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Sequential incompleteness and dynamic suboptimality in stochastic OLG economies with production

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Sequential incompleteness and dynamic suboptimality in stochastic OLG economies with production*

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Abstract

I study a stochastic overlapping generations model with production and three-period-lived agents. Agents trade bonds and risky capital. Unlike the two-period model, I show that a stationary equilibrium in which prices and allocations depend solely on the aggregate capital stock and the current shock does not exist. The recursive equilibrium becomes the relevant equilibrium concept.

For the recursive formulation of the model, markets are sequentially incomplete and hence I show that there is room for Pareto improvements in terms of intergenerational risk sharing. Finally, I examine whether the introduction of capital income taxation improves the allocation of risk.

Keywords: Overlapping generations, uncertainty, capital income taxation

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1 Introduction

In recent work, Henriksen and Spear [11] show that market incompleteness arises endogenously in an otherwise standard stochastic overlapping generations pure exchange economy, solely due to the structure of the financial markets and, in particular, the presence of a productive infinitely-lived asset in positive net supply. Market incompleteness leads to suboptimal risk sharing, which may call for government intervention. Actually they show via simulations that confiscating the dividends via taxes and distributing the proceeds to the old via lump sum transfers generates a Pareto improvement.

In this paper I show that endogenous market incompleteness is robust to the introduction of production and capital accumulation, provided that agents live at least three periods. As in Henriksen and Spear [11], overlapping generations (OLG) models with three-period-lived agents require that some portfolio and consumption decisions must be taken after the resolution of uncertainty (when middle-aged), which generates wealth effects that eventually lead to suboptimal intergenerational risk sharing. Then I show that if I tax capital income and redistribute the proceeds via lump sum transfers, then there is a Pareto improvement over the laissez-faire equilibrium.

`Related literature.` In deterministic OLG models, Kehoe and Levine [12] argue that if in the steady state the gross interest rate is greater than or equal to one (the Cass-Balasko-Shell criterion), then the stationary equilibrium is dynamically efficient, otherwise the equilibrium is inefficient. However, in stochastic OLG models with incomplete markets, there are two types of deviations from Pareto optimality. Bloise and Calciano [3] argue that, on the one hand, the OLG structure without transversality at infinitum might be responsible for capital overaccumulation or dynamic inefficiency as described above and, on the other hand, market incompleteness may inhibit full risk sharing.

In this paper I conveniently assume that agents do not accumulate too much capital in a competitive equilibrium, and hence I isolate the relationship between Pareto suboptimality and intergenerational risk sharing. In order to accomplish this, I study economies that are dynamically efficient. A similar route has been followed by Krueger and Kubler [15] and Henriksen and Spear [11].

Even though Henriksen and Spear [11] do not explicitly acknowledge this separation, one result in Bloise and Calciano [3, Lemma 3] guarantees that all equilibrium allocations that are considered by Henriksen and Spear [11] are not dynamically inefficient. In this paper I will use a similar result contained in Krueger and Kubler [15, Proposition 1], which states that if a condition tantamount to the Cass-Balasko-Shell criterion is satisfied, then the economy is not inefficient.

Besides Henriksen and Spear [11], other papers examine government interventions in stochastic OLG models with incomplete markets. Krueger and Kubler [13] find that the introduction of an unfunded social security system leads to a welfare-improving consumption allocation in the Pareto sense, because the policy reform enhances the intergenerational risk-sharing of imperfectly correlated shocks to individual endowments. However, Krueger and Kubler [15] argue that this result is no longer valid in economies with production, mainly because the policy reform reduces the aggregate capital stock.

The remainder of the paper is organized as follows. In Section 2 I lay out the three-period-lived OLG model. Section 3 shows some basic results related to the competitive equilibrium in the model. Section 4 presents the recursive representation of the equilibrium. Section 5 shows the main theoretical results on the optimality of competitive equilibrium in the model. Section 6 provides the numerical simulations and shows the Pareto improvements associated with government interventions. Finally, Section 7 concludes.

2 The model

I study a closed, stochastic overlapping generations economy with production populated by three-period-lived individuals. Time is discrete and has neither beginning nor end and hence time periods are labeled by the subscript $t = 0, \pm 1, \pm 2$, and so on. There is one individual in each cohort and no population growth. All consumers are identical, except for the date of birth, and because of this similarity I will sometimes argue there is a representative consumer. The relevant features of the environment are provided as follows.

Aggregate uncertainty

The uncertainty in the model takes the form of a simple independent and identically distributed stochastic process of a two-point support which I denote by $s \in \{h, l\}$ or, alternatively, the state of the economy is either good or bad. The probability of state h occurring is given by $0 < \pi^h < 1$. Aggregate uncertainty could be represented by an event tree, whose nodes are histories of the exogenous aggregate shock $S_t = \{s_0, s_1, \ldots, s_t\}$, where s_0 could be interpreted as the root of the tree. The generalization to a Markovian process with finite support and well-defined transition matrix Π is straightforward.

Securities

Agents in this economy can use one unit of the aggregate consumption good today to obtain one unit of the capital good next period. I will denote the investment into this technology by a_i, for $i = y, m$. In this paper, I will not assume that $a_i \geq 0$, as agents are allowed to borrow against future income. Furthermore, agents may trade a riskless bond b_i, for $i = y, m$, which pays one unit of income next period for sure and is in zero net supply. As is standard in overlapping generations models, agents will use these securities to redistribute income over time and to alter their exposure to risk. For later use, let r and q represent the gross return on investment and the price of risk-free bonds, respectively.

Firms

There is a representative firm that uses labor L and capital K in each period to produce the aggregate consumption good according to a constant returns to scale production function $G(K, L, s)$. As in Krueger and Kubler [14], I assume the following parametric form for the production function

$$G(K, L, s) = \eta(s) F(K, L) + K(1 - \delta)$$

where $\eta(\cdot)$ is the stochastic shock to productivity such that $\eta(h) > \eta(l) > 0$, δ is the deterministic depreciation rate and $F(\cdot, \cdot)$ satisfies standard properties.

Given the stock of capital at the beginning of each period, the firm decides how much labor to hire after the realization of the shock s. Therefore it does not face any aggregate uncertainty and simply maximize current period profits. Since this firm faces constant returns to scale, profits are zero in equilibrium and there is no need to specify ownership. For later use, let w be the market wage.

