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# Utility Functions of Equivalent Form and the Effect of Parameter Changes on Optimum Decision Making

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Abstract:

We derive a class of utility functions that are equivalent with respect to a well-defined functional form. We study the case of constant relative risk aversion (of some order) to investigate on different equivalence relations in order to determine the, possibly infinite, number of equivalence classes when utility functions satisfy a specific form. Then we apply our results to standard applications in economics and finance, for example, to the effect of price volatility on optimum hedging.

JEL-Classification: D81, D11, G11

Keywords: equivalence class, risk aversion, sensitivity analysis

## 1 Introduction

In economic modelling under uncertainty characteristics of the utility function which underlies the decision making play an important role (see, e.g., Chambers, Färe, and Quiggin (2004)). This holds especially when the effect upon optimum decisions is to be analyzed as a result of changes in parameter values exogenous to the model. In an article of 1985, Brockett and Golden derived the class $A_\infty$ of smooth utility functions $u(z)$ defined on $(0,\infty)$ with all derivatives alternating in sign. In their investigation decision makers' utility functions have positive marginal utility $(u'(z)>0)$, risk aversion $(u''(z)<0)$, positive prudence $(u'''(z)>0)$ and decreasing absolute risk aversion $((-u''(z)/u'(z))'<0)$. It follows that these utility functions can be represented by an exponential utility mixture.

Our analysis differs from the study of Brockett and Golden. We consider constant relative risk aversion of utility and/or marginal utility and so on, i.e., we assume $-u^{(n+1)}(z)z/u^{(n)}(z)=k, z>0, n\in \mathrm{IN}, k>0$, where $u^{(n)}(z)$ denotes the $n$th derivative of utility function $u(z)$. For appropriate and fixed $n$ and $k$ we show that the set of global utility functions is a subset of the $A_\infty$-class. This in turn implies $(-u^{(n+1)}(z)/u^{(n)}(z))'<0, n\in \mathrm{IN},$ i.e. decreasing (positive) absolute risk aversion of utility ($n=1$) and/or of marginal utility ($n=2$) and so on. Such constraint on the utility function appears in different models in economics and finance. These cases point out critical forms of utility functions for which specific effects of parameter changes on optimum values of the decision variables do not occur. The aim of our paper is to derive the number of equivalence classes with respect to the functional form of such utility functions. These equivalence classes are different from the equivalence classes with respect to the ranking of random prospects.

From the literature we can state some straightforward examples: in a model of farming with price and production uncertainty Newbery and Stiglitz (1981, chapter 6) show that there is no effect of increased risk on effort if constant relative risk aversion is equal to one, i.e., $-u''(z)z/u'(z)=1$. In Rothschild and Stiglitz (1971) a savings model shows that a mean-preserving increase in return risk does not affect optimum savings if $-u'''(z)z/u''(z)=2$, i.e. constant relative risk aversion of marginal utility or, relative prudence, is equal to two. Furthermore, Eckwert (1993) in a model regarding the neutrality of money discusses the importance of a unit constant relative risk aversion. Finally, Hadar and Seo (1992) show that relative prudence equal to two is crucial to the characterization of optimal decision making, if a mean-preserving contraction shift occurs in the probability distribution of the random variable.

Note that from the implication of sign-alternating derivatives, starting with positive marginal utility, our analysis implies proper risk aversion defined by Pratt and Zeckhauser (1987), i.e., the presence of an independent undesirable risk does not make an undesirable risk desirable. Additionally, our scenario entails standard risk aversion, i.e. positive decreasing absolute risk aversion combined with positive decreasing absolute prudence as introduced by Kimball (1993). This follows from the fact that if relative prudence is a positive constant, then positive absolute prudence must be decreasing. As Kimball points out, standard risk aversion holds ‘if every risk that has a negative interaction with a small reduction in wealth also has a negative interaction with any undesirable, independent risk’.

In the applications’ section of our paper we are interested in the economic effects of a mean-preserving spread (à la Rothschild and Stiglitz (1970)[1]) or an increase in volatility of the random variable. Our motivation comes from the well-known adverse result that, for example, in an international trade model under uncertainty a mean-preserving increase in foreign exchange risk may well *in*crease output of an exporting firm. To exclude such adverse effects the von Neumann-Morgenstern utility functions must satisfy specific conditions.

In section 2 we present some theoretical foundations for the derivation of utility functions that have constant relative risk aversion of some given order. We introduce the ‘form equivalence class’ of utility functions which is different from the rank equivalence class. In section 3 we report and offer economic applications of our theoretical findings. Section 4 concludes the paper.

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[1] See also, Hong and Herk (1996).

## 2 Theoretical Foundations

Relative risk aversion is widely used in modelling economic decision making under uncertainty to characterize attitude towards risk of the decision maker. Our study concentrates on the case of constant relative risk aversion (CRRA) which implies homothetic preferences, linear Engel curves for state-contingent securities, decreasing and convex absolute risk aversion and other convenient characteristics (see, e.g., Varian (1992) and Gollier (2001)).

In the following we derive a generalized class of CRRA-utility functions. For this purpose we make use of the so-called Stirling numbers which we introduce with the next definition. The advantage of our approach is to provide a constructive method of proving.

*Definition 1: The Stirling numbers of the first kind,* $s(m,l)$*, are given by the equation* $(x)_m = \sum_{0 \le l \le m} s(m,l)x^l$*, where* $(x)_m = x(x-1)\cdots(x-m+1)$ *and* $m \in \mathrm{IN}$*,* $l \in \mathrm{IN}_0$ (see, e.g., Comtet (1974), p. 213).[2]

Lemma 1: The Stirling numbers of the first kind satisfy the recurrence relation: $s(m,l) = s(m-1,l-1) - (m-1)s(m-1,l),\ m,l \ge 1;\ s(m,0) = s(0,l) = 0,$ except $s(0,0) = 1$ (see, Comtet (1974), p. 214).

