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Multi-choice total clan games: characterizations and solution concepts

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Abstract

This paper deals with a new class of multi-choice games, the class of multi-choice total clan games. The structure of the core of a multi-choice clan game is explicitly described. Furthermore, characterizations of multi-choice total clan games are given and bi-monotonic allocation schemes related to players' levels are introduced for such games. It turns out that some elements in the core of a multi-choice total clan game are extendable to such bi-monotonic allocation schemes via suitable compensation-sharing rules on the domain of multi-choice (total) clan games.

JEL Classification: C71.

Keywords: Multi-choice games, Clan games, Monotonic allocation schemes.

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1 Introduction

In the traditional cooperative game theory, two classes of totally balanced games have received special attention: the class of convex games (cf. Shapley (1971)) and the class of total clan games (cf. Potters et al. (1989), Muto et al. (1988) and Voorneveld, Tijs and Grahn (2002)). Recall that a traditional cooperative game is a pair $< N, v >$, where $N = \{1, \ldots, n\}$ is a non-empty finite set of players and v is a characteristic function $v : 2^N \to \mathbb{R}$, with $v(\emptyset) = 0$, where for each group S of players, called coalition, $v(S)$ is the worth guaranteed for that coalition. Given a non-empty coalition C, a game $< N, v >$ is a clan game with clan C if v is monotonic, each player in C has veto power, and a union property regarding players outside the clan is satisfied. A clan game whose clan consists of a single player is called a big boss game. Further, a clan game whose all subgames are also clan games is called a total clan game. Bi-monotonic allocation schemes (cf. Branzei, Tijs and Timmer (2001), and Voorneveld, Tijs and Grahn (2002)) are in particular appealing solution concepts for this class of games.

A more sophisticated model of cooperative games, which is a natural extension of the traditional model, called multi-choice game, was introduced by Hsiao and Raghavan (1993a, b) and reconsidered by Nouweland et al. (1995) in a more general setting. Multi-choice cooperative games have become a useful tool for modeling interaction of players in situations in which players may have different options for cooperation, varying from non-cooperation (participation level 0) to a maximal participation level which is greater than or equal to 1. Examples of various situations modeled by multi-choice cooperative games can be found in Nouweland (1993), Nouweland et al. (1995), Calvo and Santos (2000), Peters and Zank (2005), etc. Most applications of cooperative game theory to multi-choice situations have given rise to convex multi-choice games, motivating research work for new solution concepts on this class of games. We mention here the constrained egalitarian solution for convex multi-choice games (cf. Branzei et al. (2007a)) and the notion of level-increase monotonic allocation scheme (cf. Branzei, Tijs and Zarzuelo (2007b)). Multi-choice settings of economic situations modeled by (total) clan games, like production economies with indivisible goods, information acquisition and holding situations (cf. Muto et al. (1988), Potters et al. (1989), Branzei, Tijs and Timmer (2001), Tijs, Meca and Lopez (2005)) will lead to multi-choice clan games introduced and studied in this paper.

To fulfill our goal, we need some definitions, notations and results on multi-choice games, which are recalled in Section 2. The subclass of multi-choice clan games with a fixed clan is introduced and studied in Section 3. An explicit description of the core of a multi-choice clan game is given. Furthermore, we introduce the subclass of total clan games and deal with characterizations of such multi-choice games. In Section 4 we define suitable compensation-sharing rules on the class of multi-choice clan games that are additive and stable. For multi-choice total clan games we introduce the notion of bi-monotonic allocation scheme based on one-unit level-increases of non-clan members and full participation of clan members, and prove that some core elements of a multi-choice total clan game are extendable to such bi-monotonic allocation schemes. Section 5 concludes.

2 Preliminaries on multi-choice games

Let N be a set of players, usually of the form $\{1, 2, ..., n\}$, that consider cooperation in a multi-choice environment, i.e. each player $i \in N$ has a finite number of feasible participation levels whose set we denote by $M_i = \{0, 1, ..., m_i\}$, where $m_i \in \mathbb{N}$. We consider the product $\mathcal{M}^N = \prod_{i \in N} M_i$. Each element $s = (s_1, s_2, ..., s_n) \in \mathcal{M}^N$ specifies a participation profile for players and is referred to as a multi-choice coalition. So, a multi-choice coalition indicates the participation level of each player. Then, $m = (m_1, m_2, ..., m_n)$ is the players' maximal participation level profile that plays the role of the "grand coalition", whereas $0 = (0, 0, ..., 0)$ plays the role of the "empty coalition". We also use the notation $M_i^+ = M_i \setminus \{0\}$ and $\mathcal{M}_+^N = \mathcal{M}^N \setminus \{0\}$.

A cooperative multi-choice game is a triple $\langle N, m, v \rangle$, where $v : \mathcal{M}^N \to \mathbb{R}$ is the characteristic function with $v(0) = 0$ that specifies the players' potential worth, $v(s)$, when they join their efforts at any activity level profile $s = (s_1, ..., s_n)$. For $s \in \mathcal{M}^N$ we denote by (s_{-i}, k) the participation profile where all players except player i play at levels defined by s while player i plays at level $k \in M_i$. A useful particular case is $(0_{-i}, k)$, when only player i is active. We define the carrier of s by $\text{car}(s) = \{i \in N \mid s_i > 0\}$. For $s, t \in \mathcal{M}^N$ we use the notation $s \leq t$ iff $s_i \leq t_i$ for each $i \in N$ and define $s \wedge t = (\min(s_1, t_1), ..., \min(s_n, t_n))$ and $s \vee t = (\max(s_1, t_1), ..., \max(s_n, t_n))$. We denote the set of all multi-choice games with player set N and maximal participation profile m by $MC^{N,m}$. Often, we identify a multi-choice game $\langle N, m, v \rangle$ with its characteristic function v.

