



MEASURING AND TESTING FOR THE SYSTEMICALLY IMPORTANT FINANCIAL INSTITUTIONS

Carlos Castro.
Stijn Ferrari.

SERIE DOCUMENTOS DE TRABAJO

No. 101
Mayo 2011

Measuring and testing for the systemically important financial institutions

Carlos Castro*

Facultad de Economía, Universidad del Rosario, Colombia

Stijn Ferrari

National Bank of Belgium

May 26, 2011

Abstract

This paper analyzes the measure of systemic importance $\Delta CoVaR$ proposed by Adrian and Brunnermeier (2009, 2010) within the context of a similar class of risk measures used in the risk management literature. In addition, we develop a series of testing procedures, based on $\Delta CoVaR$, to identify and rank the systemically important institutions. We stress the importance of statistical testing in interpreting the measure of systemic importance. An empirical application illustrates the testing procedures, using equity data for three European banks.

1 Introduction

The 2007-2008 financial crisis has shifted the focus from the assessment of the resilience of individual financial institutions towards a more systemic approach. Hence, macro-prudential supervision and regulation will play a vital role in the new financial architecture (BIS, 2010). In particular, Basel 3 advocates financial regulation focused on limiting systemic risk. As illustrated by the crisis, an important aspect of systemic risk, which broadly speaking is the risk of a widespread crisis in the financial system, is the propagation of adverse shocks to a single institution through the rest of the system. Therefore, mitigating the risk stemming from so-called systemically important institutions, i.e., the financial institutions whose failure generates a large adverse impact on the financial system, has been identified as an important policy item. In particular, the Basel Committee and the Financial Stability Board are developing an integrated approach to deal with systemically important financial institutions. Potential

*corresponding author: carlos.castro@urosario.edu.co; Tel: +5712970200Ext.652. The findings, recommendations, interpretations and conclusions expressed in this paper are those of the authors and not necessarily reflect the view of the Department of Economics of the Universidad del Rosario or the National Bank of Belgium.

regulatory instruments that may be targeted at systemically important financial institutions in the near future include capital and liquidity surcharges, systemic levies, and contingent capital and/or bail-inable debt. The purpose of this type of regulations would be to reduce the probability of failure of systemically important institutions and to mitigate the impact of their failure if that nevertheless occurred.

Yet, a crucial step in macro-prudential supervision and regulation aimed at reducing the risk of systemically important institutions is to identify which institutions are in fact systemically important. In order to properly measure the systemic importance of a financial institution, the measure must concentrate on the institution's potential impact on the system in the event of failure or distress, which largely boils down to capturing the spillover or contagion effects from the institution in question to the rest of the system. As spillover effects operate through several channels, both direct and indirect, measuring the systemic importance of financial institutions is not a straightforward task.¹ One approach taken in the literature is to infer the impact of the failure or distress of a financial institution directly from market data, such as stock returns or CDS spreads, rather than separately modelling the various contagion channels. Within this group of measures, the so-called co-risk measures have attracted considerable attention in both academic and policy research. Intuitively, co-risk measures determine the systemic importance of a financial institution as the increase in the risk of the financial system (or other individual financial institutions in the system) when the institution in question encounters distress. Perhaps the best known co-risk measure of systemic importance is $\Delta CoVaR$ proposed by Adrian and Brunnermeier (2009, 2010), which refer to the increase in system-wide risk due to the distress of a financial institution (i.e., the estimated value of $\Delta CoVaR$) as the "systemic risk contribution" of that financial institution. While $\Delta CoVaR$ has already been extensively applied and extended both in the academic literature and by policymakers (see e.g., Fong et al., 2009; IMF, 2009; Chan-Lau, 2010a,b; Deutsche Bundesbank, 2010; Gauthier, Lehar and Souissi, 2010; Jager-Ambrozewicz, 2010; Girardi and Ergun, 2011; and Rodríguez-Moreno and Peña, 2011), statistical testing procedures to assess the significance of the findings and interpretations based on this co-risk measure have not yet been developed. In particular, the current applications of $\Delta CoVaR$ do not test whether the systemic risk contribution for a given financial institution is significant, and whether the systemic risk contribution of one financial institution is significantly larger than that of another financial institution. This is of paramount importance for drawing credible conclusions that can be used for policymaking, however.

In this paper we fill this gap by deriving, within a linear quantile regression framework, two hypothesis tests and their respective test statistics. In particular, we develop a test of significance of $\Delta CoVaR$ that allows determining

¹This may entail several identification issues, which we discuss fully in a previous survey paper (Castro and Ferrari, 2010). In particular, determining the systemic importance of a financial institution requires separating spillover or contagion effects from the effects of a systematic shock through common exposures, as well as identifying cascade or domino effects.

whether or not a financial institution can be classified as being systemically important on the basis of the estimated systemic risk contribution. In addition, we derive a test of dominance aimed at testing whether or not, according to $\Delta CoVaR$, one financial institution is more systemically important than another. To this end, we analyze $\Delta CoVaR$ within the context of a similar class of risk measures used in the risk management literature. More specifically, we study the characteristics of the statistical process underlying $\Delta CoVaR$ as a co-risk measure and relate the estimation and testing of the spillover effects to the extensive literature on treatment effects. The testing procedures developed in this literature provide a basis for the significance and dominance tests in this paper. Finally, we provide a simple example that illustrates our testing procedures and illustrates the usefulness of the tests as a tool in the process of identifying and ranking the systemically important institutions based on $\Delta CoVaR$.

Our paper builds on and contributes to the evolving literature on measuring the systemic importance of financial institutions. Within this field, studies that only use publicly available data have recently received considerable attention.² In general, the only inputs required in these approaches are market prices for the financial institutions in the system (e.g., stock returns or CDS spreads, which are available at a high frequency and have a forward-looking nature) and the financial institutions' balance sheet information. Within this class of market information-based measures, one can in general distinguish between measures that assess the impact of the failure or distress of a particular institution in terms of the likelihood of spillover effects occurring, and measures that assess the severity of the losses associated with the failure or distress of the institution. Applications of the former class of measures generally consider the probability of the failure or distress of a number of institutions in the system conditional on the failure of another institution (see e.g., Hartmann, Straetmans and de Vries, 2005; Geluk, Haan and de Vries, 2007; Segoviano and Goodhart, 2009; and Zhou, 2009). The second type of market information-based measures captures the severity of losses and consists of two types of methods: methods that infer the impact of the failure or distress of a financial institution directly from market data, without any need to quantify the overall risk in the system in advance, and methods that first quantify the overall risk in the system using a structural model of portfolio credit risk and then determine the contribution of each individual institution to system-wide risk to determine systemic importance (see e.g. Elsinger, Lehar and Summer, 2006; Huang, Zhu and Zhou, 2009; Tarashev, Borio and Tsatsaronis 2009; and Gauthier, Lehar and Souissi 2010). While having a clear interpretation of being the losses imposed on the financial system or other individual financial institutions in case of the failure or distress of the financial institution in question, these measures of systemic importance incorporate a substantial degree of model risk; small changes in the assumptions may alter not only the estimated level of systemic risk, but also the set of institutions that are identified as systemically important. Methods that infer

²IMF (2009) provides an overview of some alternative approaches, which are generally more data-demanding (such as e.g., the indicator-based approach).

the impact of the failure or distress of a financial institution directly from market data, on the other hand, have the advantage of requiring little information and using statistical methods with minimal assumptions, to obtain an estimate of a financial institution's potential impact on the system. Therefore, co-risk measures, as provided by Adrian and Brunnermeier (2009, 2010), Chan-Lau (2010a,b), Engle and Brownlees (2010), Billio et al. (2010) and White et al. (2010), can be expected to provide a more robust assessment of the systemic importance of financial institutions. However, while co-risk measures may provide an assessment of the systemically important institutions with only minimal distributional assumptions and no need to first quantify overall risk, these approaches have as a drawback the difficult interpretability of the scale of the measure of systemic importance, i.e., there seems to be no obvious answer to the question of when the impact of a financial institution on the system (or on another institution) is large enough for the institution to be considered as systemically important. The same holds for the question of whether the difference between one financial institution's impact on the system and that of another is large enough to consider the former more systemically important than the latter. While we do not aim at resolving these questions in this paper, we provide statistical tests that are crucial in the process of identifying and ranking the systemically important institutions based on $\Delta CoVaR$. In particular, our tests allow determining whether the estimated systemic risk contribution of a financial institution significantly exceeds some threshold level, and whether the systemic impact of one financial institution is significantly larger than that of another. As the most recent literature on co-risk measures (Billio et al., 2010; Engle and Brownlees, 2010; and White et al. 2010) has (to some extent) been more inclined in deriving the statistical properties of the co-risk measure in order to perform the required inference, the contribution of this paper is closer to the most recent strand of the literature.

From a methodological point of view, we relate to the literature of inference in the quantile regression framework (Koenker and Machado, 1999; Koenker and Xiao, 2002; Chernozhukov and Fernandez-Val, 2005; and Chernozhukov and Hansen, 2006). In addition, we relate to the literature on tests of stochastic dominance (Linton et al., 2005). Our approach differs from the traditional tests of stochastic dominance in two respects. First, our tests of dominance are formulated in terms of the quantile function. Second, we are interested in the conditional quantile function (or the response function) of the variable of interest, rather than an unconditional distribution or the residuals from some estimated model. More specifically, following earlier work on inference based on a quantile process, we develop a test based on the Kolmogorov-Smirnov statistic. This approach is highly attractive since the test statistic is asymptotically distribution-free.

The remainder of the paper is organized as follows. Section 2 presents a review of $\Delta CoVaR$ against the background of traditional quantile-based risk measures. Particular attention is given to the types of hypotheses regarding the systemic importance of financial institutions one may want to test in this framework. In Section 3 we develop the interpretation of $\Delta CoVaR$ in terms of

the underlying quantile theory that will form the basis of our testing procedures. In Section 4 we develop a series of testing procedures for identifying and ranking the systemically important institutions, and Section 5 provides an empirical application to illustrate the testing procedures. Section 6 concludes.

2 Review of $\Delta CoVaR$

In this section we present the definition of $\Delta CoVaR$ as proposed by Adrian and Brunnermeier (2009). In addition, we discuss how $\Delta CoVaR$ can be estimated in a parametric quantile regression framework and which type of information can be inferred from estimates of $\Delta CoVaR$. As a general background, we first provide a brief review of traditional quantile-based risk measures.

2.1 Quantile-based risk measures

The focus of risk management practice is to estimate and limit potential losses. The most commonly used risk measures are those that focus on extreme losses (i.e., the tail of the distribution): value-at-risk (VaR) and expected shortfall (ES). Let X denote a random variable with probability distribution F_X , expressing the losses of for example, a financial institution. Then the value-at-risk ($VaR_X(\tau)$) is the threshold value of losses that will only be exceeded $1 - \tau$ percent of the time on average over some predetermined interval of time:

$$VaR_X(\tau) := \inf \{x \in \mathbb{R} : F_X(x) \geq \tau\}.$$

Expected Shortfall is an alternative risk measure that considers additional information from the tail of the loss distribution, beyond the threshold value considered exclusively by the VaR risk measure:³

$$ES_X(\tau) := \frac{1}{1 - \tau} \int_{\tau}^1 VaR_X(u) du.$$

Instead of fixing a particular confidence level τ , the idea of ES is to average VaR over all levels, $u \geq \tau$, and thus obtain an average value for the tail of the distribution of X . Such risk measures are meant to represent the overall downside risk of the institution.