Now let us start with the Arrow-Pratt measure of relative risk aversion, i.e. $-u''(z)z/u'(z) > 0$ (Pratt (1964), Arrow (1965)). Constant relative risk aversion is represented by the differential equation $u''(z)z + ku'(z) = 0$, $k \in \mathrm{IR}^+$.[3] We generalize this equation to the following one: $u^{(n+1)}(z)z + ku^{(n)}(z) = 0, n \in \mathrm{IN}, k \in \mathrm{IR}^+,$ where $u^{(n)}(z)$ denotes the $n$th derivative of utility function $u(z)$. The following Lemmata 2 and 3 prove to be helpful to solve the generalized differential equation.

*Lemma 2: Let* $F(z)$ *be a function with the properties that*

(i) $F(z)$ *is continuous over an open subset* $(a,b) \subset \mathrm{IR}^+$*,*

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[2] $\mathrm{IN} = \{1,2,3,...\}$ and $\mathrm{IN}_0 = \{0,1,2,3,...\}$.

[3] $\mathrm{IR}^+ = \{z \in \mathrm{IR} : z > 0\}$.

(ii) *$F(z)$ is differentiable of order $n$.*

*Then $g(z)=F(\log z)$ is also differentiable of order $n$ and*

$$g(z)^{(n)}=\sum_{i=0}^{n} s(n,i)\frac{F^{(i)}(\log z)}{z^n}.$$

*Proof.* The proof is by induction: Assume that the claim holds for $(n-1)$.

Then

$$(g(z)^{(n-1)})'=\sum_{i=0}^{n-1} s(n-1,i)\frac{F^{(i+1)}(\log z)}{z^n}-(n-1)s(n-1,i)\frac{F^{(i)}(\log z)}{z^n}$$

$$=\sum_{i=1}^{n-1}\left(s(n-1,i-1)\frac{F^{(i)}(\log z)}{z^n}-(n-1)s(n-1,i)\frac{F^{(i)}(\log z)}{z^n}\right)$$

$$+s(n-1,n-1)\frac{F^{(n)}(\log z)}{z^n}-(n-1)s(n-1,0)\frac{F^{(0)}(\log z)}{z^n}$$

$$=\sum_{i=1}^{n-1}\frac{1}{z^n}F^{(i)}(\log z)(s(n-1,i-1)-(n-1)s(n-1,i))$$

$$+s(n,n)\frac{F^{(n)}(\log z)}{z^n},$$

since $s(n-1,n-1)=s(n,n)=1$ and $s(n-1,0)=0$. Using the recurrence relation (Lemma 1) the claim follows. Q.E.D.

*Lemma 3*: *Let $k\in \mathrm{IR}^+, n\in \mathrm{IN}$, and $n-1-k\notin \mathrm{IN}_0$ $(n-1-k\in \mathrm{IN}_0)$. Then $w_1(z)=c, c\neq 0, w_2(z)=z,\ldots,w_{n-1}(z)=z^{n-2}, w_n(z)=z^{n-1-k}$ $(w_n(z)=z^{n-1-k}\log z)$ constitute a fundamental system of the differential equation*

$$w^{(n)}(z)z+kw^{(n-1)}(z)=0.$$

*Proof.* The differential equation $w^{(n)}(z)z+kw^{(n-1)}(z)=0$ is a special form of the Euler differential equation (see, e.g., Heuser (1989), p. 240). In order to transform the Euler differential equation to a differential equation with constant coefficients we use transformation $u(z)=v(t)$, where $z=e^t$. From Lemma 2 the transformed equation reads:

$$\sum_{i=0}^{n} s(n,i)v^{(i)}(t)+k\sum_{i=0}^{n-1} s(n-1,i)v^{(i)}(t)=0.$$

It follows the characteristic equation $f(\rho)=\sum_{i=0}^{n} s(n,i)\rho^i+k\sum_{i=0}^{n-1} s(n-1,i)\rho^i$. By the definition of the Stirling numbers of the first kind we get $f(\rho)=\rho(\rho-1)\cdots(\rho-(n-2))(\rho-(n-1-k))$. Hence the roots are $\rho_1=0,\rho_2=1,...,\rho_{n-1}=n-2,$ and $\rho_n=n-1-k$. Therefore $v_1(t)=1,v_2(t)=e^t,v_3(t)=e^{2t},...,v_{n-1}(t)=e^{(n-2)t},v_n(t)=e^{(n-1-k)t}$ if $n-1-k\notin \mathrm{IN}_0$ ($v_n(t)=te^{(n-1-k)t}$ if $n-1-k\in \mathrm{IN}_0$). Q.E.D.

Let us further introduce the following definitions.

*Definition 2: Let $u(z)$ be a von Neumann-Morgenstern utility function defined on $(a,b)\subset \mathrm{IR}, a<b, a\geq 0$. We say $u(z)$ is a CRRA-utility function of order $(n;k)$, if $u'(z)>0$, $u''(z)<0$, $r^{(n)}(z)z:=\frac{-u^{(n+1)}(z)}{u^{(n)}(z)}z=k$, for some given $k$ and $n$, $n\in \mathrm{IN}, k\in \mathrm{IR}^+$.*

*Definition 3: Let $u(z)$ be a CRRA-utility function of order $(n;k)$. We say $u(z)$ is a global (local) CRRA-utility function of order $(n;k)_g$ ($(n;k)_l$), if it is (not) possible to find a CRRA-utility function $U(z)$ of order $(n;k)$ such that $U(z)\equiv u(z)$ on $(a,b)$ and $U'(z)>0, U''(z)<0$ on $(a,\infty)$.*

For example, suppose $r^{(2)}(z)z=2$, then $u(z)=-z+\log z$ is a local CRRA-utility function of order $(2;2)_l$ defined on (0,1) and $u(z)=z+\log z$ is a global CRRA-utility function of order $(2;2)_g$ defined on $(0,\infty)$. Note that, in general, $k$ may differ from $n$. Furthermore, we do not constrain the signs of all derivatives $u^{(n)}(z), n>2$.

