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# Controlled Stochastic Differential Equations under Poisson Uncertainty and with Unbounded Utility*

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Abstract:

The present paper is concerned with the optimal control of stochastic differential equations, where uncertainty stems from one or more independent Poisson processes. Optimal behavior in such a setup (e.g., optimal consumption) is usually determined by employing the Hamilton-Jacobi-Bellman equation. This, however, requires strong assumptions on the model, such as a bounded utility function and bounded coefficients in the controlled differential equation. The present paper relaxes these assumptions. We show that one can still use the Hamilton-Jacobi-Bellman equation as a necessary criterion for optimality if the utility function and the coefficients are linearly bounded. We also derive sufficiency in a verification theorem without imposing any boundedness condition at all. It is finally shown that, under very mild assumptions, an optimal Markov control is optimal even within the class of general controls.

JEL-Classification: C61

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# 1 Introduction

This paper is concerned with the optimal control of stochastic differential equations (*SDEs*) in an infinite time horizon where uncertainty is given by one or more Poisson processes. Such controlled SDEs are a standard tool in the economic literature for modeling dynamic behavior of economic variables that are hit by randomly occurring shocks and that can be controlled by an agent. They can be found (in a deterministic disguise) in quality-ladder models of growth (e.g., Grossman and Helpman (1991), Segerstrom (1998), Howitt (1999)), in the endogenous cycles and growth literature with uncertainty (e.g., Wälde (1999, 2005), Steger (2005)), in the labor market matching literature (e.g., Moen (1997), Postel-Vinay (2002)), and in finance (e.g., Merton (1971) and subsequent work), to name only a few applications. Often, Poisson processes are included as a special case in a framework with jump-diffusion, piecewise deterministic or general Markov processes, see, e.g., Aase (1984), Bellamy (2001), Framstad et al. (2001), and, in a more mathematical context, Davis (1993) or Fleming and Soner (1993).

Usually, the objective consists in finding an optimal control that maximizes (or minimizes) a certain performance criterion. The performance achieved with the optimal control is called the value function. As is well known, under certain assumptions the value function and, if existing, the optimal Markov control satisfy a partial differential equation, generally known as the Hamilton-Jacobi-Bellman (HJB) equation. On the other hand, if there is a function and a Markov control solving the HJB equation and satisfying certain terminal conditions, this function is the value function and the Markov control is optimal. Hence, the HJB equation provides both a necessary and sufficient criterion for optimality. In the economic literature, Merton (1971) was one of the first to state a HJB equation for an optimal control problem with Poisson processes. Since then it has found widespread use.

Unfortunately, the required conditions that allow the application of the HJB equation as either necessary or sufficient criterion are rather strong. In particular, besides a sufficiently smooth value function, many authors assume the utility or cost function to be bounded, see, e.g., Gihman and Skorohod (1979) for jump-diffusion processes or Dempster (1991) and Davis (1993) for piecewise deterministic processes.[1,2] The same applies for the coefficient functions in the controlled SDE, which govern the evolution of the controlled process. Other authors impose, instead of boundedness, other underlying conditions, such as a countable state and action space, cf., e.g., van Dijk (1988) for controlled jump processes. In some cases the required conditions are rather difficult to check without strong mathematical background, see, e.g., Kushner (1967) and Fleming and Soner (1993), who assume the value function to be in the domain of the infinitesimal generator of the controlled Markov process.[3] Kushner (1967) requires furthermore a certain uniform integrability condition. In other cases, precise assumptions on, for example, utility are missing, or the HJB equation is derived at a rather heuristic level, see, e.g., Kushner (1967), Malliaris and Brock (1982), Kushner and Dupuis (1992), Fleming and Soner (1993), and Dixit and Pindyck (1994).[4]

If one thinks of the frequently used class of CRRA (constant relative risk aversion) utility functions, such as $u(c) = \left(c^{1-\sigma} - 1\right) / (1 - \sigma)$, the condition on bounded utility is apparently too strong for economic modeling. Also, if one considers, for example, a budget constraint as in Merton (1971), the assumption of bounded coefficients in the controlled SDE seems to be too restrictive. Likewise, the assumption of countable state or action spaces is not convenient if one regards the continuous time modeling.

The objective of the present paper is therefore to present rigorous proofs for the necessity and sufficiency of the HJB equation under weaker boundedness assumptions than before. In particular, to show necessity, we allow the utility function and the coefficients to be linearly bounded, whereas for deriving sufficiency we nearly do not impose — apart from a terminal condition — any boundedness restrictions at all. Furthermore, since the HJB equation applies only for Markov controls, and one might feel that considering Markov controls only is too restrictive, it is also shown that the performance of Markov controls is as that good as for any other class of controls. That

[1] If the smoothness conditions are not satisfied, the value function can still be a viscosity solution of the HJB equation. This result was first derived by Crandall and Lions (1983). An excellent survey is provided by Crandall, Ishii and Lions (1992).

[2] A function $f : S \to \mathbb{R}^m$, $S \subset \mathbb{R}^n$, is said to be bounded if there exists $K \in \mathbb{R}$ such that $\|f(x)\| \leq K$ for all $x \in S$.

[3] The domain of the infinitesimal generator is given by all functions $f$ for that the limit $\lim_{h \searrow 0} \left(E_t f\left(X_{t+h}\right) - f\left(X_t\right)\right)/t$ exists where $X$ denotes the controlled process and $E_t$ the expectation conditional on information at time $t$.

[4] In both Kushner (1967) and Fleming and Soner (1993) only the necessity part is derived heuristically, wheras sufficiency is proven rigorously.

is, an optimal Markov control is also optimal within the class of general controls.