Figure 1 illustrates the notion of VaR and ES. The figure shows the probability density, F_X , of a hypothetical loss distribution of a financial institution. In addition the figure contains a series of vertical lines indicating the mean loss ($E(X)$), the $VaR_X(0.95)$ and the $ES_X(0.95)$. Since $E(X) = -2.5 < 0$, the financial institution on average expects to make a profit (negative losses imply gains).

The loss distribution is also asymmetric (skewed to the left). Therefore, even though the institution on average makes a profit, it can expect to lose much more (≈ -10) than what it can potentially gain (≈ 7.5). The $VaR_X(0.95)$ is

³Expected shortfall is also known as conditional VaR or expected tail loss.

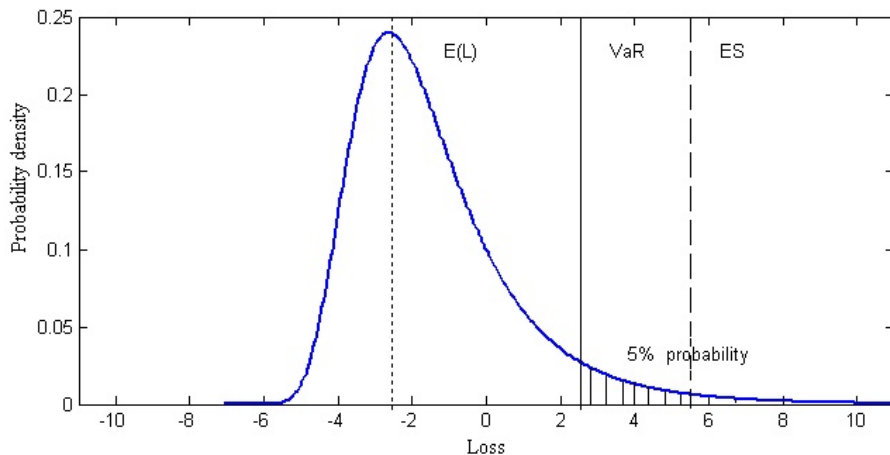


Figure 1: Value-at-risk and expected shortfall.

approximately 2.5, indicating that there is a 5% probability that the institution loses at least this amount for a given time horizon. The $ES_X(0.95)$, which considers additional information from the tail of the loss distribution, beyond the threshold value considered exclusively by the VaR risk measure (the shaded area), is approximately 5.5 in Figure 1.

The VaR risk measure is equivalent to the more general concept of the quantile function, which for a random variable X with probability distribution F_X , is defined as follows:

Definition 2.1 For $\tau \in (0,1)$ the τ -quantile function of distribution F_X is given by:

$$Q_X(\tau) := \inf \{x \in \mathbb{R} : F_X(x) \geq \tau\}.$$

Since the definition of the quantile function $Q_X(\tau)$ is equivalent to that of $VaR_X(\tau)$, VaR is a quantile-based measures of risk. In Section 3, we discuss some interesting properties of quantile functions, which do not only hold for the risk measure VaR, but also for some risk measures based on VaR, such as ES. As we will show in Section 3, this is also the case for $\Delta CoVaR$. In particular, we will show that $\Delta CoVaR$ is also a quantile function; this property of the measure will play an important role in deriving our testing procedures.

2.2 The co-risk measure $\Delta CoVaR$

Intuitively, co-risk measures determine the systemic importance of a financial institution as the increase in the risk of the financial system when the institution in question encounters distress. Co-risk measures of systemic importance

generally infer the impact of the failure or distress of a financial institution directly from market data, such as stock returns or CDS spreads, without relying on a structural credit risk model to first quantify total risk in the system. The advantage of these approaches is therefore that they require little information and make use of statistical methods with minimal assumptions, to obtain an estimate of a financial institution’s potential impact on the system. Perhaps the best known co-risk measure of systemic importance is $\Delta CoVaR$ proposed by Adrian and Brunnermeier (2009, 2010).

2.2.1 Definition

The calculation of $\Delta CoVaR$ makes use of the risk measure VaR. In Adrian and Brunnermeier (2009), $\Delta CoVaR$, is a composition of the conditional and the unconditional VaR of the financial system. First, the (unconditional) VaR from the distribution of, for instance, stock returns for an index of financial institutions (the financial system) X^{index} is computed.⁴ This represents a VaR for the financial system:

$$P(X^{index} \leq VaR_{X^{index}}(\tau)) = \tau.$$

Second, the conditional VaR (CoVaR) is computed as the VaR for the distribution of the stock returns of the index of financial institutions, conditional on the stock return of the financial institution i in question X^i being at its VaR-level (in distress):

$$P(X^{index} \leq CoVaR_{X^{index}|i(\tau_X)}(\tau) \mid X^i = VaR_{X^i}(\tau_X)) = \tau,$$

where τ_X is the confidence level at which the individual institution’s return X^i is evaluated; this may equal the confidence level τ at which the system’s return X^{index} is evaluated, but this is not necessarily the case. Without loss of generality and to simplify notation, from now on we consider the case where $\tau = \tau_X$ and suppress it from the CoVaR notation, unless otherwise stated.

The difference between CoVaR and the unconditional VaR of the system is called $\Delta CoVaR$, which is the eventual measure of systemic importance:⁵

$$\Delta CoVaR^{index|i}(\tau) = CoVaR_{X^{index}|i}(\tau) - VaR_{X^{index}}(\tau). \quad (1)$$

Adrian and Brunnermeier (2009, 2010) refer to this measure as the “systemic contribution” of financial institution i . Intuitively, it measures the increase in the risk of the financial system when the institution in question encounters distress.

⁴We in fact define our variables of interest as the negative of stock returns, so that the results can be interpreted in terms of losses.

⁵In a revised version of the paper Adrian and Brunnermeier (2010) define $\Delta CoVaR$ as the difference two conditional distributions evaluated at different points in the design space. Under this setup, the measure of systemic risk contribution is $\Delta CoVaR^{index|i}(\tau) = CoVaR_{X^{index}|i(\tau_1)}(\tau) - CoVaR_{X^{index}|i(\tau_0)}(\tau)$, where $CoVaR_{X^{index}|i(\tau_x)}(\tau)$ denotes the VaR of the system conditional on the financial institution’s return X^i being evaluated at its τ_x -th quantile, and $\tau_1 > \tau_0 = 0.5$ (e.g., $\tau_1 = 0.99$).

2.2.2 Estimation

The estimation of the co-risk measure $\Delta CoVaR$ can be accomplished in several ways. In their application of the measure, Adrian and Brunnermeier (2009, 2010) use a parametric approach based on quantile regression (QR). This parametric approach, which is followed in most of the applications of $\Delta CoVaR$, is embedded in the extensively developed linear location scale-model (Koenker, 2005). In this linear location-scale framework, the dependent variable, which in our application of $\Delta CoVaR$ is the stock returns for the index of financial institutions X^{index} , follows some factor structure

$$X_t^{index} = K_t\delta + (K_t\gamma)\varepsilon_t, \quad (2)$$

where K_t is a k -dimensional vector of factors and $t = 1 \dots T$ denotes time. The factors influencing the financial index variable in the context of $\Delta CoVaR$ typically include the stock return X_t^i for a financial institution i of interest, a constant term and possibly a set of common variables.⁶ The error term ε_t is assumed to be i.i.d with zero mean and unit variance, and is independent of K_t so that $E[\varepsilon_t | K_t] = 0$. The market variable is generated by a stochastic process within the location-scale family of distributions, implying that conditional expectation and volatility of the random variable X_t^{index} depends on the k -dimensional vector of factors, K_t . Since expression (2) represents the conditional distribution function for X_t^{index} , it can analogously be written in terms of a quantile function representation:

$$\begin{aligned} Q_{X^{index}|K}(\tau) &= K_t\delta + (K_t\gamma)Q_\varepsilon(\tau) \\ &= K_t\beta(\tau), \end{aligned} \quad (3)$$

where $\beta(\tau) = \delta + \gamma Q_\varepsilon(\tau)$. Note that in this model the quantile varying coefficients are identical up to an affine transformation. While $\tau \in (0, 1)$, we are typically interested in values of τ close to 1, since $\Delta CoVaR$ is a risk measure. The previous quantile function can be estimated via the quantile regression (see Koenker, 2005):

$$\beta(\tau) = \underset{\beta(\tau)}{\operatorname{argmin}} \sum_t \rho_\tau(X_t^{index} - K_t\beta(\tau)),$$

where $\rho_\tau(u) = u(\tau - I(u < 0))$.

In this QR-framework, the increase in system-wide risk due to the distress of financial institution i , $\Delta CoVaR^{index|i}(\tau)$, can be obtained as follows. First, equation (2) is estimated with the stock return of financial institution i excluded from the explanatory variables, i.e., with only a constant term and possibly a set of common variables included in K_t . The fitted value of this regression will result in the unconditional VaR of the financial system returns $VaR_{X^{index}}(\tau)$. Secondly, equation (2) is estimated with the stock return X_t^i of financial institution i included (in addition to a constant term and possibly a set of common

⁶Note that this model also nest the pure location shift model when $\gamma K_t = 1$.

variables) in the explanatory variables K_t . The fitted value of this regression, with X_t^i evaluated at a distressed level, say $VaR_{X^i}(\tau)$, results in the VaR of the financial system returns conditional on financial institution i being in distress, $CoVaR_{X^{index|i}}(\tau)$. From the definition of $\Delta CoVaR^{index|i}(\tau)$ in expression (1), it follows that the systemic risk contribution of financial institution i is obtained by taking the difference between the estimated values for $CoVaR_{X^{index|i}}(\tau)$ and $VaR_{X^{index}}(\tau)$.

2.2.3 Inference

Since $\Delta CoVaR$ is a co-risk measure and therefore serves as proxy for the potential impact that the failure or distress of a given financial institution may have on the financial system as a whole, it can be considered to be a useful measure for identifying and ranking systemically important financial institutions. On the basis of the $\Delta CoVaR$ methodology, systemically important financial institutions can be identified as those institutions for which $\Delta CoVaR^{index|i}(\tau)$ exceeds a given threshold level. In addition, financial institutions can be ranked in terms of systemic importance on the basis of a ranking of their $\Delta CoVaR^{index|i}(\tau)$; institutions with a larger $\Delta CoVaR^{index|i}(\tau)$ can be considered to be more systemically important. Such a ranking of financial institutions according to their systemic importance may be useful when policy instruments aimed at reducing the risk imposed on the system by financial institutions are levied in a differentiated way, with the instrument being more strict or binding for more systemically important institutions.

While this type of identifications and rankings of systemic importance have been provided in several applications (and extensions) of $\Delta CoVaR$ (see e.g., Fong et al., 2009; IMF, 2009; Chan-Lau, 2010a,b; Deutsche Bundesbank, 2010; Gauthier, Lehar and Souissi, 2010; Jager-Ambrozewicz, 2010; Girardi and Ergun, 2011; and Rodríguez-Moreno and Peña, 2011), the statistical significance of the results and interpretations based on $\Delta CoVaR$ exceeding a certain threshold or $\Delta CoVaR$ of one financial institution being larger than that of another has not been considered yet. This is of paramount importance for drawing credible conclusions that can be used for policymaking, however. We fill this gap by proposing tests for two types of hypotheses and the relevant test statistics, which we refer to as a test of significance and a test of dominance:

Significance As mentioned above, systemically important financial institutions can be identified as those institutions for which $\Delta CoVaR^{index|i}(\tau)$ exceeds a given threshold level. Without loss of generality, we set this threshold level equal to zero in the development of our hypothesis test. Hence, a hypothesis test for the identification of a systemically significant institution will have the following null hypothesis:

$$H_0 : \Delta CoVaR^{index|i}(\tau) = 0, \quad (4)$$

for a given $\tau \in (0, 1)$ or, more specifically, on a given subset of $\mathcal{T} \subset (0, 1)$. This implies that under the null hypothesis there is no statistical difference between

the empirical conditional VaR of the financial system's returns, $CoVaR_{X^{index}|i}(\tau)$, and the unconditional VaR of the financial system's returns, $VaR_{X^{index}}(\tau)$. Therefore, any change in the financial institution's individual stock return does not have a significant effect on the index for financial institutions at the given quantile τ .