Let $v \in MC^{N,m}$. We define $M := \{(i, j) \mid i \in N,\ j \in M_i\}$ and $M^+ := \{(i, j) \mid i \in N,\ j \in M_i^+\}$. A (level) payoff vector for the game $v \in MC^{N,m}$ is a function $x : M \to \mathbb{R}$, where for $i \in N$ and $j \in M_i^+$, x_{ij} denotes the payoff to player i corresponding to a change of activity level of i from $j-1$ to j, and $x_{i0} = 0$ for all $i \in N$. One can represent a payoff vector for a game v as a $\sum_{i\in N} m_i$-dimensional vector whose coordinates are numbered by the corresponding elements of M^+, where the first m_1 coordinates represent payoffs for successive levels of player 1, the next m_2 coordinates are payoffs for successive levels of player 2, and so on. For $s \in \mathcal{M}^N$, the payoff of s is $X(s) = \sum_{i\in N}\sum_{j=1}^{s_i} x_{ij}$.

A level payoff vector $x : M \to \mathbb{R}$ is called *efficient* if $X(m) = \sum_{i\in N}\sum_{j=1}^{m_i} x_{ij} = v(m)$, and is called *level-increase rational* if, for all $i \in N$ and $j \in M_i^+$, x_{ij} is at least the increase in worth that player i can obtain on his own (i.e. working alone) when he changes his activity level from $j-1$ to j, that is $x_{ij} \geq v(je^i) - v((j-1)e^i)$, or, equivalently, $x_{ij} \geq v(0_{-i}, j) - v(0_{-i}, j-1)$. A payoff vector which is both efficient and level-increase rational is called an *imputation*. We denote by $I(v)$ the set of imputations of $v \in MC^{N,m}$. The core $C(v)$ of a game $v \in MC^{N,m}$ consists of all $x \in I(v)$ that satisfy $X(s) \geq v(s)$ for all $s \in \mathcal{M}^N$, i.e.

$$C(v) = \{x \in I(v) \mid X(s) \geq v(s) \text{ for each } s \in \mathcal{M}^N\}.$$

A game whose core is non-empty is called a *balanced game*.

Let $v \in MC^{N,m}$ and let $t \in \mathcal{M}^N$. We denote by $\mathcal{M}_t^N$ the subset of $\mathcal{M}^N$ consisting of multi-choice coalitions $s \leq t$. The subgame of v with respect to t, $< N, t, v_t >$, is defined by $v_t(s) := v(s)$ for each $s \in \mathcal{M}_t^N$.

In the sequel, we introduce different notions of marginal contributions of players to the grand coalition m in a game $< N, m, v >$. First, for each player $i \in N$, we define the marginal contribution of i to m in v by $w_i(m, v) = v(m) - v(m_{-i}, 0)$. Second, since in a multi-choice game there might be players with more than two participation levels, it makes sense to consider also marginal contributions of bundles of highest levels of such players to the grand coalition m. For each $i \in N$ and $j \in \{1, \ldots, m_i - 1\}$, we define the marginal contribution $w_{ij^+}(m, v)$ of player i's levels which are higher than j to the grand coalition m in the game v by $w_{ij^+}(m, v) = v(m) - v(m_{-i}, j)$; $w_{i0^+}(m, v) = w_i(m, v)$. Finally, we introduce for each player $i \in N$ the notion of marginal contribution of each of his levels to the intermediate multi-choice coalition generated by that level given that all the other players participate at their maximal levels. For each $i \in N$ and $j \in M_i^+$ the marginal contribution of i to the coalition (m_{-i}, j) is defined by

$$w_{ij}(m, v) = v(m_{-i}, j) - v(m_{-i}, j-1).$$

We notice that, for each $i \in N$ the marginal contribution $w_i(m, v)$ of player i to m in v, is equal to the sum of all marginal contributions $w_{ij}(m, v)$ of individual levels

$j \in M_i^+$ of i to coalition m:

$$w_i(m,v) = \sum_{j=1}^{m_i} w_{ij}(m,v).$$

3 Characterizations of total multi-choice clan games

In this section we introduce a new class of multi-choice games, which we call multi-choice clan games. As in the traditional model of clan games (cf. Potters et al. (1989)), the set of players consists of two disjoint groups, a fixed (powerful) clan with "yes-or-no" choices and a group of (non-powerful) non-clan members, but in the multi-choice clan game non-clan members may have more options for cooperation. Specifically, each non-clan member can participate at any level in a given finite set, whereas each clan member can be either active or abstain from cooperation. However, the active status (i.e. participation level 1) for all clan members is a necessary condition for generating a positive reward for any coalition containing the clan and at least one non-clan member.

Let $N = (N\backslash C, C)$ be the set of players, where C stands for the clan and $N\backslash C$ for the group of non-clan members. For further use, we denote by $\mathcal{M}^{N,C}$ the set of multi-choice coalitions with player set N and fixed clan C. For each $s \in \mathcal{M}^{N,C}$ we denote its restrictions to $N\backslash C$ and C, by $s_{N\backslash C}$ and s_C, respectively. Note that, in a multi-choice clan game, the maximal participation profile is of the form $m = (m_{N\backslash C}, 1_C)$.

We also denote by $\mathcal{M}^{N,1_C}$ the set of multi-choice coalitions $s \in \mathcal{M}^{N,C}$ with $s_C = 1_C$, that is coalitions in which all clan members fully participate. We also use the notation $\mathcal{M}_+^{N,1_C} = \mathcal{M}^{N,1_C}\backslash\{0\}$. Multi-choice clan games are defined here by using the veto power of clan members, the monotonicity property of the characteristic function and a (level) union property regarding non-clan members' participation in multi-choice coalitions containing at least all clan members at participation level 1. Formally, a game $< N, (m_{N\backslash C}, 1_C), v >$ is a *multi-choice clan game* if the characteristic function $v : \mathcal{M}^{N,C} \to \mathbb{R}$ satisfies

(i) *Clan property:* $v(s) = 0$ if $s_C \neq 1_C$;

(ii) *Monotonicity property:* $v(s) \leq v(t)$ for all $s, t \in \mathcal{M}^{N,C}$ with $s \leq t$;

(iii) *(Level) Union property:* For each $s \in \mathcal{M}^{N,1_C}$

$$v(m) - v(s) \geq \sum_{i \in N\backslash C} w_{is_i^+}(m,v), \tag{3.1}$$

where for each $i \in N\backslash C$, $w_{is_i^+}(m,v)$ denotes the marginal contribution of the bundle of those levels of player i which are higher than the participation level of i in s.