Labor market

Agents are endowed with one unit of time when young that can be supplied as labor to the firm inelastically. They supply a fraction l_y when young and a fraction l_m when middle-aged, such that $l_y + l_m = 1$. Consequently, labor income in retirement is zero.

Consumers

Lifetime preferences for aggregate consumption of a typical young agent are specified by the following von Neumann-Morgestern utility function

$$EU[c_y, c_m, c_r] = E\left[u(c_y) + \beta u(c_m) + \beta^2 u(c_r)\right] \tag{1}$$

where $c = (c_y, c_m, c_r)$ is the consumption over the life cycle, which could be decomposed into young adulthood, middle age and retirement, $u(\cdot)$ is a strictly concave, strictly increasing period utility function that satisfies Inada conditions, and $\beta \in (0, 1]$ is the subjective discount factor.

The problem of the representative agent is to maximize the objective function (1) subject to the following budget constraints

$$\begin{aligned} c_y &\leq wl_y - qb_y - a_y \\ c_m &\leq wl_m + b_y + ra_y - qb_m - a_m \\ c_r &\leq b_m + ra_m \end{aligned} \tag{2}$$

plus the non-negativity constraints on consumption, where the various prices, asset holdings and consumption allocations are as yet unspecified random variables.

Since the results I develop in the next section depend on the nature of the stochastic processes followed by the equilibrium prices and interest rates, portfolio holdings and consumption allocations, I will study a competitive equilibrium in which prices and allocations, given the aggregate stock of capital, depend only on the current realization of the exogenous shock. Notice that this type of equilibrium is not new in the literature, see for example early work by Lucas [16] or Spear [19], who defines a steady state equilibrium as one in which the stochastic process of equilibrium prices is measure isomorphic to the driving stochastic process, i.e. the components of the equilibrium price sequence depend on the current state and not on state histories.

3 Non-existence of competitive equilibrium

When the economy is stationary, it is possible to simplify the characterization of the equilibrium considerably. Following the terminology of Spear [19], the simplest possible type of equilibrium, given the assumption that aggregate shocks are independent and identically distributed, is a strongly stationary rational expectations equilibrium in which asset prices and interest rates, given the aggregate stock of capital K, depend only on the current realization of the shock s so that, for instance, $q_t^{s_t} = q^s [K_t]$, $r_t^{s_t} = r^s [K_t]$ and $w_t^{s_t} = w^s [K_t]$, with $s \in \{h, l\}$. The full characterization is as follows

Definition 1 (Strongly stationary equilibrium) *A strongly stationary equilibrium is a collection of consumption allocations $\left(c_y^s [K_t], c_m^s [K_t], c_r^s [K_t]\right)$, investment decisions $\left(a_y^s [K_t], a_m^s [K_t]\right)$, bond holdings $\left(b_y^s [K_t], b_m^s [K_t]\right)$, and prices $\left(q^s [K_t], r^s [K_t], w^s [K_t]\right)$ such that*

1. *Given prices, demand functions are generated as solutions to the expected utility maximization problem*

$$\begin{aligned} \max\ EU [c_y, c_m, c_r] \quad = \quad & u\left(c_y^s\right) + \beta\left[\pi^h u\left(c_m^{sh}\right) + \pi^l u\left(c_m^{sl}\right)\right] + \\ & \beta^2 \left\{\pi^h \left[\pi^h u\left(c_r^{hh}\right) + \pi^l u\left(c_r^{hl}\right)\right] + \pi^l \left[\pi^h u\left(c_r^{lh}\right) + \pi^l u\left(c_r^{ll}\right)\right]\right\} \end{aligned}$$

subject to

$$
\begin{aligned}
c_y^s &\le w^s l_y - q^s b_y^s - a_y^s, \text{ for } s \in \{h,l\} \\
c_m^{ss'} &\le w^{s'} l_m + b_y^s + r^{s'} a_y^s - q^{s'} b_m^{s'} - a_m^{s'}, \text{ for } (s,s') \in \{h,l\}^2 \\
c_r^{s's''} &\le b_m^{s'} + r^{s''} a_m^{s'}, \text{ for } (s',s'') \in \{h,l\}^2
\end{aligned}
$$

plus the non-negativity constraints on consumption. In this problem, I let

$$
\begin{aligned}
c_y^s &= \text{young agent's first period allocation when current state is } s \\
c_m^{ss'} &= \text{allocation of middle-age agent who faced event } s \text{ when young,} \\
&\quad \text{when the current event is } s', \text{ and} \\
c_r^{s's''} &= \text{allocation of retired agent who faced event } s' \text{ when middle-aged,} \\
&\quad \text{when the current event is } s''.
\end{aligned}
$$

for $(s,s',s'') \in \{h,l\}^3$.

2. *Given prices and the capital stock inherited from the previous period K_t, the firm maximizes profits*

$$V(K_t, L_t^s, s) = \eta(s) F(K_t, L_t^s) + K_t(1-\delta) - r_t^s K_t - w_t^s L_t^s$$

for $s \in \{h,l\}$.

3. *Markets (labor, capital and bonds) clear in every period*

$$
\begin{aligned}
L_t^s &= 1 \\
K_{t+1}^s &= a_y^s [K_t] + a_m^s [K_t] \\
0 &= b_y^s [K_t] + b_m^s [K_t]
\end{aligned}
$$

for $s \in \{h,l\}$.

Note that by Walras' law market clearing in these three markets implies market clearing in the consumption goods market. On the other hand, notice that I index consumption in the second and third period with both the current and lagged states, because agents' investment decisions will depend generally on the state in which the security is purchased. In equilibrium, however, the capital stock inherited from the previous period will capture the effect of the lagged state.

One of the deepest results of this paper is that the competitive equilibrium as defined above does not exist. This is the content of the following proposition.

