INTEGRAL TRANSFORMS WITH THE HOMOTOPY PERTURBATION METHOD AND SOME APPLICATIONS

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ABSTRACT. This paper applies He's homotopy perturbation method to compute a large variety of integral transforms. As illustration, the paper gives special attention to the Esscher transform, the Fourier transform, the Hankel transform, the Mellin transform, the Stieljes transform and some applications.

Key Words: He's homotopy method; integral transforms; linear equations; Type G and spherical distributions.

1. INTRODUCTION

Approximation theory covers some important topics in applied analysis, and its application serves many fields in science and engineering such as fluid mechanics, electromagnetism, diffraction theory, statistics and economics. Although it is an old subject, dating back to Laplace, new methods and applications continue to appear in various publications. There is now a need to provide new methods in the other main area of asymptotic theory, namely, the asymptotic approximation of integrals. In fact, complicated integrals are difficult to solve, and cannot be expressed in terms of elementary functions or analytical formulae. The purpose of this paper is to fulfil this need.

The application of the homotopy perturbation method (HPM) in mathematical problems is highly considered by scientists, because without demanding a small parameter in equations, HPM continuously transforms a complex problem which is not easy to solve into a simple problem. The homotopy perturbation method [12] was first proposed by He in 1998. It is in fact a coupling of the traditional perturbation method and homotopy in topology. The HPM was further developed and improved by He [12]-[20] and applied to asymptotology [17], bifurcation for non-linear problems [18], strongly non-linear equations [19] and many other subjects. The method yields a very rapid convergence of the solution series in most cases, and usually only a few iterations lead to very accurate solutions. Although the goal of He' s homotopy perturbation method was to find a technique

to unify linear and nonlinear, ordinary or partial differential equations for solving initial and boundary value problems, here we deal only with the simple first-order differential equation.

Generally, the solutions of the first-order differential equations can be transformed into integrals. In the case where such integrals are difficult to estimate numerically or analytically, new methods are welcome. For example, Hardar et al.[8] applied Adomian's decomposition method to calculate certain integrals. In Bobolian et al.[5], the so-called Adomian decomposition method is used for the computation of the Laplace transform. The shortcoming of Adomian's method [4], [26] is a complicated calculation of the so-called Adomian polynomials. Even if Momani et al.[24] showed that the variational iteration method [21, 22] would allow the overcoming of this difficulty, a serious alternative approach to overcoming the shortcoming arising in the Adomian method is the homotopy perturbation method. Note that, [19] and some references therein found that the shortcomings arising in Adomian method can be completely eliminated by the variational iteration method. Recently, Abbasbandy [1]-[3] suggested the application of HPM in order to compute the Laplace transform .

In this paper, we use He's homotopy perturbation method to compute the general integral transform

$$(1.1) \qquad I(z,x) = \int_0^x f(t) \, \exp\left(\int_t^x g(s,z) \, ds\right) dt.$$

Note that many important integral transforms, including the Esscher, Fourier, Hankel, Hilbert, Laplace, Mellin and Stieljes transforms, can be put in the form of (1.1). Indeed, consider the first-order differential equation

$$(1.2) \qquad \begin{cases} v'(x) = g(x,z) \, v(x) + f(x), & x > 0 \\ v(0) = 0, \end{cases}$$

where the function $f(t)$ on $(0,\infty)$ is locally integrable and the function $g(x,z)$ is defined in $(0,\infty) \times D$ and $D \subset \mathbf{C}$. After solving the homogeneous equation of (1.2) and using the "method of variation of parameters", the analytic solution of (1.2) is given by

$$(1.3) \qquad \begin{aligned} v(x) &= \int_0^x f(t) \, \exp\left(\int_t^x g(s,z)ds\right) dt \\ &= \exp\left(\int_0^x g(s,z)ds\right) \int_0^x f(t) \, \exp\left(-\int_0^t g(s,z)ds\right) dt, \end{aligned}$$

and therefore

$$v(x) \exp\left(-\int_0^x g(s,z) \, ds\right) = \int_0^x f(t) \, \exp\left(-\int_0^t g(s,z)ds\right) dt.$$

2. Homotopy perturbation method

The homotopy perturbation method provides an alternative approach to introducing an expanding parameter.

To illustrate the basic ideas of He's homotopy perturbation method, He [12] considers the following differential equation

$$(2.4) \qquad A(v) = F(\mathbf{r}), \qquad \mathbf{r} \in \Omega$$

with boundary conditions

$$B(v) = B\left(v, \frac{\partial v}{\partial n}\right), \qquad \mathbf{r} \in \Gamma$$

where A is a general differential operator, B is a boundary operator, $F(\mathbf{r})$ is a known analytic function, Γ is the boundary of the domain Ω and $\frac{\partial v}{\partial n}$ denotes the differentiation along the normal vector drawn outwards from Ω. The operator A can be divided into two parts L and N, where L is a linear operator, while N is a linear or non-linear operator. Therefore (2.4) can be rewritten as follows:

$$(2.5) \qquad L(v) + N(v) - F(\mathbf{r}) = 0.$$

He [12], [13] constructed a homotopy $w(\mathbf{r}, p) : \Omega \times [0,1] \to \mathbb{R}$ which satisfies

$$(2.6) \qquad \mathcal{H}(w,p) = (1-p)\left[L(w) - L(v_0)\right] + p\left[A(w) - F(\mathbf{r})\right] = 0$$

where v_0 is the initial value of (2.4) and $p \in [0,1]$ is an embedding parameter. Hence, it is obvious that

$$\mathcal{H}(w,0) = L(w) - L(v_0)$$

and

$$\mathcal{H}(w,1) = A(w) - F(\mathbf{r}).$$

The changing process of p from zero to unity is simply that of $w(\mathbf{r}, p)$ from v_0 to $F(\mathbf{r})$. In topology, this is called deformation, and $L(w) - L(v_0)$, $A(w) - F(\mathbf{r})$ are homotopic. Applying the perturbation technique, due to the fact that $0 \leq p \leq 1$ can be considered as a small parameter, we can assume that the solution of (2.6) can be expressed as a series in p, as follows:

$$(2.7) \qquad w = \sum_{i=0}^{\infty} p^i w_i,$$

when $p \to 1$ and since $A := L + N$, (2.7) becomes the approximate solution of (2.5), therefore

$$(2.8) \qquad v = \lim_{p \to 1} \sum_{i=0}^{\infty} p^i w_i.$$

The convergence of the series in eq. (2.8) is discussed by He [12, 13].