Remark 3.1 The (level) union property is a natural extension for multi-choice games of the union property for traditional cooperative games. Note that the right side of (3.1) consists of two types of level-based contributions: full marginal contributions of those non-clan members with participation level 0 in s, and partial marginal contributions for all non-clan members $i \in \mathrm{car}(s_{N\setminus C})$, corresponding to the bundle of levels greater than s_i. In formula,

$$\sum_{i\in N\setminus C} w_{is_i^+}(m,v) = \sum_{i\notin \mathrm{car}(s_{N\setminus C})} w_{i0^+}(m,v) + \sum_{i\in \mathrm{car}(s_{N\setminus C})} w_{is_i^+}(m,v) =$$

$$= \sum_{i\notin \mathrm{car}(s_{N\setminus C})} w_i(m,v) + \sum_{i\in \mathrm{car}(s_{N\setminus C})} w_{is_i^+}(m,v).$$

In the sequel, we simply use v instead of $< N, (m_{N\setminus C}, 1_C), v >$. For further use, we denote by $MC_C^{N,m}$ the set of multi-choice games with a fixed non-empty and finite set of players N, fixed non-empty clan C, and maximal participation profile m. We notice that $MC_C^{N,m}$ is a convex cone in $MC^{N,m}$, that is for all $v, w \in MC_C^{N,m}$ and for all $p, q \in \mathbb{R}_+$, $pv + qw \in MC_C^{N,m}$, where $\mathbb{R}_+$ denotes the set of non-negative real numbers.

The next theorem gives an explicit description of the core of a multi-choice clan game.

Theorem 3.1 Let $v \in MC_C^{N,m}$. Then

$$C(v) = \{x : M \to \mathbb{R}_+ | X(m) = v(m);\ \sum_{k=j}^{m_i} x_{ik} \le v(m) - v(m_{-i}, j-1), \forall i \in N\setminus C,\ j \in M_i^+\}.$$

Proof. We denote by $B(v)$ the set in the right-hand side of the above equality. We prove first that $C(v) \subset B(v)$. Let $v \in MC_C^{N,m}$ and let $x \in C(v)$. Note that non-negativity of x follows from the clan property and the monotonicity property; so, $x_{ij} \ge 0$ for all $i \in N$ and $j \in M_i^+$. Clearly, the efficiency condition holds true. To prove the upper boundness of the accumulated payoffs for each bundle of highest levels of each non-clan member, we note first that the payoff of each multi-choice coalition $\tilde{s} = (m_{-i}, j-1) \in \mathcal{M}^{N,1_C}$, $i \in N\setminus C$, $j \in \{1, \ldots, m_i\}$, can be expressed as

$$X(\tilde{s}) = X(m) - \sum_{k=j}^{m_i} x_{ik} = v(m) - \sum_{k=j}^{m_i} x_{ik},$$

where the second equality follows from the efficiency of x.

Now, the stability condition $X(\tilde{s}) \geq v(\tilde{s})$ implies that $v(m) - \sum_{k=j}^{m_i} x_{ik} \geq v(m_{-i}, j-1)$, or, equivalently, $\sum_{k=j}^{m_i} x_{ik} \leq v(m) - v(m_{-i}, j-1)$. Hence, $x \in B(v)$.

Now, we prove the converse inclusion. Let $x \in B(v)$. The level-increase rationality of x follows from the clan property, since $x_{ij} \geq 0 = v(0_{-i}, j) - v(0_{-i}, j-1)$ for all $i \in N$ and $j \in M_i^+$. We only need to prove that $X(s) \geq v(s)$ for each $s \in \mathcal{M}^{N,C}$.

Clearly, for each $s \in \mathcal{M}^{N,C}$ with $s_C \neq 1_C$, we have, by the clan property, $X(s) = \sum_{i \in N} \sum_{j=1}^{s_i} x_{ij} \geq 0 = v(s)$.

We focus now on stability conditions for multi-choice coalitions $s \in \mathcal{M}^{N,C}$ with $s_C = 1_C$.

By the (level) union property, we obtain

$$v(m) - v(s) \geq \sum_{i \in N \setminus C} w_{is_i^+}(m, v).$$

Further,

$$X(s) + \sum_{i \in N \setminus C} \sum_{k=s_i+1}^{m_i} x_{ik} - v(s) = X(m) - v(s) = v(m) - v(s) \geq \sum_{i \in N \setminus C} w_{is_i^+}(m, v),$$

implying that

$$X(s) \geq v(s) + \sum_{i \in N \setminus C} (w_{is_i^+}(m, v) - \sum_{k=s_i+1}^{m_i} x_{ik}).$$

Note that $w_{is_i^+}(m, v) = v(m) - v(m_{-i}, s_i) \geq \sum_{k=s_i+1}^{m_i} x_{ik}$, for each $i \in N \setminus C$, since $x \in B(v)$. Therefore, we obtain $w_{is_i^+}(m, v) - \sum_{k=s_i+1}^{m_i} x_{ik} \geq 0$, for each $i \in N \setminus C$. Hence, $X(s) \geq v(s)$ for each $s \in \mathcal{M}^{N,1_C}$. □

Corollary 3.1 Let $v \in MC_C^{N,m}$ and let $t \in \mathcal{M}^{N,1_C}$. If the subgame v_t is a clan game, then its core is described by

$$C(v_t) = \{x : M^t \to \mathbb{R}_+ | X(t) = v(t); \sum_{k=j}^{t_i} x_{ik} \leq v(t) - v(t_{-i}, j-1),\ \forall i \in \mathrm{car}(t_{N \setminus C}), j \in M_i^t\},$$

where $M_i^t = \{1, \ldots, t_i\}$ and $M^t = \{(i, j) | i \in \mathrm{car}(t),\ j \in M_i^t\}$.

