Anomalies in net present value calculations. A solution*

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Abstract

The so-called anomalies that arise in the computation and interpretation of the Net Present Value (NPV) and the Internal Rate of Return (IRR) can be easily overcome if the properties of the NPV function are taken into account and it is clearly defined what is an investment and what is a credit. All the roots of the NPV function have economic meaning and, when there is at least one IRR, the NPV and the IRR criteria agree.

Keywords: Net Present Value, Internal Rate of Return, Investment Analysis, Project Evaluation.

JEL Classification: Q28, D92.

1. Introduction
1.1. Background

A review of existing literature and the usual practice between professionals and experts, shows that there exists some lack of knowledge on the characteristics of the Net Present Value (NPV) and of the Internal Rate of Return (IRR). In addition, as the results that provide these decision criteria are not always they are intuitive, a number of conceptual errors may arise when carrying out the economic interpretation.

Several authors, as Peumans (1974), Weston and Brigham (1984), Brealey and Myers (1985) and Belli (1996) among others, recognize the difficulty to apply the criterion of the IRR, due to the lack of good properties of this indicator of project desirability. Thus, Gronchi

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The internal rate of return can be unambiguously used in decision-making procedures only if it is unique. Due to these difficulties one may opt for leaving aside the IRR and work only with the NPV, which is the main conclusion of Oehmke (2000) and the starting point for Castelo (2001): if the researcher is quite sure of the appropriate discount rate to use, then there is no real issue: either the NPV is positive at that rate or is not. According to Ross (1995), the IRR is not a good decision criterion at all, because it does not lead to the same decision that NPV: In fact, it is not uncommon to spend a considerable amount of time in class making sure that the student understand all the wrong ways of thinking about investment decision making —from the IRR rule to payback period. Wrong, of course, because they don’t coincide with the NPV rule.

The difficulty in the economic interpretation of the IRR’s and the apparent disagreement with the NPV application, is an old problem that has a very simple explanation. It is evident that the yield of an investment project must diminish when increasing the costs and, in particular, when increasing the price of the capital \( r \). This desirable characteristic is preserved only if the yield is measured with a decreasing monotonic function with respect to the discount rate \( r \), and as the NPV function does not have this property, there non-intuitive results appears. In particular, the lack of monotonicity may cause that the NPV function exhibit several and different real roots —see Hirshleifer (1958)— and, in this case, one can wonder which one is the relevant IRR to be apply the criterion in a specific case and if all of them has or not economic sense.

The problem has been studied from diverse approaches. For Massé (1962) all roots of the NPV function are meaningful, which constitutes an important step to interpret them and to apply in a correct way the criterion of the IRR. Teichroew et al. (1965a, 1965b), distinguish between pure and mixed investments, opening the way to understand a project as the aggregation of several subprojects of different type that, altogether, can behave as an investment or as a credit, on the basis of the relative weight of each one of them.

1.2. Goals

The aim of this paper is to modify the framework of analysis of the NPV and IRR criteria, with the purpose of obtaining a more general decision method, that allows to apply these criteria both in conventional cases and in supposedly anomalous ones. We also intent to provide an economic explanation of the results that obtain when applying the NPV and IRR criteria, no matter which case is analyzed. The analysis is limited to real numbers, with discount rates and IRR’s greater than minus one.

1.3. Framework

Section 2 shows the basic definitions. The projects are classified as investment and credit, by means of a new definition, and a rule is show that allows find the IRR relevant in the case of multiple roots. In section 3 some of the properties of the NPV function are examined and it is proven that both the IRR and the NPV criteria coincide. Section 4 presents a method
to interpret the results of the application of the NPV and IRR criteria, through the analysis of two classic cases that appear in literature as example of the non-validity of the IRR criterion. Finally, in section 5, we offer some concluding remarks.