3. Integral Transforms with the HPM

In this section, we consider the homotopy perturbation method with

$$L(v(x, z)) = -g(x, z)\, v(x, z)$$

and

$$N(v(x, z)) = \frac{\partial v}{\partial x}(x, z).$$

Following (2.5) and (2.6), we construct the simple homotopy

$$(3.9) \quad (1 - p)\left[-g(x, z)[w(x, z) - v_0(x)]\right] + p\left[\frac{\partial w}{\partial x}(x, z) - g(x, z)w(x, z) - f(x)\right] = 0,$$

or

$$(3.10) \qquad -g(x, z)[w(x, z) - v_0(x)] + p\left[\frac{\partial w}{\partial x}(x, z) - g(x, z)\, v_0(x) - f(x)\right] = 0.$$

Substituting (2.7) into (3.10), and equating the terms with the identical powers of p, we have

(3.11)

$$-g(x, z)\left[\sum_{i=0}^{\infty} p^i w_i(x, z) - v_0(x)\right] - p\, g(x, z)v_0(x) + p\left[\sum_{i=0}^{\infty} p^i \frac{\partial w_i}{\partial x}(x, z) - f(x)\right] = 0$$

$$
\begin{aligned}
p^0 &: & L(w_0 - v_0) &= 0 \\
p^1 &: & L(w_1 + v_0) + N(w_0) - f(x) &= 0 \\
p^2 & & L(w_2) + N(w_1) &= 0 \\
p^3 & & L(w_3) + N(w_2) &= 0 \\
&\vdots & \vdots & \\
p^{n+1} & & L(w_{n+1}) + N(w_n) &= 0.
\end{aligned}
$$

In other words

$$
\begin{aligned}
p^0 &: & -g(x, z)\left[w_0(x, z) - v_0(x)\right] &= 0 \\
p^1 &: & -g(x, z)\left(w_1(x, z) + v_0(x)\right) + \frac{\partial w_0}{\partial x}(x, z) - f(x) &= 0 \\
p^2 & & -g(x, z)\, w_2(x, z) + \frac{\partial w_1}{\partial x}(x, z) &= 0 \\
p^3 & & -g(x, z)\, w_3(x, z) + \frac{\partial w_2}{\partial x}(x, z) &= 0 \\
&\vdots & \vdots & \\
p^{n+1} & & -g(x, z)\, w_{n+1}(x, z) + \frac{\partial w_n}{\partial x}(x, z) &= 0.
\end{aligned}
$$

Theorem 3.1. *The solution provided by the HPM is*

$$v(x) = \int_0^x f(t) \exp\left(\int_t^x g(s,z)\,ds\right) dt$$

$$(3.12) \qquad = \lim_{p\to 1} \sum_{i=0}^{\infty} p^i\, w_i(x,z).$$

Since

$$\exp\left(-\int_0^x g(s,z)ds\right) \int_0^x f(t) \exp\left(\int_t^x g(s,z)\,ds\right) dt = \int_0^x \exp\left(-\int_0^t g(s,z)ds\right) f(t)\,dt,$$

therefore, according to (1.4) and (2.8), we introduce the following definition theorem:

Definition 3.2. *If f is a locally integrable function on $[0,\infty($, the so-called incomplete Sadefo transform IS_g of f is the function of z defined by*

$$(3.13) \qquad IS_g(z) = \int_0^x \exp\left(-\int_0^t g(s,z)ds\right) f(t)dt$$

$$= \exp\left(-\int_0^x g(s,z)ds\right) \lim_{p\to 1} \sum_{i=0}^{\infty} p^i\, w_i(x,z)$$

and the so-called Sadefo transform S_g of f is the function of z defined by

$$(3.14) \qquad S_g(z) = \int_0^\infty \exp\left(\int_0^t g(s,z)\,ds\right) f(t)\,dt$$

$$= \lim_{x\to\infty}\left[\exp\left(-\int_t^x g(s,z)ds\right) \lim_{p\to 1}\sum_{i=0}^{\infty} p^i\, w_i(x,z)\right],$$

where g is given such that the integrals $S_g^X(z)$ and $IS_g(z)$ converge, and the approximation of $IS_g(z)$ and $S_g(z)$ are obtained via the application of the homotopy perturbation method to the equation (1.2).

Example 3.3. In the calculation of flux integrals associated with double-scattering waves, Servadio [25] has investigated the asymptotic behavior of the integral

$$(3.15) \qquad I_F(z) = \int_0^\infty f(t)\, F(z\,t)\, dt$$

where

$$(3.16) \qquad F(t) = \int_t^\infty e^{-i\tau^2}\, d\tau.$$

For a given f, the estimation of $I_F(z)$ is reduced to the estimation of (1.1) where

$$g(s,z) = z\,\frac{e^{-i\,s^2 z^2}}{\int_{sz}^\infty e^{-i\,t^2}\, dt},$$

and the components $w_i(x,z)$ are given by (3.9). In fact

$$(3.17) \qquad I_F(z) \; = \; \int_0^\infty f(t)\, F(z\,t)\, dt$$

$$= \; F(0) \int_0^\infty f(t)\, \exp\left(ln(F(z\,t)/F(0))\right) dt$$

$$= \; F(0) \int_0^\infty f(t)\, \exp\left(-\int_0^t -z\frac{F'(z\,s)}{F(z\,s)}\,ds\right) dt$$

$$(3.18) \qquad = \; F(0) \int_0^\infty f(t)\, \exp\left(-\int_0^t g(s,z)\,ds\,ds\right) dt$$

where

$$g(s,z) = -z\,\frac{F'(z\,s)}{F(z\,s)} = z\,\frac{e^{-i\,s^2 z^2}}{\int_{sz}^\infty e^{-i\,t^2}\,dt}.$$

Corollary 3.4. Following the example that precedes, it is easy to see that the convolution integral

$$(3.19) \qquad I_h(z) = \int_0^\infty f(t)\, h(z\,t)\, dt,$$

and the more symmetric convolution integral

$$(3.20) \qquad f*g(z) = \int_0^\infty f(t)\, g(z\,t^{-1})\, t^{-1}\, dt$$

can be obtained via the HPM, when $I_h(z)$ and $f*g(z)$ converge.