Proof. It follows straightforwardly from Theorem 3.1, by taking into account that $t = (t_{N \setminus C}, 1_C)$ is the "grand coalition" in the subgame v_t. □

Remark 3.2 The definition of the set $B(v)$ contains $\sum_{i\in N\setminus C} m_i$ inequalities which are the multi-choice version of the conditions $x_i \leq M_i(N,v) = v(N) - v(N\setminus\{i\})$, $i \in N\setminus C$, for traditional clan games.

Remark 3.3 The monotonicity property and the clan property imply together the non-negativity of the characteristic function, i.e. $v(s) \geq 0$ for each $s \in \mathcal{M}^{N,C}$, as well as the non-negativity of all types of marginal contributions defined in Section 2. In particular, $w_i(m,v) \geq 0$ for each $i \in N\setminus C$, and it holds true $w_{ij}(m,v) \geq 0$, $w_{ij^+}(m,v) \geq 0$, for each $i \in N\setminus C$ and for each $j \in M_i^+$. Note that the bundle of conditions $w_{ij^+}(m,v) \geq 0$, $j \in M_i^+$, for each $i \in N\setminus C$, is the multi-choice version of the condition $M_i(N,v) \geq 0$ for traditional clan games.

Remark 3.4 Let $v \in MC_C^{N,m}$. For each $s \in \mathcal{M}^{N,C}$ it holds

$$v(m) - v(s) \geq \sum_{i\notin \text{car}(s_{N\setminus C})} w_i(m,v).$$

Proof. By Remark 3.1

$$\sum_{i\in N\setminus C} w_{is_i^+}(m,v) = \sum_{i\notin \text{car}(s_{N\setminus C})} w_i(m,v) + \sum_{i\in \text{car}(s_{N\setminus C})} w_{is_i^+}(m,v).$$

Now, we use the (level) union property together with the non-negativity of $w_{is_i^+}(m,v)$ for each $i \in \text{car}(s_{N\setminus C})$ (see Remark 3.3). □

Multi-choice clan games for which the clan consists of one player are called multi-choice big boss games; we notice that in case $m_{N\setminus C} = 1_{N\setminus C}$ these games are big boss games in the terminology of Branzei, Tijs and Timmer (2001). The model of a multi-choice clan game where the clan consists of at least two members is an extension of the model of a clan game (cf. Voorneveld, Tijs and Grahn (2002)).

In the rest of this paper, we focus on total clan games with multi-choice coalitions. A game $v \in MC_C^{N,m}$ is a *multi-choice total clan game* with clan C if all its subgames v_t, $t \in \mathcal{M}^{N,1_C}$, are clan games with clan C. We denote the set of all multi-choice total clan games $v \in MC_C^{N,m}$ by $TMC_C^{N,m}$.

The next theorem yields a characterization of multi-choice total clan games.

Theorem 3.2 Let $< N, m, v >$ be a multi-choice game and let $C \in 2^N\setminus\{\emptyset\}$ with $m_C = 1_C$. The following assertions are equivalent:

(i) $< N, m, v >$ is a total clan game with clan C;

(ii) $< N, m, v >$ is monotonic, each player $i \in C$ is a veto player, and for all $s, t \in \mathcal{M}^{N,1_C}$ with $s \leq t$

$$v(t) - v(s) \geq \sum_{i \in \mathrm{car}(t_{N \setminus C})} w_{is_i^+}(t, v); \tag{3.2}$$

(iii) $< N, m, v >$ is monotonic, each player $i \in C$ is a veto player, and for each $i \in \mathrm{car}(s_{N \setminus C})$ and for all $s, t \in \mathcal{M}^{N,1_C}$ with $s \leq t$ and $s_i = t_i$:

$$v(t) - v(t - e^i) \leq v(s) - v(s - e^i). \tag{3.3}$$

Proof. (i)$\leftrightarrow$(ii): Relation (3.2) simply writes out the (level) union property of multi-choice subgames. In the sequel, we refer to relation (3.2) as *the total (level) union property* of v.

(ii)$\rightarrow$(iii): It suffices to prove that (3.2) implies (3.3). We note that inequality (3.3) expresses a *total concavity property* of v which reflects the fact that the same one-unit level decrease of a non-clan member in coalitions containing at least all clan members at participation level 1, could be more beneficial in smaller such coalitions than in larger ones. Let $s \in \mathcal{M}^{N,1_C}$ and let $i \in \mathrm{car}(s_{N \setminus C})$. Consider the coalition $s + e^k$ obtained from s when one of the other non-clan members, namely k, increases his participation with one unit. We prove first

$$w_{is_i}(s, v) \geq w_{is_i}(s + e^k, v). \tag{3.4}$$

$$\begin{aligned} w_{is_i}(s, v) + w_{k,s_k+1}(s + e^k, v) &= (v(s) - v(s_{-i}, s_i - 1)) + \\ &\quad + (v(s + e^k) - v(s_{-k}, s_k)) \\ &= v(s + e^k) - v(s_{-i}, s_i - 1). \end{aligned} \tag{3.5}$$

The total union property (with $s + e^k$ in the role of t and $(s_{-i}, s_i - 1)$ in the role of s) yields

$$v(s + e^k) - v(s_{-i}, s_i - 1) \geq w_{is_i}(s + e^k, v) + w_{k,s_k+1}(s + e^k, v). \tag{3.6}$$

From (3.5) and (3.6) we conclude that (3.4) holds true. Denote by $\{i_1, \ldots, i_q\}$ the set of levels which are involved in t but not in s. Repeated application of (3.4) yields

$$w_{is_i}(s, v) \geq w_{is_i}(s + e^{i_1}, v) \geq \ldots \geq w_{is_i}(s + (e^{i_1} + \ldots + e^{i_q})) = w_{is_i}(t, v).$$

Hence, the total concavity property (3.3) holds true.