The following Proposition 1 gives the generic form of utility function $u(z)$ that satisfies $r^{(n-1)}(z)z=k, n\in \mathrm{IN}, n>1, k\in \mathrm{IR}^+$.

*Proposition 1: Let $u(z)$ be a CRRA-utility function of order $(n-1;k), n\in \mathrm{IN}, n>1, k\in \mathrm{IR}^+$. Then*

$$u(z)=\lambda_1 w_1(z)+\lambda_2 w_2(z)+\cdots+\lambda_n w_n(z),$$

*where $\lambda_i\in \mathrm{IR}, i=1,...,n.$*

*Proof.* Any solution of the differential equation $u^{(n)}(z)z+ku^{(n-1)}(z)=0$ is a linear combination of functions $w_i(z), i = 1,\ldots,n$, because the $w_i(z), i = 1,\ldots,n,$ are linearly independent. Q.E.D.

*Proposition 2:* Let $u(z)$ be a CRRA-utility function of order $(1;k)_g, k \in \mathrm{IR}^+$. Then with Arrow-Pratt constant relative risk aversion $k$ we have:

$$\begin{aligned} &\text{(i)}\ u(z) = \lambda_1 + \lambda_2 z^{1-k}, \lambda_1 \in \mathrm{IR}, \lambda_2 > 0, k < 1; \\ &\text{(ii)}\ u(z) = \lambda_1 + \lambda_2 \log z, \lambda_1 \in \mathrm{IR}, \lambda_2 > 0, k = 1; \\ &\text{(iii)}\ u(z) = \lambda_1 + \lambda_2 z^{1-k}, \lambda_1 \in \mathrm{IR}, \lambda_2 < 0, k > 1. \end{aligned}$$

*Proof.* Since $n = 2$ with Lemma 3 and Proposition 1 we get $u(z) = \lambda_1 + \lambda_2 z^{1-k}$, letting $c = 1$ w.l.o.g. With $u'(z) > 0$ and $u''(z) < 0$ we derive the solution (i) from $k < 1$ with $\lambda_2 > 0$. The solutions (ii) and (iii) follow from $k = 1$ and $k > 1$, respectively. Q.E.D.

Proposition 2 gives a well-known result. Note, however, that our method of finding utility functions with special characteristics is applicable to constant relative risk aversion of order $(n;k)$, i.e., $r^{(n)}(z)z = k$, and to constant absolute risk aversion of order $(n;k)$, i.e., $r^{(n)}(z) = k, n \in \mathrm{IN}, k \in \mathrm{IR}^+$. For example, $r^{(2)}(z)z = 2$ gives the global utility functions of the form $u_1(z) = \log z$ and $u_2(z) = z + \alpha \log z$, $\alpha > 0$, and $r^{(2)}(z) = 2$ is related to the utility functions $u_1(z) = -e^{-2z}$ and $u_2(z) = z - \alpha e^{-2z}, \alpha > 0$, the so-called one-switch utility function.

In order to analyze how many utility functions are equivalent with respect to optimum decisions, on one hand, and how many utility functions exhibit the same functional form, on the other hand, we set the following equivalence relations.

*Definition 4: The global (local) CRRA-utility functions $u(z)$ and $v(z)$ of order $(n;k)_g$ $((n;k)_l)$ are equivalent with regard to the ranking of random prospects, i.e., $u(z) \sim_{\mathrm{R}} v(z)$, if $v(z) = \alpha + \beta u(z), \alpha \in \mathrm{IR}, \beta > 0$, where $u(z)$ and $v(z)$ are defined on the same interval. Let $[u(z)]_g^R$ $([u(z)]_l^R)$ denote the rank equivalence class for the global* *(local) CRRA-utility function* $u(z)$ *and let* $\#[r^{(n)}(z)z = k]^R_{g(l)}$ *denote the number of different equivalence classes.*

*Definition 5:* Two global (local) CRRA-utility functions $u(z)$ and $v(z)$ of order $(n;k)_g$ $((n;k)_l)$ such that $\lambda_1 \equiv 0$[4] are equivalent with regard to their functional form, i.e., $u(z) \sim_F v(z)$, if $u(z) = \sum_{i=1}^{p} \lambda_{j_i} w_{j_i}(z)$ and $v(z) = \sum_{i=1}^{p} \mu_{j_i} w_{j_i}(z)$, where $\lambda_{j_i}, \mu_{j_i} \neq 0,\ j_i \in \mathrm{IN}, 1 < j_i \leq n+1\ \forall i, i \in \mathrm{IN},\ j_i \neq j_q,$ if $i \neq q,\ p \leq n,\ j \in \mathrm{IN}$. Let $[u(z)]^F_g$ $([u(z)]^F_l)$ denote the form equivalence class for the global (local) CRRA-utility function $u(z)$ and let $\#[r^{(n)}(z)z = k]^F_{g(l)}$ denote the number of different equivalence classes.

Note that $u(z) \sim_F v(z)$ does not imply that the utility functions are defined on the same interval. Furthermore, $w_{n+1}(z)$ is always contained in a CRRA-utility function of any order. Therefore, if $p = 1$, then $j_1 = n+1$.