Dominance In order to establish some form of ranking across the institutions according to their systemic importance, the magnitude of the estimated $\Delta CoVaR$ could be compared for different pairs of financial institutions i and j . Since the unconditional VaR of the system, $VaR_{X^{index}}(\tau)$, appears in both $\Delta CoVaR^{index|i}(\tau)$ and $\Delta CoVaR^{index|j}(\tau)$, this boils down to comparing $CoVaR_{X^{index}|i}(\tau)$ and $CoVaR_{X^{index}|j}(\tau)$. Therefore, a hypothesis test to test whether financial institution i is statistically more systemically important than institution j will have the following null hypothesis:

$$H_0 : CoVaR_{X^{index}|i}(\tau) > CoVaR_{X^{index}|j}(\tau), \quad (5)$$

for a given $\tau \in (0, 1)$ or, more specifically, on a given subset of $\mathcal{T} \subset (0, 1)$. As we will show in the next section, this test is equivalent to a test of stochastic dominance between two conditional distributions (or equivalently, quantile functions); we therefore refer to this hypothesis test as a test of dominance.

3 Underlying quantile theory

Before presenting our testing procedures for the abovementioned types of hypotheses, we develop in this section the interpretation of $\Delta CoVaR$ in terms of the underlying quantile theory that will form the basis of the testing procedures. In particular, we discuss the property that $\Delta CoVaR$ is a quantile function, which plays an important role in deriving our testing procedures. Second, we relate $\Delta CoVaR$ to the well-known concept of quantile treatment effects. Inference procedures developed in the quantile treatment literature will serve as a basis for the testing procedures that we will develop for $\Delta CoVaR$ in the next section.

3.1 Quantile functions

In this subsection, we show that $\Delta CoVaR$, being the difference between a conditional and an unconditional quantile function, is a quantile function.

3.1.1 Properties of quantile functions

As explained in the previous section, $\Delta CoVaR$ is based on the quantile-based risk measure VaR, which is equivalent to the more general concept of the quantile

function. There are some interesting properties of quantile functions that play an important role in deriving our testing procedures⁷:

Theorem 3.1 (Properties of the quantile function) *Let $a_0, a_1 \in \mathbb{R}$, $a_1 > 0$. Then for $w, \tau \in [0, 1]$*

1. $Q_X(\tau)$ is a left continuous and non-decreasing function of τ .
2. $-Q_X(1 - \tau) = Q_X(\tau)$, when $F(X)$ is symmetric.
3. If $Q_X(\tau)$ and $Q_{X'}(\tau)$ are quantile functions then $Q_X(\tau) + Q_{X'}(\tau)$ is also a quantile function.
4. If $Q_X(\tau)$ and $Q_{X'}(\tau)$ are quantile functions and non-negative then $Q_X(q) * Q_{X'}(\tau)$ is also a quantile function.
5. If $Q_X(\tau)$ and $Q_{X'}(\tau)$ are quantile functions then $wQ_X(\tau) + (1-w)Q_{X'}(\tau)$ is also a quantile function.
6. The quantile function for $\frac{1}{X}$ is $\frac{1}{Q_X(1-\tau)}$.
7. If $Y = a_0 + a_1X$, then $Q_Y(\tau) = a_0 + a_1Q_X(\tau)$.

Property 1. defines the quantile function (note that the quantile function is the inverse of the distribution function: $Q_X = F_X^{-1}$), 2. implies the reflection principle, 3. and 4. indicate that quantile functions are closed under addition and multiplication, 5. implies there is some intermediate value, 6. is the reciprocal and 7. indicates scale and location equivariance. With respect to the last point, quantile functions enjoy another equivariance property, one much stronger than the one presented under point 7. This additional property may be denoted as equivariance to monotone transformations and has important consequences for conditional quantiles:

Theorem 3.2 *Let $g(x)$ denote a left continuous and non-decreasing function, then the quantile functions of $g(X)$ is,*

$$Q_{g(X)}(\tau) = g(Q_X(\tau)).$$

The quantiles of the transformed random variable $g(X)$ are simply the transformed quantiles of the original random variable X .

Corollary 3.3 *For a pair of random variables (X, Y) , the conditional quantile representation is,*

$$Q_{Y|X=x}(\tau) = Q_Y(s),$$

where $s = Q_{F_{Y|X=x}}(\tau)$, naturally $s \in [0, 1]$. To compute s note that,

$$\begin{aligned} \tau &= F_{F_{Y|X=x}}(s) = P[F_Y \leq s \mid X = x] \\ &= P[Y \leq Q_Y(s) \mid X = x] = F_{Y|X=x}(Q_Y(s)). \end{aligned}$$

⁷For a detailed discussion of the properties of quantile functions and proofs, see Parzen (1979, 2004).

3.1.2 $\Delta CoVaR$ as a quantile function

Given that $\Delta CoVaR$ is a composition of a conditional VaR and an unconditional VaR, it can also be expressed as the difference between a conditional and an unconditional quantile function:

$$\Delta CoVaR^{index|i}(\tau) = Q_{X^{index}|X^i}(\tau) - Q_{X^{index}}(\tau).$$

As presented in the previous section, we use a linear function to represent the relationship between the random variables (X^{index}, X^i) . Assuming without loss of generality that in the remainder of the paper K only includes X^i and a constant term, the conditional quantile function for the response variable X^{index} given X^i can be defined as:⁸

$$\begin{aligned} Q_{X^{index}|X^i}(\tau) &= CoVaR_{X^{index}|i}(\tau) \\ &= \beta_0(\tau) + X^i\beta_1(\tau) \end{aligned} \quad (6)$$

The empirical counterpart of the conditional quantile function or the expected value of the quantile response function (see Appendix A) is defined as:

$$\widehat{Q}_{X^{index}|X^i}(\tau) = \widehat{\beta}_0(\tau) + X^i\widehat{\beta}_1(\tau), \quad (7)$$

where $\widehat{\beta}_0(\tau)$ and $\widehat{\beta}_1(\tau)$ describe the quantile regression process.

Similarly, the unconditional quantile function and its empirical counterpart can be defined as:

$$\begin{aligned} Q_{X^{index}}(\tau) &= VaR_{X^{index}}(\tau) \\ &= \beta_0^u(\tau) \end{aligned} \quad (8)$$

and

$$\widehat{Q}_{X^{index}}(\tau) = \widehat{\beta}_0^u(\tau). \quad (9)$$

Finally, we show that $\Delta CoVaR^{index|i}(\tau)$ is a quantile function as well. This property plays an important role in deriving our testing procedures.

Theorem 3.4 *Let (X^{index}, X^i) denote a pair of random variables with support on \mathbb{R} , with conditional and unconditional probability distributions $F_{X^{index}|X^i}$, $F_{X^{index}}$, respectively. Then for a given $\tau \in [0, 1]$, $\Delta CoVaR^{index|i}(\tau)$ is a quantile function.*

Proof

The proof for the case where $F_{X^{index}|X^i}$ and $F_{X^{index}}$ are symmetric distributions is trivial. In particular, using the reflection property of quantile functions

$$\begin{aligned} \Delta CoVaR^{index|i}(\tau) &= Q_{X^{index}|X^i}(\tau) - Q_{X^{index}}(\tau) \\ &= Q_{X^{index}|X^i}(\tau) + Q_{X^{index}}(1 - \tau). \end{aligned}$$

⁸Without loss of generality, in the remainder of the paper we drop the common variables Z_t from the vector of explanatory variables K_t .

Since quantile functions are closed under addition, the equation implies that $\Delta CoVaR^{index|i}(\tau)$ is a quantile function.

However, for our purpose the case of symmetric distribution functions $F_{X^{index}|X^i}$ and $F_{X^{index}}$ is not the interesting case, since loss distributions are generally not symmetric (see Figure 1). For the case where both $F_{X^{index}|X^i}$ and $F_{X^{index}}$ are not symmetric, we appeal directly to the properties of a quantile function. First, we know that the quantile function is closed under any monotone transformation. Therefore, for any linear location-scale model, which represents exclusively a monotone transformation, the resulting conditional function will still be a non-decreasing function of τ in $0 \leq \tau \leq 1$.⁹ Second, and perhaps more important, we must guarantee that the difference between the conditional quantile function and the unconditional quantile function is also a quantile function. However, even in relevant extreme case where X^{index} is orthogonal to X^i , and therefore $\Delta CoVaR^{index|i}(\tau) = 0 \forall \tau$ we still have a non-decreasing function of τ in $0 \leq \tau \leq 1$.¹⁰

3.2 Quantile treatment effects

In this subsection we relate $\Delta CoVaR$ to the well-known concept of quantile treatment effects.¹¹ In particular, $\Delta CoVaR$ can be interpreted as a two-sample quantile treatment effect where the unconditional distribution represents the control group and the conditional distribution reflects the treatment group. In Section 4 we will show how this interpretation can be exploited to derive proper statistical inference for the systemic risk contribution measure $\Delta CoVaR$.

3.2.1 Two-sample treatment effects

The general model for two-sample treatment effects considers that the responses for the treatment and control group follow a given distribution G and F , respectively. In order to determine if the treatment is unambiguously beneficial then we must test whether G is stochastically larger than F . However, there are situations where the treatment might be beneficial for some individual and neutral or detrimental to others (crossing across the distribution functions). In this two-sample case the quantile treatment effect is given by the following expression:

$$\varrho(\tau) = G^{-1}(\tau) - F^{-1}(\tau),$$

where G^{-1} and F^{-1} are the quantile functions of distributions G and F , respectively. Note that we can recover the mean treatment effect by simply

⁹This is often the case when the conditional distribution represents a stressed version (stressed by the conditioning variable) of the unconditional distribution.

¹⁰The same argument holds for the second definition of $\Delta CoVaR$, the main difference involves the composition of two conditional quantile functions rather than the conditional and the unconditional quantile functions as stated in Theorem 3.1.

¹¹The following section takes important elements from section 2.1 of Koenker (2005) on quantile treatment effects.

integrating the quantile treatment effect over τ ,

$$\bar{\varrho} = \int_0^1 G^{-1}(u)du - \int_0^1 F^{-1}(u)du = \mu(G) - \mu(F),$$

where $\mu(G)$ and $\mu(F)$ are the mean of distributions G and F , respectively. Figure 2 presents a simple location-scale example, that illustrates the distribution, density and quantile function of hypothetical losses in the financial system. The probability distribution of system-wide losses is plotted as the solid line in the figure. Similarly, it is possible to obtain a probability distribution of the stock return of the system, conditional on a given financial institution being in distress (the dashed line). Therefore, the impact of the failure or distress of the financial institution on the system, i.e., its systemic importance, could be obtained from the difference in the VaR for the conditional and the unconditional loss distribution. Note that at the median ($\tau = 0.5$) there is a positive treatment effect ($\varrho(\tau) > 0$) that becomes larger as we move upward into the right tail of the distribution; but in the left tail the treatment is detrimental ($\varrho(\tau) < 0$). That is, while the distress of the financial institution in question increases the system-wide losses for higher levels of the confidence level τ , the opposite is true for lower values of τ . Note that, under some conditions, the quantile treatment effect can be interpreted as a quantile function: $\varrho(\tau)$ is a left continuous and non-decreasing function of τ . However, this is not necessarily the case, if the treatment alters the skewness of the distribution from highly left-skewed to highly right-skewed, then this results in a U-shaped quantile treatment effect that cannot be considered as a quantile function.