For discrete time and in a deterministic environment, Rincón-Zapatero and Rodriguez-Palmero (2003) and Le Van and Morhaim (2002) are concerned with a similar problem. They show for unbounded utility that the HJB equation possess a unique solution and that this solution is the value function. In this paper, the proofs follow the proceeding given in Kushner and Dupuis (1992) and Fleming and Soner (1993). This means in particular, the HJB equation is derived via the dynamic programming approach, where the main tool is the change of variables formula.[5] Crucial for showing the necessity property of the HJB equation is that the value function belongs to the domain of the infinitesimal generator of the controlled process, what, e.g., Fleming and Soner (1993) simply assumed. Herein lies a major improvement compared to the literature. Whereas this condition was so far almost trivially satisfied due to the boundedness assumption for the utility and coefficient functions, we show that it holds as well in the more general case where these functions are linearly bounded. The well-known result on the performance of Markov controls was derived by, e.g., Gihman and Skorohod (1972) and Fleming and Soner (1993), but under stronger assumptions, as mentioned above. For our proof we adapt the proof presented in Øksendal (2000), who derived an analogous result for controlled diffusion processes.

As an illustration of the proofs and results presented in this paper, an optimum consumption and investment problem with labor income is given in the accompanying paper Sennewald and Wälde (2005). A reader that is not interested in the proofs can directly refer to this paper and use it as a toolbox for own modeling.

The organization of this paper is as follows. The subsequent section gives some general assumptions and definitions concerning the formal background. In section 3 we establish the control problem with the necessary assumptions. Then, section 4 provides useful properties of the controlled state process and the value function. In section 5 we present the main results of the paper, the HJB equation as optimality criterion. The proofs are given in section 6, and the last section, finally, concludes.

[5] In a framework with Brownian motion the change of variables formula is also known as Itô's formula.

## 2 General definitions and assumptions

We start by stating briefly some general assumptions and definitions. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_t, t \geq 0\}$. A filtration is an increasing sequence of sub-$\sigma$-algebras of $\mathcal{F}$, that is $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $0 \leq s < t$. The $\sigma$-algebra $\mathcal{F}_t$ can be thought of as the set of information available at time $t$.

Let $\{X_t(\omega), t \geq s\}$ be a $n$-dimensional stochastic process starting at time $s \geq 0$. Then it is said to be adapted (to the filtration $\{\mathcal{F}_t, t \geq s\}$) if $X_t(\cdot)$ is $\mathcal{F}_t$ -measurable for each $t \geq s$. In the following we omit the stochastic argument $\omega$, and we write shortly $X$ for $\{X_t(\omega), t \geq s\}$, whenever there is no risk of confusion. $X$ is called cádlág if its paths are continuous from the right with left limits.[6] The left limit of $X$ at time $t$, $\lim_{\tau \nearrow t} X_\tau$, is denoted by $X_{t_-}$, where $X_{s_-} := 0$. Trivially, $X_{t_-}$ coincides with $X_t$ if $X$ possess continuous paths. Note that, if $X$ is cádlág, the paths of the process $X_-$ defined by $(X_-)_t := X_{t_-}$ for each $t \geq s$ are continuous from the left with right limits.[7] In the following the expression cádlág is also used for any real-valued function $f(x)$ that is continuous from the right with left limits in its argument $x$. If $f : \mathbb{R}^n \to \mathbb{R}^m$, $n, m \in \mathbb{N}$, is such a cádlág function, and the process $X$ adapted and cádlág, the process $f(X)$ becomes adapted and cádlág, too, and we denote the left limit in $t$, $\lim_{\tau \nearrow t} f(X_\tau)$, by $f(X_t)_-$.[8] Then, if $f$ is continuous, $f(X_t)_- = f(X_{t_-})$.

Let $x, y \in \mathbb{R}^n$. Then $x \cdot y := \sum_{i=1}^n x_i y_i$ stands for the standard scalar product and $\|x\| := \left(\sum_{i=1}^n x_i^2\right)^{1/2}$ for the Euclidean norm. $\mathcal{C}^1$ denotes the space of once continuously differentiable functions.

## 3 The Control Problem

Let $C$ be a $r$-dimensional adapted cádlág process and $N^1, ..., N^d$ independent adapted Poisson processes with arrival rates $\lambda^1, \ldots, \lambda^d$. Then the $n$-dimensional *state process* $X$ controlled by the process $C$ and starting at time $s$ in point $x \in \mathbb{R}^n$ is supposed to

[6] The expression cádlág is an acronym from the french expression "continu á droite, limites á gauche". Any Poisson process, for example, is cádlág.

[7] In analogy to cádlág, a process continuous from the left with right limits is called cáglád.

[8] From the assumption of the piecewise continuity of $f$ we can easily deduce that $f$ is measurable, which in turn ensures that the process $f(X)$ is adapted.

obey a SDE of the form

$$X_t = x + \int_s^t \alpha_0(\tau, X_\tau, C_\tau)\, d\tau + \sum_{k=1}^{d} \int_s^t \alpha_k(\tau, X_{\tau_-}, C_{\tau_-}) dN_\tau^k, \tag{1}$$

with continuous *coefficient functions* $\alpha_0, \ldots, \alpha_d : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$.[9] The coefficient function $\alpha_0$ describes the time continuous evolvement of the state process $X$, whereas for each $k = 1, \ldots, d$ the function $\alpha_k$ gives the magnitude of the jump in $X$ whenever Poisson process $N^k$ jumps. Both the time continuous behavior and the jump size are controlled by the choice of the control process $C$. In the following it is always assumed that SDE (1) possess a unique adapted solution, which is denoted by $X^{C,s,x}$. A detailed analysis of SDEs with sufficient conditions for the existence of such a unique solution can be found in, e.g., Protter (1995).

According to requirements in many economic models, we introduce a state space constraint by assuming that the state process $X$ is allowed to range only within a certain connected concave space $\Theta \subset \mathbb{R}^n$, which is called the *state space*. We require furthermore that, if at time $t$ state $z \in \Theta$ is observed, the control at this time, $C_t$, can take only values in a certain connected *control space* $\Gamma_{t,z} \subset \mathbb{R}^r$. Let $\Gamma := \cup_{(t,z)\in[0,\infty)\times\Theta}\Gamma_{t,z}$ be the union of all possible control spaces. A control $C$ with $C_t \in \Gamma_{t,X_t^{C,s,x}}$ for all $t \geq s$ and for that the corresponding state process $X^{C,s,x}$ remains in $\Theta$ is called *admissible control*.