3.1. **The case where $g(x,z)$ is homogeneous.** Here we consider the particular case where the function g doesn't depend of x, namely

$$(3.21) \qquad g(x,z) = h(z).$$

So the integral (1.4) becomes

$$(3.22) \qquad v(x) \; = \; \int_0^x f(t)\, \exp(-h(z)\,(t-x))\, dt$$

$$= \; \exp(h(z)\,x) \int_0^x f(t)\, \exp(-h(z)\,t)\, dt.$$

If f is an homogeneous function of order 1 (ie: $\forall u \in \mathbb{R}$, h(u z)= u h(z)), then (3.22) becomes

$$(3.23) \qquad v(x)\exp(-h(z\,x)) = \int_0^x f(t)\, \exp(-h(z\,t))\, dt.$$

We then have the n-th approximation of the integral (3.23) denoted by $I_1(z)$ as follows

$$(3.24) \qquad I_1(z) \;=\; \int_0^\infty f(t)\, \exp(-h(z\,t))\, dt$$

$$\;=\; \lim_{x\to\infty}\lim_{p\to 1}\sum_{i=0}^\infty p^i w_i(x,z)\exp(-h(z\,x)).$$

Example 3.5. *Let $h(z) = z$ and $f(x) = e^{-x^\nu}$, where $x \geq 0$ and the parameter $\nu \in \mathbb{R}_+^*$. By choosing $v_0(x) = 0$, we have*

$$w_0(x,z) = 0$$
$$w_1(x,z) = -\frac{f(x)}{z}$$
$$w_2(x,z) = -\frac{f'(x)}{z^2}$$
$$w_3(x,z) = -\frac{f''(x)}{z^3}$$
$$\vdots$$
$$w_n(x,z) = -\frac{f^{(n-1)}(x)}{z^n}.$$

with $f^{(n)}$ is the n-derivative of the function $f(x)$. So for $Re(z) > 0$,

$$\int_0^\infty \exp(z\,t - t^\nu)\, dt \;=\; -\lim_{x\to\infty}\lim_{p\to 1}\sum_{i=0}^\infty p^i\,\frac{b_{i,\nu}(x)}{z^i}\,\exp(-z\,x - x^\nu))$$

$$(3.25) \qquad\qquad \approx\; -\lim_{x\to\infty}\sum_{i=0}^n \frac{b_{i,\nu}(x)}{z^i}\,\exp(-z\,x - x^\nu))$$

where

$$b_{0,\nu}(x) = 0, \quad b_{1,\nu}(x) = \nu\, x^{-1+\nu}, \quad b_{2,\nu}(x) = (1 - \nu + \nu\, x^\nu)x^{-2+\nu}\nu,$$

$$b_{3,\nu}(x) = x^{-3+\nu}\,(-2+\nu)(-1+\nu)\,\nu + x^{-3+2\nu}\,(-1+\nu)\,\nu^2 - x^{-3+3\nu}\,\nu^3 + x^{-3+2\nu}\,\nu^2\,(-2+2\nu),$$

$$b_{4,\nu}(x) = \left(6 - 11\nu\,(1 - x^\nu) + (6 - 18x^\nu + 6x^{2\nu})\nu^2 + (-1 + 7x^\nu - 6x^{2\nu})\nu^3 + x^{3\nu}\right)\, x^{-4+\nu}\,\nu,$$

$$\vdots$$

4. Application for the Esscher, Fourier, Hankel, Mellin and Stieljes transforms

In this section, with a change of suitable variables, we show how to compute the Esscher, Fourier, Hankel, Laplace, Mellin and Stieljes transforms.

4.1. **Computation of the Fourier Transform.** Chosing

$$(4.26) \qquad\qquad\qquad g(s,z) = i\,z$$

for all $s \in [t, x]$, the Fourier transform of f is

$$(4.27) \qquad \mathcal{F}[f,z] = e^{-i\,x\,z}\left[\int_0^\infty f(-t)e^{i\,t\,z}dt + \int_0^\infty f(t)\,e^{-i\,t\,z}\,dt\right]$$

4.2. **Computation of the Laplace Transform.** In the case where we consider $h(z) = z$ which is homogeneous of the order 1, in the integral (3.15), the Laplace transform of the function $f(x)$, denoted by $\mathcal{L}[f, z]$, is defined by the integral

$$(4.28) \qquad \mathcal{L}[f,z] = \int_0^\infty f(t)\,\exp(-z\,t)\,dt$$

$$= \lim_{x\to\infty}\lim_{p\to 1}\sum_{i=0}^\infty p^i w_i(x,z)\exp(-z\,x),$$

where $Re(z) > 0$. The functions $f(x)$ and $\mathcal{L}[f, z]$ are called a Laplace transform pair. Even though the approximation of the Laplace transform has been the subject of [1], we proposed the following theorem:

Definition 4.1. *For all $x \in (0,\infty)$, the incomplete Laplace transform of f as*

$$(4.29) \qquad I\mathcal{L}[f,z](x) = \int_0^x f(t)\,\exp(-z\,t)\,dt$$

$$= \exp(-z\,x)\lim_{p\to 1}\sum_{i=0}^\infty p^i\,w_i(x,z),$$

where $Re(z) > 1$.

Theorem 4.2. *Using some properties of the Laplace transform, we introduce the following results:*

(1) *If an integrer $n > 0$ and $\lim_{x\to\infty} f(x)e^{-sx} = 0$, then, for $x > 0$,*

$$\mathcal{L}[f^{(n)}, s] = s^n\lim_{x\to\infty}\left[\exp(-s\,x)\lim_{p\to 1}\sum_{i=0}^\infty p^i\,w_i(x,z)\right] - \sum_{i=0}^{n-1}s^{n-1-i}f^{(i)}(0),$$

   where $f^{(0)} \equiv f$ and $f^{(j)}$ is the j-th derivative of f.