A natural non-parametric estimator of the treatment effect is:

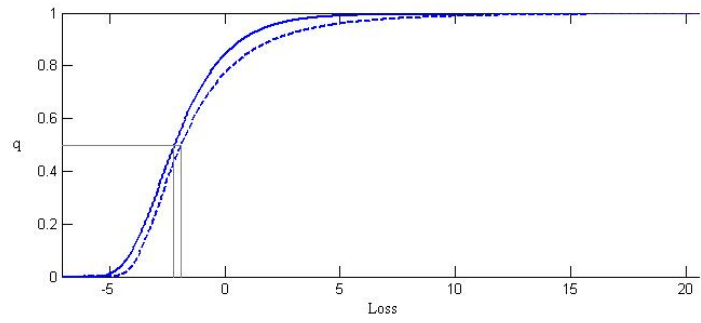
$$\hat{\varrho}(\tau) = \hat{G}_n^{-1}(\tau) - \hat{F}_m^{-1}(\tau),$$

where \hat{G}_n and \hat{F}_m denote the empirical distribution functions of the treatment and control observations, based on n and m observations, respectively. The direct derivation of the empirical distribution for the quantile treatment effect captures all the differences across the treatment and control distributions. In a parametric approach (quantile regression) the treatment effect is entirely determined by the effect over the location and scale parameters of the distribution of interest. For example, in the binary treatment problem the quantile treatment effect can be estimated from the following QR equation:

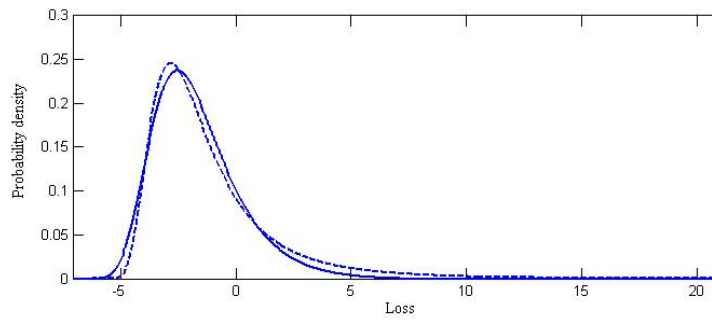
$$\begin{aligned} \hat{Q}_{Y|D^i=1}(\tau) &= \hat{\alpha}(\tau) + \hat{\varrho}(\tau)D_i \\ &= \hat{\alpha}(\tau) + \hat{\varrho}(\tau) \\ &= \hat{G}_n^{-1}(\tau), \end{aligned}$$

where Y is the response variable and $D^i = 1$ is the treatment indicator. Note that when $D^i = 0$, $\hat{Q}_{Y|D^i=0}(\tau) = \hat{\alpha}(\tau) = \hat{F}_m^{-1}(\tau)$ refers to the control group.

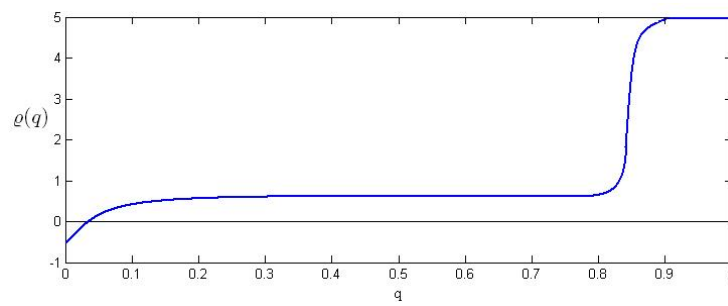
In this context the most common types of hypothesis tests that are considered in the literature on quantile treatment effects are the following:



(a) CDF



(b) PDF



(c) Quantile F

Figure 2: Quantile treatment effect

1. Hypothesis of no effect: $\varrho(\tau) = 0$ for all $\tau \in (0, 1)$.
2. Constant effect hypothesis: $\varrho(\tau) = \varrho$ for all $\tau \in (0, 1)$.
3. Dominance hypothesis: $H_0 : \varrho(\tau) \geq 0$ for all $\tau \in (0, 1)$ versus $H_a : \varrho(\tau) < 0$ for some $\tau \in (0, 1)$.

3.2.2 $\Delta CoVaR$ as a quantile treatment effect

Using the relationships between quantile and distribution functions, the definition of $\Delta CoVaR$ for a given level of τ can be formulated as follows:

$$\begin{aligned} \widehat{\Delta CoVaR}^{index|i}(\tau) &= \widehat{Q}_{X^{index}|X^i}(\tau) - \widehat{Q}_{X^{index}}(\tau) \\ &= \widehat{F}_{X^{index}|X^i}^{-1}(\tau) - \widehat{F}_{X^{index}}^{-1}(\tau), \end{aligned} \quad (10)$$

where $\widehat{F}_{X^{index}|X^i}$ and $\widehat{F}_{X^{index}}$ denote the empirical conditional and unconditional distributions functions obtained from the stock market returns for the index of financial institutions and the individual financial institution i , respectively. From this formulation, we can easily see the equivalence between $\Delta CoVaR$ and two-sample treatment effects. In particular, $\widehat{F}_{X^{index}|X^i}^{-1}(\tau) = \widehat{G}_n^{-1}(\tau)$ and $\widehat{F}_{X^{index}}^{-1}(\tau) = \widehat{F}_m^{-1}(\tau)$.

As a consequence, we can relate our hypothesis tests, as formulated in Section 2.2.3, to the hypothesis tests 1.-3. considered in the literature on quantile treatment effects. In particular, the hypothesis of significance given by equation (4) relates to hypothesis test 1. (hypothesis of no effect) of the quantile treatment effects literature:

$$H_0 : \Delta CoVaR^{index|i}(\tau) = 0,$$

for a given $\tau \in (0, 1)$ or, more specifically, on a given subset of $\mathcal{T} \subset (0, 1)$. The hypothesis of dominance in equation (5) is similar to hypothesis test 3. (dominance hypothesis) of the quantile effects literature:

$$H_0 : CoVaR_{X^{index|i}}(\tau) > CoVaR_{X^{index|j}}(\tau),$$

for a given $\tau \in (0, 1)$ or, more specifically, on a given subset of $\mathcal{T} \subset (0, 1)$.

As indicated, in the case of $\Delta CoVaR$ we are not interested in the entire domain of $\tau \in (0, 1)$, like in hypotheses 1.-3. in the quantile treatment effects literature, but rather in a particular quantile ($\tau = 0.95, \tau = 0.99$) or on a given subset $\mathcal{T} \subset (0, 1)$.¹² Since our interest is mainly a downside risk measure this subset will generally be defined as $\mathcal{T} := (0.90, 0.99)$, the lower tail of the conditional distribution of the random variable of interest (losses, returns). In the next section, we will use the inference procedures developed in the quantile treatment literature for testing hypotheses 1.-3. as a basis for the testing

¹²This is an important difference with respect to the standard statistical test for stochastic dominance.

procedures that we develop for the two abovementioned hypothesis tests in the context of $\Delta CoVaR$. In particular, the tests that we develop are based on testing the difference between a conditional and an unconditional distribution or quantile function (significance) and whether one of two conditional distributions or quantile functions stochastically dominates the other (dominance), respectively, in the domain of interest for τ .

4 Testing for the systemic importance of a financial institution

Testing procedures for the hypothesis of significance and dominance are entirely determined by the underlying statistical model. In a parametric approach the differences between the conditional and unconditional distribution for the system or institution's losses will be entirely determined by the location and scale parameters or linear functions of such parameters. In other words, the statistics used in the hypothesis test are linear function of the location and scale parameters. In the next subsections we derive two test of significance and one test for dominance.

4.1 A simple test of significance for $\Delta CoVaR$

In the location-scale family of distributions (e.g. normal, student-t, uniform, among others) the quantile function can be derived analytically. Recall that since VaR, ES and CoVaR are themselves quantile functions this implies that they have a simple analytical form (Jargen-Ambrozewicz, 2010).¹³ Let X_0 denote a standardized centered random variable from the location-scale family with probability distribution F_{X_0} , then the quantile function of such random variable is given by $Q_{X_0}(\tau) = F_{X_0}^{-1}$. From the properties of the quantile function (Theorem 3.1), for any random variable from the location-scale family, $X \sim (\mu_x, \sigma_x)$, the quantiles for such random variable is given by $Q_X(\tau) = \mu_x + \sigma_x Q_{X_0}(\tau)$. In other words, the unconditional quantile is a linear function of $Q_{X_0}(\tau)$. In general, we denote the conditional quantile functional or the quantile response function for a given linear relationship between the pair of random variables (X, Y) by $Q_{Y|X=x}(\tau) = X\beta(\tau)$.¹⁴

As mentioned in Section 2.2.2, estimation of $\Delta CoVaR$ can be accomplished using various methods. Let the stock return of the index of financial institutions $X^{index} \sim (\mu_{index}, \sigma_{index})$ denote a random variable having a probability

¹³In other families of distribution it is also possible to obtain expression for VaR and ES. An example is the family of alpha-stable distributions (Stoyanov et al. 2006). The alpha-stable family is an important set of distributions for modeling financial variables since it provides a flexible environment for replicating the stylized facts of the probability distributions for financial variables (e.g. asymmetry and heavy tails).

¹⁴See appendix A for the details.

distribution function from the location scale family, and the stock return of financial institution of interest, X^i , the single covariate in addition to a constant term. An estimate of $CoVaR_{X^{index}|i}(\tau)$ can then be obtained from the following quantile regression:

$$Q_{X^{index}|X^i}(\tau) = \beta_0(\tau) + X^i\beta_1(\tau),$$

where $\beta_0(\tau), \beta_1(\tau)$ are the unknown parameters of the quantile regression. An expression for $\Delta CoVaR^{index|i}(\tau)$ can be derived as follows,

$$\begin{aligned} \Delta CoVaR^{index|i}(\tau) &= \beta_0(\tau) + X^i\beta_1(\tau) - Q_{X^{index}}(\tau) \\ &= [\beta_0(\tau) - \mu_{index}] + [X^i\beta_1(\tau) - \sigma_{index}Q_{X_0}(\tau)], \end{aligned}$$

where $\Delta CoVaR^{index|i}(\tau)$ reflects the difference between the conditional and the unconditional quantile functions of the random variable X^{index} . In the next paragraph, we provide a result indicating that the hypothesis test on the significance of the $\Delta CoVaR$ is equivalent to testing the a similar hypothesis with respect to the parameter $\beta_1(\tau)$ of the quantile regression. A similar result is found in Chernozhukov and Umantsev (2001) in the context of testing whether the (unconditional) VaR model is statistically different from the conditional VaR model.