Notice that in the economic literature SDEs appear often in differential notation. In this somewhat shorter notation, SDE (1) reads

$$dX_t = \alpha_0(t, X_t, C_t)\, dt + \sum_{k=1}^{d} \alpha_k\left(t, X_{t_-}, C_{t_-}\right) dN_t^k, \quad X_s = x.$$

This expression might appear more intuitive since it seems to show more clearly what the (infinitesimal) change of $X$ at time $t$ is driven by. Nevertheless, the differential notation is only an abbreviation of the integral form, and both notations have the same meaning. Throughout this paper, we shall always use the integral notation.

Let $u : [0, \infty) \times \Theta \times \Gamma \to \mathbb{R}$ (the "*instantaneous utility function*") and $\rho : [0, \infty) \to$

[9] Notice that, due to the continuity of the coefficient functions, we can write $\alpha_k(\tau, X_{\tau_-}, C_{\tau_-})$ for $\alpha_k(\tau, X_\tau, C_\tau)_-$.

$\mathbb{R}_+$ (the "*time preference rate*") be continuous functions. Suppose that for all admissible controls,

$$(2) \quad E_s \int_s^\infty e^{-\int_s^t \rho(\tau)d\tau} \left| u\left(t, X_t^{C,s,x}, C_t\right) \right| dt < \infty,$$

where $E_s$ denotes the conditional expectation with respect to $\mathcal{F}_s$. Then the objective is to find an admissible control that maximizes the *performance criterion* ("*expected lifetime utility*")[10]

$$(3) \quad W^C(s,x) := E_s \int_s^\infty e^{-\int_s^t \rho(\tau)d\tau} u\left(t, X_t^{C,s,x}, C_t\right) dt.$$

Such a control is called *optimal control* for the starting point $(s,x)$. We can now consider $W^C$ as a function of the initial point $(s,x) \in [0,\infty) \times \Theta$. Then $W^C$ is called *performance function.*

There exist various types of controls that may be considered. Following Øksendal (2000), these are, e.g.,

- Processes that are adapted to the Filtration $\{\mathcal{M}_t, t \geq s\}$ where $\mathcal{M}_t$ denotes the $\sigma$-algebra generated by $\left\{X_\tau^{s,x,C}, s \leq \tau \leq t\right\}$. That is, the choice of the control value at time $t$ depends on the whole history of $X_t^{s,x,C}$. These controls are called *feedback or closed loop controls.*

- *Deterministic* or *open loop controls.* These are controls that do not depend on $\omega$, i.e., they are deterministic.

- Controls whose value at time $t$ is given as a function of current time and state. That is, $C_t(\omega) = \phi(t, X_t^{s,x,C}(\omega))$ for some function $\phi : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^r$. Such controls are called *Markov controls* since the corresponding state process, $X_t^{s,x,C}$, becomes a Markov process.

In applied optimization problems, Markov controls present the most practical class of controls since they "say clearly" what to do if at a certain time a certain state is

[10]In some cases one may wish to minimize $W^C$, for example, if $u$ stands for a cost rate. Then one equivalently maximizes $-W^C$, where $u$ in (3) is replaced with $-u$ and the following proceeding remains the same.

observed. Moreover, the HJB equation provides a powerful tool to characterize and verify optimal Markov controls, as we shall see in theorems 5.1 and 5.3. It even turns out that, under very mild assumptions, one obtains as good a performance with a Markov control as with any other admissible control, see theorem 5.5. Hence, it is justified if we work in our analysis only with Markov controls.[11] The following definitions give the necessary tools to formulate our control problem precisely:

(i) A cádlág function $\phi : [0,\infty) \times \mathbb{R}^n \to \Gamma$, $(t,z) \longmapsto \phi(t,z)$ is called a *policy*. Let $X_t$ be an adapted cádlág process. Then a Markov control $C^\phi$ induced by a policy $\phi$ via $C_t^\phi := \phi(t, X_t)$ is adapted and cádlág, too. Thus, the integrals in the controlled SDE (1) are well-defined if the state is controlled by a Markov control with policy $\phi$. For SDE (1) we write then

$$(4) \qquad X_t = x + \int_s^t \alpha_0^\phi(\tau, X_\tau)\, ds + \sum_{k=1}^d \int_s^t \alpha_k^\phi(\tau, X_\tau)_- dN_\tau^k,$$

where $\alpha_k^\phi(t,x) := \alpha_k(t,x,\phi(t,x))$. The unique solution is denoted by $X^{\phi,s,x}$. Furthermore, the performance function, defined according to (3), is indicated by the superscript $\phi$ (instead of $C$) and reads with $u^\phi(t,x) := u(t,x,\phi(t,x))$ and $\overline{\rho}_s(t) := \frac{1}{t-s}\int_s^t \rho(\tau)\, d\tau$ (the "average time preference rate" from $s$ to $t$):

$$(5) \qquad W^\phi(s,x) = E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} u^\phi\left(t, X_t^{\phi,s,x}\right) dt.$$

(ii) A policy $\phi$ is called *admissible* if $\phi(t,z) \in \Gamma_{t,z}$ for all $(t,z) \in [0,\infty) \times \Theta$ and if for any starting point $(s,x) \in [0,\infty) \times \Theta$ the controlled process $X^{\phi,s,x}$ never leaves $\Theta$, i.e., $X_t^{\phi,s,x} \in \Theta$ for all $t \geq s$. The space of admissible policies is denoted by $\Pi$.

(iii) If the supremum is finite for all $(s,x) \in [0,\infty) \times \Theta$, we call the function $V : [0,\infty) \times \Theta \to \mathbb{R}$ given by

$$(6) \qquad V(s,x) := \sup_{\phi \in \Pi} W^\phi(s,x)$$

[11] Restricting ourselves only to deterministic controls is clearly not sufficient since in a stochastic environment it is not likely that a deterministic control is optimal.

the *value function*.

(iv) An admissible policy $\phi^* \in \Pi$ is called *optimal policy* if its performance function is equal as the value function (6) for all $(s,x) \in [0,\infty) \times \Theta$. That is,

$$W^{\phi^*}(s,x) = V(s,x) \quad \forall (s,x) \in [0,\infty) \times \Theta.$$

Notice that the optimal policy does not depend on the initial point $(s,x)$.