Proposition 1 *For an open and dense set of OLG economies with production and aggregate uncertainty, there is no strongly stationary competitive rational expectations equilibrium.*

Proof. Without loss of generality, I will assume in this proof that both β and δ are equal to unity. Notice that from the market clearing conditions, the stock of capital tomorrow depends on the current state

$$K_{t+1}^{s} = a_y^s \left[K_t\right] + a_m^s \left[K_t\right]$$

Using this fact, a young agent's first-order conditions take the form

$$\begin{aligned}
-u'\left(c_y^h \left[K_t\right]\right) q^h \left[K_t\right] + \pi^h u'\left(c_m^{hh}\left[K_{t+1}^h\right]\right) + \pi^l u'\left(c_m^{hl}\left[K_{t+1}^h\right]\right) &= 0 \\
-u'\left(c_y^h \left[K_t\right]\right) + \pi^h u'\left(c_m^{hh}\left[K_{t+1}^h\right]\right) r^h\left[K_{t+1}^h\right] + \pi^l u'\left(c_m^{hl}\left[K_{t+1}^h\right]\right) r^l\left[K_{t+1}^h\right] &= 0 \\
-u'\left(c_y^l \left[K_t\right]\right) q^l \left[K_t\right] + \pi^h u'\left(c_m^{lh}\left[K_{t+1}^l\right]\right) + \pi^l u'\left(c_m^{ll}\left[K_{t+1}^l\right]\right) &= 0 \\
-u'\left(c_y^l \left[K_t\right]\right) + \pi^h u'\left(c_m^{lh}\left[K_{t+1}^l\right]\right) r^h\left[K_{t+1}^l\right] + \pi^l u'\left(c_m^{ll}\left[K_{t+1}^l\right]\right) r^l\left[K_{t+1}^l\right] &= 0
\end{aligned}$$

On the other hand, a middle-age agent's first-order conditions take the form

$$\begin{aligned}
-u'\left(c_m^{sh}\left[K_{t+1}^s\right]\right) q^h\left[K_{t+1}^s\right] + \pi^h u'\left(c_r^{hh}\left[K_{t+2}^h\right]\right) + \pi^l u'\left(c_r^{hl}\left[K_{t+2}^h\right]\right) &= 0 \\
-u'\left(c_m^{sh}\left[K_{t+1}^s\right]\right) + \pi^h u'\left(c_r^{hh}\left[K_{t+2}^h\right]\right) r^h\left[K_{t+2}^h\right] + \pi^l u'\left(c_r^{hl}\left[K_{t+2}^h\right]\right) r^l\left[K_{t+2}^h\right] &= 0 \\
-u'\left(c_m^{sl}\left[K_{t+1}^s\right]\right) q^l\left[K_{t+1}^s\right] + \pi^h u'\left(c_r^{lh}\left[K_{t+2}^l\right]\right) + \pi^l u'\left(c_r^{ll}\left[K_{t+2}^l\right]\right) &= 0 \\
-u'\left(c_m^{sl}\left[K_{t+1}^s\right]\right) + \pi^h u'\left(c_r^{lh}\left[K_{t+2}^l\right]\right) r^h\left[K_{t+2}^l\right] + \pi^l u'\left(c_r^{ll}\left[K_{t+2}^l\right]\right) r^l\left[K_{t+2}^l\right] &= 0
\end{aligned}$$

for $s \in \{h, l\}$. From the optimality conditions of the firm, the equilibrium prices satisfy

$$\begin{aligned}
w^s\left[K_t\right] &= \eta\left(s\right) F_L(K_t, 1) \\
r^s\left[K_t\right] &= \eta\left(s\right) F_K(K_t, 1)
\end{aligned}$$

for $s \in \{h, l\}$. Market clearing requires that

$$b_y^s\left[K_t\right] + b_m^s\left[K_t\right] = 0, \text{ for } s \in \{h, l\}$$

These equations have several implications. Note first that since the expected marginal utility expressions in each of the first-order conditions of the middle-aged is independent of the lagged state, it must be true that

$$c_m^{hl}\left[K_{t+1}^h\right] = c_m^{ll}\left[K_{t+1}^l\right] \equiv c_m^l\left[K_{t+1}\right]$$

and

$$c_m^{hh}\left[K_{t+1}^h\right]=c_m^{lh}\left[K_{t+1}^l\right]\equiv c_m^h\left[K_{t+1}\right]$$

Furthermore, since first-period consumption only depends on the current state and the aggregate stock of capital, the resource constraint at period t requires that

$$c_r^{s's}\left[K_t^{s'}\right]=\eta\left(s\right)F(K_t,1)-K_{t+1}^s\left[K_t\right]-c_y^s\left[K_t\right]-c_m^s\left[K_t\right]$$

so that consumption when retired can only depend on the current shock realization and the capital stock, that is

$$\begin{aligned}c_r^{hl}\left[K_t^h\right] &= c_r^{ll}\left[K_t^l\right]\equiv c_r^l\left[K_t\right]\\ c_r^{hh}\left[K_t^h\right] &= c_r^{lh}\left[K_t^l\right]\equiv c_r^h\left[K_t\right]\end{aligned}$$

The budget constraints now take the form (supressing the dependence of capital on the shock realization)

$$\begin{aligned}c_y^s\left[K_t\right] &= w^s\left[K_t\right]l_y-q^s\left[K_t\right]b_y^s-a_y^s\\ c_m^{ss'}\left[K_{t+1}\right] &= w^{s'}\left[K_{t+1}\right]l_m+b_y^s+r^{s'}\left[K_{t+1}\right]a_y^s-q^{s'}\left[K_{t+1}\right]b_m^{s'}-a_m^{s'}\\ c_r^{s's''}\left[K_{t+2}\right] &= b_m^{s'}+r^{s''}\left[K_{t+2}\right]a_m^{s'}\end{aligned}$$

Via the budget constraints above, I can show explicitly that the bond and capital holdings in the model must be state independent. To see this, consider

$$c_m^{hh}\left[K_{t+1}\right]=w^h\left[K_{t+1}\right]l_m+b_y^h+r^h\left[K_{t+1}\right]a_y^h-q^h\left[K_{t+1}\right]b_m^h-a_m^h$$

and

$$c_m^{lh}\left[K_{t+1}\right]=w^h\left[K_{t+1}\right]l_m+b_y^l+r^h\left[K_{t+1}\right]a_y^l-q^h\left[K_{t+1}\right]b_m^h-a_m^h$$

Since $c_m^{hh}\left[K_{t+1}\right]=c_m^{lh}\left[K_{t+1}\right]$, this implies that

$$b_y^h+r^h\left[K_{t+1}\right]a_y^h=b_y^l+r^h\left[K_{t+1}\right]a_y^l$$

Similarly, since $c_m^{hl}\left[K_{t+1}\right]=c_m^{ll}\left[K_{t+1}\right]$,

$$b_y^h+r^l\left[K_{t+1}\right]a_y^h=b_y^l+r^l\left[K_{t+1}\right]a_y^l$$

Subtracting the second equation from the first, I get

$$a_y^h\left(r^h\left[K_{t+1}\right]-r^l\left[K_{t+1}\right]\right)=a_y^l\left(r^h\left[K_{t+1}\right]-r^l\left[K_{t+1}\right]\right)$$

From this expression, I have either $a_y^h = a_y^l$ or $r^h = r^l$. Now I will analyze each case separately

1. In the first case, I also infer that $b_y^h = b_y^l$. Consequently, I am left with a system of 10 equations (4 first-order conditions for the young, 4 first-order conditions for the middle-aged, and 2 market-clearing conditions) in 8 unknowns, namely $q^h, q^l, r^h, r^l, a_y, a_m, b_y$ and b_m.