(2) *If $\lim_{x\to\infty}\left[e^{-sx}\int_0^x f(\xi)\,d\xi\right] = 0$, then*

$$\mathcal{L}\left[\int_0^x f(\xi)d\xi), s\right] = \frac{1}{s}\lim_{x\to\infty}\left[\exp(-s\,x)\lim_{p\to 1}\sum_{i=0}^\infty p^i\,w_i(x,z)\right],$$

(3)

$$\mathcal{L}\left[e^{-ax}f(x),s\right] = \lim_{x\to\infty}\left[\exp(-(s+a)\,x)\lim_{p\to 1}\sum_{i=0}^{\infty}p^i\,w_i(x,z)\right],$$

(4) *The Laplace transform of the convolution $f*g = g*f$ of two functions $f(x)$ and $g(x)$, has the following approximation*

(4.30)
$$\mathcal{L}\left[f*g(x),s\right] = \lim_{x\to\infty}\left[\lim_{p\to 1}\sum_{i=0}^{\infty}p^i\,w_i(x,z)\sum_{j=0}^{\infty}p^j\,v_j(x,z)\right]\exp(-2z\,x)$$

where

$$\mathcal{L}[f,z] = \lim_{x\to\infty}\lim_{p\to 1}\sum_{i=0}^{\infty}p^i\,w_i(x,z)\exp(-z\,x),$$

$$\mathcal{L}[g,z] = \lim_{x\to\infty}\lim_{p\to 1}\sum_{j=0}^{\infty}p^j\,v_j(x,z)\ \exp(-z\,x),$$

and where the convolution function $f*g$ is defined by the integral

$$f*g(x) = \int_0^x (f(x-\xi)g(\xi)\,d\xi.$$

(5) *If f is a summable function over all finite intervals, and there is a constant c for which*

$$\int_0^{\infty}|f(x)|\,e^{-cx}\ dx$$

is finite, then the Laplace transform is accomplished for the analytic function $\mathcal{L}[f,z]$ of order $O(z^{-k})$, with $k > 1$, by means of the inverse integral

(4.31)
$$f(x) = \frac{1}{2\,\pi\,i}\int_{\gamma-i\infty}^{\gamma+i\infty}\mathcal{L}[f,z]e^{xz}dz$$

$$= \frac{1}{2\,\pi\,i}\int_{\gamma-i\infty}^{\gamma+i\infty}\exp(z\,x)\left[\lim_{x\to\infty}\lim_{p\to 1}\sum_{i=0}^{\infty}p^i\,w_i(x,z)\exp(-z\,x)\right]\,dz$$

where γ is a real constant that exceeds the real parts of all the singularities of $\mathcal{L}[f,z]$.

Example 4.3. *Consider the integral*

$$S(x) = \int_0^{\infty}\frac{t^{1/2}}{1+t}\,sin(x\,t)\,dt = Im\left(\int_0^{\infty}\frac{t^{1/2}}{1+t}\,\exp(i\,x\,t)\,dt\right),$$

where $i^2 = -1$ and $Im(z)$ denotes the imaginary part of the complex z. To approximate the integral

$$\int_0^{\infty}\frac{\sqrt{t}}{1+t}\,\exp(i\,z\,t)\,dt$$

via the HPM, it suffices to replace in (1.2), g by

$$g(s, z) = -\imath\, z, \qquad f(t) = \frac{\sqrt{t}}{1 + t}.$$

4.3. **Computation of the Mellin Transform.** To get the Mellin transform, we choose the function

$$g(s, z) = \frac{1 - z}{s}.$$

If we replace g in (3.22), we get

$$(4.32) \qquad \mathcal{M}[f, z] \;\; = \;\; \int_0^\infty f(t)\, t^{z-1}\, dt$$

$$(4.33) \qquad\qquad\qquad = \;\; \lim_{x \to \infty} \lim_{p \to 1} \sum_{i=0}^\infty p^i\, w_i(x, z)\, x^{z-1}.$$

Note that $a < Re(z) < b$ is an infinite strip such that the Mellin transform of $f(t)$ denoted by $\mathcal{M}[f, z]$ converges (See [28] for more details).

Proposition 4.4. *For all $x \in (0, \infty)$, the incomplete Mellin transform of f is defined as*

$$(4.34) \qquad I\mathcal{M}[f, z](x) \;\; = \;\; \int_0^x f(t)\, t^{z-1}\, dt$$

$$= \;\; x^{z-1} \lim_{p \to 1} \sum_{i=0}^\infty p^i\, w_i(x, z),$$

where $Re(z) > 1$.

Theorem 4.5. *In using the basic properties of the Mellin transform, we get the following results:*

(1) *If $\lim_{x \to \infty} x^{s-r-1} f^{(r)}(x) = 0, \quad r = 0, 1, \ldots, n-1$*

-

$$\mathcal{M}[f^{(n)}, z] = (-1)^n \frac{\Gamma(z)}{\Gamma(z - n)} \lim_{x \to \infty} \lim_{p \to 1} \sum_{i=0}^\infty p^i\, w_i(x, z - n)\, x^{z-n-1},$$

-

$$\mathcal{M}[x^n\, f^{(n)}, z] = (-1)^n \frac{\Gamma(z + n)}{\Gamma(z)} \lim_{x \to \infty} \lim_{p \to 1} \sum_{i=0}^\infty p^i w_i(x, z)\, x^{z-1},$$

(2) *Denoting the nth repeated integral of $f(x)$ by $I_n(f(x))$, where*

$$I_n[f(x)] = \int_0^x I_{n-1}[f(u)]\, du,$$

(4.35) $$\mathcal{M}[I_n[f(x)], z] = (-1)^n \frac{\Gamma(z)}{\Gamma(z+n)} \lim_{x \to \infty} \lim_{p \to 1} \sum_{i=0}^{\infty} p^i w_i(x, z+n)\, x^{z+n-1}$$

(3) *The transform exists provided the integral*

$$\int_0^\infty |f(x)|\, x^{k-1}\, dx$$

is bounded for some $k > 0$, and then the inversion of the Mellin transform is accomplished by means of the inversion integral

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{M}[f, z] x^{-z} dz \\
(4.36) \qquad &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} x^{-z} \left[\lim_{x \to \infty} \lim_{p \to 1} \sum_{i=0}^{\infty} p^i w_i(x, z)\, x^{z-1} \right] dz
\end{aligned}
$$

where $c > k$.

Proposition 4.6. *The inversion formula for the Mellin transform is given by*

$$
\begin{aligned}
(4.37) \qquad f(t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-z}\, \mathcal{M}[f, z]\, dz \\
&= \frac{1}{2\pi i} \lim_{x \to \infty} \lim_{p \to 1} \sum_{i=0}^{\infty} p^i \int_{c-i\infty}^{c+i\infty} w_i(x, z)\, x^{z-1}\, t^{-z}\, dz.
\end{aligned}
$$

where $a < c < b$.