2. Basic Definitions

2.1. Project. A project, \( P \), is a sequence of dated quantities, \( C_t \), beginning at period \( M \) ending at period \( M + T \), and depending on a discount rate \( r \):

\[
P = \{(C_t, r), t = 0,..., M,..., M + T\}
\]

2.2. Net present value (NPV). It measures the variation that takes place in the present wealth (period 0) due to the project:

\[
NPV = N(C_t, r) = \sum_{t=0}^{M+T} \frac{C_t}{(1+r)^t}
\]

with \( r \neq -1 \). The function \( NPV = N(C_t, r) \) admits continuous derivatives from any order with respect to \( r \).

2.3. Additivity. NPV of the project A plus NPV of the project B is equal to NPV of the sum of both projects:

\[
N(C_t^A, r) + N(C_t^B, r) = N(C_t^{A+B}, r)
\]

2.4. Internal Rate of Return (IRR). The IRR’s \( r_j^* \), \( j = 1,..., J \), measures the rate variation of the wealth generated by the project by period time. They are defined as:

all those \( r_j^* \) such that \( N(C_t, r_j^*) = 0 \)

2.5. Basic condition for all indicator of project desirability: the indicator improves as result of any improvement of the project.

2.6. Project types:

a) Investment: all projects whose quantities have positive and negative signs, behaves as an investment in the interval \((r^a, r^b)\), if \( \delta N(C_t, r)/\delta r < 0 \) in this interval.

b) Credit: all projects whose quantities have positive and negative signs, behaves as a credit in the interval \((r^a, r^b)\), if \( \delta N(C_t, r)/\delta r > 0 \) in this interval.

c) Gift: all projects with non-negative flows and at least one strictly positive.

d) Loss: all projects with non-positive flows and at least one strictly negative.

2.7. Acceptance criteria:

a) Project acceptance.
The acceptance rule for $NPV$, whatever the type of project, is $\frac{N(C_t, r)}{c_1} \geq 0$.

The IRR criterion is applicable when there exists at least one root of $N(C_t, r)$. If the IRR ($r^*$) is unique, the project is accepted if the IRR is not lower (resp. not higher) than the discount rate ($r^0$) when it is an investment (resp. credit):

$$r^* \geq r^0 \text{ if the project behaves as an investment} \quad [5]$$
$$r^* \leq r^0 \text{ if the project behaves as a credit} \quad [6]$$

In order to determine if a project behaves as an investment or as a credit, we have to look at the sign of the derivative of $N(C_t, r)$ at the point $r^* + \varepsilon, \varepsilon \to 0$, if $r^* < r_0$ and at $r^* - \varepsilon$ when $r^* > r_0$.

If there exists more than one IRR the criterion is applied in the same way, once selected the relevant IRR in each case, as follows. Let $r_1^* < r_2^* < \ldots < r_n^*$ the $n$ real roots of the $NPV$ function and $r_0$ the cost of the capital, the relevant IRR $r^*$ is:

$$r_1^* \text{ if } r^0 \leq r_1^* \quad [7]$$
$$r_2^* \text{ and } r_{k+1}^* \text{ if } r_k^* \leq r^0 \leq r_{k+1}^* \quad [8]$$
$$r_n^* \text{ if } r^0 \geq r_n^* \quad [9]$$

b) Choice between two projects, $X$ and $Y$. It is agreed that $X$ is preferable or indifferent to $Y$ if and only if the difference project ($X - Y$), whose flows are $(C_t^X - C_t^Y)$, is acceptable.

### 3. Some properties of the $N(C_t, r)$ function

3.1. If the project is a gift (resp. loss), then the $N(C_t, r)$ function is monotonic decreasing (resp. increasing) with respect to $r$ and there exists no IRR. The converse does not hold, for example, the project with flows $\{11, -40, 40\}$ and executed at period 0 has a $NPV > 0 \forall r$, and exhibits a minimum at $r = 1$.

3.2. If $N(C_t, r)$ is monotonic increasing (resp. decreasing) and there exists IRR, this one is unique because $N(C_t, r)$ is a continuous function, but the unicity of the IRR does not imply monotonicity.