(iii)→(ii): We simply prove that (3.3) implies (3.2). Let $s, t \in \mathcal{M}^{N,1_C}$ with $s \leq t$. Denote by $\{i_1, \ldots, i_q\}$ the set of levels which are involved in t but not in s.

Then

$$v(t) - v(s) = \sum_{i \in \text{car}(t_{N \setminus C})} \sum_{k=1}^{q} w_{i,i_k}(s + (e^{i_1} + \ldots + e^{i_k}), v) \geq$$

$$\geq \sum_{i \in \text{car}(t_{N \setminus C})} \sum_{k=1}^{q} w_{i,i_k}(t, v) = \sum_{i \in \text{car}(t_{N \setminus C})} w_{is_i^+}(t, v),$$

where the inequality follows from the total concavity property (cf. (3.6)). Hence, the total union property (3.2) holds true. □

Remark 3.5 Let $v \in TCM_C^{N,m}$ and let $s, t \in \mathcal{M}^{N,1_C}$ such that $s \leq t$. Then, for each $i \in \text{car}(s_{N \setminus C})$ such that $s_i = t_i$, it holds

$$w_i(t, v) \leq w_i(s, v). \tag{3.7}$$

Proof. Let $s, t \in \mathcal{M}^{N,1_C}$ with $s \leq t$, and let $i \in \text{car}(s_{N \setminus C})$ such that $s_i = t_i$. Repeated application of the total concavity property (relation (3.3)), yields

$$v(t_{-i}, s_i) - v(t_{-i}, s_i - 1) \leq v(s_{-i}, s_i) - v(s_{-i}, s_i - 1),$$
$$\vdots$$
$$v(t_{-i}, 1) - v(t_{-i}, 0) \leq v(s_{-i}, 1) - v(s_{-i}, 0).$$

By summing these inequalities, we obtain $v(t) - v(t_{-i}, 0) \leq v(s) - v(s_{-i}, 0)$. Hence, (3.7) holds. □

4 Bi-monotonic allocation schemes for multi-choice clan games

Inspired by Sprumont (1990) (see also Hokari (2000)), who introduced and studied the interesting notion of population monotonic scheme (pmas) for traditional cooperative games, Branzei, Tijs and Zarzuelo (2007b) introduced for multi-choice games the notion of level-increase monotonic allocation scheme (limas) and proved that convexity is a sufficient condition for the existence of a limas. It turned out that for convex multi-choice games each element of the Weber set (cf. Nouweland et al. (1995)) is extendable to a limas. Also inspired by Sprumont (1990), Branzei, Tijs and Timmer (2001) introduced the notion of bi-monotonic allocation scheme for a total big boss game, and Voorneveld, Tijs and Grahn (2002) extended this notion for total clan games.

Recall that a bi-mas for a traditional total clan game $< N, v >$ with fixed clan $C \neq \emptyset$ is an allocation scheme $a = [a_{iS}]_{i\in S, S\supset C}$ such that

(i) *stability condition:* $(a_{iS})_{i\in S} \in C(v_S)$ for each $S \in 2^N$ with $S \supset C$;

(ii) *bi-monotonicity condition:* For all $S, T \in 2^N$ with $S, T \supset C$, $S \subset T$ we have:

- $a_{iS} \leq a_{iT}$, for each $i \in S \cap C$, and
- $a_{iS} \geq a_{iT}$, for each $i \in S \backslash C$.

Note that in coalitions containing all clan members each clan member is better off in larger coalitions than in smaller ones, whereas each non-clan member is worse off when more other non-clan members join him and the clan members.

Since multi-choice total clan games are natural extensions of traditional total clan games, we introduce here the notion of bi-(level-increase) monotonic allocation scheme (bi-limas) for multi-choice total clan games.

Definition 4.1 Let $v \in TMC_C^{N,m}$ and let $t \in \mathcal{M}^{N,1_C}$. A scheme, $[a_{ij}^t]_{i\in N, j\in M_i^t}^{t\in\mathcal{M}^{N,1_C}}$ where $M_i^t = \{1, \ldots, t_i\}$, is called a *bi-(level-increase) monotonic allocation scheme (bi-limas)* if

(i) *stability condition:* $a^t \in C(v_t)$ for all $t \in \mathcal{M}^{N,1_C}$; and

(ii) *bi-monotonicity w.r.t. one-unit level-increases:* For all $s, t \in \mathcal{M}^{N,1_C}$ with $s \leq t$, we have:

- $a_{i1}^s \leq a_{i1}^t$, for each $i \in C$;
- $a_{ij}^s \geq a_{ij}^t$, for each $i \in \text{car}(s_{N\backslash C})$ and each $j \in \{1, \ldots, s_i\}$.

Note that such a bi-limas is a defective $|\mathcal{M}_+^{N,1_C}| \times |M^+|$-matrix, whose rows correspond to multi-choice coalitions where all clan members are active, and whose columns correspond to elements in M^+ arranged according with the natural ordering for players, and for successive levels within each player. In each row t there is a core element of the multi-choice subgame v_t, with "$*$" for all components x_{ij}, with $i \in N\backslash C$ and $j \in \{t_i + 1, \ldots, m_i\}$. The (one-unit level-increase) monotonicity condition implies that, if the scheme is used as regulator for the (level) payoff distributions, clan members are paid more in larger coalitions with all clan members active than in smaller ones, whereas each non-clan member is weakly worst off in larger coalitions with all clan members active than in smaller such coalitions.