The control problem consists in finding an optimal admissible policy and can be tackled with the HJB equation. As mentioned before, we do not limit ourselves to a bounded utility function or bounded coefficients to ensure application to more general modeling. Nevertheless, to show the necessity of the HJB equation for optimality in theorem 5.1 we assume at least the following conditions to be satisfied. For the sufficiency part in theorem 5.3 they are *not* required.

(H1) We say that $u$ satisfies a *linear boundedness condition* if there exists a constant $m > 0$ such that for all $(t,z) \in [0,\infty) \times \Theta$ and $c \in \Gamma_{t,z}$,

$$(7) \quad |u(t,z,c)| \leq m\,[1 + \|z\| + \|c\|],$$

where $\|\cdot\|$ denotes the Euclidean norm.[12]

(H2) The coefficient function $\alpha_k$ satisfies a *linear growth condition* if for each $t \geq 0$ there exist *boundedness coefficients* $a_k(t) \geq 0$ and $b_k(t) \geq 0$ such that for all $z \in \Theta$ and $c \in \Gamma_{t,z}$,

$$(8) \quad |\alpha_k(t,z,c)| \leq a_k(t) + b_k(t)\|z\|,$$

and the mappings $t \mapsto a_k(t)$ and $t \mapsto b_k(t)$ are cádlág. Notice that this condition must hold uniformly over the control variable $c$.

(H3) Define for any $s \in [0,\infty)$

$$(9) \quad P_s(t) := \frac{1}{t-s}\int_s^t \left( b_0(\tau) + \sum_{k=1}^{d} \lambda_k b_k(\tau) \right) d\tau, \quad t \geq s,$$

[12] For the definition of the Eucledian norm see section 2.

and

$$(10)\quad Q_s(t) := \int_s^t e^{-P_s(\tau)(\tau-s)} \left( a_0(\tau) + \sum_{k=1}^{d} \lambda_k a_k(\tau) \right) d\tau, \quad t \geq s.$$

If for some $k$ there exists a $t^* \geq 0$ with $a_k(t^*) > 0$, the right continuity of $a_k$ implies that $Q_0(t) > 0$ for all $t > t^*$, and we say that the *regularity condition* is satisfied if

$$(11)\quad A := \int_0^\infty e^{-[\overline{\rho}_0(t) - P_0(t)]t} Q_0(t)\, dt < \infty.$$

If, in contrast, in the degenerated case, for each $k \in \{0, 1, \ldots, d\}$ the boundedness coefficient $a_k(t)$ is equal as 0 for all $t \geq 0$ , then $Q_0(t) = 0$ and the regularity condition is said to be satisfied if

$$(12)\quad B := \int_0^\infty e^{-[\overline{\rho}_0(t) - P_0(t)]t} dt < \infty.$$

(H4) If there exists an optimal policy $\phi^*$, the expected present value of the corresponding Markov control discounted with the time preference rate is finite. That is,

$$(13)\quad E_s \int_s^\infty e^{-\overline{\rho}_s(t)t} \left\| \phi^* \left( t, X_t^{s,x,\phi^*} \right) \right\| dt < \infty, \quad \forall (s,x) \in [0,\infty) \times \Theta.$$

Let us give a quick preview of the results presented in the subsequent section to explain why and where we shall have need of the conditions stipulated in (H1)-(H4). The linear growth condition (8) gives an upper bound for the growth rate of the controlled process $X^{\phi,s,x}$. It allows to derive a finite upper bound for the expectation of $X_t^{\phi,s,x}$, which can be expressed in terms of the initial state $x$, see lemma 4.1. Regularity conditions (11) and (12), respectively, make sure that the expected present value of the controlled process is finite for any admissible policy $\phi$, see corollary 4.3. Then, together with the linear boundedness condition (7) and condition (13), we can deduce that the value function is linearly bounded with respect to the initial state $x$, see lemma 4.4. This result will be used to show that the value function belongs to the domain of the infinitesimal generator of the controlled process (see lemma 6.3), which in turn is crucial for deriving the HJB equation as a necessary criterion for optimality in theorem 5.1. Notice that the regularity conditions (11) and (12), as well as condition (13), are only satisfied for a sufficiently high time preference rate. This can also be seen in part (ii) of the following remark.

**Remark 3.1** *(i) The following conclusion will be helpful for the proofs in section 6. In the non-degenerated case, where there exist some* $k$ *and* $t^*$ *with* $a_k(t^*) > 0$, *regularity condition (11) implies* $B < \infty$, *where* $B$ *is defined as in (12). This result is derived in appendix A. On the other hand, if* $a_k(t) = 0$ *for all* $k \in \{0, 1, \ldots, d\}$ *and* $t \geq 0$, *we obtain immediately* $A = 0$. *Thus, in either case we have* $A < \infty$ *and* $B < \infty$.
*(ii) If the linear boundedness coefficients and the time preference rate are constants, i.e.,* $a_k(t) := a_k$, $b_k(t) := b_k$, *and* $\rho(t) := \rho$ *for all* $t \geq 0$, *then regularity conditions (11) and (12), respectively, hold if and only if* $\rho > b_0 + \sum_{k=1}^{d} \lambda_k b_k$.

# 4 Properties of the state process and the value function

This section serves as preparation for the derivation of the HJB equation as a necessary condition for optimality in the subsequent sections. It provides some useful properties of the controlled state process, the objective and the value function if the assumptions in (H1)-(H4) from the preceding section are met. The proofs are given in section 6. The first lemma shows that the expectation of $||X_t^{\phi,s,x}||$ is linearly bounded with respect to the initial value $x$. This property holds uniformly over all admissible policies $\phi \in \Pi$.

**Lemma 4.1** *If the coefficients* $\alpha_0, \ldots, \alpha_d$ *satisfy the linear growth condition (8), then for all admissible policies* $\phi \in \Pi$,

$$E\left\|X_t^{\phi,s,x}\right\| \leq e^{P_s(t)(t-s)}\left[\|x\| + Q_s(t)\right],$$

*where* $P_s(t)$ *and* $Q_s(t)$ *are defined as in (9) and (10), respectively.*

From lemma 4.1 we deduce the following corollary.