Roughly speaking, utility functions of equivalent form only differ in coefficients of the linear combination of *given* functional types in the summation terms. For example, Definitions 2-5 imply for $r^{(2)}(z)z = 2$ that the global utility functions $u_2^i(z) = z + \alpha_i \log z, \alpha_i > 0,$ and $u_2^j(z) = z + \alpha_j \log z, \alpha_j > 0$, where $\alpha_i \neq \alpha_j$, do not belong to the same equivalence class of ranking, i.e., $u_2^i(z) \nsim_R u_2^j(z)$, but both utility functions belong to the same equivalence class of form, i.e., $u_2^i(z) \sim_F u_2^j(z)$. Indeed, the rank equivalence class in our example has an infinite number of global utility functions since $\alpha \in \mathrm{IR}^+$, whereas only two form equivalence classes exist: $[\log z]^F_g$ and $[z + \log z]^F_g$.[5]

*Corollary 1:* *Consider CRRA-utility functions of order* $(n-1;k), n > 1$. *Then the number of form equivalence classes does not exceed* $2^{n-2}$.

*Proof.* From Proposition 1 the solution of the differential equation

[4] *See, Proposition 1.*

$u^{(n)}(z)z + ku^{(n-1)}(z) = 0$, $n \in \mathrm{IN}, n > 1$, $k \in \mathrm{IR}^{+}$, is given by

$$u(z) = \sum_{i=1}^{n} \lambda_i w_i(z).$$

Taking $\lambda_i$ to be zero or non-zero for $i = 2,3,...,n-1$, the maximum number of non-equivalent functional forms is determined by all combinations of the $(n-2)$ polynomial coefficients. Q.E.D.

*Corollary 2: Let $u(z)$ be a CRRA-utility function of order $(n-1;k), n > 1$. Then $\#[r^{(n-1)}(z)z = k]_g^R = 1,$ if $n = 2,$ and $\#[r^{(n-1)}(z)z = n-1]_g^R = \infty,$ if $n > 2, n \in \mathrm{IN}$.*

*Proof.* The claim follows from Lemma 3 and the fact that $w_2(z) + w_n(z)$ is not equivalent to $w_2(z) + \lambda w_n(z)$, regarding the ranking of random prospects, for any $\lambda > 0$, $\lambda \neq 1$. Q.E.D.

Corollary 2 shows that there is only one rank equivalence class of global utility functions if we consider order $(1;k)$. On the other hand, the number of rank equivalence classes of global utility functions becomes infinite if constant relative risk aversion is of order $(n-1;k)$ for $k = n-1$ and $n$ is greater than two. For example, if $n = 3$ we have the case of constant relative risk aversion of marginal utility or, relative prudence, equal to 2. Hence in the savings model of Rothschild and Stiglitz (1971) a mean-preserving increase in return risk does not affect optimum savings. Nevertheless, we have an infinite number of global utility functions which differ with respect to the ranking of consumption alternatives, i.e. optimum savings.

Before we present, in detail, a selection of economic models with utility functions that exhibit constant relative risk aversion of some given order we relate our results to standard risk aversion (Kimball (1993)) and exponential-mixture utility functions (Brockett and Golden (1985)).

*Lemma 4: Let $u(z)$ be a CRRA-utility function of order $(m;k)$, $m \in \mathrm{IN}, k \in \mathrm{IR}^{+}$. Then $u(z)$ is also a CRRA-utility function of order $(m+1;k+1), m \in \mathrm{IN}, k \in \mathrm{IR}^{+}$.*

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[5]Or, $u_1(z) = 0 \cdot z + \alpha \log z$, $\alpha = 1$ w.l.o.g.; $u_2(z) = 1 \cdot z + \alpha \log z, \alpha > 0$.

*Proof.* By differentiating $-u^{(m+1)}(z)z = ku^{(m)}(z)$ we get $-u^{(m+2)}(z)z - u^{(m+1)}(z) = ku^{(m+1)}(z)$. Hence $-u^{(m+2)}(z)z = (k+1)u^{(m+1)}(z)$ and the claim follows. Q.E.D.

*Lemma 5:* Let $u(z)$ be a CRRA-utility function of order $(m;k), m \in \mathrm{IN}, k \in \mathrm{IR}^+$. Then $|u^{(m-1)}(z)|$ is less risk averse (in the sense of absolute risk aversion) than $|u^{(m)}(z)|$ for every $m$.

*Proof.* From Lemma 4 it follows that, if $r^{(m)}(z)z = k$, then $r^{(m+1)}(z)z = k+1$. Therefore $r^{(m+1)}(z) - r^{(m)}(z) = 1/z > 0$. Since $r^{(m)}(z)$ measures absolute risk aversion of $u^{(m-1)}(z)$, where $u^{(m-1)}(z)$ is the $(m-1)$th derivative of utility function $u(z)$, the claim follows. Q.E.D.

Another way of presenting Lemma 5 is to say that positive absolute risk aversion increases with its order, i.e., $0 < r^{(1)}(z) < r^{(2)}(z) < r^{(3)}(z) < ....$ Note that Lemmata 4 and 5 hold for local and global CRRA-utility functions as well. The following claim states the main result of this paper.