3.3. A variation of the period of execution of the project does not modify the value of the IRR but it produces changes in the intervals of monotonicity of the $N(C_t, r)$ function. If $N_0$ is the net present value of the project beginning in $t = 0$ and $N_M$ the corresponding one when executed at $t = M$, computed both at $t = 0$, then:

$$N_M = N_0 (1+r)^{-M} \quad [10]$$

and it is evident that the slopes of $N_0$ and $N_M$ may have different signs:

$$\frac{\partial N_M}{\partial r} = (1+r)^{-M}(\frac{\partial N_0}{\partial r} - MN_0(1+r)^{-1}) \quad [11]$$

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For example, if the project with flows \{-10, 11\} is executed at period \(M = 0\) the \(NPV\) function is monotonic decreasing. Nevertheless, the same project executed at period \(M = 1\) gives rise to a \(NPV\) function which it is not monotonic (see figure 1).

![Figure 1. The \(NPV\) function for the project with flows \{-10, 11\} executed at period \(M = 0\) is monotonic, but if it is executed at moment \(M = 1\) is not](image1)

3.4. The change of execution period transforms the project into another different one. For example, the project \{-10, 4\} is an investment if it is executed at period zero, but it behaves as a credit for all \(r > 0\) if it is executed at period \(M = 2\) (see figure 2) \(^5\).

![Figure 2. The project with flows \{-10, 4\} executed at period \(M = 0\), behaves as an investment, but if is executed at period \(M = 2\) behaves as a credit](image2)
3.5. The existence of multiple distinct real roots is a sufficient condition for not monotonicity of \( N(r) \) but it is not a necessary condition. For example, the project with flows \{-1, ..., -1, 30\} defined in the temporal interval \([0, 20]\) has an unique IRR, \( r^* = 3.72\% \), but \( M(C, r) \) is decreasing only until point \( \pi = 14.4\% \).

3.6. If \( N(C, r) \) has \( n \) distinct real roots, because of its differentiability, the derivate of \( M(C, r) \) presents at least \((n-1)\) changes of sign by the Rolle’s theorem. The converse does not hold.

3.7. When the flows of a project exhibit more than one change of sign, then: a) \( N(C, r) \) may be monotonic or not and, b) the existence and unicity of the IRR are not guaranteed 6:

a) The project \([-1, 6, -12, 8]\) with three changes of sign presents a monotonic \( N(C, r) \) function. In contrast, the \( N(C, r) \) corresponding to the project \([10, -31, 20]\) is not monotonic and has two sign changes.

b) Projects \( A = \{1, -40, 40\} \), \( B = \{10, -40, 40\} \) and \( C = \{11, -40, 40\} \) have two sign changes, however, while \( N^A(C, r) \) has two roots (2.6 percent y 3.8 percent), \( N^B(C, r) \) has a double root (100 percent) and \( N^C(C, r) \) it has none.

3.8. Equivalence of the \( NPV \) and IRR decision criterion.

**Proposition.** If \( N(C, r) \) has at least one IRR, then the acceptance project criteria \( NPV \) and IRR coincide.

**Proof.**

Let \( r_1^* < ... < r_n^* \) be the real roots of \( N(C, r) \), \( r^l \), with \( r^l > r_n^* \), a discount rate arbitrarily high and \( r^l \), with \(-1 < r^l < r_1^* \), a discount rate arbitrarily low. If the discount rate agrees with some root, then the proposition is trivially true since the acceptance condition is satisfied with equality both for \( NPV \) and IRR. Otherwise, the discount rate will belong to one of the open intervals \((r^l, r_1^*)\), \((r_1^*, r_2^*)\), ..., \((r_n^*, r^l)\).