Remark 4.7. Note that one of the two convolution integral transforms, including the Laplace, Fourier, Hankel and Stieljes transforms, can be put in the form of

(4.38) $$I(x) = \int_0^\infty f(t)\, h(x\, t)\, dt.$$

Since $\mathcal{M}[I, z] = \mathcal{M}[f, 1-z]\, \mathcal{M}[h, z]$, then by inversion we obtain

(4.39) $$I(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-z}\, \mathcal{M}[h, z]\, \mathcal{M}[f, 1-z]\, dz$$

where $c > k$ and the Mellin transform in the integral (4.39) can be expressed as in (4.32).

4.4. **Computation of the Hankel Transform.** The Hankel transform (of order zero) is an integral transform equivalent to a two-dimensional Fourier transform with a radially symmetric integral kernel and is also called the Fourier-Bessel transform. It is defined as

$$g(u,v) = \mathcal{F}[f(r)](u,v)$$
$$(4.40) \qquad = \int_{-\infty}^{\infty} f(r)e^{-2\pi i(ux+vy)} \, dx \, dy,$$

where $r = \sqrt{x^2 + y^2}$.

Let $x = r\cos(\theta)$, $y = r\sin(\theta)$, $q = \sqrt{u^2 + v^2}$, $u = q\cos(\phi)$ and $v = q\sin(\phi)$, then

$$g(q) = \int_0^{\infty} \int_0^{2\pi} f(r)e^{-2\pi i(\cos(\theta)\cos(\phi)+\sin(\theta)\sin(\phi))}r \, dr \, d\theta$$
$$= \int_0^{\infty} \left[\int_0^{2\pi} f(r)e^{-2\pi i\cos(\theta)} \, d\theta \right] r \, dr$$
$$(4.41) \qquad = 2\pi \int_0^{\infty} f(r)J_0(2\pi qr)r\,dr,$$

where J_0 is a zero-order Bessel function of the first kind. Therefore, the Hankel transform pairs are

$$g(q) = 2\pi \int_0^{\infty} f(r)J_0(2\pi qr)r \, dr$$
$$f(r) = 2\pi \int_0^{\infty} g(q)J_0(2\pi qr)q \, dq.$$

The Hankel transform of n-th order is defined by

$$(4.42) \qquad \mathcal{H}_n[f(t),\phi] = \int_0^{\infty} t \, J_n(\phi \, t) \, f(t) \, dt$$

where J_n is a Bessel function of the first kind (Bronshtein et al.[6], p. 706).

We continue and propose the following definition

Definition 4.8. *For all $u \in (0,\infty)$ and for a given function f, the incomplete Hankel transform of order n is*

$$(4.43) \qquad I\mathcal{H}_n[f,z](u) = \int_0^{u} t \, J_n(z \, t) \, f(t) \; dt$$

where the Bessel function of the first kind $J_n(x)$ is defined as the solution of the Bessel differential equation

$$(4.44) \qquad x^2 \frac{\partial^2 y}{\partial x^2} + x\frac{\partial y}{\partial x} + (x^2 - n^2)y = 0$$

which is non-singular at the origin and n is a fix real or complex number.

Remark 4.9. If we replace the function $g(x, z)$ in (1.4) by

$$(4.45) \qquad g(x, z) = -\frac{1}{s} - s \frac{J'_n(s\,z)}{J_n(s\,z)},$$

we get

$$\begin{aligned}
v(x) &= \frac{1}{x\,J_n(x\,z)} \left[\int_0^u t\,J_n(z\,t)\,f(t)\;dt \right] \\
(4.46) &= \frac{1}{x\,J_n(x\,z)} I\mathcal{H}_n[f, z](x).
\end{aligned}$$

Since the solution v of (1.2) has been approximate via the homotopy perturbation method, we then propose the following theorem:

Theorem 4.10. *For a given function f and $\forall x \in)0, \infty($, the incomplete Hankel transform is given by*

$$(4.47) \qquad I\mathcal{H}_n[f, z](x) = \frac{1}{x\,J_n(x\,z)} \lim_{p \to 1} \sum_{i=0}^{\infty} p^i\,w_i(x, z),$$

where the Bessel function of the first kind $J_n(x)$ is defined as the solution of (4.44).

We also propose the following corollary

Corollary 4.11. *For a given function f, the Hankel transform is given by*

$$\begin{aligned}
(4.48) \qquad \mathcal{H}_n[f, z] &= \lim_{x \to \infty} \left[\frac{1}{x\,J_n(x\,z)} \lim_{p \to 1} \sum_{i=0}^{\infty} p^i\,w_i(x, z) \right] \\
&= \int_0^{\infty} t\,J_n(z\,t)\,f(t)\,dt
\end{aligned}$$

where the Bessel function of the first kind $J_n(x)$ is defined as the solution of (4.44).

Example 4.12. There are some problems of high energy nuclear physics (see Glauber [10] and Gabutti et al.[9]) that involve integrals of the form

$$(4.49) \qquad I_g(x) = \int_0^{\infty} e^{-t^2} J_0(xt)\,g(t^2)\,t\,dt,$$

where $J_0(t)$ is the Bessel function of order zero and g is a continuous function in $(0, \infty)$.

Clearly these integrals are special cases of the Hankel transform formula (4.48), where

$$f(t) = e^{-t^2} g(t^2)$$

and $n = 0$.

4.5. **Computation of the Stieljes Transform.** We need to approximate the following integral

$$(4.50) \qquad GS_{m,p}[f, z](u) = \int_0^u (t^m + z^m)^{-p} f(t) \, dt$$

where $u \in (0, \infty)$, $m > 0$ and $p \geq 1$. Note that in the case where $p = 1$, $m = 1$ and $u \to \infty$, we have the Stieljes transform of f.

Remark 4.13. If we replace the function $g(x, z)$ in (1.4) by

$$(4.51) \qquad g(x, z) = p \, m \, \frac{t^{m-1}}{x^m + z^m},$$

we get

$$(4.52) \qquad v(x) \;=\; (x^m + z^m) \, GS_{m,p}[f, z, u].$$

We therefore obtain the following definition theorem:

Definition 4.14. *For all $x \in [0, \infty)$, the incomplete generalized Stieljes transform of a locally integrable function f on $[0, \infty)$ is defined by*

$$IGS_{m,p}[f, z](x) \;=\; \int_0^x f(t) \, (t^m + z^m)^{-p} \, dt$$

$$(4.53) \qquad\qquad\qquad = \; (x^m + z^m) \lim_{p \to 1} \sum_{i=0}^{\infty} p^i \, w_i(x, z),$$

where z is a complex variable in the cut plane $|\arg z| < \pi$.