Theorem 4.1 *Testing the hypothesis $H_0 := \beta_1(\tau) = 0$ is equivalent to testing the hypothesis $H_0 := \Delta CoVaR_{X^{index}|i}(\tau) = 0$, for a given τ .*

Proof

If $\beta_1(\tau) = 0$ then,

$$\begin{aligned} \Delta CoVaR^{index|i}(\tau) &= \beta_0(\tau) - \mu_{index} - \sigma_{index}Q_{X_0}(\tau) \\ &= \beta_0(\tau) - Q_{X^{index}}(\tau), \end{aligned}$$

for $\Delta CoVaR^{index|i}(\tau) = 0$, $\beta_0(\tau)$ must be equal to the unconditional quantile function of X^{index} (also known as the $VaR^{index}(\tau)$). Therefore if $\beta_1(\tau) = 0$ this necessarily implies that there is no difference between the conditional and unconditional quantile function of the random variable X^{index} and hence by definition $\Delta CoVaR^{index|i}(\tau) = 0$.¹⁵ \square

The respective statistic for such a test is obtained directly from the asymptotic distribution of the quantile regression estimator (Koenker, 2005):

$$\sqrt{T}(\hat{\beta}(\tau) - \beta(\tau)) \sim N(0, V)$$

¹⁵In Adrian and Brunnermeier (2010), $\Delta CoVaR$ is defined as follows:

$$\Delta CoVaR_{X^{index}|i}(\tau) = \beta_1(\tau)(X^i(\tau_1) - X^i(\tau_0)) \quad (11)$$

where $\tau_1 > \tau_0 = 0.5$. Under this setup it is trivial to see that if $\hat{\beta}(\tau) = 0$ then $\Delta CoVaR^{index|i}(\tau) = 0$ for some τ . Therefore, within the location scale family of models testing for the significance of $\Delta CoVaR^{index|i}(\tau)$ is equivalent to testing for the significance of $\beta_1(\tau) = 0$.

where the covariance matrix $V = \tau(1-\tau)D(\tau)^{-1}\Omega D(\tau)^{-1}$ with $\Omega = T^{-1} \sum_{t=1}^T (x_t^i)(x_t^i)'$ and $D(\tau) = T^{-1} \sum_{t=1}^T (x_t)(x_t^i)' f_t(\xi_t(\tau))$.¹⁶ The term Ω is introduced so as to account for the presence of autocorrelation in a non-iid setting. The term $f_t(\xi_t(\tau))$ denotes the conditional density of the response X^{index} evaluated at the τ -th conditional quantile. In the iid case the f_t 's are identical and the expression is simpler (Koenker, 2005).¹⁷

4.2 Test for significance and dominance using the quantile response function

In this subsection we derive two new tests for significance and dominance in the linear quantile regression framework. We first introduce the general linear testing framework in quantile regressions. In particular, we present the elements that highlight the distribution of a linear functional of $\beta(\tau)$, and discuss that the distribution derived from the general linear hypothesis test, critically depends on the function $r(\tau)$. Next, we derive specific testing procedures for testing the specific hypotheses in the context of $\Delta CoVaR$.

4.2.1 General linear testing framework

Consider for a given quantile τ a two-sided test of hypothesis for the general linear hypothesis test:

$$H_0 : R\beta(\tau) = r(\tau), \tau \in \mathcal{T} \quad (12)$$

Where $\beta(\tau)$ is p dimensional and q is the rank of matrix R , ($q \leq p$). The distribution under the null of the Wald statistic or process will depend on the characteristics of the function $r(\tau)$. If $r(\tau) = r \forall \tau$, then the test statistic is:

$$W_T = T(R\hat{\beta}(\tau) - r)'(RV R')^{-1}(R\hat{\beta}(\tau) - r) \quad (13)$$

where $V = \tau(1-\tau)D(\tau)^{-1}\Omega D(\tau)^{-1}$. For a single given τ , the statistic W_T is asymptotically χ_q^2 under H_0 .

If $r(\tau)$ is a process indexed by τ , then under the null hypothesis we have the following Wald process:

$$v_n(\tau) = \sqrt{n}\varphi_0(\tau)(RV R')^{-1/2}(R\hat{\beta}(\tau) - r(\tau)) \quad (14)$$

where $\varphi_0(\tau) = f_0(F_0^{-1}(\tau))$. This process converges weakly to a q -dimensional Brownian Bridge. For a sequence of local alternatives defined for the hypothesis test

$$H_n : R\beta(\tau) - r(\tau) = \frac{\zeta(\tau)}{\sqrt{n}}, \tau \in \mathcal{T} \quad (15)$$

the statistic

$$\mathcal{B}_n(\tau) = \frac{\|v_n(\tau)\|}{\sqrt{\tau(1-\tau)}}, \quad (16)$$

¹⁶See appendix B for the details.

¹⁷In appendix B we include a short introduction on statistical inference for $\hat{\beta}(\tau)$. Furthermore, we use these results in deriving the distribution of the quantile response function.

behaves asymptotically like a non-central Bessel process with non-centrality parameter,

$$\eta(\tau) = \frac{\zeta'(RVR')^{-1}\zeta}{\tau(1-\tau)} \quad (17)$$

(see Koenker and Machado, 1999). The Critical values, that we will use in the sequel and that are given for the supremum of the simpler (central) Bessel process of order q , $\sup B_q^2(\tau)$, have been tabulated by DeLong (1981) and Andrews (1993, 2003) by simulation methods, and more recently by exact methods by Estrella (2003) and Anatolyev and Kosenok (2011).

4.2.2 A test of significance

We need to explicitly derive a two-sample and two-sided test that will allow defining a test of significance for the following null hypothesis:

$$H_0 : \Delta CoVaR^{index|i}(\tau) = 0.$$

Furthermore, recall that $\Delta CoVaR$ can be interpreted as the difference between an conditional and a unconditional quantile function, which are themselves a linear functional of $\beta(\tau)$,

$$\begin{aligned} \hat{Q}_{X^{index}|X^i}(\tau) &= \mathbf{X}\hat{\beta}(\tau) \\ &= \hat{\beta}_0(\tau) + X^i\hat{\beta}_1(\tau) \end{aligned} \quad (18)$$

For such test, we define R under the null ((12)) as some lower dimensional form of the design matrix (for example evaluated at the centroid $R = \bar{\mathbf{X}}$ or a particular quantile $R = \mathbf{X}(\tau)$). In addition, once we start considering a test for dominance across a set of quantile response functions, we require a different statistic and in particular a one-sided test of hypothesis. One possible approach is to consider an inequality constrained version of the Wald test (see Gouriéroux et al. 1982, Wolok 1989). Under the null hypothesis, the statistic proposed by these authors is distributed as a mixture of χ^2 distributions. However, we do not follow this approach rather we look for a test based on the Kolmogorov-Smirnov (KS) type statistic. KS type test are highly attractive since they are asymptotically distribution free.¹⁸ The KS test provides a natural way to measure the discrepancy between distributions (Abadie, 2002). Furthermore, variants of the two-sample KS test have been widely used for inference based on a quantile process, such as those considered in section 3.2.1. However, it is not a straight forward task to guarantee that the statistic is still distribution free, since we are dealing with a conditional (parametric) distribution.

In order to perform inference on the quantile response function we consider the first part of expression (40) in Theorem 2 as a parametric empirical process¹⁹

$$V_n(\tau) = \sqrt{n}(\hat{Q}_{\mathbf{Y}}(\tau | \mathbf{X}) - Q_{\mathbf{Y}}(\tau | \mathbf{X})) \quad (19)$$

¹⁸In distribution free type test we can tabulate the distribution under the null, of the statistic, without specifying the underlying distribution of the data. The distribution free property, of a statistic, is a key property of many non-parametric procedures.

¹⁹See appendix B for the details.

where the quantile response function contains the parameter(s) $\hat{\beta}(\tau)$. The KS statistic would be:

$$K_n = \sup_{\tau \in \mathcal{T}} \| V_n(\tau) \| . \quad (20)$$

The existence of the unknown parameter(s) $\beta(\tau)$ is problematic because under the null hypothesis (of say a one-sample goodness-of-fit test) the statistic does not have the desired KS distribution.²⁰ Fortunately, Koenker and Xiao (2002) offer a solution to such problem in the context of quantile regression and in particular, for the linear location scale model (Koenker , 2005)²¹:

$$y_i = x_i' \delta + (x_i' \gamma) u_i \quad (21)$$

where $u_i \sim i.i.d F_0$. Under this specification the corresponding parameter of interest, $\beta(\tau)$, of the quantile response function, is a vector function $\beta(\tau) = \delta + \gamma F_0^{-1}(\tau)$ (a vector function with p components). Therefore, the quantile varying coefficients are identical up to a affine transformation. A test of hypothesis for the general linear hypothesis test has the following statistic:

$$\hat{v}_n(\tau) = \sqrt{n} \varphi_0(\tau) (R V R')^{-1/2} (R \hat{\beta}(\tau) - \hat{r}(\tau)) \quad (22)$$

where $\varphi_0(\tau) = f_0(F_0^{-1}(\tau))$. This process is asymptotically Gaussian but the introduction of r disrupts (introduces a drift component) the asymptotic distribution-free character of the limiting theory. Therefore, some martingale transformation is required to eliminate the drift component (Khmaldze, 1981). In practice this requires replacing $\hat{v}_n(\tau)$ by the residual process obtained by a continuous time recursive least-square regression of $\hat{v}_n(\tau)$ on the score function defining the estimator \hat{r} . After transformation $\tilde{v}_n = Q_g \hat{v}_n \Rightarrow w_0$ we obtain a standard Brownian motion, with $p - 1$ independent coordinates (p stands for the discretization of the quantile τ). Test based on the KS type statistic $K_n = \sup_{\tau \in \mathcal{T}} \| \tilde{v}_n(\tau) \|$, have easily simulated critical values (Andrews, 1993). As suggested in (Koenker , 2005) in some situations it is desirable to restrict the interval of estimation to a closed subinterval $[\tau_0, \tau_1]$ of $(0, 1)$. This can easily be accommodated by considering the renormalized statistic

$$K_n = \sup_{\tau \in [\tau_0, \tau_1]} \| \tilde{v}_n(\tau) - \tilde{v}_n(\tau_0) \| / \sqrt{\tau_1 - \tau_0} \quad (23)$$

This test statistic will allow us to determine if the conditional and the unconditional quantile functions (or between two conditional quantile functions) that make up $\Delta CoVaR$ are statistically different from each other. If not then $\Delta CoVaR$ is not statistically different than zero and hence the such financial institution according to this measure is not systemically important.

In order to test for dominance, in addition to the two-sample test we require a one sided test for the general linear hypothesis. Such test will give us the

²⁰This is known in the literature as the Durbin problem.

²¹Recall from section 2.2.2, the linear location scale model: the class of models that are of interest for the $\Delta CoVaR$ measure.

opportunity to test the following null hypothesis,

$$H_0 : CoVaR^{X^{index}|X^j=x^j}(\tau) > CoVaR^{X^{index}|X^i=x^i}(\tau),$$

In the two sample case a test for stochastic dominance (first order) of the quantile response function under two continuous treatment effects (different covariates) can be implemented using the two sample one-sided, positive part KS statistic. Let X_1, \dots, X_n denote a randomly treated sample with empirical distribution function $G(x)$ and Z_1, \dots, Z_n denote another randomly treated sample with empirical distribution function $F(z)$. We are interested in testing whether one treatment stochastically dominates the other. For example the hypothesis test is as follows $H_0 : G^{-1}(\tau) > F^{-1}(\tau) \forall \tau \in (0, 1)$. Therefore the null hypothesis of such test states that sample X stochastically dominates sample Z. The variant of the KS type test is such that the test is formulated in terms of the empirical quantile function, rather than the empirical distribution function. In a simple unconditional case the test statistic is:

$$J_n = \sup_{\tau \in \mathcal{T}} (G_n^{-1}(\tau) - F_n^{-1}(\tau)) \quad (24)$$

Generality, \mathcal{T} is considered as a closed subinterval of $(0, 1)$, typically $[\epsilon, 1 - \epsilon]$, for some $\epsilon \in (0, 1/2)$. Our approach differs from the previous one, since we consider a conditional distribution, rather than an unconditional distribution and in particular the the quantile response function of a linear model.