Let the intervals delimited by two roots \((r_s^*, r_{s+1}^*)\), \( s = 1, ..., n-1 \), as \( M(C, r_s^*) = N(C, r_{s+1}^*) = 0 \) then \( N(C, r) \), as a result of the Bolzano’s theorem, it is either positive or negative in that interval; if in the point \( r_s + \varepsilon, \varepsilon > 0, \varepsilon \rightarrow 0, \delta N(C, r)/\delta r \) is lower (resp. greater) than zero then, the \( N(C, r) \), is negative (resp. positive) in this interval and its derivate in the point \( r_{s+1} - \varepsilon \) is greater (resp. lower) than zero, because \( N(C, r) \) is a continuous function.

Consequently, the project would be rejected (resp. accepted) with the \( NPV \) criterion. The same occurs if the criterion employed is the IRR if root \( r_s \) is taken, because at \( r_s + \varepsilon \) the project behaves as an investment (resp. credit) and \( r_s \) is lower (resp. higher) than the discount rate, and also if the root \( r_{s+1} \) is considered because at \( r_{s+1} - \varepsilon \) the project behaves as a credit (resp. investment) and \( r_{s+1} \) is higher (resp. lower) than the discount rate.

Consider now intervals containing a single root \((r^l, r_1^*)\) and \((r_n^*, r^l)\). Within these intervals, the function does not change of sign because it is continuous; if \( N(C, r) > 0 \) (resp. < 0) in the interval \((r^l, r_1^*)\) the project should be accepted (resp. rejected) with the \( NPV \) criterion.
and as $N'(C_t, r_1^* - \epsilon) < 0$ (resp. $> 0$) the project behaves as an investment (resp. credit) and, since $r^b < r_1^*$ (resp. $r^b > r_1^*$), following the criterion of the IRR would yield the same result. In the interval $(r_a^*, r_b^*)$, if $N(C_t, r) > 0$ (resp. $< 0$) the project should be accepted (resp. rejected) following the $NPV$ criterion and as $N'(C_t, r_1^* + \epsilon) > 0$ (resp. $< 0$) the project behaves as a credit and, since $r^a > r_1^*$ (resp. $r^a < r_1^*$) it would also be accepted (resp. rejected) using the IRR criterion.

4. Interpretation

As a result of executing a project a positive or negative change in wealth takes place. $NPV$ calculates the increase in present wealth due to execution of the project and, therefore, the higher the $NVP$, the better. IRR measures the rate of change of the capital per period, which means that, if it is an investment, the higher the IRR the better, and if it is a credit the opposite occurs. Although it is demonstrated that when the IRR exists it provides the same qualitative result that $NPV$, it is important to consider the economic explanation that, in this case, arises with facility when analyzing some classic examples.

There is no need to have multiple roots to find counter-intuitive results, as it happens with a project with flows $\{-1, 4, -4\}$. It seems evident that it is not a good project (see figure 3) because the sum of the flows is negative. However, only one IRR exists and its value is high, $r^* = 100\%$, which seems contradictory with the value of $NPV$ that it is never positive. In order to solve the apparent contradiction between the $NPV$ and IRR criteria, it is enough to use the definition of investment and credit (see 2.6) and apply both criteria in the usual way.

![Figure 3](image-url)  

This project does not seem very attractive. However shows a high IRR:  

$r^* = 100\%$
There are three possible cases as a function of the value attained by the discount rate $r^0$, $r^0 < r^*$, $r^0 = r^*$ and $r^0 > r^*$. In the first case, $r^0 < r^*$, the $NPV$ function has a positive slope, which means that the project is coherent with the credit definition, the IRR measures the cost of this credit and as this cost outweighs the reference ones, $r^0$, the project is rejected following the IRR criterion. If $r^0 > r^*$, then the slope of the $NPV$ function is negative, the project behaves as an investment and as the yield measured by the IRR is lower than the capital cost $r^0$, the project is rejected. Finally, if $r^0 = r^*$, the coincidence between the recommendations of the $NPV$ and IRR criteria is obvious. In summary, in the three possible cases the $NPV$ and IRR criteria agree.