*Proposition 3: Let $u(z)$ be a global CRRA-utility function of order $(m;k)_g$ with appropriate $m \in \mathrm{IN}$ and $k \in \mathrm{IR}^+$. Then:*

(i) $u(z) \in A_\infty$, *where* $A_l := \{u(z) : (-1)^j u^{(j+1)}(z) \geq 0, z > 0,$ *for* $j = 0, 1, ..., l-1\}$;

(ii) $\#[r^{(m)}(z)z = k]_g^R = \begin{cases} 1, \text{if } m = 1, \\ \infty, \text{if } m > 1; \end{cases}$

(iii) $\#[r^{(m)}(z)z = k]_g^F = \begin{cases} 1, \text{if } m = 1, \\ 2, \text{if } m > 1; \end{cases}$

(iv) $u(z)$ *has standard risk aversion.*

*Proof.* (i) Let $m = 1$. Then Proposition 3 follows from Proposition 2. (ii) Let $m > 2$. We sketch the prove by considering $m = k$. Then from Proposition 1 it follows that

$$u(z) = \lambda_1 w_1(z) + \lambda_2 w_2(z) + \lambda_3 w_3(z) + \cdots + \lambda_{m+1} w_{m+1}(z).$$

From Lemma 3 we get

$$u'(z) = \lambda_2 + 2\lambda_3 z + \cdots + (m-1)\lambda_m z^{m-2} + \lambda_{m+1}\frac{1}{z}$$

and

$$u''(z) = 2\lambda_3 + 6\lambda_4 z + \cdots + (m-1)(m-2)\lambda_m z^{m-3} - \lambda_{m+1}\frac{1}{z^2}.$$

Now we show that the utility function becomes

$$u(z) = \lambda_1 w_1(z) + \lambda_2 w_2(z) + \lambda_{m+1} w_{m+1}(z).$$

Assume there exists an $l, 2 < l < m+1$, such that $\lambda_l \neq 0$. Define $\hat{l}$ by $\hat{l} := \{\max_{2<l<m+1} l : \lambda_l \neq 0\}$. Then $\hat{\lambda} := \lambda_{\hat{l}} \neq 0$ and sign $\hat{\lambda}$ = sign $u'(z)$ = sign $u''(z)$ for sufficiently large values of $z$ (Bronstein and Semendjajew (1985), p. 170). But then a contradiction occurs, since we assume $u'(z) > 0$ and $u''(z) < 0$. Hence we must have $\hat{\lambda} = 0$. Regarding the form of the utility function this fact implies from Lemma 3

$$u(z) = \lambda_1 c + \lambda_2 z + \lambda_{m+1} \log z,$$

where $\lambda_1 \in \mathrm{IR}, \lambda_2 \in \mathrm{IR}^+ \cup \{0\}$ and $\lambda_{m+1} \in \mathrm{IR}^+$. Thus, $u(z)$ has standard risk aversion, since $r^{(i)}(z)\,(i = 1,2)$ are positive and decreasing. Q.E.D.

Note that for the case $m - k \notin \mathrm{IN}_0$ we obtain the global CRRA-utility function $u(z) = \lambda_1 c + \lambda_2 z + \lambda_{m+1} z^{m-k}, \lambda_1 \in \mathrm{IR}, \lambda_2 \in \mathrm{IR}^+ \cup \{0\}$, if with $m - 1 < k < m$ we set $\lambda_{m+1} \in \mathrm{IR}^+$ or with $k > m$ we set $-\lambda_{m+1} \in \mathrm{IR}^+$.

Proposition 3 (i) shows that every global CRRA-utility function is infinitely often differentiable and has derivatives alternating in sign. Now let us relate our result to an observation of Kimball (1993) concerning the relationship of positive absolute risk aversion and prudence for global utility functions.

*Corollary 3: For global CRRA-utility functions of order* $(m;k)_g$, *positive absolute risk aversion of order* $m$ *decreases (i.e.,* $(r^{(m)}(z))' < 0$*) if and only if positive absolute risk aversion of order* $m+1$ *decreases (i.e.,* $(r^{(m+1)}(z))' < 0$*).*

*Proof.* (i) Sufficiency follows from Proposition 3 of Kimball (1993) and from Theorem 148 of Hardy, Littlewood and Pólya (1934), by shifting up the necessary number of derivatives in their proofs over the intervall $(a,\infty)$. By induction it follows that $(r^{(m+1)}(z))' < 0$ implies $(r^{(m)}(z))' < 0$ for $m \in \mathrm{IN}$. (ii) Necessity follows from Lemma 5 and the proof of Lemma 4 because $(r^{(m)}(z))' < 0$ implies $r^{(m+1)}(z) > 0, m \in \mathrm{IN}$. Since $r^{(m)}(z)z = k$ implies $r^{(m+1)}(z)z = k+1$, we get $(r^{(m+1)}(z))' < 0$. Q.E.D.

Taking $m=1\,(m=2)$ Corollary 3 shows that the monotonically increasing, concave utility function $u(z)$ has decreasing absolute prudence (temperance[6]) over the intervall $(a,\infty)$ if and only if $u(z)$ has decreasing absolute risk aversion (prudence) over the intervall $(a,\infty)$. Hence standard risk aversion regarding $u(z)\,(u^{'}(z))$ is necessary and sufficient for CRRA-utility functions of order $(1;k)_g$ $((2;k)_g)$ to be global utility functions.

*Corollary 4:* *Let $u(z)$ be a global CRRA-utility function of order $(m;k)_g$. Then $u(z)$ can be represented by a mixture of exponential utilities.*

*Proof.* Since global CRRA-utility functions of order $(m;k)_g$ have derivatives of alternating sign (see Proposition 3 (i)) starting by positive marginal utility, Corollary 4 follows from Theorem 1 of Brockett and Golden (1985). Q.E.D.

Note that the exponential-mixture class of utility functions includes among others a subset of the HARA-utility functions, i.e., utility functions with hyperbolic absolute risk aversion,[7] namely the HARA-utility functions which exhibits decreasing absolute risk aversion.

## 3 Economic Applications

Let us present some economic models concerning the effects of parameter changes such as increasing taxation and risk on the optimal decision. We focus on global CRRA-utility functions of order $(m;k)_g$ for which $k=m,\ m\in \mathrm{IN}$, holds.