We study this kind of bi-monotonic allocation schemes by means of suitably defined compensation-sharing rules on the class of multi-choice (total) clan games.

Definition 4.2 Let $N\backslash C = \{1,\dots,q\}$, $C = \{q+1,\dots,n\}$, $\alpha \in [0,1]^q$ and $\beta \in \Delta(C) = \Delta(\{q+1,\dots,n\}) = \{z_+^{n-q} | \sum_{i=q+1}^{n} z_i = 1\}$. The *compensation-sharing rule* based on α and β, $\psi^{\alpha,\beta} : MC_C^{N,m} \to \mathbb{R}^{M^+}$, is defined by

$$\psi_{ij}^{\alpha,\beta}(v) = \begin{cases} \alpha_i(v(m) - v(m_{-i},0)) & , \ i \in N\backslash C,\ j=1; \\ 0 & , \ i \in N\backslash C,\ j \in M_i^+\backslash\{1\}; \\ \beta_i[v(m) - \sum_{j\in N\backslash C} \alpha_j(v(m)-v(m_{-j},0))] & , \ i \in C,\ j=1 \end{cases} \quad (4.1)$$

for each $i \in N$ and $j \in M_i^+$.

The i-th coordinate of the compensation vector α indicates that level 1 of non-clan member i gets as payoff, in view of its decisive role for multi-choice cooperation, the part $\alpha_i w_i(m,v)$ of the marginal contribution of this player to the grand coalition m. Then, the remainder, $v(m) - \sum_{j\in N\backslash C} \alpha_j w_j(m,v)$, is distributed over the clan members. For each clan member i, the i-th coordinate β_i of the sharing vector β determines the share $\beta_i[v(m) - \sum_{j\in N\backslash C} \alpha_j w_j(m,v)]$.

Theorem 4.1 Let $MC_C^{N,m}$ be the cone of multi-choice clan games with clan C. Then,

(i) $\psi^{\alpha,\beta}$ is additive, for each $\alpha \in [0,1]^{N\backslash C}$ and each $\beta \in \Delta(C)$; and

(ii) $\psi^{\alpha,\beta}$ is stable, that is $\psi^{\alpha,\beta}(v) \in C(v)$, for each $v \in MC_C^{N,m}$.

Proof.

(i) Let $v \in [0,1]^{N\backslash C}$ and $\beta \in \Delta(C)$. For all $v, w \in MC_C^{N,m}$ and all $p, q \in \mathbb{R}_+$, where $\mathbb{R}_+$ stands for the set of non-negative real numbers, it holds

$$\psi^{\alpha,\beta}(pv+qw) = p\psi^{\alpha,\beta}(v) + q\psi^{\alpha,\beta}(w).$$

Hence, $\psi^{\alpha,\beta}$ is additive on the cone of multi-choice clan games.

(ii) Let $v \in MC_C^{N,m}$. From Theorem 3.1 and $\sum_{i\in C} \beta_i = 1$ we obtain that $\psi^{\alpha,\beta}(v) \in C(v)$.

□

In Theorem 4.2 we prove that compensation-sharing rules defined by (4.1) play a key role for the existence of bi-limas for a subclass of multi-choice total clan games. But, first we need to establish some useful results.

Lemma 4.1 Let $v \in TMC_C^{N,m}$ and $s, t \in \mathcal{M}^{N,1_C}$ with $s \leq t$. Then,

$$v(t) - v(s) \geq \sum_{i \in \text{car}((t-s)_{N\setminus C})} w_i(t, v).$$

Proof. Note that $s \leq t$ implies $\text{car}(s_{N\setminus C}) \subset \text{car}(t_{N\setminus C})$. We denote $\text{car}(t_{N\setminus C}) - \text{car}(s_{N\setminus C})$ by $\text{car}((t-s)_{N\setminus C})$. From the total (level) union property we obtain

$$v(t) - v(s) \geq \sum_{i \in \text{car}(t_{N\setminus C})} w_{is_i^+}(t, v) = \sum_{i \in \text{car}(s_{N\setminus C})} w_{is_i^+}(t, v) + \sum_{i \in \text{car}((t-s)_{N\setminus C})} w_{i0^+}(t, v) =$$

$$= \sum_{i \in \text{car}(s_{N\setminus C})} (v(t) - v(t_{-i}, s_i)) + \sum_{i \in \text{car}((t-s)_{N\setminus C})} w_{is_i^+}(t, v) \geq \sum_{i \in \text{car}((t-s)_{N\setminus C})} w_{is_i^+}(t, v),$$

where the last inequality holds true because, by monotonicity of v, $v(t) - v(t_{-i}, s_i) \geq 0$ for each $i \in \text{car}(s_{N\setminus C})$.

Lemma 4.2 Let $v \in TMC_C^{N,m}$ and $s, t \in \mathcal{M}^{N,1_C}$ such that $s \leq t$. Then

$$v(t) - v(s) - \sum_{i \in \text{car}((t-s)_{N\setminus C})} \alpha_i w_i(t, v) \geq 0.$$

Proof. First, by non-negativity of α_i and $w_i(t, v)$, and since $\alpha_i \leq 1$, for each $i \in N\setminus C$ and $t \in \mathcal{M}^{N,1_C}$, we have $\alpha_i w_i(t, v) \leq w_i(t, v)$, implying that

$$- \sum_{i \in \text{car}((t-s)_{N\setminus C})} \alpha_i w_i(t, v) \geq - \sum_{i \in \text{car}((t-s)_{N\setminus C})} w_i(t, v).$$

Second, by Lemma 4.1 we have

$$v(t) - v(s) - \sum_{i \in \text{car}((t-s)_{N\setminus C})} w_i(t, v) \geq 0.$$

Therefore,

$$v(t) - v(s) - \sum_{i \in \text{car}((t-s)_{N\setminus C})} \alpha_i w_i(t, v) \geq v(t) - v(s) - \sum_{i \in \text{car}((t-s)_{N\setminus C})} w_i(t, v) \geq 0.$$

□

It turns out that for a subclass of multi-choice total-clan games compensation-sharing rules $\psi^{\alpha,\beta}$ with $\alpha \in [0,1]^{N\setminus C}$ and $\beta \in \Delta(C)$ generate bi-(level-increase) monotonic allocation schemes.