The most interesting case appears when there are multiple different real roots. Consider the project examined by Hirshleifer (1958), denoted by H hence forth. Its flows are $\{-1, 6, -11, 6\}$ and it shows three IRRs, $r_1^* = 0\%$, $r_2^* = 100\%$ and $r_3^* = 200\%$. The corresponding $N(C_t, r)$ function has a negative slope until arriving at a minimum at $r = 24\%$, then, goes on with a positive slope until reaching a maximum at $r = 146\%$, and finally, it has a negative slope from this point on, can be observed in figure 4.

Taking advantage of the additive property of the $NPV$, the H project will be divided into three subprojects, $H_1 = \{-1, 2\}$, $H_2 = \{4, -8\}$ and $H_3 = \{-3.6\}$, all of them with a same IRR, $r^* = 100\%$. H Project is formed with H1 project executed at period 0, (H10), plus H2 executed at period 1, (H21), and H3 executed at period 2, (H32), as can seen in table 1 and figure 5.

If an investment (resp. credit) improves, the IRR increases (resp. decreases) reflecting a higher yield (resp. lower cost). If we improved the H project increasing the value of the flow in period 3 in one hundredth, we obtain the improved $H^+$ project with flows $\{-1; 6; -11; 54\}$. 

![Figure 4. Case of project with multiple roots, $r_1^* = 0\%$, $r_2^* = 100\%$ and $r_3^* = 200\%$ --taken from Hirshleifer (1958)](image-url)
that also has three roots, \( r_1^+ = 0.5\% \), \( r_2^+ = 99\% \) and \( r_3^+ = 200.5\% \). As a result of the improvement, the IRRs \( r_1^* \) and \( r_3^* \) increase whereas \( r_2^* \) decreases, which indicates that H is a project that behaves in fact as if would be the result of adding qualitatively different projects.\(^{10}\)

If instead of improving H we worsen it, for example, diminishing in one hundredth the flow that takes place in period 3, the resulting H* project will have flows \((-1; 6; -11; 5.99)\), with the three roots \(^{11}\) \( r_1^- = -0.5\% \), \( r_2^- = 101\% \) and \( r_3^- = 199.5\% \), and the result is a decrease of \( r_1^* \) and \( r_3^* \), together with an increase of \( r_2^* \). Table 2 summarizes the process.

Once we know the basic properties of the \( NPV \) and the own characteristics of the H project, the economic analysis is immediate. We know that any investment (resp. credit) is better (resp. worse) the lower is the cost of capital, which is reflected in the negative (resp. positive) sign of the slope of \( NPV \) and, in addition, that the corresponding IRR measures the rate of yield (resp. cost) per period and per unit of capital.

### Table 1

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>IRR (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_{10} )</td>
<td>-1</td>
<td>2</td>
<td></td>
<td></td>
<td>100</td>
</tr>
<tr>
<td>( H_{21} )</td>
<td>4</td>
<td>-8</td>
<td></td>
<td></td>
<td>100</td>
</tr>
<tr>
<td>( H_{32} )</td>
<td>-3</td>
<td>6</td>
<td></td>
<td></td>
<td>100</td>
</tr>
<tr>
<td>( H = H_{10} + H_{21} + H_{32} )</td>
<td>-1</td>
<td>6</td>
<td>-11</td>
<td>6</td>
<td>0, 100, 200</td>
</tr>
</tbody>
</table>

**Figure 5.** Decomposition of H project in three subprojects, \( H_{10}, H_{21} \) and \( H_{32} \), executed at periods 0, 1 y 2 respectively.
On the other hand, H project is equivalent to a set of projects, some of which diminish their value with the discount rate, the investments, while the desirability of others increases when the cost of capital \( r \), increases, the credits. As a result —the original H project— will behave as an investment or as a credit according to the weight of each subproject, which in turn depends on the rate \( r \). Hence, H will have to be analyzed as an investment or as a credit when in a determined interval, H behaves in one or another way, that is to say, in according with the negativeness or positiveness of the slope of \( \text{NPV} \) function.