***3.1 Terminal Wealth Tax*** Consider an investor who has initial wealth $W_0>0$ and who wishes to allocate it between present consumption $C_0$ and risky investment leading to random terminal wealth $\tilde{W}_1$. The investor decides on the savings rate $s$ and, therefore, on risky investment $W_0 s$. Risky investment yields a per dollar rate of return of $\tilde{R}$, where $\mathrm{Prob}(\tilde{R}>-1)=1$. Terminal wealth is proportionally taxed at rate $t,\ 0\le t<1$. Maximizing expected (time separable, increasing and concave) utility over present consumption and terminal wealth with time patience parameter $\tau$,

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[6]See, Kimball (1992).

[7]Or, the reciprocal of absolute risk aversion is linear in the argument of the utility function.

$\max_s u(C_0) + \tau Eu(\tilde{W}_1), \tau > 0$, where $C_0 = W_0(1-s)$, $\tilde{W}_1 = \tilde{X}W_0 s(1-t)$, $\tilde{X} = 1+\tilde{R}$, yields the following necessary and sufficient first-order condition:

$$-u^{'}(C_0^*) + \tau E(u^{'}(\tilde{W}_1^*)\tilde{X}(1-t)) = 0, \tag{1}$$

where $u^{'}(\cdot)$ is marginal utility and an asterisk indicates optimum level. Because of risk aversion $u^{''}(C_0^*)W_0 + \tau E(u^{''}(\tilde{W}_1^*)\tilde{X}^2 W_0(1-t)^2) < 0.$ Hence we obtain

$$\text{sign}\frac{\partial s^*}{\partial t} = -\text{sign}E(\tilde{X}(u^{'}(\tilde{W}_1^*) + u^{''}(\tilde{W}_1^*)\tilde{W}_1^*)).$$

Therefore, increasing terminal wealth taxation will increase (not alter, decrease) optimum savings, if $u^{'}(W_1) + u^{''}(W_1)W_1 < (=,>)\ 0$ or, $r^{(1)}(W_1)W_1 > (=,<)\ 1$, for all $W_1$. For example, if the utility function is a CRRA-utility function of order $(1;k), k<1$, then savings will increase if and only if taxation decreases.

From Proposition 3 we know that the number of rank and form equivalence classes equals one, respectively. That is to say, there is only one representative type of utility function in each equivalence class for which a change in the tax rate of terminal wealth does not affect savings, namely the logarithmic utility function. This result is in conflict with the observation of Stiglitz (1969), where in an atemporal model a proportional wealth tax leaves unchanged the demand for risky asset as the investor has constant relative risk aversion, in general. In our intertemporal model with time separability this is true only when decision making is myopic. It is well-known that a logarithmic utility function implies this behavior. In this case the utility function is a global CRRA-utility function of order $(1;1)_g$.

***3.2 Exchange Rate Volatility and Trade*** We consider a trade model in which we study the effect of changes in exchange rate volatility and the firm's decision whether or not to export. The firm is competitive and risk-averse and produces a commodity to be allocated to the domestic and a foreign market. The future foreign exchange spot rate is a random variable, $\tilde{e}$, which we define to be $\tilde{X}$ times the present spot rate of foreign exchange, $e_0$. To use similar notation as in the previous section, w.l.o.g. we set $\tilde{e} = \tilde{X}e_0 = (1+\tilde{R})e_0, e_0 \equiv 1.$ Hence the random variable $\tilde{R}$ represents the random percentage change in the present foreign exchange spot rate.

The international firm is a price-taker in the sense that its action does not influence the goods prices at home and abroad. The production process adopted gives rise to a cost function $C(y)$, where $y$ is the quantity of output. We assume that $C(0)=0, C(y)$ is strictly convex, increasing and differentiable, $C'(0)=0$ and $C'(y)\to\infty$ as $y\to\infty$. Hence the firm always produces a positive amount. The firm's random revenues in domestic currency are $px+\tilde{X}q(y-x)$, where $p$, $q$ are the goods prices at home and abroad, respectively, $y$ is total production, $x$ domestic supply and $y-x$ is export volume. That is, production is fixed in the sense that it must be chosen before the spot exchange rate is observed; the allocation decision is variable and can be made conditional on the realization of the exchange rate. Hence the firm has export flexibility.

Let us denote by $\bar{X}$ and $\gamma^2$ expected value and variance of the random spot exchange rate, respectively. Then we may write

$$\tilde{X}=\bar{X}+\gamma\tilde{\varepsilon}, \quad E\tilde{\varepsilon}=0, \operatorname{var}(\tilde{\varepsilon})=1. \tag{2}$$

Note that $\gamma$ is bounded as we assume $\text{Prob}(\tilde{X}>0)=1$, i.e. a positive foreign exchange spot rate in the future. An increase in $\gamma$ (in its relevant range) leads to an increased spread of the probability distribution around the constant mean, and this will be regarded as a definition of an increase in volatility of the foreign exchange spot rate. Finally, we assume that $\bar{X}=p/q$.

The firm's profit at date 1 is given by $\Pi=px+Xq(y-x)-C(y)$. The optimal decision rule at date 1 is found by maximizing profit $\Pi$ with respect to the optimal allocation of production for given $X$ and $y$. With our assumptions, for all realization $\varepsilon>0$ the firm's exports are equal to total production. There are no exports for all realizations $\varepsilon\le 0$.

