219
(2)	38.900	6.644	6.289	44.957	0.487
(3)	38.739	1.523	3.541	18.480	0.398
(4)	38.768	1.934	4.054	23.124	0.414
(5)	38.769	1.949	4.072	23.294	0.415
(6)	38.768	1.946	4.068	23.258	0.414
(7)	38.771	1.994	4.124	23.819	0.416
(8)	38.779	2.206	4.356	26.191	0.423

Table 3.1: Results - moments of the distribution of individual capital holdings

this framework generates results similar to those found in Thomas and Worrall (1990), Lucas (1992) and Atkeson and Lucas (1992).

By contrast, scenarios (4) to (7) deliver very similar results. Their common feature is that there is at least one grid point near or at the value of 39 for mean capital holdings, where the mean of the ergodic distribution is located. Scenario (8) is there to show that if one has a particular reason for being economical with grid points, four of them spread unevenly on the baseline interval can deliver a solution reasonably close to the solutions obtained on the basis of denser grids. On the other hand, comparison with scenario (3) suggests that increasing the density of the grid in important regions of the state space may be preferable to a general increase in the density of the grid.

To make sense of these results, one has to look at individual decision functions. Figure 3.1 plots the obtained solutions for individual decision functions in three dimensions. It is clear that there is significant curvature in these decision functions in the aggregate capital direction (labelled 'km'). It is important to capture the curvature in regions of the state space that are of interest. Figures 3.2 and 3.3 plot the individual decision functions obtained by cubic interpolation between the grid points at the value of 39 for aggregate capital. It is easy to see that there are significant differences across scenarios, and it is straightforward to figure out how these differences get translated to the results reported in Table 3.1 above. Under scenario (2), the decision functions consistently indicate more saving and less dissaving for the wealthier types than under scenario (3) or (5). At the other end (shown in the Figure 3.3), we see the less wealthy saving less or dissaving more. In the long run, this implies significantly more inequality in wealth compared with either of the alternatives. Scenario (1) is somewhat less obvious. Figure 3.2 indicates that wealthy individuals will be dissaving under most circumstances, whilst Figure 3.3 suggests the poor individuals will be saving more or dissaving less in all states of nature.[7] This leads to the more equal distribution reported in Table 3.1. However, these functions are significantly different from the ones obtained with finer grids.

These results allow us to propose that the results obtained under parameterizations (1) and (2), and partly also the baseline parameterization, are more of a consequence of the inaccuracy in the numerical solution instead of reflecting a fundamental reason. To test this claim formally, we conducted the dynamic Euler equation accuracy tests used in Den Haan (2010). The results displayed in Table 3.2 indeed confirm that grids that are finer around the mean of the ergodic distribution produce the most accurate individual decision functions among the scenarios examined.[8]

[7] We also see that under scenario (3), the behaviour of the very wealthy can be similar to their behaviour under the other alternatives, but unless one generates such people in the initial guess of the distribution, such types will not exist in the economy. This is to say that the solution under scenario (3) can also be sensitive to the initial guess, whilst this is generally not the case under the alternatives considered.

[8] The results reported are differences between the values of the capital and consumption paths generated with the individual policy functions and the values of the paths that are obtained when each period the values of capital and consumption that are implied by the explicitly calculated conditional expectation are used. We used the same series of shocks as in the above simulations. Errors for capital are reported relative to mean individual capital holdings.

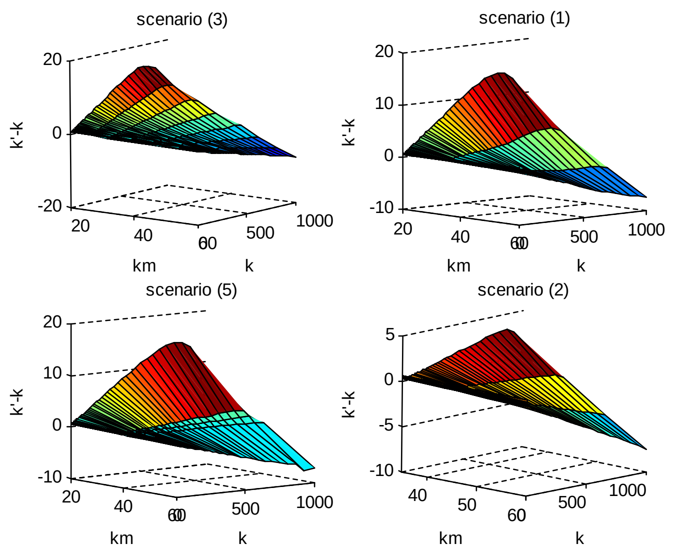

Figure 3.1: Individual decision functions and the mean capital direction (‘good’ aggregate and ‘employed’ idiosyncratic state)