In order to analyze the desirability of the H project it is necessary to consider different intervals based on the roots of H. Consider the interval formed by a rate \( r_b \) arbitrarily low, which can be assumed higher than minus 100\% \((r > -1)\), and the TIR \( r_1^* \); the unique IRR lying in this interval is \( r_1^* = 0 \); at this point the slope of \( \text{NPV} \) is negative, H behaves therefore as an investment and H is profitable in this interval since \( r_1^* > r_b \). In this interval \( \text{NPV} \) is always positive.

The second interval is \((r_1^*, r_2^*)\) and \( \text{NPV} \) is negative in the interval. If the first IRR is taken, \( r_1^* \), H behaves as a non-profitable investment because \( r_1^* < r \). If the second IRR is taken, \( r_2^* \), as the slope of \( \text{NPV} \) in \( r_2^* \) is positive H behaves as a credit and is not interesting either because the discount rate is lower than \( r_2^* \). In the interval \((r_2^*, r_3^*)\) \( \text{NPV} \) is always positive. Taking the IRR \( r_2^* \), H appears as a desirable credit since the discount rate \( r > r_2^* \); if the IRR, \( r_3^* \), is considered, then H represents a good investment since the slope of \( \text{NPV} \) at this point turns again to be negative, and \( r < r_3^* \).

Finally, taking a rate \( r_a \) high and arbitrarily greater than \( r_3^* \), we have the interval \((r_3^*, r_a)\) in which \( \text{NPV} \) is always negative. The only IRR of the interval is \( r_3^* \), and H behaves as a non-profitable investment because \( r_a > r_3^* \).

As we have observed, the acceptance conditions for the IRR’s always agree with the rule \( \text{NPV} \geq 0 \), see Table 3.

In order to complete the economic interpretation, it is necessary to verify the coherence of a high yield rate, measured by an IRR, with a low value for \( \text{NPV} \). If the discount rate is \( r = \)}
150%, for example, the project behaves as an investment, the relevant IRR is $r^* = 200\%$, a high value, while the corresponding NPV is low, NPV ($r = 1.5$) = 0.024. As H project is equivalent to a mixture of investments and credits, the net capital inverted in average is low; for this reason a low absolute yield is compatible with a high relative yield. Given a discount rate ($r = 1.5$ in this case), one can always construct an h project of the same duration than H, the same values for the IRR and NPV and only two non-null flows, a capital $K$ at initial period 0 and an amount $B$ in the last period, $T$. The desired values of $K$ and $B$ are such that

\[
K + B / (1 + r)^T = NPV(r) \quad \text{[12]}
\]

\[
K + B / (1 + r^*)^T = 0 \quad \text{[13]}
\]

from these equations, the amount of net capital involved in average is obtained, as expected, is low: $K = -0.033$. The next table shows an example for a different rate for each interval of the NPV function of H project, the agreement of the NPV and IRR criteria is verified for all the IRR’s of H project, and where the amount of capital $K$ involved is exhibited.

### Table 3
Coincidence of the acceptance rule of IRR and NPV

<table>
<thead>
<tr>
<th>condition</th>
<th>N'</th>
<th>at</th>
<th>type</th>
<th>IRR acceptance</th>
<th>NPV acceptance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r \leq r^*$ 1</td>
<td>&lt; 0</td>
<td>$r^*$</td>
<td>investment</td>
<td>$r^* \geq r$ yes</td>
<td>$\geq 0$ yes</td>
</tr>
<tr>
<td>$r^* &lt; r &lt; r^*$ 2</td>
<td>&lt; 0</td>
<td>$r^*$</td>
<td>investment</td>
<td>$r^* &lt; r$ no</td>
<td>&lt; 0 no</td>
</tr>
<tr>
<td>$r^* \leq r \leq r^*$ 3</td>
<td>&gt; 0</td>
<td>$r^*$</td>
<td>credit</td>
<td>$r^*$ ≥ r yes</td>
<td>$\geq 0$ yes</td>
</tr>
<tr>
<td>$r^* &lt; r$ 4</td>
<td>&lt; 0</td>
<td>$r^*$</td>
<td>investment</td>
<td>$r^* &lt; r$ no</td>
<td>&lt; 0 no</td>
</tr>
</tbody>
</table>