**Corollary 4.2** *If the coefficients* $\alpha_0, \ldots, \alpha_d$ *satisfy the linear growth condition (8), then for all admissible policies* $\phi \in \Pi$,

$$E \sup_{s \leq \tau \leq t} \left\| X_\tau^{\phi,s,x} \right\| \leq e^{P_s(t)(t-s)} \left[ \|x\| + Q_s(t) \right].$$

The next corollary shows that, for any admissible policy $\phi$, the expected present value of the controlled process $X^{\phi,s,x}$ discounted with the time preference rate is finite and linearly bounded with respect to the initial state $x$.

**Corollary 4.3** *If the coefficients* $\alpha_0, \ldots, \alpha_d$ *satisfy the linear growth condition (8) such that regularity conditions (11) and (12), respectively, hold, then for all admissible policies* $\phi \in \Pi$,

$$E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} \left\| X_t^{\phi,s,x} \right\| dt \leq A(s) + B(s) \|x\| < \infty,$$

*where*

$$\text{(14)} \quad A(s) := \int_s^\infty e^{-(\overline{\rho}_s(t) - P_s(t))(t-s)} Q_s(t)\, dt$$

*and*

$$\text{(15)} \quad B(s) := \int_s^\infty e^{-(\overline{\rho}_s(t) - P_s(t))(t-s)} dt,$$

*and* $P_s(t)$ *and* $Q_s(t)$ *are defined as in (9) and (10), respectively.*

If the utility function $u$ is linearly bounded in the sense of (7), we derive from the preceding results the following theorem 4.4. It shows that the value function, as well, is linearly bounded with respect to the initial state $x$.

**Theorem 4.4** *Let the utility function* $u$ *satisfy the linear boundedness condition (7) and the coefficients* $\alpha_0, \ldots, \alpha_d$ *the linear growth condition (8), such that regularity conditions (11) and (12), respectively, hold. Assume that there exists an optimal* *policy $\phi^*$ satisfying (13). Then for all $(s,x) \in [0,\infty) \times \Theta$,*

$$|V(s,x)| \leq K(s) + mB(s)\|x\| + mE_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} \left\| \phi^* \left(t, X_t^{\phi^*,s,x}\right) \right\| dt < \infty,$$

*where $B(s)$ is defined as in (15), and $K(s)$ is a deterministic value that depends on the boundedness coefficients $m$, $a_0, \ldots a_d$, and $b_0, \ldots, b_d$.*

From theorem 4.4 we can deduce immediately that the performance function is linearly bounded, too.

**Corollary 4.5** *If the conditions of theorem 4.4 are satisfied, then for any admissible policy $\phi$,*

$$\left|W^\phi(s,x)\right| \leq K(s) + mB(s)\|x\| + mE_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} \left\| \phi^* \left(t, X_t^{\phi,s,x}\right) \right\| dt < \infty.$$

# 5 The Hamilton-Jacobi-Bellman equation

This section presents the main results of the paper, the HJB equation as a necessary and sufficient criterion for optimality. To achieve a shorter notation, we define at first the following differential operator $D$ associated with the controlled SDE (4). For a $C^1$-function $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ let

$$(16)\quad D^c f(s,x) := f_t(s,x) + \alpha_0(s,x,c) \cdot f_x(s,x) + \sum_{k=1}^d \lambda_k [f(s, x + \alpha_k(s,x,c)) - f(s,x)],$$

where $f_t$ denotes the partial derivative with respect to the time argument $t$, and $f_x$ stands for the gradient with respect to the state argument $x$.[13] Then the necessity part is given in the following theorem.

**Theorem 5.1** *Assume that for any $(t,z) \in [0,\infty) \times \Theta$ and $c \in \Gamma_{t,z}$ there exists an admissible policy $\phi$ with $\phi(t,z) = c$. Let the utility function $u$ satisfy the linear boundedness condition (7), and the coefficients $\alpha_0, \ldots, \alpha_d$ the linear growth condition (8), such that regularity conditions (11) and (12), respectively, hold. Assume that*

[13] Recall from section 2 that the operator "·" denotes the standard scalar product.

*an optimal policy $\phi^*$ satisfying (13) exists. If furthermore the value function $V$ is once continuously differentiable with bounded first derivatives, the following equation is satisfied for all $(s,x) \in [0,\infty) \times \Theta$:*

$$(17) \quad \rho(s) V(s,x) = \max_{c \in \Gamma_{s,x}} \{u(s,x,c) + D^c V(s,x)\},$$

*and the maximum is achieved by $\phi^*(s,x)$.*

Equation (17) is called the HJB equation. Theorem 5.1 says that under the stipulated conditions the HJB equation must be necessarily satisfied by the value function and the optimal policy. Based on the fact that the optimal policy maximizes the right-hand side of (17), we derive the following corollary.

**Corollary 5.2** *Let the conditions of theorem 5.1 be satisfied, and let furthermore $u$ be differentiable with respect to $c$. Then, for all $(s,x) \in [0,\infty) \times \Theta$ for that $\phi^*(s,x)$ lies in the inner of $\Gamma_{s,x}$ the following first-order condition holds:*

$$(18) \quad \frac{\partial}{\partial c_i} u(s,x,\phi^*(s,x)) = -\frac{\partial}{\partial c_i} D^{\phi^*(s,x)} V(s,x), \quad i = 1,\ldots,r.$$

If the value function and the optimal policy are unknown, equation (18) can be used to do further analysis. For example, starting from (18) it is possible to derive a Keynes-Ramsey rule for optimum-consumption problems, see, e.g., Wälde (1999) and the accompanying paper Sennewald and Wälde (2005) or, for the case of Brownian motion, Turnovsky (2000). In some cases, one may even derive explicit expressions for candidates of both the value function and the optimal policy.

So far, we know only that the HJB equation is necessary. That it is also a sufficient condition for optimality is shown in the subsequent theorem.