At date 0, the firm maximizes expected utility of profit by choosing total production $y$ given the probability distribution of $\tilde{\varepsilon}$ and the profit-maximizing allocation of production at date 1. Thus, the decision problem can be written:

$$\max_{y} E_{\varepsilon>0}(u((p+\gamma q\tilde{\varepsilon})y - C(y))) + \text{Prob}(\tilde{\varepsilon} \le 0)u(py - C(y)).$$

The necessary and sufficient first-order condition for optimal output at date 0 reads:

$$E_{\varepsilon>0}(u^{'}(\tilde{\Pi}^{*})(p+\gamma q\tilde{\varepsilon} - C^{'}(y^{*}))) + \text{Prob}(\tilde{\varepsilon} \le 0)u^{'}(\Pi^{*})(p - C^{'}(y^{*})) = 0. \qquad (3)$$

From condition (3) we can show that with sufficiently low relative risk aversion a positive effect of exchange rate volatility on production and international trade exists. The firm's production is increasing in exchange rate volatility, i.e., $\partial y^{*}/\partial\gamma > 0$, if the level of relative risk aversion is less than or equal to one.

This result can be obtained by differentiating implicitly condition (3). We get:

$$\text{sign}\frac{\partial y^{*}}{\partial\gamma} = \text{sign}\,\Gamma,$$

where $\Gamma := E_{\varepsilon>0}(q\tilde{\varepsilon}u^{'}(\tilde{\Pi}^{*})(1 - r^{(1)}(\tilde{\Pi}^{*})y^{*}(p+\gamma q\tilde{\varepsilon} - C^{'}(y^{*}))))$. Note that $C(y^{*}) < y^{*}C^{'}(y^{*}),\ y^{*} > 0$, holds by the strict convexity of the cost function. Thus,

$$\Gamma = E_{\varepsilon>0}(q\tilde{\varepsilon}u^{'}(\tilde{\Pi}^{*})(1 - r^{(1)}(\tilde{\Pi}^{*})\tilde{\Pi}^{*} - r^{(1)}(\tilde{\Pi}^{*})(C(y^{*}) - y^{*}C^{'}(y^{*})))) > E_{\varepsilon>0}(q\tilde{\varepsilon}u^{'}(\tilde{\Pi}^{*})(1 - r^{(1)}(\tilde{\Pi}^{*})\tilde{\Pi}^{*})). \qquad (4)$$

Our assertion then follows from inequality (4) since absolute risk aversion $r^{(1)}(\Pi) > 0$, for all $\Pi$. Hence increasing the volatility of the exchange rate while holding the expected exchange rate constant has a positive effect on production if relative risk aversion $r^{(1)}(\Pi)\Pi \le 1$, for all $\Pi$. For example, global CRRA-utility functions of order $(1;k)$ for $k \le 1$ satisfy this condition. If $k = 1$ we obtain the logarithmic utility function.

***3.3 Savings under Uncertainty*** In this section we mention a model in which $r^{(2)}(z)z = 2$ represents a critical value. We analyze the effect of risk in the rate of return on savings (Rothschild and Stiglitz (1971)). In this model, a consumer with positive initial wealth $W_0$, wishes to allocate it between present and future consumption $C_0$ and $C_1$, respectively. Again, the random return per dollar invested is $\tilde{R}$, where Prob($\tilde{X} = 1 + \tilde{R} > 0) = 1$. The amount invested is given by $W_0 s$, where $s$ is the savings rate. Maximizing expected (time separable, increasing and concave) utility over intertemporal consumption, $\max_s u(C_0) + \tau Eu(\tilde{C}_1), \tau > 0$, where $C_0 = W_0(1-s)$ and $\tilde{C}_1 = \tilde{X}W_0 s$, yields the following necessary and sufficient first-order condition:

$$u^{'}(C_0^*) = \tau E(u^{'}(\tilde{C}_1^*)\tilde{X}). \tag{5}$$

As Rothschild and Stiglitz point out, increasing variability while holding the expected rate of return constant will increase or decrease optimum savings $s^*$ whether $u^{'}(C_1)X$ is convex or concave in $X$. Hence, if the return risk increases savings increase (remain constant, decrease) if $r^{(2)}(C_1)C_1 > (=,<) 2$, for all $C_1$. Relative prudence equal to two, i.e., $r^{(2)}(C_1)C_1 = 2$, for all $C_1$, is equivalent of saying that the global CRRA-utility function is of order $(2;2)_g$. Thus, from Proposition 3 it follows that the number of rank equivalence classes equals infinity, whereas the number of form equivalence classes equals two. That is, there exists two representative forms of utility functions for which the level of return risk has no impact on the savings rate: $u_1(C_1) = \log C_1$ and $u_2(C_1) = C_1 + \alpha \log C_1, \alpha > 0.$

Rothschild and Stiglitz show that an increase in the risk of the random return does not affect the savings rate if the utility function is logarithmic in consumption. However, this condition is only a sufficient condition. The same result also holds for the utility function $C_1 + \alpha \log C_1$, $\alpha > 0$, i.e. for a much broader class of utility functions. In fact, there exists an infinite number of such utility functions which are not equivalent with respect to the ranking of random consumption.

***3.4 Hedging of Exchange Rate Risk*** Let us consider a competitive exporting firm under exchange rate risk. Export production gives rise to a cost function $C(x)$ with properties given in section 3.2, where $x$ denotes export level. Exports can be sold at a given world market price $p$ per unit. Currency forwards are available.

Besides optimal level of export $x^*$ the firm chooses its optimal hedging policy $z^*$ at a given forward (random spot) exchange rate $e_f$ $(\tilde{e})$. The risk-averse firm maximizes expected utility of profit, where random profit in domestic currency from exports and hedging reads $\tilde{\Pi} = \tilde{e}px - C(x) + (e_f - \tilde{e})z$. Since optimum export level satisfies the separation property, we have $C'(x^*) = e_f p$. Hence the degree of risk aversion does not affect optimum export (see, e.g., Broll, Wahl and Zilcha (1995), Broll, Wahl and Zilcha (1999)).