Suppose we have two different (at least one column is different) design matrices \mathbf{X} and \mathbf{Z} . This setup includes the case where we have two different continuous treatment effects applied to the same population Y , as well as a set of common control variables, all within the framework of a linear model that relates Y with the X and Z covariates. The respective empirical quantile response functions are a follows:

$$\hat{Q}_{\mathbf{Y}}(\tau | \mathbf{X}) = \mathbf{X}\hat{\beta}_n^x(\tau) \quad (25)$$

and

$$\hat{Q}_{\mathbf{Y}}(\tau | \mathbf{Z}) = \mathbf{Z}\hat{\beta}_n^z(\tau) \quad (26)$$

Without loss of generality, we consider equal amount of observations n throughout the design space. Therefore, we have the following parametric empirical process:

$$W_n(\tau) = \sqrt{n}(\hat{Q}_{\mathbf{Y}}(\tau | \mathbf{X}) - \hat{Q}_{\mathbf{Y}}(\tau | \mathbf{Z})) \quad (27)$$

$$= \sqrt{n}(\tilde{\mathbf{X}}\hat{\beta}_n^x(\tau) - \tilde{\mathbf{Z}}\hat{\beta}_n^z(\tau)) \quad (28)$$

Where $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Z}}$ implies the quantile response function is evaluated at any point of the design space. This point can be the centroid $(\bar{\mathbf{X}}, \bar{\mathbf{Z}})$ or an extreme quantile of interest.

Following, Koenker and Xiao (2005) both of the underlying quantile response functions come from the following linear location scale models.

$$y_i = \mathbf{X}_i\delta^x + (\mathbf{X}_i\gamma^x)u_i \quad (29)$$

$$y_i = \mathbf{Z}_i \delta^z + (\mathbf{Z}_i \gamma^z) u_i \quad (30)$$

therefore, as seen before, $\beta^x(\tau) = \delta^x + \gamma^x F_0^{-1}(\tau)$ and $\beta^z(\tau) = \delta^z + \gamma^z F_0^{-1}(\tau)$. From Theorem 1 in Koenker and Xiao (2005) we can derive a statistic for the two sample one-sided, test of hypothesis, embedded in the general linear hypothesis frameset.

$$\hat{w}_n(\tau) = \sqrt{n} \Psi_0(\tau) (RV^{-1} R')^{-1/2} (R \hat{\beta}(\tau) - \hat{r}(\tau)) \quad (31)$$

where Ψ_0 denotes the difference between the scalar sparcity function of both models. Under the null hypothesis $R = [\tilde{\mathbf{X}}, -\tilde{\mathbf{Z}}]$, $\beta(\tau) = [\beta^x(\tau), \beta^z(\tau)]'$, and $r(\tau) = 0$.

$$\hat{w}_n(\tau) = \sqrt{n} \Psi_0(\tau) (RV^{-1} R')^{-1/2} (R \hat{\beta}(\tau)) \quad (32)$$

where the KS type one-sided statistic is $K_n = \sup_{\tau \in \mathcal{T}} (\hat{w}_n(\tau))$. A two-sided version of the KS type statistic can also be derived $K_n = \sup_{\tau \in \mathcal{T}} |\hat{w}_n(\tau)|$ however, this would only be indicative of the statistical difference between the two empirical quantile functions but not of any sort of dominance that could possible arise between the conditional distributions. Since $r(\tau) = 0$ the test statistic is still distribution free, therefore there is no need to perform the martingale transform. Koenker and Machado (1999) show that the nuisance parameters existing in Ψ_0 and V can be replaced by consistent estimates without jeopardizing the distribution-free character of the test. Furthermore, as in Andrews(1999) the critical values for the statistic, $\sup_{\tau \in \mathcal{T}} (\hat{w}_n(\tau))$ are based on the asymptotic null distribution of $\sup_{\tau \in \mathcal{T}} (\mathcal{Q}_p(\tau))$. Where $\sup_{\tau \in \mathcal{T}} (\mathcal{Q}_p(\tau))$ represents the process to which the statistic converges in distribution.²² By definition the critical values c_α satisfies $P(\sup_{\tau \in \mathcal{T}} (\mathcal{Q}_p(\tau)) > c_\alpha) = \alpha$. Most of the tables associated to the critical values depend on a subset of \mathcal{T} , $[\tau_0, \tau_1]$, such that the distribution of the critical values of $\sup_{\tau \in [\tau_0, \tau_1]} (\mathcal{Q}_p(\tau))$ depend on τ_0 and τ_1 only through the parameter $\lambda = \frac{\tau_1(1-\tau_0)}{\tau_0(1-\tau_1)}$. Instead of using the simulated critical values derived by Andrews (1993) we use the exact asymptotic p-values obtained from Anatolyev and Kosenok (2011).²³

As mentioned by Koenker (2005) the importance of the previous results are that they provide partial orderings of the conditional distributions under stochastic dominance, which is precisely our object of interest.

5 Empirical application

To illustrate the tests described in the previous sections we consider a sample of three European banks, denote them as bank A, B and C (Table 1). The market information that we use to characterize the institutions and the system are the

²²See Theorem 3 in Andrews (1993)

²³We thank, Anatolyev and Kosenok (2011) for providing the source code in GAUSS of their methodology.

daily stock returns (1986-2010) of each institution and the returns on the index on financials.²⁴

Table 1: Size and $\Delta CoVaR$ of three European banks

Bank	Assets (millions)	Quantile Regression Results	$\Delta CoVaR$
A	€1,571,768	$X^{index A}(0.99) = 0.026 + 0.526X^A(0.99)$	1.38
B	€102,185	$X^{index B}(0.99) = 0.042 + 0.231X^B(0.99)$	1.18
C	€10,047	$X^{index C}(0.99) = 0.037 + 0.028X^C(0.99)$	0.03

Each bank that we consider is approximately ten times bigger than the other. Table 1 also reports the quantile regression results as well as the $\Delta CoVaR$ measure based on Adrian and Brunnermeier (2009)'s approach as an indicator of systemic importance. We find that the systemic risk contribution $\Delta CoVaR$ is largest for bank A, and smallest for bank C. However, two natural questions come to mind. The first is whether the systemic risk contributions obtained for the three banks are large enough to consider the banks as systemically important. A common form of identifying the systemically important banks is to establish an arbitrary threshold above which the measure indicates the institution is systemically significant; without loss of generality, we set this level at zero. A second question is whether institution A is truly more systemically significant than institution B, and B is truly more systemically important than institution C, as indicated by the estimated values of $\Delta CoVaR$ in Table 1. These are precisely the two types of questions we aim to answer.

In testing for significance we look at two types of tests. The first is a simple test based on the slope coefficient of the regression quantile as developed in Section 4.1. The second test is based on the two-sided version of the KS statistic developed in Section 4.2; this test will allow us to determine if the conditional and the unconditional quantile functions that make up $\Delta CoVaR$ are statistically different from each other. If not, then $\Delta CoVaR$ is not statistically different from zero and hence, such financial institution is not systemically important according to $\Delta CoVaR$.

Both tests indicate that the measure of systemic importance is significant for bank A and B. On the other hand, the measure is not significant for bank C. Figures 3 and 4 confirm the findings. In both figures, the left side panel illustrates the conditional and unconditional quantile functions at the extremum quantiles and the right side panel is a plot of the respective densities. Note that the densities are the typical illustration found in most articles to illustrate $\Delta CoVaR$ (recall figure 2). Figure 3 illustrates the important difference between the conditional and unconditional quantile functions of bank A. In contrast, Figure 4 shows no significant difference between the conditional and unconditional

²⁴We switch the sign of return in order to recover the interpretation of positive losses, in particular when we are analysing any graph. This obviously changes the sign of the $\Delta CoVaR$ measure.

Table 2: Test of significance for three European banks

Bank	$\Delta CoVaR$	$H_0 : \beta(0.99) = 0$	$H_0 : \Delta CoVaR(0.99) = 0$
A	1.38	0.000	0.000
B	1.18	0.039	0.000
C	0.03	0.782	0.424

The second column is the value of the $\Delta CoVaR$ measure. The third and fourth column are the p-values of the statistic. The quantile response function-based test in the fourth column is evaluated at the extreme quantiles $[\tau_0, \tau_1] = [0.90, 0.99]$.

quantile function for bank C. It is important to recall that in the conditional functions that will be presented in the figures, the data from each bank (A, B, C) represent the conditioning variable in the quantile regression.

In testing for dominance we perform the test based on the one-sided version of the KS statistic as outlined in Section 4.2. The test will allow us to determine if the conditional quantile function of bank A stochastically dominates the conditional quantile function of bank B, for a given subset $\mathcal{T} \subset (0, 1)$ defined as $[\tau_0, \tau_1]$. A sequence of such stochastic dominance test, on the pairs of banks, will give a partial ordering on the systemic importance of the set of banks A, B, and C, that follow from the $CoVaR$ measure.

Table 3: Test of dominance for three European banks

Banks	$\Delta CoVaR$	$[\tau_0, \tau_1] = [0.90, 0.99]$	$[\tau_0, \tau_1] = [0.10, 0.99]$
AB	1.38	0.000	0.913
AC	1.18	0.000	0.874
BC	0.03	0.000	0.482

The second column is the value of the $\Delta CoVaR$ measure. The third and fourth column are the p-values of the statistic.

The results indicate that by looking at the extremum quantiles $[\tau_0, \tau_1] = [0.90, 0.99]$, we reject the null hypothesis of strict stochastic dominance, hence we are not able to establish a partial ordering. However, if we extend the subset of quantile on which we evaluate the statistic to $[\tau_0, \tau_1] = [0.10, 0.99]$, we are able to accept the null hypothesis. Hence, we can determine that the following ordering, $A \succ B \succ C$, for that set of quantiles. Furthermore, we find that the largest value of the statistic (the point at which we are able to accept the null hypothesis) is reached at the following quantiles 0.475, 0.43, 0.607 for the bank pairs AB, AC and BC, respectively.

Figures 5 and 6 are in line with the findings, because they indicate that the largest deviations in the quantile functions are not in the extremum quantiles, but rather around the center of the distributions, whereas in the extreme right

tail we barely notice a difference between the conditional and unconditional distribution. These results indicate that perhaps the linear relationship, between the variables of interest (X^{index}, X^i) that is at the core of the CoVaR measure, is too restrictive. The affine transformation that characterizes the construction of the conditional distribution of the variable (X^{index}) , is heavily stressed, by construction, at the center of the distribution rather than at the extreme.

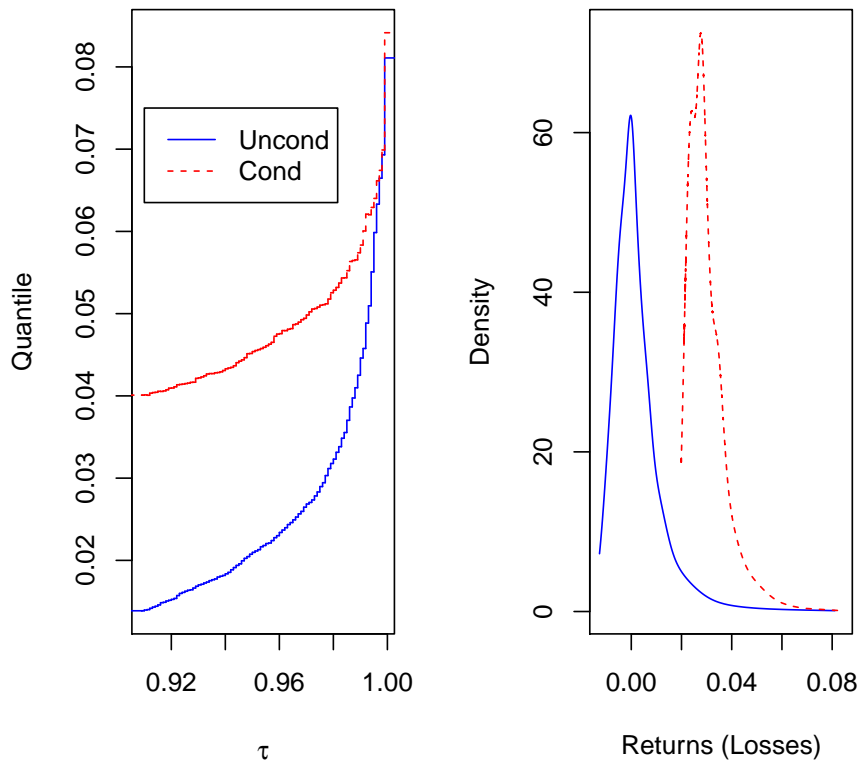


Figure 3: Conditional vs unconditional quantile and density functions for bank A.