Theorem 4.2 Let $v \in TMC_C^{N,m}$ be such that, for each $s,t \in \mathcal{M}^{N,1_C}$ with $s \leq t$ and each $i \in \text{car}(s_{N\setminus C})$,

$$v(t) - v(t_{-i},0) \leq v(s) - v(s_{-i},0). \tag{4.2}$$

Then, for each $\alpha \in [0,1]^{N\setminus C}$ and $\beta \in \Delta(C)$ the compensation-sharing rule $\psi^{\alpha,\beta}$, defined by (4.1), generates a bi-limas for v, namely

$$[\psi_{ij}^{\alpha,\beta}(v_t)]_{i\in N, j\in M_i^t}^{t\in\mathcal{M}^{N,1_C}}.$$

Proof. Let $\alpha \in [0,1]^{N\setminus C}$ and $\beta \in \Delta(C)$. For each $t \in \mathcal{M}^{N,1_C}$, in the subgame v_t, the α-based compensation (regardless of β) for each non-clan member $i \in \text{car}(t_{N\setminus C})$, $\alpha_i w_i(t,v)$, is fully assigned as payoff to level 1 of that player. So, the α-based compensation for each other level of each non-clan member $i \in \text{car}(t_{N\setminus C})$ is simply equal to 0. The amount left for the clan, $v(t) - \sum_{j\in\text{car}(t_{N\setminus C})} \alpha_j w_j(t,v)$, is shared based on β. For each clan member $i \in C$, the β-based share in the subgame v_t is $\beta_i[v(t) - \sum_{j\in\text{car}(t_{N\setminus C})} \alpha_j w_j(t,v)]$, with $\sum_{i\in C} \beta_i = 1$.

Since v_t is a clan game, by Theorem 4.1(ii) and Corollary 3.1, we have $\psi^{\alpha,\beta}(v_t) \in C(v_t)$.

Now, we focus on the bi-monotonicity property. Let $s,t \in \mathcal{M}^{N,1_C}$ with $s \leq t$. First, we prove that, for each player $i \in \text{car}(s_{N\setminus C})$ and for each level $j \in \{1,\ldots,s_i\}$ the compensation $\psi_{ij}^{\alpha,\beta}(v_s)$ is weakly better than the compensation $\psi_{ij}^{\alpha,\beta}(v_t)$. Clearly, $\psi_{ik}^{\alpha,\beta}(v_s) = 0 = \psi_{ik}^{\alpha,\beta}(v_t)$ for $k \in \{2,\ldots,s_i\}$. Further, by (4.2) and non-negativity of α_i, we obtain $\psi_{i1}^{\alpha,\beta}(v_s) = \alpha_i w_i(s,v) \geq \alpha_i w_i(t,v) = \psi_{i1}^{\alpha,\beta}(v_s)$, for each $i \in \text{car}(s_{N\setminus C})$.

In the sequel, we cope with the monotonicity condition regarding shares of clan members. Denote by $R_\alpha(v_t)$ the α-based remainder for the clan members in the "grand coalition" $(t_{N\setminus C}, 1_C)$ of the multi-choice clan game v_t. We prove first that, for each $s,t \in \mathcal{M}^{N,1_C}$ with $s \leq t$,

$$R_\alpha(v_t) \geq R_\alpha(v_s). \tag{4.3}$$

$$R_\alpha(v_t) - R_\alpha(v_s) = (v(t) - \sum_{i\in \text{car}(t_{N\setminus C})} \alpha_i w_i(t,v)) - (v(s) - \sum_{i\in \text{car}(s_{N\setminus C})} \alpha_i w_i(s,v))$$

$$= (v(t) - v(s) - \sum_{i\in \text{car}((t-s)_{N\setminus C})} \alpha_i w_i(t,v)) + \sum_{i\in \text{car}(s_{N\setminus C})} \alpha_i (w_i(s,v) - w_i(t,v))$$

$$\geq v(t) - v(s) - \sum_{i\in \text{car}((t-s)_{N\setminus C})} \alpha_i w_i(t,v) \geq 0,$$

where the first inequality follows from (4.2) and the non-negativity of α_i, and the last inequality follows from Lemma 4.2.

Hence, relation (4.3) holds true. Now, from the non-negativity of β_i and (4.3) we obtain $\psi_{i1}^{\alpha,\beta}(v_t) \geq \psi_{i1}^{\alpha,\beta}(v_s)$, for each $i \in C$. □

Example 4.1 Consider the multi-choice total clan game with $N\setminus C = \{1,2\}$, $C = \{3,4\}$ and $m = (2,2;1,1)$ where $v(s,0) = v(0,s) = 2$, for $s \in \{1,2\}$, and $v(s,t) = 3$, for all $s,t > 0$. We note that relation (4.2) holds true. Theorem 4.2 guarantees the existence of a bi-limas via compensation-sharing rules $\psi^{\alpha,\beta}(v)$ with $\alpha \in [0,1]^2$ and $\beta \in \Delta(C)$.

For arbitrary multi-choice total clan games bi-limas does not necessarily exist, even if we consider a weaker version of bi-limas where the (level) monotonicity condition regarding the non-clan members is defined by:

For all $s,t \in \mathcal{M}^{N,1_C}$ with $s \leq t$, we have $a_{ij}^s \geq a_{ij}^t$ for each $i \in \text{car}(s_{N\setminus C})$ such that $s_i = t_i$ and each $j \in \{1,\ldots,s_i\}$.