**Theorem 5.3** *Let a $C^1$ - function $J : [0,\infty) \times \Theta \to \mathbb{R}$ satisfy for all $(s,x) \in [0,\infty) \times \Theta$ inequality*

$$(19) \quad \rho(s) J(s,x) \geq u(s,x,c) + D^c J(s,x), \quad \forall c \in \Gamma_{s,x},$$

*and suppose that there exists an admissible policy $\phi^*$ such that*

$$(20) \quad \rho(s) J(s,x) = u^{\phi^*}(s,x) + D^{\phi^*(s,x)} J(s,x), \quad \forall (s,x) \in [0,\infty) \times \Theta.$$

*If furthermore for all* $(s,x) \in [0,\infty) \times \Theta$ *the limiting condition*

$$(21)\quad \lim_{t\to\infty} E\left[e^{-\overline{\rho}_s(t)t} J(t, X_t^{\phi^*,s,x})\right] = 0$$

*and the limiting inequality*

$$(22)\quad \lim_{t\to\infty} E\left[e^{-\overline{\rho}_s(t)t} J(t, X_t^{\phi,s,x})\right] \geq 0, \quad \forall \phi \in \Pi,$$

*are satisfied, then* $J$ *is the value function* $V$ *and the policy* $\phi^*$ *is optimal.*

The HJB equation from theorem 5.1 is here divided into inequality (19) and equation (20). The theorem tells us that, if there exist a $C^1$-function and a policy such that this policy maximizes the HJB equation and terminal conditions (21) and (22) are satisfied, then this policy is optimal and the function is the value function. Thus, one can use theorem 5.3 to verify whether a given function and a given policy (which were, for example, found by "guessing" or via the first-order conditions in corollary 5.2)[14] coincide with the value function and the optimal policy. Such theorems are therefore also called verification theorems. Notice that the conditions in theorem 5.3 are much milder than in the necessity theorem 5.1. In particular, one can show that the linear boundedness and growth conditions, (7) and (8), together with regularity conditions (11) and (12) and condition (13) are sufficient for both terminal conditions, (21) and (22), to be satisfied.

Limiting condition (21) generalizes the boundary condition for final time in finite time horizon settings, see, e.g., Kushner and Dupuis (1992). In a deterministic framework, Michel (1982) and later Kamihigashi (2001) show that such terminal (or transversality) conditions may even be necessary conditions. In many control problems, the utility function $u$ is assumed to be nonnegative. Then limiting inequality (22) holds obviously since only candidates $J$ for the value function with $J(s,x) \geq 0$ for all $(s,x) \in [0,\infty) \times \Theta$ are sensible.

The following corollary shows that, under certain conditions and making use of the fact that a concave function can have only a unique maximum point, the verification can be undertaken quite easily.

[14] The method of "guessing" the value function and then verifying it has first been applied by Merton (1971). He showed that, if the utility function $u$ is of the HARA class, then the value function can easily be guessed since it is of similar form as the utility function $u$.

**Corollary 5.4** *Let the instantaneous utility function $u$ be nonnegative as well as strictly concave and differentiable in the control variable $c$. Assume furthermore that also the coefficients $\alpha_0, \dots \alpha_d$ are concave $c$.*[15] *Then, if a concave $C^1$ - function $J : [0, \infty) \times \Theta \to \mathbb{R}$ and an admissible policy $\phi^*$ satisfy equation (20) and the first-order condition*

$$(23) \quad \frac{\partial}{\partial c_i} u(s, x, \phi^*(s, x)) = -\frac{\partial}{\partial c_i} D^{\phi^*(s,x)} J(s, x), \quad i = 1, \dots, r,$$

*and if furthermore limiting condition (21) holds, $\phi^*$ is an optimal policy and $J$ is the value function $V$.*

The following theorem tells us that an optimal Markov control is even optimal within the class of general admissible controls under very mild assumptions.

**Theorem 5.5** *Suppose that an optimal Markov policy $\phi^*$ exists. Let the value function $V$ be once continuously differentiable and satisfy for all $(s, x) \in [0, \infty) \times \Theta$ inequality*

$$(24) \quad \rho(s) V(s, x) \geq u(s, x, c) + D^c V(s, x), \quad \forall c \in \Gamma_{s,x}.$$

*Furthermore, assume that the following limiting inequality holds for all admissible controls $C$:*

$$(25) \quad \lim_{t \to \infty} E_s \left[ e^{-\overline{\rho}_s(t)t} V(t, X_t^{C,s,x}) \right] \geq 0.$$

*Define the supremum of the performance function over all general admissible controls by $V^a(s, x) := \sup \{ W^C(s, x) : C \text{ admissible control} \}$. Then, $V(s, x) = V^a(s, x)$ for all $(s, x) \in [0, \infty) \times \Theta$.*

The result in theorem 5.5 is actually not surprising since, regarding the “implicit” Markov nature of the controlled SDE (1), one can guess that Markov controls represent, so to speak, the natural class of controls, and no wider class has to be taken into account. Note that the HJB equation is sufficient for inequality (24) to be satisfied. That is, under the conditions of theorems 5.1 and 5.3, inequality (24) holds, and only limiting condition (25) has to be verified.

[15] Note that $\alpha_0, \dots, \alpha_d$ can be linear in the control variable as well.

## 6 Proof of results

This part presents the proofs for the statements made in the sections 4 and 5. Before starting, we repeat some useful properties of the stochastic integral with respect to Poisson processes. We are given a Poisson process $N$ with arrival rate $\lambda$ and a cádlág process $X$. Both processes are assumed to be adapted. Then, since $N$ has paths of finite variation, the stochastic integral $\int_s^t X_{\tau_-} dN_\tau$, if existing, coincides with the Lebesgue-Stieltjes integral, computed path by path, see, e.g., Protter (1990, theorem II. 17).[16] Hence, any stochastic integral in this paper can be considered pathwise, instead, as usually, in the Itô-sense, where the integral is only known in probability. Furthermore, it follows from the martingale property of the compensated Poisson process, $N_t - \lambda t$, that for any $0 \leq r \leq s < t$

$$(26) \quad E_r\left[\int_s^t X_{\tau_-} dN_\tau\right] = \lambda E_r\left[\int_s^t X_\tau d\tau\right],$$

see, e.g., Garcia and Griego (1994, theorems 3.5 and 5.3).

We turn now to the proofs and present at first some preliminary results. The central tool is the change of variables formula, given in the following theorem.