### Table 4
Example of application with rates –10, 40, 145 and 230 percent

<table>
<thead>
<tr>
<th>condition</th>
<th>N'</th>
<th>at</th>
<th>capital</th>
<th>type</th>
<th>IRR ≥ r acceptance</th>
<th>NPV ($r$) acceptance</th>
</tr>
</thead>
<tbody>
<tr>
<td>–0.1 &lt; 0</td>
<td>&lt; 0</td>
<td>$r^*$</td>
<td>–0.853</td>
<td>investment</td>
<td>0 &gt; –0.1 yes</td>
<td>0.317 &gt; 0 yes</td>
</tr>
<tr>
<td>0 &lt; 0.4 &lt; 1</td>
<td>&lt; 0</td>
<td>$r^*$</td>
<td>–0.2201</td>
<td>investment</td>
<td>0 &lt; 0.4 no</td>
<td>–0.1399 &lt; 0 no</td>
</tr>
<tr>
<td></td>
<td>&gt; 0</td>
<td>$r^*$</td>
<td>0.0730</td>
<td>credit</td>
<td>1 &gt; 0.4 no</td>
<td></td>
</tr>
<tr>
<td>1 &lt; 1.45 &lt; 2</td>
<td>&gt; 0</td>
<td>$r^*$</td>
<td>0.0535</td>
<td>credit</td>
<td>1 &lt; 1.45 yes</td>
<td>0.0244 &gt; 0 yes</td>
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<td>$r^*$</td>
<td>–0.0292</td>
<td>investment</td>
<td>2 &gt; 1.45 yes</td>
<td></td>
</tr>
<tr>
<td>2 &lt; 2.3</td>
<td>&lt; 0</td>
<td>$r^*$</td>
<td>–0.1005</td>
<td>investment</td>
<td>2 &lt; 2.3 no</td>
<td>–0.025 &lt; 0 no</td>
</tr>
</tbody>
</table>
5. Conclusions

The NPV function is not monotonic except in particular cases, which causes the improperly named anomalies that make difficult the interpretation of results. A project can behave as an investment or as a credit depending on the prevailing discount rate and the period of execution, and can be understood as the result of the aggregation of several subprojects of different type. All the real roots of the NPV function have an economic meaning, some ones measure the yield of the project as an investment and others compute the cost of the project as a credit. In order to understand the information that provides the multiple roots of the NPV function, is enough to adopt a new definition of investment and credit. In this way, we can interpret correctly the counter-intuitive results due to the lack of monotonicity of the NPV function, it is not necessary to modify the hypotheses of the standard model since, contrary to what is usually assumed, the decisions based on IRR criterion always agree with the ones based on NPV.

It is needless to say that if some of the hypotheses of the NPV model are relaxed, as the one of perfect capital market, the results lose validity and it is necessary to use more general evaluation models. For example, when the investment and reinvestment rate do not agree, it is necessary to apply a more general model, as the one by Montllor (1978). On the other hand, choice of projects problem has not been here. The interested reader can fruitfully consult Cantor and Lippman (1995) and Herreolen et al. (1995).

Notes

1. Rate r is known as the price of capital, and discount rate, among other denominations. That r responds to one or another denomination depends on the specific case under study — see Hirshleifer (1958) and Souto (2001).