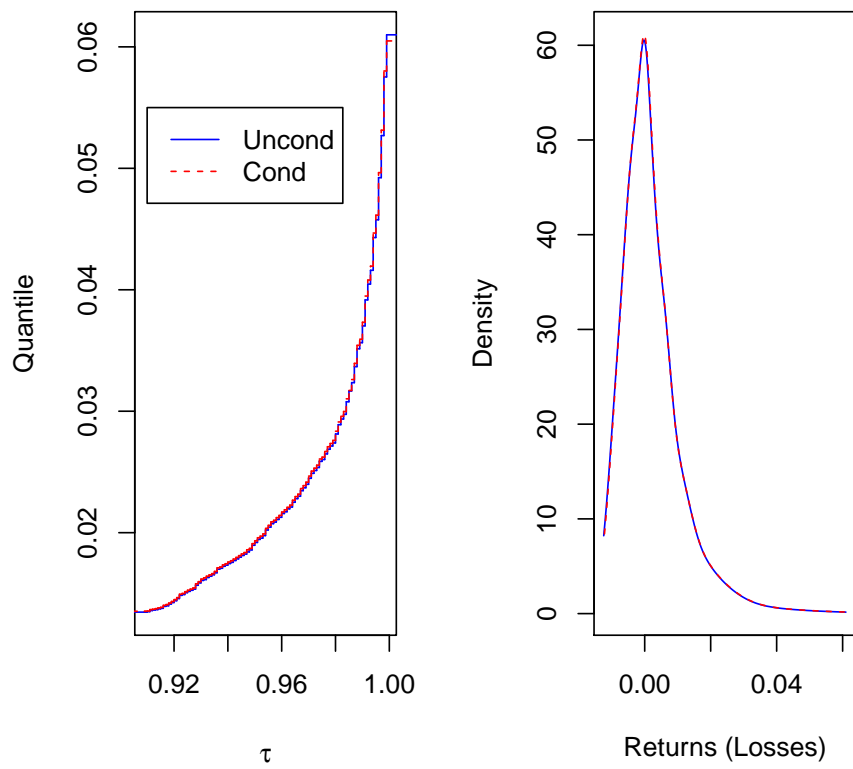


Figure 4: Conditional vs unconditional quantile and density functions for bank C.

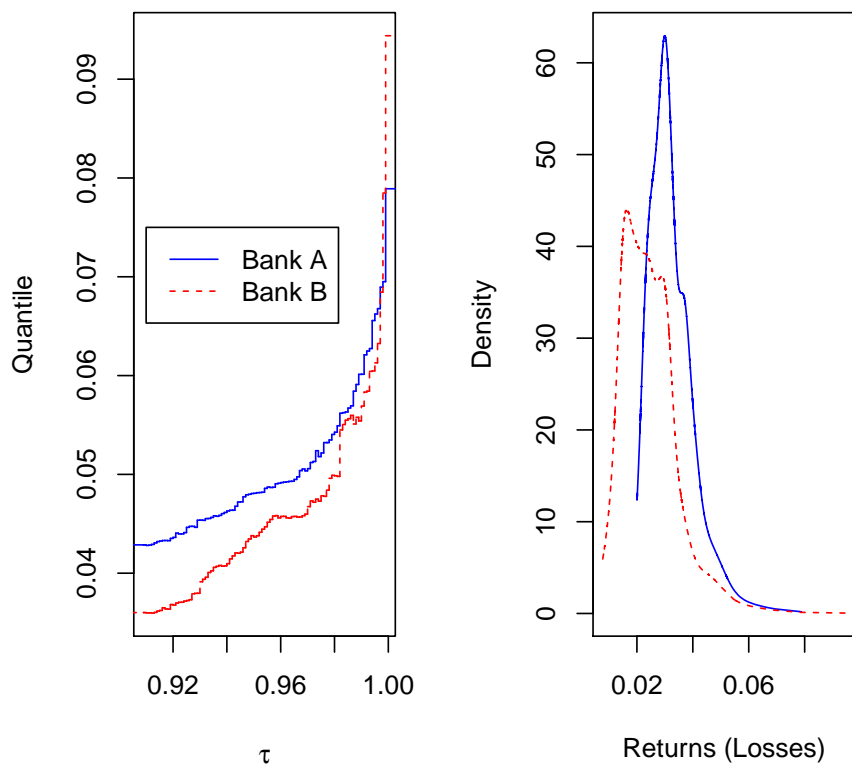


Figure 5: Conditional quantile and density functions of Bank A vs Bank B.

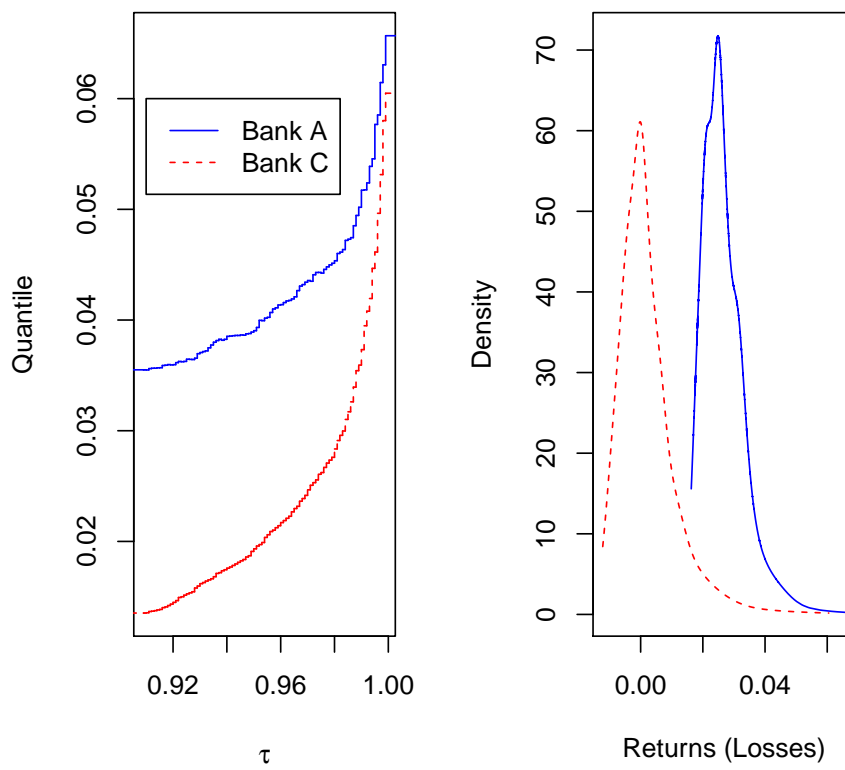


Figure 6: Conditional quantile and density functions of Bank A vs Bank C.

6 Conclusions

In response to the 2007-2008 financial crisis, the Basel Committee and the Financial Stability Board are developing an integrated approach to deal with systemically important financial institutions. Potential regulatory instruments that may be targeted at systemically important financial institution in the near future include capital and liquidity surcharges, systemic levies, and contingent capital and/or bail-inable debt. However, in order for such endeavors to be operational supervisors and Central Banks have to come up with methodologies to properly identify those systemically important financial institutions. Currently there are many competing methodologies that have been proposed to assess the systemic importance of financial institutions.

In this paper we take one of such proposed methodologies that has already been extensively applied, namely $\Delta CoVaR$ proposed by Adrian and Brunnermeier (2009, 2010). $\Delta CoVaR$ is an interesting tool for measuring systemic importance through market information (equity returns, CDS spreads), as it allows estimating the institution's potential impact on the system in the event of failure or distress. Our view is, however, that statistical testing on the estimated systemic risk contributions is required before interpreting the results. We exploit the analogies of $\Delta CoVaR$ with quantile-based measures and, in particular, inference procedures for quantile treatment effects, in order to derive, within a linear quantile regression framework, a series of testing procedures that will help in identifying and ranking the systemically important financial institutions.

First, we derive a test of significance of the $\Delta CoVaR$ measure that allows to determine whether or not a financial institution can be identified as systemically important. Second, we derive a test of dominance that may be a useful tool for testing whether one financial institution is more systemically important than another, according to $\Delta CoVaR$. We illustrate the possible use for the tests using equity data for three European banks. We find that the tests provide a proper interpretation of the $\Delta CoVaR$ measure proposed by Adrian and Brunnermeier (2009, 2010). On the one hand, the significant test indicates for which banks the estimated systemic risk contribution is statistically significant. On the other hand, the dominance test gives a ranking of the institution according systemic importance indicated by the measure. However, this ranking is not perfect: whether or not we can reject the null hypothesis of stochastic dominance depends at which range of quantiles we evaluate the distributions. In addition, since the test is pairwise, there is nothing that guarantees that transitivity will hold across the tested sample (as it does in the example).

That is, while the testing procedures developed in this paper entail a first step in the right direction, further work is required in order to (i) determine the power of the test; and (ii) to adjust the asymptotics for some of the extremal regression quantiles that are used in such quantile based measures (see Chernozhukov, 1999; Chernozhukov and Umantsev, 2001). A medium term goal of this research agenda is to develop proper stochastic dominance test at the extremum for a general class of conditional and unconditional quantile functions. Such type of test are of interest for a much needed inferential-based analysis

that will hopefully allow to statistically compare loss distributions in risk management.

References

- [1] ABADIE, A.: Bootstrap test for distribution treatment effects in instrumental variable models, *Journal of the American Statistical Association*, Vol. 97(457), pp.284-92, (2002).
- [2] ADRIAN, T., BRUNNERMEIER, M.: CoVaR, working paper, Federal Reserve Bank of New York, (2009).
- [3] ANATOLYEV, S. KOSENOK, G.: Another numerical method of finding critical values for the Andrews stability test, *Econometric Theory*, Forthcoming, (2011).
- [4] ANDREWS, D.P.: Test for Parameter instability and structural change with unknown change point, *Econometrica*, Vol. 61(4), pp.821-856, (1993).
- [5] ANDREWS, D.P.: Test for Parameter instability and structural change with unknown change point: A corrigendum, *Econometrica*, Vol. 71(1), pp.395-397, (2003).
- [6] BANK FOR INTERNATIONAL SETTLEMENTS: Basel III: A global regulatory framework for more resilient banks and banking systems, Report, December, (2010).
- [7] BASSET, G., KOENKER, R.: An empirical quantile function for linear model with iid errors, *Journal of the American Statistical Association*, Vol. 77(378), pp. 407-15, (1982).
- [8] BROWNLEES C., ENGLE, R.: Volatility, correlation, and tails for systemic risk measurement, working paper, NYU Stern, (2010).
- [9] BILLIO, M., GETMANSKY, M., LO, A., PELIZZON, L.: Econometric measures of systemic risk in finance and insurance sectors, working paper, NBER, 16223, (2010).
- [10] CAI, Z., WANG, X.: Nonparametric estimation of conditional VaR and expected shortfall, *Journal of Econometrics*, 147, pp 120-130 (2008).
- [11] CASTRO, C., FERRARI, S.: Measuring the systemic importance of financial institutions using market information, *Financial Stability Review*, National Bank of Belgium, June , pp 127-141 (2010).
- [12] CHAN-LAU, J.: Balance sheet network analysis of too-connected-to-fail risk in global and domestic banking systems, IMF Working Papers, April (2010).