The generalized Stieljes transform of a locally integrable function f is defined by

$$GS_{m,p}[f, z] \;=\; \int_0^{\infty} f(t) \, (t^m + z^m)^{-p} \, dt$$

$$(4.54) \qquad\qquad\qquad = \; \lim_{x \to \infty} \left[(x^m + z^m) \lim_{p \to 1} \sum_{i=0}^{\infty} p^i \, w_i(x, z) \right],$$

where $m > 0$, $p \geq 1$ and z is a complex variable in the cut plane $|\arg z| < \pi$.

4.6. **Computation of the Esscher Transform.** The Esscher transform was developed to approximate the aggregate claim amount distribution around a point of interest x_0, by applying Edgeworth series to the transformed distribution with the parameter h chosen such that the mean is equal to x_0.

We defined the generalized Esscher transform of an appropriate chosen function f : $]0, \infty(\to \mathbb{R}$ by

Proposition 4.15.

$$(4.55) \qquad \mathcal{E}s[f,z] \;=\; \frac{1}{M_z} \int_0^\infty x \exp(z\,x)\, f(x)dx$$

$$=\; \frac{1}{M_z}\, \mathcal{L}[g,-z]$$

$$(4.56) \qquad =\; \frac{1}{M_z}\, \lim_{x\to\infty} \lim_{p\to 1} \sum_{i=0}^\infty p^i w_i(x,-z)\, \exp(z\,x)$$

where $M_z = \int_0^\infty \exp(z\,t)\, f(t)dt = \mathcal{L}[f,-z]$, $g(x) = x\, f(x)$, $\mathcal{L}$ denotes the Laplace transform defined in (4.28), and the real constant $Re(z) > 0$.

Example 4.16. *In the following example we choose*

$$f(x) = e^{-x^3}$$

where $x \in \mathbb{R}^+$. Using Gradshteyn and Ryzhik [11], we get

$$M_z \;=\; \int_0^\infty \exp(z\,t)\,\exp(-t^3)\,dt$$

$$(4.57) \qquad =\; \frac{\sqrt{z}}{18}\left(4\pi\left(J_{-\frac{1}{3}}\left(\frac{2z^{3/2}}{3\sqrt{3}}\right) - J_{\frac{1}{3}}\left(\frac{2z^{3/2}}{3\sqrt{3}}\right)\right) + 3z^{3/2}\,_1F_2\left[1;\frac{4}{3},\frac{5}{3};-\frac{z^3}{27}\right]\right),$$

$$\int_0^\infty xe^{z\,x}\,e^{-x^3}\,dx \;=\; \frac{z}{27}\left(2\pi\sqrt{3}\left(J_{-\frac{2}{3}}\left(\frac{2z^{3/2}}{3\sqrt{3}}\right) - J_{\frac{2}{3}}\left(\frac{2z^{3/2}}{3\sqrt{3}}\right)\right) - 9\,_1F_2\left[1;\frac{2}{3},\frac{4}{3};-\frac{z^3}{27}\right]\right),$$

and

$$(4.58) \qquad \mathcal{E}s[f,z] = \frac{18\sqrt{z}\left(2\pi\sqrt{3}\left(J_{-\frac{2}{3}}\left(\frac{2z^{3/2}}{3\sqrt{3}}\right) - J_{\frac{2}{3}}\left(\frac{2z^{3/2}}{3\sqrt{3}}\right)\right) - 9\,_1F_2\left[1;\frac{2}{3},\frac{4}{3};-\frac{z^3}{27}\right]\right)}{27\left(4\pi\left(J_{-\frac{1}{3}}\left(\frac{2z^{3/2}}{3\sqrt{3}}\right) - J_{\frac{1}{3}}\left(\frac{2z^{3/2}}{3\sqrt{3}}\right)\right) + 3z^{3/2}\,_1F_2\left[1;\frac{4}{3},\frac{5}{3};-\frac{z^3}{27}\right]\right)}.$$

We then deduce that

$$\lim_{x->\infty} \lim_{p\to 1} \sum_{i=0}^\infty p^i w_i(x,-z)\, e^{z\,x} = \frac{z}{27}\left(2\pi\sqrt{3}\left(J_{-\frac{2}{3}}\left(\frac{2z^{3/2}}{3\sqrt{3}}\right) - J_{\frac{2}{3}}\left(\frac{2z^{3/2}}{3\sqrt{3}}\right)\right) - 9\,_1F_2\left[1;\frac{2}{3},\frac{4}{3};-\frac{z^3}{27}\right]\right)$$

where, for $n = 0, 1, \ldots$, we set

$$w_n(x,-z) = e^{-x^3}\,\frac{a_n(x)}{z^n},$$

where $a_n(x)$ is a polynomial function of order $2(n-1)$ that admits the following ten first terms, obtained via **Mathematica** *software:*

$$a_0(x) = 0,\; a_1(x) = 1,\; a_2(x) = 3x^2,\; a_3(x) = (6x - 9x^4),\; a_4(x) = (6x - 54x^3 + 27x^6),$$

$$a_5(x) = -180x^2 + 324x^5 - 81x^8,\; a_6(x) = -360x + 2160x^4 - 1620x^7 + 243x^{10},$$

$$a_7(x) = -360 + 9720x^3 - 1780x^6 + 243x^{10} + 7290x^9 - 729x^{12},$$

$$a_8(x) = 30240x^2 - 136080x^5 + 119070x^8 - 30618x^{11}x^{10} + 2187x^{14}$$

$$a_9(x) = 640480x - 771120x^4 + 1360800x^7 - 694008x^{10} + 122472x^{13} - 6561x^{16}$$