Then, by Remark 3.5, the (level) monotonicity condition for non-clan members as defined above holds true. However, the (level) monotonicity condition for clan members does not necessarily hold, because the remainder for the clan in larger subgames v_t might be less than the remainder for the clan in smaller subgames v_s. We illustrate this issue in the following example.

Example 4.2 Consider the multi-choice total clan game with $N\setminus C = \{1,2\}$, $C = \{3,4\}$ and $m = (2,2;1,1)$ where $v(1,0) = v(0,1) = 2, v(2,0) = v(0,2) = 3, v(1,1) = 4, v(1,2) = v(2,1) = 5$, and $v(2,2) = 6$.

Note that this game does not belong to the subclass of multi-choice total clan games considered in Theorem 4.2 because, for $s = (1,0)$ and $t = (2,0)$, we should have $v(2,0) - v(0,0) \leq v(1,0) - v(0,0)$, however $3 > 2$. Hence, condition (4.2) does not hold.

The scheme with $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{3}, \beta_3 = \frac{1}{3}$, and $\beta_4 = \frac{2}{3}$ does not satisfy the monotonicity condition for player 2 (a non-clan member) because, for $s = (0,1)$ and $t = (0,2)$, $a_{21}^{(0,1)} = \frac{2}{3} < 1 = a_{21}^{(0,2)}(v_t)$, consequently, it is not a bi-limas.

Let $v \in TMC_C^{N,m}$. A (level) payoff vector $b \in C(v)$ is *bi-limas extendable* if there exists a bi-limas $[a_{ij}^t]_{j\in N, j\in M_i^t}^{t\in\mathcal{M}^{N,1_C}}$ such that $b_{ij} = a_{ij}^m$, for each $i \in N$ and $j \in M_i^+$.

Theorem 4.3 Let $v \in TMC_C^{N,m}$ be such that inequality (4.2) holds, for all $s, t \in \mathcal{M}^{N,1_C}$ such that $s \leq t$ and for each $i \in \text{car}(s_{N\backslash C})$. Then, there exist (level) payoff vectors $b \in C(v)$ which are extendable to a bi-limas.

Proof. By Theorem 3.1 each (level) payoff vector $b \in C(v)$ is of the form

$$b_{ij} = \begin{cases} \alpha_{im_i}(v(m) - v(m_{-i}, m_i - 1)), \ i \in N\backslash C, \ j = m_i; \\ \alpha_{ij}(v(m_{-i}, j) - v(m_{-i}, j-1)) - \alpha_{i,j+1}(v(m_{-i}, j+1) - v(m_{-i}, j)), \\ \qquad\qquad\qquad\qquad\qquad\qquad i \in N\backslash C, \ j \in M_i^+\backslash\{m_i\}; \\ \beta_i[v(m) - \sum_{i\in N\backslash C} \alpha_{i1}(v(m) - v(m_{-i}, 0))], \ i \in C, \ j = 1. \end{cases}$$

where $\alpha_{ij} \in [0,1]$, for each $i \in N\backslash C$ and $j \in M_i^+$, and $\beta \in \Delta(C)$, i.e. $\beta_i \geq 0$, for each $i \in C$ and $\sum_{i\in C} \beta_i = 1$.

Consider the particular matrix $\alpha = (\alpha_{ij})_{i\in N\backslash C, j\in M_i^+}$ with $\alpha_{ij} = 0$, for each $i \in N\backslash C$ and each $j \in M_i^+\backslash\{1\}$. Denote $(\alpha_{i1})_{i\in N\backslash C}$ by $\tilde{\alpha}$. Consider a core element $\tilde{b} \in C(v)$ corresponding to $\tilde{\alpha}$ and β. Note that $\tilde{b} = \psi^{\alpha,\beta}(v)$, where $\psi^{\alpha,\beta}(v)$ is defined by (4.1). Define for each $s \in \mathcal{M}^{N,1_C}$, each $i \in \text{car}(s_{N\backslash C})$ and each $j \in M_i^s$:

$$a_{ij}^s = \begin{cases} \alpha_{i1}(v(s) - v(s_{-i}, 0)) & , \ i \in \text{car}(s_{N\backslash C}), \ j = 1; \\ 0 & , \ i \in \text{car}(s_{N\backslash C}), \ j \in \{2, \ldots, s_i\} \\ \beta_i[v(s) - \sum_{k\in N\backslash C} \alpha_{k1}(v(s) - v(s_{-k}, 0))] & , \ i \in C, \ j = 1. \end{cases} \tag{4.6}$$

By Theorem 3.1, $[a_{ij}^s]_{i\in N\backslash C, j\in M_i^+}^{s\in\mathcal{M}^{N,1_C}}$ is a bi-limas. Now, note that $\tilde{b}_{ij} = a_{ij}^m$ for each $i \in N\backslash C$ and each $j \in M_i^+$. Hence, $\tilde{b}$ is bi-limas extendable. □

Clearly, in case $m_{N\backslash C} = 1_{N\backslash C}$ a bi-limas coincides with a bi-mas; the subclass of multi-choice total clan games considered in Theorem 4.2 coincides with the class of traditional total clan games; and Theorem 4.3 says (as Theorem 3 in Voorneveld, Tijs and Grahn, 2002) that each core element of such game is extendable to a bi-mas.

5 Concluding remarks

In this paper the class of multi-choice total clan games is introduced and characterized. An explicit description of the core of a multi-choice clan game plays a key role in defining suitable compensation-sharing rules on this class of games. The notion of bi-(level-increase) monotonic allocation scheme (bi-limas) is introduced, and it is shown that for a subclass of multi-choice total clan games particular allocations in the core of each game in that class are extendable to such bi-monotonic allocation schemes. This is the multi-choice version of the extendability of each core element of a traditional total clan game to a bi-mas (cf. Voorneveld, Tijs and Grahn (2002)) and the extendability of each core element of a fuzzy total clan game of a bi-pamas (cf. Tijs et al. (2004)).

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