2. In the latter case, I must have $\eta(h) = \eta(l)$, but this contradicts the fact that $\eta(h) > \eta(l)$.

Then I end up with a non-linear system with more equations than unknowns. It is possible to show that there cannot be generally a strongly stationary equilibrium (i.e. a solution) using techniques similar to those developed by Spear [19] or Citanna and Siconolfi [5]. While the actual proof requires complex mathematical tools, the intuition is simple: when a system of simultaneous equations has more independent equations than unknowns, the solution to some square subsystem of equations will not generally solve the remaining equations. ■

This non-existence result is crucial for the argument I will develop later in the paper. Basically, Proposition 1 implies that in order to deal with the sequential markets equilibrium, I need to work with a different equilibrium concept that may include lagged, endogenous variables as state variables. The existence of such a concept, the Markovian equilibrium, was first studied by Spear and Srivastava [21], and Duffie, Geanakoplos, Mas-Colell and McLennan [8].

Fortunately, Citanna and Siconolfi [6] have recently shown that it is possible to consider a time homogenous Markov equilibrium, also known as recursive equilibrium, in which the state space is reduced to the exogenous shock, the aggregate stock of capital and the lagged wealth distribution of the agents. When the representative agent lives more than two periods, the recursive competitive equilibrium handles the histories that arise because of the rich interactions among agents at different stages in life.

Notice however that in two-period models, histories do not matter in equilibrium and hence it is not necessary to encompass lagged, additional endogenous state variables that embody the prior and current information of the economy as the following remark shows

Remark 1 *With two-period lived economies with production, a strongly stationary equilibrium* does *exist. The first-order conditions together with the market-clearing conditions suffice to compute prices and allocations solely in terms of the exogenous shock and the aggregate capital stock.*

Put it differently, the two-period model with separable preferences and one good initially studied by Spear [19] differs from the three-period model to the extent that only the former admits a strongly stationary equilibrium. This will affect the completeness of markets as I will show in the next section.

4 Recursive equilibrium

I study a recursive equilibrium where the distribution of capital holdings and bond holdings of the young, the current aggregate capital stock and the realization of the productivity shock constitute the state space. In principle, I could add the bond holdings and capital holdings by the middle-aged, but these can be readily disregarded because of the market-clearing conditions. As in Henriksen and Spear [11] and Krueger and Kubler [14], it may well be assumed that all transition functions are smooth and that the state space lies in a known compact set. I will be looking for a functional rational expectations equilibrium, following the terminology of Spear [19]. The specific details of the computation are provided in the appendix of this paper.

For later use, it is convenient to understand what market completeness means and, in particular, what sequential completeness implies in stochastic OLG models. In this paper, sequential completeness is defined as follows

Definition 2 (Sequential market completeness) *Sequential market completeness is a market arrangement in which, given the overall state of the economy today, there are sufficiently many financial instruments for individuals in the economy to transfer wealth between states of the world tomorrow. Alternatively, as in Demange [9], markets are sequentially complete if, for each state, the space that is spanned by the returns of the financial securities is of dimension equal to the number of states tomorrow that follow the state today.*

Henriksen and Spear [11] argue that markets are not sequentially complete in the recursive formulation. The proof follows from the contruction of the recursive equilibrium. In this model, the vector $\sigma = [b_{y,t-1}, a_{y,t-1}, K_t, s_t] \in \Sigma \subset \mathbb{R}^4$ is the state of the economy. If I fixed the state variables today at their equilibrium values, then I would take the realizations of past bond and capital holdings, the current aggregate capital stock and the contemporaneous productivity shock as fixed. The states tomorrow would consist of the current bond and capital holdings of the young, the new aggregate capital stock, together with the two possible productivity realizations. Because equilibrium prices and allocations tomorrow do depend on the lagged state variables, there are necessarily more than two future states. But agents today have only the two financial instruments with which to transfer wealth across states, so the markets are necessarily sequentially incomplete. (This incompleteness extends to any Markovian representation that depends on lagged endogenous variables.)

As in the previous section, the fact that the representative agent lives three periods is crucial. Stochastic OLG models with three-period-lived agents require that some portfolio and consumption decisions must be taken after the resolution of uncertainty (when middle-aged). Since capital pays a rate of return that is state contingent, there are wealth effects that preclude the sequential markets equilibrium from being strongly stationary. Nevertheless, the following remark shows that the two-period model with production and aggregate uncertainty has a strongly stationary representation

Remark 2 *Since two-period-lived models admit a strongly statationary equilibrium, then there is no need to rely on the recursive representation that requires lagged endogenous variables within the state space. Consequently, models of that type are sequentially complete.*

Economists have been using recursive methods in stochastic OLG models with finitely many periods at least since Rios-Rull [17] and [18]. More recent examples that implement the recursive equilibrium for numerical simulations include Geanakoplos, Magill and Quinzii [10], Storesletten, Telmer and Yaron [22] and Henriksen and Spear [11].

The absence of sequential completeness is the key result of this paper, since Demange [9] shows that when markets are sequentially complete, then the equilibrium allocation is short-run interim Pareto optimal. Short-run interim optimality implies that living agents trade optimally among themselves, and no goverment intervention is necessary to improve intergenerational risk sharing[1]. The discussion on Pareto optimality is the content of the next section.