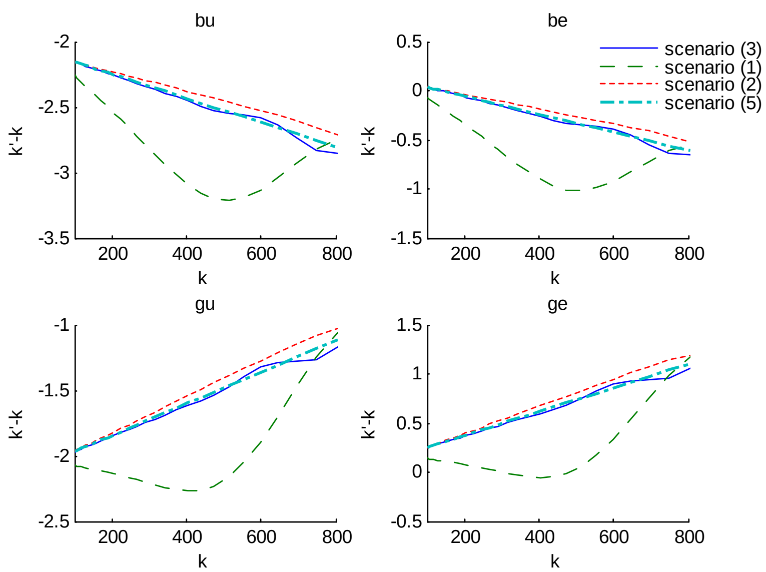

Figure 3.2: Individual decision functions of the wealthy at mean capital of 39

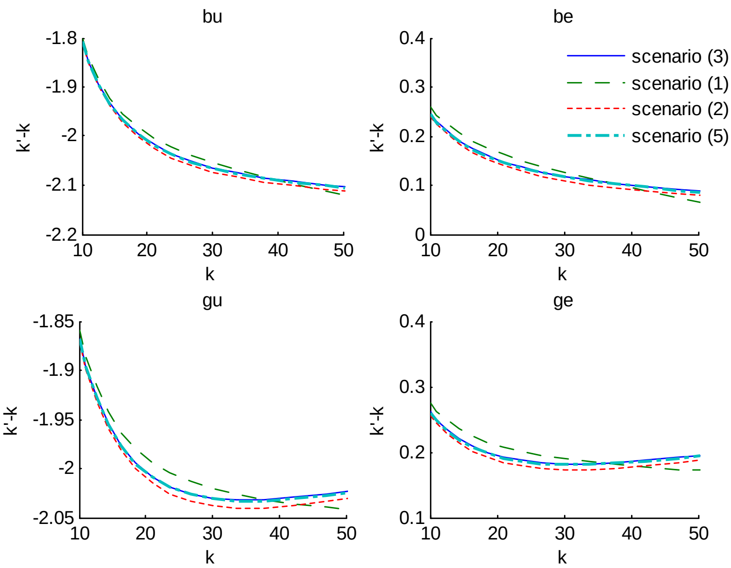

Figure 3.3: Individual decision functions of the poor at mean capital of 39

Scenario	Capital (scaled error)		Consumption (% error)	
	average	maximum	average	maximum
baseline	0.02201	0.03400	0.00212	0.00654
(1)	0.65466	0.93123	0.08139	0.16867
(2)	0.03544	0.06254	0.00293	0.01926
(3)	0.01256	0.01980	0.00121	0.00600
(4)	0.00041	0.00066	0.00006	0.00319
(5)	0.00017	0.00038	0.00007	0.00314
(6)	0.00013	0.00031	0.00005	0.00308
(7)	0.00086	0.00137	0.00010	0.00299
(8)	0.00593	0.01073	0.00066	0.00266

Table 3.2: Dynamic Euler equation accuracy test results

4. Concluding remarks

We have shown that the parameterization of the grid in the aggregate asset dimension is an important factor in delivering accurate solutions to models with incomplete markets and aggregate uncertainty solved using stochastic simulations. Lack of density of the grid in this direction, especially in the neighbourhood of the mean of the ergodic distribution of assets, can lead to inaccurate approximate decision functions that imply significantly different results.

The findings in this paper have important wider implications. The general lesson is that when parametric changes examined in the context of a model in the given class lead to changes in the mean of the ergodic distribution of asset holdings, the grid for the aggregate asset has to be adjusted accordingly to maintain consistency across results. For example, this can be the case when one considers tax reform experiments that are common in models without aggregate uncertainty, or in models with bounded rationality where changes in perceptions might drive the economy into an equilibrium with a new aggregate capital level. In the absence of an appropriate adjustment to the grid, one might easily draw the incorrect conclusion that the observed change in the distribution of asset holdings is a direct result of the parametric change. In reality, the observed change would be a combination of fundamental shifts and changes in the accuracy of the solution.

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Kaushik Mitra
Castlecliffe, School of Economics and Finance
University of St Andrews
Fife, UK, KY16 9AL
Email: km91@at-andrews.ac.uk; Phone: +44 (0)1334 462443; Fax: +44 (0)1334 462444.