**Theorem 6.1** *Let $X$ be a $n$-dimensional adapted cádlág process and $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ a $C^1$ - function. Then the process $\{f(t, X_t^{\phi,s,x}) : t \geq s\}$ is adapted and cádlág, too, and it obeys*

$$\begin{aligned} f\left(t, X_t^{\phi,s,x}\right) &= f(s,x) + \int_s^t \left[f_t\left(\tau, X_\tau^{\phi,s,x}\right) + \alpha_0\left(\tau, X_\tau^{\phi,s,x}\right) \cdot f_x\left(\tau, X_\tau^{\phi,s,x}\right)\right] d\tau \\ &\quad + \sum_{k=1}^d \int_s^t \left[f\left(\tau, X_{\tau_-}^{\phi,s,x} + \alpha_k^\phi(\tau, X_{\tau_-}^{\phi,s,x})\right) - f\left(\tau, X_{\tau_-}^{\phi,s,x}\right)\right] dN_\tau^k, \end{aligned}$$

*where $f_t$ denotes the partial derivative of $f$ with respect to $t$ and $f_x$ stands for the gradient of $f$ with respect to $x$.*

This formula allows to describe the evolvement of processes induced by a $C^1$-mapping of the time-state process $\{(t, X_t^{\phi,s,x}) : t \geq s\}$. We omit the proof since it

[16] This does not apply if the integrator, such as Brownian motion, does not have paths of finite variation.

is a simple conclusion of Garcia and Griego (1994, p. 344), who consider stochastic differential equations driven by Poisson processes. One has only to make sure that $X_t^{\phi,s,x}$ is cádlág and that the stochastic integrals in the controlled SDE (4) are Lebesgue-Stieltjes integrals. But as mentioned above at the beginning of this section, any integral in this paper can be considered as a Lebesgue-Stieltjes integral. The cádlág property of $X_t^{\phi,s,x}$ follows immediately from SDE (4).

For the reader's convenience we recall the following result from real analysis. It can be proven using the $(\varepsilon, \delta)$ - definition of continuity at point $t$. A proof can be found in many textbooks on real analysis as in, e.g., Browder (1996).

**Lemma 6.2** *Let the function $f : [0, \infty) \to \mathbb{R}$ be integrable and right continuous at point $t \in [0, \infty)$. Then,*

$$\lim_{h \searrow 0} \frac{1}{h} \int_t^{t+h} f(\tau) d\tau = f(t).$$

We turn now to the proof of lemma 4.1, which shows that the expectation of $||X_t^{\phi,s,x}||$ is linearly bounded with respect to the initial state $x$.

**Proof of lemma 4.1.** Using a comparison principle as, e.g., Bassan et al. (1993, corollary 3.5), we deduce from the linear growth condition (8) that $\left\| X_t^{\phi,s,x} \right\| \leq Z_t^{s,x}$, where $Z_t^{s,x}$ denotes the unique solution of [17]

$$(27) \quad Z_t = \|x\| + \int_s^t \left[ a_0(\tau) + b_0(\tau) Z_\tau \right] d\tau + \sum_{k=1}^d \int_s^t \left[ a_k(\tau_-) + b_k(\tau_-) Z_{\tau_-} \right] dN_\tau^k.$$

Hence,

$$(28) \quad E_s \left\| X_t^{\phi,s,x} \right\| \leq E_s Z_t^{s,x}.$$

We compute now $E_s Z_t^{s,x}$. Taking expectation on SDE (27) and using the martingale property (26) yields

$$(29) \quad E_s Z_t^{s,x} = \|x\| + E_s \int_s^t \left[ a_0(\tau) + b_0(\tau) Z_\tau + \sum_{k=1}^d \lambda_k \left[ a_k(\tau) + b_k(\tau) Z_\tau \right] \right] d\tau.$$

[17] With Protter (1990, theorem V.6) one can show easily that (27) has a unique solution, which is cádlág and has finite expectation.

Then, by interchanging expectation and integral due to the theorem of bounded convergence,[18]

$$E_s Z_t^{s,x} = \|x\| + \int_s^t \left[ a_0(\tau) + \sum_{k=1}^{d} \lambda_k a_k(\tau) + \left( b_0(\tau) + \sum_{k=1}^{d} \lambda_k b_k(\tau) \right) E_s Z_\tau^{s,x} \right] d\tau.$$

This deterministic linear differential equation in $E_s Z_t^{s,x}$ has the unique solution

$$(30) \quad E_s Z_t^{s,x} = e^{P_s(t)(t-s)} \left[ \|x\| + Q_s(t) \right],$$

where $P_s(t)$ and $Q_s(t)$ are defined as in (9) and (10), respectively. This together with (28) finishes the proof. ■

The preceding proof immediately implies the subsequent proof of corollary 4.2.

**Proof of corollary 4.2.** Since the boundedness coefficients $a_0, \ldots, a_d$ and $b_0, \ldots, b_d$ are nonnegative, $Z^{s,x}$ has increasing paths. Remember from the proof of lemma 4.1 that $\left\| X_t^{\phi,s,x} \right\| \leq Z_t^{s,x}$ for all $t \geq s$. Thus, $\sup_{s \leq \tau \leq t} \left\| X_\tau^{\phi,s,x} \right\| \leq \sup_{s \leq \tau \leq t} Z_\tau^{s,x} = Z_t^{s,x}$ and hence, $E_s \sup_{s \leq \tau \leq t} \left\| X_\tau^{\phi,s,x} \right\| \leq E_s Z_t^{s,x}$, which together with (30) yields corollary 4.2. ■

**Proof of corollary 4.3.** From the proof of lemma 4.1 we know that $\left\| X_t^{\phi,s,x} \right\| \leq Z_t^{s,x}$. Thus, $E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} \left\| X_t^{\phi,s,x} \right\| dt \leq E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt$. Using (30) and assuming for the moment that $A(s)$ and $B(s)$ defined as in (14) and (15), respectively, are finite, we can now apply the theorems of bounded and monotone convergence to interchange expectation and integral on the right-hand side, which yields[19]

$$(31) \quad E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} \left\| X_t^{\phi,s,x} \right\| dt \leq A(s) + B(s) \|x\|.$$

It remains to be shown that $A(s)$ and $B(s)$ are finite. For this purpose we use that

$$(32) \quad A(s) \leq e^{[\overline{\rho}_0(s) - P_0(s)]s} A$$

[18] See appendix B to see how to use the theorem of bounded convergence in this case.