Example 4.17. *It is very difficult to get the exact analytic Esscher transform of the function*

$$f(x) = e^{-x^\nu},$$

where the parameter $\nu \in \mathbb{R}_+^*$. In this example we propose to find the Esscher transform of f. Since $M_z = \mathcal{L}[f, -z]$, by using (4.28), we have

$$(4.59) \qquad M_z = \int_0^\infty e^{sz} e^{-s^\nu} \, ds = \lim_{x \to \infty} \lim_{p \to 1} \sum_{i=0}^\infty p^i w_i(x, -z) \exp(zx),$$

where, for $n = 0, 1, \ldots$, we set

$$w_n(x, -z) = e^{-x^\nu} \frac{b_n^\nu(x)}{z^n},$$

*where $b_n^\nu(x)$ is a polynomial function that admits the following five first terms, obtained via **Mathematica** software:*

$$b_0^\nu(x) = 0, \quad b_1^\nu(x) = \nu\, x^{-1+\nu}, \quad b_2^\nu(x) = (1 - \nu + \nu\, x^\nu) x^{-2+\nu}\nu,$$

$$b_3^\nu = x^{-3+\nu}(-2+\nu)(-1+\nu)\nu + x^{-3+2\nu}(-1+\nu)\nu^2 - x^{-3+3\nu}\nu^3 + x^{-3+2\nu}\nu^2(-2+2\nu),$$

$$b_4^\nu(x) = \left(6 - 11\nu(1 - x^\nu) + (6 - 18x^\nu + 6x^{2\nu})\nu^2 + (-1 + 7x^\nu - 6x^{2\nu})\nu^3 + x^{3\nu}\right) x^{-4+\nu}\nu,$$

$$\vdots$$

5. Some applications for probability

5.1. **Approximation of type G and spherical distributions.** The Type G family of processes is a subclass of the Lévy processes, which allows retaining some of the Gaussian properties of the Wiener process and makes it possible to incorporate processes with jumps and infinite variance. Marcus [23] initially introduced the concept of Type G random variables and processes. Following the theorem 1 of Fotopoulos [7], we propose the following corollary:

Corollary 5.1. *Let ν_0 be a σ-finite measure on $\mathbf{B}_0(\mathbb{R}^d)$ and $\mathbf{X} \in \mathbb{R}^d$ a random vector of type G with no Gaussian component such that*

$$(5.60) \qquad \mathbb{E}\left(i(x, \mathbf{X})\right) = \exp\left(\int_{\mathbb{R}^d} (\exp(-x, \Sigma^{1/2}y >^2 /2) - 1)\, \nu_0(dy)\right) = \exp(-\vec{\Phi}(|x|_\Sigma)),$$

where $|x|_\Sigma = <x, \Sigma x>$ with Σ being the positive definite symmetric matrix, and $\vec{\Phi}$: $\mathbb{R}^d \to \mathbb{R}_+$ is a continuous function such that $\Phi(0) = 0$. Then there exists a positive σ-finite measure on $\mathbf{B}_0(\mathbb{R}_+)$ denoted by μ such that the cumulant transform of $\mathbf{X} \in \mathbb{R}^d$ is

$$(5.61) \quad \vec{\Phi}(|x|_\Sigma) = C \frac{1}{\Gamma\left(\frac{d}{2}\right)} \left(\frac{\Gamma\left(\frac{d-1}{2}\right)}{2\pi^{\frac{d-1}{4}}}\right)^2 \int_0^\infty \left[1 - {}_1F_1\left(\frac{1}{2}, \frac{d}{2}; -\frac{1}{2}|x|_\Sigma^2 r^2\right)\right] \mu(dr)$$

where $C = \int_{S_{d-1}} \lambda(ds)$ and λ is a σ-finite measure on $\mathbf{B}_0(S_{d-1})$.

Proof. Theorem 1 of Fotopoulos [7] proposes

$$(5.62) \quad \vec{\Phi}(|x|_\Sigma) = C \frac{\Gamma\left(\frac{d-1}{2}\right)}{4\pi^{d/2}} \int_0^\infty \mu(dr) \int_0^1 (1 - e^{t\,|x|_\Sigma^2\, r^2/2} t^{-1/2}(1-t)^{\frac{d-1}{2}-1}\, dt.$$

Since, after some calculations, we have

$$\int_0^1 (1 - e^{t\,|x|_\Sigma^2\, r^2/2} t^{-1/2}(1-t)^{\frac{d-1}{2}-1}\, dt = \sqrt{\pi}\frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left[1 - {}_1F_1\left(\frac{1}{2}, \frac{d}{2}; -\frac{1}{2}|x|_\Sigma^2 r^2\right)\right],$$

if we replace the preceded integral in (5.62), we have proved corollary (5.61). QED

Example 5.2. *Consider the polar decomposition of the measure ν_0 is expressed as*

$$\nu_0(dy) = r^{-1-\alpha}\, \lambda(ds)\, dr,$$

where $0 < \alpha < 2$ and λ is the Borel measure on S_{d-1}, the unit sphere of $\mathbb{R}^d$. Substituting ν_0 in (5.61) gives

$$\vec{\Phi}(|x|_\Sigma) = C \frac{1}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{\Gamma\left(\frac{d-1}{2}\right)}{2\pi^{\frac{d-1}{4}}}\right)^2 \int_0^\infty r^{-1-\alpha}\left[1 - {}_1F_1\left(\frac{1}{2}, \frac{d}{2}; -\frac{1}{2}|x|_\Sigma^2 r^2\right)\right]\, dr$$

$$(5.63) \qquad = \frac{1}{\sqrt{\pi}}\, \mathcal{M}\left[f(u), -\frac{\alpha}{2}\right]$$

$$(5.64) \qquad = \lim_{y \to \infty} y^{-\frac{\alpha}{2}-1}\left[\lim_{p \to 1} \sum_{i=0}^\infty p^i w_i(y)\right]$$

$$(5.65) \qquad = \frac{2^{-2-\frac{\alpha}{2}}\,\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d+\alpha}{2}\right)}\, |x|_\Sigma^\alpha$$

where

$$f(u) = 1 - {}_1F_1\left(\frac{1}{2}, \frac{d}{2}; -\frac{1}{2}|x|_\Sigma^2 u\right),$$

$\lambda(ds) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$ and $\mathcal{M}$ is the Mellin transform as defined in (4.32) .