5 Pareto suboptimality

In order to examine optimality issues, I assume there is an artificial period 0 chosen arbitrarily. Following Bohn [4], Pareto optimal allocations are obtained by solving the social planning problem at period 0 given welfare weights $\omega_t > 0$ subject to the resource constraint. The planner's problem is to maximize a welfare function W_0 (to simplify the notation, I omit the dependence on the stochastic shock where appropriate and assume that there is full capital depreciation)

$$W_0 = E_0 \left[\sum_{t=0}^{\infty} \left(\prod_{i=0}^{t} \omega_i \right) \left[u(c_{y,t}) + \beta u(c_{m,t+1}) + \beta^2 u(c_{r,t+2}) \right] \right]$$

subject to

$$c_{y,t} + c_{m,t} + c_{r,t} + K_{t+1} = G(K_t, 1, s_t)$$

The unconstrained objective function is then

$$W_0 = E_0 \left[\sum_{t=0}^{\infty} \left(\prod_{i=0}^{t} \omega_i \right) \left[u(G(K_t, 1, s_t) - c_{m,t} - c_{r,t} - K_{t+1}) + \beta u(c_{m,t+1}) + \beta^2 u(c_{r,t+2}) \right] \right]$$

[1] Furthermore, in presence of a productive infinitely-lived asset, the short-run interim Pareto optimal allocation is also Pareto optimal.

The first-order conditions for an interior solution are

$$\begin{aligned} c_{m,t} &: \quad \omega_t u'(c_{y,t}) = \beta u'(c_{m,t}) \\ c_{r,t} &: \quad \omega_t \omega_{t-1} u'(c_{y,t}) = \beta^2 u'(c_{r,t}) \\ K_{t+1} &: \quad E_t \left[\omega_{t+1} \frac{u'\left(c_{y,t+1}\right)}{u'\left(c_{y,t}\right)} G_K(K_{t+1}, 1, s_{t+1}) \right] = 1 \end{aligned} \tag{3}$$

These allocations are efficient in an interim sense. The conditions in (3) are similar to the ones that appear in Bohn [4], and are consistent with both optimal risk sharing and efficient accumulation of capital. The first two conditions characterize the division of consumption between different generations in each state of nature. The planner transfers resources across generations until the marginal utility of one generation equals the marginal utility of some other generation times the welfare weight. The last condition is just a version of the well-known Euler equation.

When welfare weights are time invariant, the equilibrium allocations derived from the planner's problem, given the aggregate capital stock, depend only on the current realization of the shock s. (A similar characterization of the equilibrium has been developed by Aiyagari and Peled [1].) Put it differently, the optimal allocations ignore any endogenous fluctuations, regardless of the optimality concept at hand (e.g. interim optimality). This basic result is the content of the following proposition

Proposition 2 (Henriksen-Spear) *Perfect risk-sharing implies a strongly stationary consumption sequence.*

Proof. See Henriksen and Spear [11, Theorem 1]. ■

From Proposition 2 it is clear that the equilibrium allocations of the model laid out in Section 4 (the laissez-faire allocations) offer some room for Pareto improvements. As acknowledged in Section 1, there are two fields in which these improvements could occur: risk sharing or capital accumulation. In this paper I will consider those economies in which capital is not accumulated inefficiently, and study government policies that improve intergenerational risk sharing. Krueger and Kubler [13, Proposition 1] provides a condition that guarantees dynamic efficiency (or adequate capital accumulation), which will be verified numerically in the next section[2]. In other words, I will show that the laissez-faire competitive equilibrium allocation is not consistent with perfect intergenerational risk-sharing. Thus there may be

[2]Henriksen and Spear [11] follow a similar strategy: they evaluate Pareto improvements in terms of risk sharing in a dynamically efficient economy. More specifically, Bloise and Calciano [3, Lemma 3] show that all equilibrium allocations considered by Henriksen and Spear [11] are not inefficient because there is a productive asset that yields a non-negligible share of the aggregate endowment. However, there may exist a substancial welfare improvement by a mere reallocation of risk.

reallocations among the current young and middle-aged that can improve welfare. This result in turn implies that there is no short-run interim optimality, which happens to be a necessary condition for Pareto optimality.

Because there is an initial starting period 0 (in which there is a fixed population of initial agents who are either middle-aged or old), now it is possible to make different steady-state allocations in the model Pareto comparable in the sense that I can move from such allocation to another in ways that make all agents no worse off. In particular, following Henriksen and Spear [11] I wish to exclude the possibility of reallocations that make all future generations better off at the expense of the initial generations. In models of this type, if there is no initial period, only steady-state by steady-state optimality comparisons matter (see also the work of Weiss [23] and Benveniste and Cass [2]). I will follow this strategy in the next section.

6 Numerical simulations

Now I will provide a quantitative assessment of the impact of a reallocation of risk on welfare. For the numerical work, I assume that the period utility function is of the CRRA type

$$u(x) = \frac{x^{1-\sigma}}{1-\sigma}$$

where x is a generic argument, and σ is the coefficient of relative risk aversion. On the other hand, the production function, following Kubler and Kubler [14], takes the form

$$G(K, L, s) = \exp(s)K^{\alpha}L^{1-\alpha} + (1-\delta)K$$

where $\exp(s)$ captures the shock to total factor productivity, δ is the depreciation rate and α measures the capital intensity.

6.1 Parameterization

In this subsection I describe the parameters that will be used in the numerical simulations. Except for the shocks to total factor productivity, most parameter values are fairly standard in the literature.

Risk aversion

The value of σ is equal to 2, as in Henriksen and Spear [11].

Discount factor

Since each period in this model consists of 20 years, I follow Geanakoplos, Magill and Quinzii [10] and set β equal to 0.5. This 20-year discount factor corresponds to an annual discount factor of 0.965.

Labor supply

I assume that $l_y = 3/8$ and $l_m = 5/8$ in any state $s \in \{h, l\}$, as labor supply is completely inelastic in this model. Notice that these values imply that the representative agent is somehow more productive when middle-aged.

Shocks to total factor productivity

Since $\exp(h) > \exp(l)$, I assume that $s = \{0.95, 1.05\}$. These values are arbitrary, and all results presented in the next subsection just require that $h > l$.

Capital intensity

As is standard in production functions with labor, I assume that $\alpha = 1/3$.

Depreciation rate

Following Geanakoplos, Magill and Quinzii [10], I set $\delta = 0.5$, which implies a depreciation of the order of 3 percent a year.