2. Here it is understood, that a project always improves whenever the value of any flow is increased or it is added (resp. suppressed) a positive (resp. negative) flow \( x \in R \). If \( r > 0 \), the project improves if a positive (resp. negative) flow is put forward (resp. backward), whereas if \( r < 0 \) it happens the opposite. When an improvement occurs, the NPV always increases. If there exists at least one root, when an improvement of the project takes place, the IRR increases its value when the project behaves as an investment, and diminishes its value in the case of a credit.

3. The condition \( NPV \geq 0 \) can be interpreted as a no rejection rule, and it does not hold necessarily in problems of project selection — see Cantor and Lippman (1995) — since a project can behave both as an investment or as a credit (see 2.6 and 3.4).

4. It assumes that IRR and discount rate are greater than minus one.

5. That a (bad) typical investment such as \{-10, 4\} behaves as a credit if the execution period is changed, it is a non desirable characteristic of the NPV function, that it is necessary to know and remind to avoid erroneous interpretations of the results.

6. Descartes’s Rule: The number of positive zeros of a polynomial, \( p(x) \), is equal to the number of sign changes in the sequence of its coefficients, or to a lower number that it differs of this one in a positive par number. The number of negative zeros it obtains in a similar form from the polynomial \( p(-x) \).

7. Opened intervals are defined to include those cases in which the \( N(r) \) function is tangent to the r axis, those IRRs \( r_j^* \) with \( \delta N(r)/\delta r_j^* = 0 \). The demonstration also includes the case of lack of monotony within an interval.

9. The negative sum of the flows implies that \( NPV \) calculated to a zero rate is also negative, but cannot affirm that \( \sum C_t < 0 \Rightarrow NPV < 0 \), \( \forall r > 0 \). For example, with the flows \( \{-0.5; 4; -4\} \) results \( \sum C_t < 0 \), but \( NPV > 0 \) if \( 0.172 < r < 5.83 \).

10. This specific form to divide \( H \) project has been chosen by regularity and explanatory capacity, but there are others, being valid all of them. The immediate form is \( H = A_0 + B_2 \), let the flows of \( A = \{-1, 6\} \) and of \( B = \{-11, 6\} \), with \( A \) executed at period 0 and \( B \) at 2. As \( A_0 \) behaves as an investment and \( B \) as a credit, if it is executed at period 2, \( H \) can be explained as the aggregation of two opposed projects, \( A_0 \) an investment and \( B_2 \) a credit.

11. As \( \eta^r \) is negative and the other two are positive, there exists the temptation to deny the economic sense to \( \eta^r \). Nevertheless, if the TIR \( \eta^*_1 \) of \( H \) project makes sense, the \( \eta^r \) of the \( H \), that arises as a result of a small modification of \( H \) must also have it. That the relative yield of an investment will be negative is not a good news if the discount rate is positive, although does not suppose a problem if, against the habitual way, the discount rate is negative and lower than the IRR. That the IRR of a credit will be negative does not have too much sense, because it would mean that the capital is not a good, but something non-desirable; nevertheless, cases of negative types of real interest have occurred during long time in some countries, reason why it is not pointless to consider this case.

12. The result of the monotony analysis of the \( NPV \) function, made here, contrasts with the recent contribution of Saak and Hennessy (2001), for the continuous case. In special cases the \( NPV \) function is monotonic, as it is demonstrated in Arrow and Levhari (1969).

References


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**Resumen**

Las mal llamadas anomalías que surgen en el cálculo y la interpretación del Valor Actual Neto (VAN) y la Tasa Interna de Rendimiento (TIR) son fácilmente superables, si se tienen en cuenta las propiedades de la función VAN y se define adecuadamente lo que es una inversión y un crédito. Todas las raíces de la función VAN tienen significado económico y, cuando existe por lo menos una TIR, los criterios VAN y TIR coinciden.

**Palabras clave**: Valor Actual neto, Tasa Interna de Rendimiento, Análisis de inversiones, Evaluación de proyectos.

**Clasificación JEL**: Q28, D92.