Theorem 5.3. *Suppose that the characteristic function of the random vector* $\mathbf{X} \in \mathbb{R}^d$
satisfies the

$$\mathbb{E}(\exp(i < x, X >)) = exp(-\Phi(|x|_\Sigma))$$

where $|x|_\Sigma^2 = < x, \Sigma x >$, $\exp(-\Phi(\cdot)) \in L(\mathbb{R}^d)$, *and* Φ *satisfies the same conditions as in*
theorem 1. Then, the joint probability distribution function f *of* $\mathbf{X} \in \mathbb{R}^d$ *has the following*
expression

$$(5.66) \qquad f(x) \quad = \quad \frac{|x|_\Sigma^{-\frac{d-2}{2}}}{(2\pi)^{d/2}} \, \mathcal{H}_{\frac{d-2}{2}}[h, |x|_\Sigma]$$

$$(5.67) \qquad\qquad = \quad \frac{|x|_\Sigma^{-\frac{d-2}{2}}}{(2\pi)^{d/2}} \lim_{y\to\infty} \left[\frac{1}{y \, J_n(y\,|x|_\Sigma)} \lim_{p\to 1} \sum_{i=0}^{\infty} p^i w_i(x, y) \right]$$

where

$$h(z) = z^{\frac{d}{2}-1} e^{-\Phi(z)}$$

and $\mathcal{H}_{\frac{d-2}{2}}$ *is the Hankel transform as defined in (4.48).*

Proof. Under the conditions of (5.4),we have the joint probability distribution of the
random vector $\mathbf{X} \in \mathbb{R}^d$ as follows

$$f(x) = \frac{|x|_\Sigma^{-\frac{d-2}{2}}}{(2\pi)^{d/2}} \int_0^{\infty} \rho^{\frac{d}{2}} \, e^{-\Phi(\rho)} J_{\frac{d-2}{2}}\left(|x|_\Sigma \, \rho\right) \, d\rho.$$

Following (4.48), we have

$$\mathcal{H}_{\frac{d-2}{2}}[h, |x|_\Sigma] = \int_0^{\infty} \rho^{\frac{d}{2}} \, e^{-\Phi(\rho)} J_{\frac{d-2}{2}}\left(|x|_\Sigma \, \rho\right) \, d\rho,$$

where

$$h(z) = z^{\frac{d}{2}-1} e^{-\Phi(z)}.$$

Therefore

$$f(x) \quad = \quad \frac{|x|_\Sigma^{-\frac{d-2}{2}}}{(2\pi)^{d/2}} \, \mathcal{H}_{\frac{d-2}{2}}[h, |x|_\Sigma]$$

$$(5.68) \qquad\qquad = \quad \frac{|x|_\Sigma^{-\frac{d-2}{2}}}{(2\pi)^{d/2}} \lim_{y\to\infty} \left[\frac{1}{y \, J_n(y\,|x|_\Sigma)} \lim_{p\to 1} \sum_{i=0}^{\infty} p^i w_i(x, y) \right].$$

We have then proved theorem (5.4).   QED

Example 5.4. *When the characteristic function has the form (5.63), in view of theorem (5.4), [7] proposes the following joint probability density function of $\mathbf{X} \in \mathbb{R}^d$:*

$$f(x) = \frac{|x|_\Sigma^{-\frac{d-2}{2}}}{(2\pi)^{d/2}} \int_0^\infty \rho^{\frac{d}{2}} e^{-c\,\rho^\alpha} J_{\frac{d-2}{2}} \left(|x|_\Sigma\,\rho\right)\,d\rho$$

(5.69)
$$= \frac{|x|_\Sigma^{-\frac{d-2}{2}}}{(2\pi)^{d/2}} \mathcal{H}_{\frac{d-2}{2}}[f, |x|_\Sigma].$$

where

$$f(u) = u^{\frac{d}{2}-1} e^{-c\,u^\alpha},$$

and the positive constant

$$c = \frac{2^{-2-\frac{\alpha}{2}}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{d+\alpha}{2}\right)}.$$

Because it is difficult to get the explicit expression of the joint density probability function, we propose to write this as a series, by using He's homotopy perturbation method. Since the density function can be estimated via the Hankel transform, it suffices to find w_i as indicated in (2.7), as the term of the series that approximate the solution of (1.2), with the function g given in (4.51) and $f(u) = u^{\frac{d}{2}-1} e^{-c\,u^\alpha}$.

5.2. **Integral transform of random variable.**

Definition 5.5. *If F is a proper or defective probability distribution concentrated on $[0, \infty($, for a given function g, the so-called incomplete Sadefo transform IS_g of F is the function defined by*

(5.70)
$$IS_g(z, x) = \int_0^x \exp\left(\int_t^x g(s, z)ds\right) dF(t)$$

and the so-called Sadefo transform $\mathcal{S}_g$ of F is the function defined by

(5.71)
$$\mathcal{S}_g(z) = \int_0^\infty \exp\left(\int_t^\infty g(s, z)ds\right) dF(t).$$

Definition 5.6. The Sadefo transform $\mathcal{S}_g^X$ of a non-negative random variable $X \geq 0$ with the probability density function $f(x)$ is defined as

(5.72)
$$\mathcal{S}_g^X(z) = \mathbb{E}\left(\exp\left(\int_X^\infty g(s, z)\,ds\right)\right)$$
$$= \int_0^\infty \exp\left(\int_t^\infty g(s, z)ds\right) f(t)\,dt,$$

where $\mathbb{E}$ denotes the expectation probability of X. We assume that g is given such that the integral $\mathcal{S}_g^X(z)$ converges.

Example 5.7. *Consider a random variable X with the probability density function*

$$f(x) = \begin{cases} C(\nu)\ exp(-x^\nu), & if; x \geq 0 \\ 0, & if\ x \leq 0 \end{cases}$$

Remark 5.8. Here it is understood that the interval is closed and may be replaced by $)-\infty, \infty($. Whenever, without loss of generality, we speak of the integral transforms (e.g. the Mellin transform of a distribution F, it is tacitly understood that F is concentrated on $[0, \infty($.

Remark 5.9. To estimate the n-th moment $\mathbb{E}\left(\mathbf{X}^n\right)$ of the random variable $\mathbf{X}$ with the distribution function F, where the integer $n > 1$, it suffices to choose the function g as follows

$$g(s, t) = -\frac{n}{s},$$

where $s > 0$. If the and the n-th moment exists, its expression is given by

(5.73) $$\mathbb{E}\left(\mathbf{X}^n\right) = \lim_{x \to \infty} \lim_{p \to 1} \sum_{i=0}^{\infty} p^i w_i(x)\ x^n.$$

6. Conclusion

In this work, we apply HPM to estimate a large variety of integral transforms (e.g. so called *incomplete Sadefo transform*). We also show that some known integral transforms such as the Esscher transform, the Fourier transform, the Hankel transform, the Mellin transform and the Stieljes transform are particular cases of our integral transforms family. The method is interesting in order to derive new integration formulae to approximate certain difficult integrals, to calculate the expectation of certain nonlinear functions of random variables and to approximation of type G and spherical distributions. The HPM requires only simple differentiation in order to deduce the integral formulae. In the next paper, we will suggest an application of the HPM in order to serve probability, statistics and mathematical finance.

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