The parameters are summarized in Table 1

Table 1

Parameter	Description	Value
σ	risk aversion	2
β	discount factor	0.5
l_y	labor supply young	3/8
l_m	labor supply adult	5/8
$\exp(h)$	good productivity shock	exp(1.05)
$\exp(l)$	bad productivity shock	exp(0.95)
α	capital intensity	1/3
δ	depreciation rate	0.5

6.2 Results

After solving for the laissez-faire equilibrium, using a projection method described in the appendix, I simulate the model 2,000 times and disregard the first 500 observations. In Figure 1 I present the typical sequences of consumption for each stage of life extracted from the simulation, together with the associated series of shock realizations.

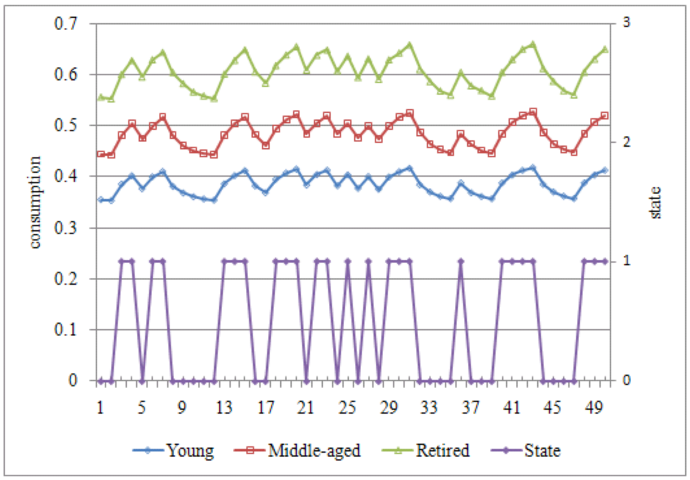

Figure 1. Typical simulated consumption series for each stage of life, and state realizations (good=1, bad=0).

Unlike Henriksen and Spear [11], who deal with a pure-exchange economy, consumption does not alternate in Figure 1 between two regimes (high consumption and low consumption). With production, consumption changes gradually, because state realizations affect the level of aggregate capital. Also, since there is partial depreciation in this model ($\delta < 1$), aggregate capital adjusts with a lag.

Table 2 depicts the mean and standard deviation of the simulated consumption series and portfolio composition in each state $s \in \{h, l\}$.

Table 2
Summary statistics, benchmark

	c_y	c_m	c_r	b_y	b_m	a_y	a_m
Low	0.3697	0.4651	0.5833	0.0187	-0.0187	-0.0008	0.1979
(SD)	0.0101	0.0143	0.0184	0.0028	0.0028	0.0008	0.0163
High	0.4050	0.5071	0.6360	0.0095	-0.0095	-0.0004	0.1010
(SD)	0.0118	0.0164	0.0227	0.0095	0.0095	0.0007	0.0995

Table 2 clearly illustrates that the standard deviation of consumption is increasing in age (actually the standard deviation of consumption for the old in the good state is an order of magnitude higher than that of the young). In equilibrium, the market imposes more risk on the old than it does on the young or middle-aged, because the young and middle-age agents can rebalance their portfolios, whereas the old cannot. As in Krueger and Kubler [15], the representative agent in the laissez-faire equilibrium saves for retirement mostly in physical capital. Because it carries high return and risk, the agent acquires a significant amount of capital to guarantee a certain amount of consumption when retired. On the other hand, the hump-shaped pattern in equity ownership and the fact that the middle-aged have a short position in the risk-free asset are both in line with the findings of Storesletten, Telmer and Yaron [22]. Finally, I estimate the expected utility of the representative agent from the simulated data using the fact that the equilibrium allocations follow an ergodic process (such that time-series and cross-sectional averages give the same information). In this case, I find that $EU = -4.0637$.

Now I will provide a quantitative assessment of the impact of a reallocation of risk on welfare, since the state dependent marginal rate of substitution for the middle-aged is not equal to that of the retired in the laissez-faire equilibrium. From first principles, this suboptimality stems from the fact that capital pays a rate of return that is state contingent, which generates wealth effects that preclude the equilibrium from being strongly stationary.

In order to generate a Pareto improvement, I will try to neutralize these wealth effects by imposing a tax on capital income and use the proceeds from this tax to give lump sum transfers ζ_i, for $i = y, m, r$ to the households. Even though this paper does not fit ideally in the literature on optimal taxation, there is evidence in favor of capital income taxes in stochastic OLG models with production (see for example Conesa, Kitao and Krueger [7]).

In presence of capital taxation, the adjusted gross return on capital in period t is now written as

$$r_t \equiv 1 + (F_K(K_t, 1) - \delta)(1 - \tau_K)$$

where τ_K is the constant tax rate. The government's budget is balanced in each period and therefore the budget constraint can be written as

$$\zeta_y + \zeta_m + \zeta_r = \tau_K(F_K(K_t, 1) - \delta)K_t$$

I compute the recursive competitive equilibrium of the tax-transfer model for tax rates of 25 percent (equivalent to an annual tax of 1.1 percent), under the assumption that two thirds of the lump sum transfer go to the young and one third goes to the old. Table 3 shows the average and standard deviation of consumption at each age. With respect to the benchmark case, notice that the standard deviation of consumption for the old and middle-aged is lower, the average consumption for the young is higher, and the average consumption of the old is lower. The expected utility for this tax level is now $EU = -3.7119$, which proves that the introduction of capital income tax has led to a reallocation of risk and a corresponding welfare improvement of 9%.

Table 3
Summary statistics, capital tax

	c_y	c_m	c_r
Low	0.4370	0.4659	0.5163
(SD)	0.0109	0.0138	0.0174
High	0.4805	0.5072	0.5590
(SD)	0.0119	0.0146	0.0184

Could this policy reform, in which government taxes heavily the middle-aged and mainly benefits the young, occur in the real world? Yes, it could. Actually, Henriksen and Spear [11] argue that in the U.S. there are Social Security transfers that help reduce consumption variance in old age, as well as substantial income transfers that benefit younger households, which are paid for through taxes levied on older (and wealthier) households.

7 Concluding remarks

This paper shows that endogenous market incompleteness is robust to the introduction of production and capital accumulation, provided that agents live at least three periods. With this result at hand, it should be worth while re-examining the concept of Pareto optimality in stochastic OLG models of the type examined in this paper that claim that markets are complete (e.g. Rios-Rull [17] and [18], and Storesletten, Telmer and Yaron [22]).