Now we study a mean-preserving spread in the foreign exchange spot rate and its impact on hedging. Since the separation theorem holds, we only have to consider the hedging policy of the firm. Optimum hedging satisfies the first-order condition

$$E(u'(\tilde{\Pi}^*)(e_f - \tilde{e})) = 0. \tag{6}$$

Suppose backwardation $e_f < E\tilde{e}$, positive relative prudence $0 < r^{(2)}(\Pi)\Pi \leq 2$ with $\Pi > 0$ and $e_S$ to be a mean-preserving spread of $e$. Since the firm's export and hedging decision satisfies $e_f z^* - C(x^*) > 0$,[8] defining $\tilde{\Pi}_S^* = \tilde{e}_S(px^* - z^*) + e_f z^* - C(x^*)$, it follows that

$$Eu'(\tilde{\Pi}_S^*) > Eu'(\tilde{\Pi}^*) \quad \text{and} \quad E(u'(\tilde{\Pi}_S^*)\tilde{e}_S) < E(u'(\tilde{\Pi}^*)\tilde{e}),$$

due to the fact that backwardation requires an underhedge position $px^* - z^* > 0$. Using these inequalities and equation (6) we obtain

$$E(u'(\tilde{\Pi}_S^*)(e_f - \tilde{e}_S)) > 0.$$

It follows that $z_S^* > z^*$ since $f(z) = E(u'(\tilde{e}_S(px^* - z) + e_f z - C(x^*))(e_f - \tilde{e}_S))$ is strictly decreasing in $z$. Hence, in our scenario, a mean-preserving spread of the foreign exchange spot rate increases the hedge position of the firm.

If we consider CRRA-utility functions of order $(2;k)$, then $k \leq 2$ is sufficient for the speculative position $px^* - z^* > 0$ to decrease, when a mean-preserving spread occurs. Note that risk aversion of the firm is not sufficient for this intuitive result to hold in general.

***3.5 Redistribution of returns*** In this section we we analyze a scenario in which

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[8]This holds for positive profits if the foreign exchange spot rate can assume arbitrarily small values.

$r^{(3)}(z)z = 3$ represents a critical value and, therefore, entails CRRA-utility functions of order $(3;3)$. Let us start by equation (5) and let us consider a redistribution of the rate of return such that the savings rate $s^*$ remains unchanged. Total return $\tilde{X}$ consists of two parts, $\tilde{X} = \tilde{X}_0 + \sum_i \alpha_i \tilde{X}_i$. The distribution of total return is determined by the contractual design $\alpha \in \mathrm{IR}^n$. We get

$$E((u'(\tilde{C}_1^*) + u''(\tilde{C}_1^*)\tilde{C}_1^*)d\tilde{X}) = 0, \tag{7}$$

where $d\tilde{X} = \sum_i \tilde{X}_i d\alpha_i$ is assumed to be positive. A mean-preserving spread in return $\tilde{X}_0$ with a given change in contractual design will not alter optimum savings if $u'(C_1) + u''(C_1)C_1$ is linear in $C_1$, i.e., $r^{(3)}(C_1)C_1 = \frac{-u^{(4)}(C_1)}{u^{(3)}(C_1)}C_1 = 3$. The number of form equivalence classes is two, the number of ranking equivalence classes equals infinity. We get the following form equivalence classes of global CRRA-utility functions of order $(3;3)_g$: $u_1(C_1) = \log C_1$ and $u_2(C_1) = C_1 + \alpha \log C_1, \alpha > 0$.[9] Besides we obtain the local CRRA-utility functions of order $(3;3)_l$: $u_3(C_1) = \delta_1 C_1^2 + \delta_2 \log C_1$ and $u_4(C_1) = C_1 + \delta_3 C_1^2 + \delta_4 \log C_1$, with appropriate parameters $\delta_i$ such that marginal utility is positive and decreasing.

If $r^{(3)}(C_1)C_1 > [<] 3$, the savings will have to decrease (increase) if $\tilde{X}$ becomes more risky (à la Rothschild and Stiglitz) *before* redistribution in order to satisfy equation (7). This is due to the concavity of expected utility in $s$ and the first-order condition (5). Note that derivatives alternate in sign for global CRRA-utility functions of order $(3;3)_g$, i.e., $u'(C_1) > 0, u''(C_1) < 0, u^{(3)}(C_1) > 0, u^{(4)}(C_1) < 0$.

## 4 Conclusions

Constant relative risk aversion (CRRA) is widely used in economic and finance modelling to characterize decision makers' attitude to risk. We have derived an equivalence class of CRRA-utility functions which we defined as the form equivalence class, i.e., a class of utility functions that are equivalent with respect to a well-defined functional form. CRRA-utility functions of order $(m;k)$ for $k = m$, $m \in \mathrm{IN}$, proved to be important for modelling decision making when asking the question how changes in parameters of the model, e.g., a mean-preserving spread in the random variable, affect the optimum values of the decision variables. We have shown that although the number of form equivalence classes may be considerably small, at the same time the number of rank equivalence classes, i.e., the class of utility functions which are equivalent with respect to the ranking of alternatives, can be infinite. Therefore, assuming critical utility functions which imply no parameter effects on optimal decisions may nevertheless allow for an infinite number of utility functions which differ regarding the ranking of random prospects.

Some avenues are possible for future research: allowing for state-dependent utility functions as in Zilcha (1987); working out economic and finance models that require higher-order critical CRRA-utility functions than $(2;k)$, $k \in \mathrm{IR}^+$.

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[9]Globality implies that $r^{(m)}(z)z = m$ describes an equivalent critical value.

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