- [13] CHERNOZHUKOV, V.: Conditional Extremes and Near-extremes: Concepts, Estimation and Economic applications, Standfod Ph.D Dissertation (2000).
- [14] CHERNOZHUKOV, V., FERNANDEZ-VAL, I.: Subsampling inference on quantile regression process, *The Indian Journal of Statistics*, Vol.6 (2), pp 253-276 (2005).
- [15] CHERNOZHUKOV, V., HANSEN, C.: Instrumental quantile regression inference for structural and treatment effect models, *Journal of Econometrics*, Vol. 132, pp 491-525 (2006).
- [16] CHERNOZHUKOV, V., UMANTZEV, L.: Conditional Value-at-Risk: Aspects of modelling and estimation, *Empirical Economics*, 26, pp 271-292 (2001).
- [17] DE JONGHE, O.: Back to the Basics of Banking? A Micro-Analysis of Banking System Stability, *Journal of Financial Intermediation*, forthcoming, working paper, (2008).
- [18] ESTRELLA, A.: Critical values and p-values of Bessel process distributions: Computation and application to structural break tests, *Econometric Theory*, 19, pp. 1128-1143, (2003).
- [19] ELSINGER, H., LEHAR, A., SUMMER, M.: Using market information for banking system risk assesment, *International Journal of Central Banking*, 2(1), pp. 137-165, (2006).
- [20] JAGER-AMBROZEWIEZ, M.: Systemic risk and CoVaR in a Gaussian setting, Working Paper, September, (2010).
- [21] GAIL, M., SYLVAN, G.: A generalization of the one-sided two-sample Kolmogorov-Smirnov statistic for evaluating diagnostic test, *Biometrics*, 32, pp. 561-70, (1976).
- [22] GAIL, M., SYLVAN, G.: Critical values for the one-sided two-sample Kolmogorov-Smirnov statistic, *Journal of the American Statistical Association*, Vol. 71(355), pp. 757-60, (1976).
- [23] GAUTIER, C., LEHAR, A., SOUSSI, M.: Macroprudential regulation and systemic capital requirements, Bank of Canada Working Paper, April, (2010).
- [24] GELUK, J., DE HAAN, L., DE VRIES, C.: Weak and strong financial system fragility, Tinbergen Institute, Discussion Paper, 023/2, (2007).
- [25] GIRARDI, G., ERGUN, A.T.: Systemic Risk Measurement: Multivariate GARCH Estimation of CoVaR, Working Paper, April, (2011).
- [26] GILCHRIST, W.: Statistical modelling with quantile functions, Chapman & Hall/CRC (2000).

- [27] GOURIEROUX, C., LAURENT, J., SCAILLET, O.: Sensitivity analysis of Value at Risk, *Journal of Empirical Finance*, 7, 225-245, (2000).
- [28] HARTMANN, P., STRAETMANS, S., DE VRIES, C.: Banking system stability: A cross atlantic perspective, ECB Working Paper, 527, (2005).
- [29] HUANG, X., ZHOU, H., ZHU, H.: Assessing the systemic risk of a heterogeneous portfolio of banks during the recent financial crisis, Federal Reserve Board Finance and Economics Discussion Series, 44, (2009).
- [30] INTERNATIONAL MONETARY FUND: Financial Stability Report, April, pp 74-109, (2009).
- [31] KOENKER, R.: Quantile Regression, Econometric Society Monographs, Cambridge University Press (2005).
- [32] KOENKER, R., XIAO, Z.: Inference on the quantile regression process, *Econometrica*, Vol. 70(4), pp. 1583-1612, (2002).
- [33] KOENKER, R., MACHADO, J.: Goodness of fit and related inference process for quantile regression, *Journal of the American Statistical Association*, Vol. 94(448), pp. 1296-1310, (1999).
- [34] KOYLUOGLU, U., STOKER, J.: Honour your contribution, *Risk*, April, (2002).
- [35] KURTH, A., TASCHE, D.: Contributions to credit risk, *Risk*, March, (2003).
- [36] LINTON, O., MAASOUMI, E., WHANG, Y.: Consistent testing for stochastic dominance under general sampling schemes, *Review of Economic Studies*, 72, pp. 735-765, (2005).
- [37] PARZEN, E.: Nonparametric statistical data modeling, *Journal of the American Statistical Association*, Vol. 74(365), pp 105-121, (1979).
- [38] PARZEN, E.: Quantile probability and statistical data modeling, *Statistical Science*, Vol. 19(4), pp 652-662, (2004).
- [39] MASON, D., SCHUENEMEYER J.: A modified Kolmogorov-Smirnov test sensitive to tail alternatives, *The Annals of Statistics*, Vol. 11(3), pp.933-46 (1983).
- [40] MCNEIL, A., FREY, R., EMBRECHTS, P.: Quantitative Risk Management, Princeton University Press, (2005).
- [41] SCAILLET, O.: Nonparametric estimation and sensitivity analysis of expected shortfall, *Mathematical Finance*, January, No. 1 (2004).
- [42] SEGOVIANO, M., GOODHART, C.: Bank Stability Measures, IMF Working paper, No. 4 (2009).

- [43] STOYANOV, S., SAMORODNITSKY, G., RACHEV, S., ORTOBELLI, S.: Computing the portfolio conditional value-at-risk in the α -stable case, *Probability and Mathematical Statistics*, Vol. 26(1) pp 1-22 (2006).
- [44] TARASHEV, N., BORIO, C., TSATSARONIS K.: The systemic importance of financial institutions, *BIS Quartely review*, September, (2009).
- [45] TASCHE, D.: Risk contributions and perfomence measurement, working paper, TU Munchen, (2000).
- [46] WHITE, H., KIM, TAE-HWAN, K., MANGANELLI, S.: VAR for VaR: Measuring systemic risk using multivariate regression quantiles, working paper, UC San Diego,(2010).
- [47] ZHOU, C.: Are banks too big to fail, DNB working paper, 232,(2009).

7 Appendix A

Let $\mathbf{Y} = (Y_1, \dots, Y_n)$ denote a vector of independent random variables and a design matrix \mathbf{X} of size $n \times p$. Denote $\hat{\beta}(\tau)$ as the quantile regression process, such that:

$$\hat{\beta}_n(\tau) = \operatorname{argmin}_{\beta \in \mathbb{R}^p} \sum_{i=1}^n \rho_\tau(y_i - \mathbf{x}_i \beta) \quad (33)$$

where $\rho_\tau(u) = u(\tau - I(u < 0))$ and $\tau \in (0, 1)$ (Koenker, 2005).

The conditional quantile function for the response variable \mathbf{Y} given \mathbf{X} can be defined as

$$Q_{\mathbf{Y}}(\tau | \mathbf{X}) = \mathbf{X} \beta_n(\tau) \quad (34)$$

Some important equivariance properties, with respect to scale, location and reparametrization of the design matrix, for the conditional quantile function can be found in Theorem 2.3 of Basset and Koenker (1982).

The empirical counterpart of the conditional quantile function or the expected value of such response function is defined as:

$$\hat{Q}_{\mathbf{Y}}(\tau | \mathbf{X}) = \mathbf{X} \hat{\beta}_n(\tau) \quad (35)$$

Note that like a quantile treatment effect, the empirical conditional quantile function will not necessarily satisfy the fundamental monotonicity requirement of a quantile function (i.e. that the function is non decreasing in τ). The estimated conditional quantile function is subject to possible quantile crossings. As pointed out in Theorem 2.5 of Koenker (2005) these crossings are generally confined to the outlying regions of the design space. Therefore in the centroid of the design space $\bar{\mathbf{X}}$ the estimated conditional quantile function

$$\hat{Q}_{\mathbf{Y}}(\tau | \bar{\mathbf{X}}) = \bar{\mathbf{X}} \hat{\beta}_n(\tau) \quad (36)$$

is more likely to remain monotone in τ . Hence also the expectation of the response function evaluated at the centroid of the design space is monotone with respect to τ .

8 Appendix B

For a general form of the linear quantile regression model, the independent random variables Y_1, \dots, Y_n will have distribution functions F_1, \dots, F_n , respectively. The conditional distribution functions will be denoted as follows:

$$Q_{Y_i}(\tau | x_i) = F_{Y_i}^{-1}(\tau | x_i) \equiv \xi_i(\tau) \quad (37)$$

We state Theorem 4.1 of Koenker (2005), in order to derive the distribution of the $\hat{\beta}_n(\tau)$ and in the sequel the distribution of the quantile response function or conditional quantile function, for a given value of τ . Before restating the theorem we need a series of regularity conditions:

- **Condition 1:** The distribution functions F_i are absolutely continuous, with continuous densities $f_i(\xi)$ uniformly bounded away from 0 and ∞ at the points $\xi_i(\tau)$.
- **Condition 2:** There exist positive definite matrices Q and $D(\tau)$ such that:
 1. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n x_i x_i' = \Omega$.
 2. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f_i(\xi_i(\tau)) x_i x_i' = D(\tau)$.
 3. $\max_{i=1, \dots, n} \frac{\|x_i\|}{\sqrt{n}} \rightarrow 0$.

Theorem 1: Under conditions 1 and 2

$$\sqrt{n}(\hat{\beta}_n(\tau) - \beta(\tau)) \sim N(0, \tau(1 - \tau)D^{-1}\Omega D^{-1}) \quad (38)$$

In the i.i.d. error model:

$$\sqrt{n}(\hat{\beta}_n(\tau) - \beta(\tau)) \sim N(0, \omega\Omega^{-1}) \quad (39)$$

where $\omega = \frac{\tau(1-\tau)}{f^2(\xi_i(\tau))}$.

Theorem 2: From Theorem 1 and let us define some continuous mapping $g(\theta(\tau)) = \mathbf{X}\beta(\tau)$, where this mapping defines the quantile response function, evaluated at some point in the design space.

$$\sqrt{n}(\hat{Q}_{\mathbf{Y}}(\tau | \mathbf{X}) - Q_{\mathbf{Y}}(\tau | \mathbf{X})) \sim N(0, \tau(1 - \tau)\mathbf{X}D^{-1}\Omega D^{-1}\mathbf{X}') \quad (40)$$

Proof:

Direct application of the Delta Method such that:

$$\sqrt{n}(\mathbf{X}\hat{\beta}_n(\tau) - \mathbf{X}\beta(\tau)) \sim N(0, \tau(1 - \tau)\mathbf{X}D^{-1}\Omega D^{-1}\mathbf{X}'). \quad (41)$$

Hence $\hat{Q}_{\mathbf{Y}}(\tau | \mathbf{X})$ is weakly consistent for $Q_{\mathbf{Y}}(\tau | \mathbf{X})$ ²⁵.

Theorem 2 serves as a first step toward introducing additional inference problems, based on the quantile response function, beyond what is known as the fundamental problem, in quantile inference, that involves testing for the equality of the slope parameters across quantiles²⁶.

²⁵A stronger form of consistency of the conditional quantile function requires more stringent regularity conditionals and it is explored in Basset and Koenker (1982)

²⁶Such hypothesis test is also known as the constant effect hypothesis, see Chernozhukov and Fernandez-Val (2005)