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Appendix: Description of the algorithm

The steps are:

1. Choose orders of approximation N_{b_y}, N_{k_y} and N_{k_m} and compute the $N_{b_y}+1$, $N_{k_y}+1$ and $N_{k_m}+1$ Chebyshev nodes on $[-1,1]$ as follows

   (a) $x_{b_y}^{i_{b_y}} = \cos\left(\frac{(2i-1)\pi}{2(N_{b_y}+1)}\right)$ for $i_{b_y} = 1, ..., N_{b_y}+1$

   (b) $x_{k_y}^{i_{k_y}} = \cos\left(\frac{(2i-1)\pi}{2(N_{k_y}+1)}\right)$ for $i_{k_y} = 1, ..., N_{k_y}+1$

   (c) $x_{k_m}^{i_{k_m}} = \cos\left(\frac{(2i-1)\pi}{2(N_{k_m}+1)}\right)$ for $i_{k_m} = 1, ..., N_{k_m}+1$

2. Choose approximation intervals $[\underline{b}_y, \overline{b}_y]$, $[\underline{k}_y, \overline{k}_y]$ and $[\underline{k}_m, \overline{k}_m]$. Map the nodes in 1. into these intervals as follows

   (a) $b_y^{-1,i_{b_y}} = \underline{b}_y + \left(x_{b_y}^i + 1\right)\frac{\overline{b}_y - \underline{b}_y}{2}$ for $i_{b_y} = 1, ..., N_{b_y}+1$

   (b) $k_y^{-1,i_{k_y}} = \underline{k}_y + \left(x_{k_y}^i + 1\right)\frac{\overline{k}_y - \underline{k}_y}{2}$ for $i_{k_y} = 1, ..., N_{k_y}+1$

   (c) $k_m^{-1,i_{k_m}} = \underline{k}_m + \left(x_{k_m}^i + 1\right)\frac{\overline{k}_m - \underline{k}_m}{2}$ for $i_{k_m} = 1, ..., N_{k_m}+1$

3. At each node $\eta_{i_{b_y} i_{k_y} i_{k_m} i_z} \equiv \left(b_y^{-1,i_{b_y}}, k_y^{-1,i_{k_y}}, k_m^{-1,i_{k_m}}, z^{i_z}\right)$ for $i_{b_y} = 1, ..., N_{b_y}+1$, $i_{k_y} = 1, ..., N_{k_y}+1$, $i_{k_m} = 1, ..., N_{k_m}+1$, and $i_s \in \{h, l\}$ compute

$$\begin{aligned} b_y^t\left(\eta_{i_{b_y} i_{k_y} i_{k_m} i_s}, \theta\right) &= \sum_{j_{b_y}=0}^{N_{b_y}} \sum_{j_{k_y}=0}^{N_{k_y}} \sum_{j_{k_m}=0}^{N_{k_m}} \theta_{j_{b_y} j_{k_y} j_{k_m}}^{b_y, i_s} T_{j_{b_y}}\left(b_y^{-1,i_{b_y}}\right) \\ &\quad \times T_{j_{k_y}}\left(k_y^{-1,i_{k_y}}\right) T_{j_{k_m}}\left(k_m^{-1,i_{k_m}}\right) \end{aligned}$$

$$\begin{aligned} k_y^t\left(\eta_{i_{b_y} i_{k_y} i_{k_m} i_s}, \theta\right) &= \sum_{j_{b_y}=0}^{N_{b_y}} \sum_{j_{k_y}=0}^{N_{k_y}} \sum_{j_{k_m}=0}^{N_{k_m}} \theta_{j_{b_y} j_{k_y} j_{k_m}}^{k_y, i_s} T_{j_{b_y}}\left(b_y^{-1,i_{b_y}}\right) \\ &\quad \times T_{j_{k_y}}\left(k_y^{-1,i_{k_y}}\right) T_{j_{k_m}}\left(k_m^{-1,i_{k_m}}\right) \end{aligned}$$

$$\begin{aligned} k_m^t\left(\eta_{i_{b_y} i_{k_y} i_{k_m} i_s}, \theta\right) &= \sum_{j_{b_y}=0}^{N_{b_y}} \sum_{j_{k_y}=0}^{N_{k_y}} \sum_{j_{k_m}=0}^{N_{k_m}} \theta_{j_{b_y} j_{k_y} j_{k_m}}^{k_m, i_s} T_{j_{b_y}}\left(b_y^{-1,i_{b_y}}\right) \\ &\quad \times T_{j_{k_y}}\left(k_y^{-1,i_{k_y}}\right) T_{j_{k_m}}\left(k_m^{-1,i_{k_m}}\right) \end{aligned}$$

$$
\begin{aligned}
q^t\left(\eta_{i_{b_y} i_{k_y} i_{k_m} i_s}, \theta\right) &= \sum_{j_{b_y}=0}^{N_{b_y}} \sum_{j_{k_y}=0}^{N_{k_y}} \sum_{j_{k_m}=0}^{N_{k_m}} \theta^{q,i_s}_{j_{b_y} j_{k_y} j_{k_m}} T_{j_{b_y}}\left(b_y^{-1,i_{b_y}}\right) \\
&\quad \times T_{j_{k_y}}\left(k_y^{-1,i_{k_y}}\right) T_{j_{k_m}}\left(k_m^{-1,i_{k_m}}\right)
\end{aligned}
$$

And calculate:

$$
\begin{aligned}
r^t &= \exp(s_t)\alpha\left(k_y^{-1,i_{k_y}} + k_m^{-1,i_{k_m}}\right)^{\alpha-1} \\
w^t &= \exp(s_t)\left(1-\alpha\right)\left(k_y^{-1,i_{k_y}} + k_m^{-1,i_{k_m}}\right)^{\alpha} \\
w_y^t &= (3/8)\, w^t \\
w_m^t &= (5/8)\, w^t \\
w_r^t &= 0
\end{aligned}
$$

4. Using the market clearing condition for bonds, compute:

$$
b_m^t\left(\eta_{i_{b_y} i_{k_y} i_{k_m} i_s}, \theta\right) = -b_y^t\left(\cdot\right)
$$

Also, compute:

$$
\begin{aligned}
c_y^t\left(\eta_{i_{b_y} i_{k_y} i_{k_m} i_s}, \theta\right) &= w_y^t - k_y^t\left(\cdot\right) - q^t b_y^t\left(\cdot\right) \\
c_m^t\left(\eta_{i_{b_y} i_{k_y} i_{k_m} i_s}, \theta\right) &= w_m^t + b_y^{-1,i_{b_y}} + (1 + r^t - \delta) k_y^{-1,i_{k_y}} - k_m^t\left(\cdot\right) - q^t\left(\cdot\right) b_m^t\left(\cdot\right) \\
c_r^t\left(\eta_{i_{b_y} i_{k_y} i_{k_m} i_s}, \theta\right) &= w_r^t + b_m^{-1,i_{b_m}} + (1 + r^t - \delta) k_m^{-1,i_{k_m}}
\end{aligned}
$$

5. For $s_{t+1} \in \{h, l\}$ and at each vector:

$$
\left[b_y^t\left(\eta_{i_{b_y} i_{k_y} i_{k_m} i_s}\right), k_y^t\left(\eta_{i_{b_y} i_{k_y} i_{k_m} i_s}\right), k_m^t\left(\eta_{i_{b_y} i_{k_y} i_{k_m} i_s}\right)\right]
$$

compute:

$$
\begin{aligned}
b_y^{t+1,s_{t+1}}\left(\eta_{i_{b_y} i_{k_y} i_{k_m} i_s}, \theta\right) &= \sum_{j_{b_y}=0}^{N_{b_y}} \sum_{j_{k_y}=0}^{N_{k_y}} \sum_{j_{k_m}=0}^{N_{k_m}} \theta^{b_y,s_{t+1}}_{j_{b_y} j_{k_y} j_{k_m}} T_{j_{b_y}}\left(b_y^t\left(\cdot\right)\right) \\
&\quad \times T_{j_{k_y}}\left(k_y^t\left(\cdot\right)\right) T_{j_{k_m}}\left(k_m^t\left(\cdot\right)\right)
\end{aligned}
$$

$$k_y^{t+1,s_{t+1}}\left(\eta_{i_{b_y}i_{k_y}i_{k_m}i_s},\theta\right) = \sum_{j_{b_y}=0}^{N_{b_y}}\sum_{j_{k_y}=0}^{N_{k_y}}\sum_{j_{k_m}=0}^{N_{k_m}}\theta_{j_{b_y}j_{k_y}j_{k_m}}^{k_y,s_{t+1}}T_{j_{b_y}}\left(b_y^t(\cdot)\right)\times T_{j_{k_y}}\left(k_y^t(\cdot)\right)T_{j_{k_m}}\left(k_m^t(\cdot)\right)$$

$$k_m^{t+1,s_{t+1}}\left(\eta_{i_{b_y}i_{k_y}i_{k_m}i_s},\theta\right) = \sum_{j_{b_y}=0}^{N_{b_y}}\sum_{j_{k_y}=0}^{N_{k_y}}\sum_{j_{k_m}=0}^{N_{k_m}}\theta_{j_{b_y}j_{k_y}j_{k_m}}^{m_m,s_{t+1}}T_{j_{b_y}}\left(b_y^t(\cdot)\right)\times T_{j_{k_y}}\left(k_y^t(\cdot)\right)T_{j_{k_m}}\left(k_m^t(\cdot)\right)$$

$$q^{t+1,s_{t+1}}\left(\eta_{i_{b_y}i_{k_y}i_{k_m}i_s},\theta\right) = \sum_{j_{b_y}=0}^{N_{b_y}}\sum_{j_{k_y}=0}^{N_{k_y}}\sum_{j_{k_m}=0}^{N_{k_m}}\theta_{j_{b_y}j_{k_y}j_{k_m}}^{q,s_{t+1}}T_{j_{b_y}}\left(b_y^t(\cdot)\right)\times T_{j_{k_y}}\left(k_y^t(\cdot)\right)T_{j_{k_m}}\left(k_m^t(\cdot)\right)$$

And analogously to the procedure in 4., calculate $b_m^{t+1,s_{t+1}}(\cdot)$, r^{t+1}, $c_y^{t+1,s_{t+1}}(\cdot)$, $c_m^{t+1,s_{t+1}}(\cdot)$ and $c_r^{t+1,s_{t+1}}(\cdot)$.

6. Compute the residuals:

$$R_1\left(\eta_{i_{b_y}i_{k_y}i_{k_m}i_s},\theta\right) = -u_y(\cdot)\,q^t(\cdot)+\beta\sum_{z_{t+1}=\{h,l\}}\pi^{s_{t+1}}u_{m,z_{t+1}}(\cdot)$$
$$R_2\left(\eta_{i_{b_y}i_{k_y}i_{k_m}i_s},\theta\right) = -u_m(\cdot)\,q^t(\cdot)+\beta\sum_{z_{t+1}=\{h,l\}}\pi^{s_{t+1}}u_{r,z_{t+1}}(\cdot)$$
$$R_3\left(\eta_{i_{b_y}i_{k_y}i_{k_m}i_s},\theta\right) = u_y(\cdot)+\beta\sum_{s_{t+1}=\{h,l\}}\pi^{s_{t+1}}\left(1+r^{t+1}-\delta\right)u_{m,s_{t+1}}(\cdot)$$
$$R_4\left(\eta_{i_{b_y}i_{k_y}i_{k_m}i_s},\theta\right) = u_m(\cdot)+\beta\sum_{s_{t+1}=\{h,l\}}\pi^{s_{t+1}}\left(1+r^{t+1}-\delta\right)u_{r,s_{t+1}}(\cdot)$$

where $\pi^{s_{t+1}}\equiv\Pr(s_{t+1}=s)$ for $s\in\{h,l\}$.

7. Choose $\hat{\theta}$ so that:

$$R_p\left(\eta_{i_{b_y}i_{k_y}i_{k_m}i_s},\hat{\theta}\right)\approx 0 \text{ for every } \eta_{i_{b_y}i_{k_y}i_{k_m}i_s} \text{ and } p=1,...,4$$

Notice that we have a system of $4\times\left(N_{b_y}+1\right)\times\left(N_{k_y}+1\right)\times\left(N_{k_m}+1\right)\times 2$ equations in the same number of unknowns.