[19] See appendix C.

and

(33) $B(s) \leq e^{[\overline{\rho}_0(s)-P_0(s)]s} B.$

But since we know from remark 3.1 (i) that due to regularity conditions (11) and (12), respectively, $A$ and $B$ are always finite, the result follows. ■

We proceed with the proof of theorem 4.4, which shows that the value function is linearly bounded with respect to the initial value $x$.

**Proof of theorem 4.4.** Using the linear boundedness condition (7), we can find the following upper bound for the value function:

$$
\begin{aligned}
(34)\quad |V(s,x)| &= \left|W^{\phi^*}(s,x)\right| \\
&\leq E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} \left|u^{\phi^*}\left(t, X_t^{\phi^*,s,x}\right)\right| dt \\
&\leq m \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} dt + m E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} \left\|X_t^{\phi^*,s,x}\right\| dt \\
&\quad + m E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} \left\|\phi^*\left(X_t^{\phi^*,s,x}\right)\right\| dt.
\end{aligned}
$$

Since $B(s)$ is an upper bound for $\int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} dt$ and $B(s)$ is finite according to (33) and remark 3.1 (i), the first term on the right-hand side is finite, too. The second term is finite according to corollary 4.3, whereas the third term is finite by assumption (13). Hence, if we define now

$$K(s) := m \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} dt + mA(s) < \infty,$$

it follows altogether

$$|V(s,x)| \leq K(s) + mB(s)\|x\| + mE_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} \left\|\phi^*\left(X_t^{\phi^*,s,x}\right)\right\| dt < \infty,$$

which is what was to be shown. ■

To simplify the notation in the following, we drop the explicit time argument by introducing the time-state process

(35) $\left\{Y_t^{\phi,y} = \left(s+t, X_{s+t}^{\phi,s,x}\right), t \geq 0\right\}, \quad Y_0^{\phi,y} := y := (s,x).$

Then the state space corresponding to this process is $\tilde{\Theta} := [0, \infty) \times \Theta \subset \mathbb{R}^{n+1}$, and $Y_t^{\phi,y}$ solves the transformed SDE

$$(36)\quad Y_t = y + \int_0^t \tilde{\alpha}_0^{\phi}(Y_\tau)\, d\tau + \sum_{k=1}^{d} \int_0^t \tilde{\alpha}_k^{\phi}(Y_\tau)_-\, d\tilde{N}_\tau^k,$$

where the coefficients are given by $\tilde{\alpha}_0^{\phi}(t,x) := \left(1, \alpha_0^{\phi}(t,x)\right)'$ and $\tilde{\alpha}_k^{\phi}(t,x) := \left(0, \alpha_k^{\phi}(t,x)\right)'$, $k = 1, \ldots, d$, and for each $k = 1, \ldots, d$ the process $\tilde{N}^k$ defined by $\tilde{N}_\tau^k := N_{s+\tau}^k - N_s^k$ forms a Poisson process. The corresponding filtration is $\left\{\tilde{\mathcal{F}}_t, t \geq 0\right\}$, where $\tilde{\mathcal{F}}_t := \mathcal{F}_{s+t}$. The performance function we rewrite by time transformation as

$$(37)\quad W^{\phi}(y) = \tilde{E}_0 \int_0^{\infty} e^{-\tilde{\rho}_s(\tau)\tau} u^{\phi}\left(Y_t^{\phi,y}\right) dt,$$

where $\tilde{\rho}_s(t) := \frac{1}{t}\int_0^t \rho(s+r)\, dr = \overline{\rho}_s(s+t)$, and $\tilde{E}_t$ denotes the conditional expectation with respect to $\tilde{\mathcal{F}}_t$.

Altogether, by deriving (36) and (37), we have transformed the general control problem into a time-autonomous one. The corresponding differential operator $D$ is the same as in (16) and reads adapted to the time-autonomous setup

$$(38)\quad D^c f(y) = \tilde{\alpha}_0(y,c) \cdot f_y(y) + \sum_{k=1}^{d} \lambda_k [f(y + \tilde{\alpha}_k(y,c)) - f(y)],$$

where $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is a $C^1$ - function and $f_y$ denotes the gradient of $f$.

The following lemma shows that the value function $V$ belongs to the domain of the infinitesimal generator of the controlled process $X^{\phi,s,x}$ for any admissible policy $\phi$. This result is crucial for deriving the necessity of the HJB equation in theorem 5.1. Whereas the proof is almost trivial if utility (or value function)[20] and the coefficients are bounded, it becomes more complex for the more general case with linearly bounded utility and coefficient functions.

**Lemma 6.3** *Let the conditions of theorem 5.1 be satisfied. Then for any admissible*

[20] As one can show easily, a bounded utility function implies that the value function is bounded as well.

*policy* $\phi$,

$$\lim_{h \searrow 0} \frac{1}{h} \tilde{E}_0 \left[ e^{-\tilde{\rho}_s(h)h} V(Y_h^{\phi,y}) - V(y) \right] = D^{\phi(y)} V(y) - \rho(s) V(y).$$

**Proof.** Applying the change of variable formula from theorem 6.1 to the $C^1$ - function $f(v) = f(t, z) = e^{-\tilde{\rho}_s(t)t} V(v)$ yields

$$\begin{aligned}
& e^{-\tilde{\rho}_s(h)(h)} V\left(Y_h^{\phi,y}\right) - V(y) \\
= & \int_0^h \left[ \tilde{\alpha}_0^\phi \left(Y_\tau^{\phi,y}\right) \cdot e^{-\tilde{\rho}_s(\tau)\tau} V_y\left(Y_\tau^{\phi,y}\right) - \rho(s+\tau) e^{-\tilde{\rho}_s(\tau)\tau} V(Y_\tau^{\phi,y}) \right] d\tau \\
& + \sum_{k=1}^d \int_0^h \left[ e^{-\tilde{\rho}_s(\tau)\tau} V\left( Y_{\tau_-}^{\phi,y} + \tilde{\alpha}_k^\phi \left(Y_\tau^{\phi,y}\right)_- \right) - e^{-\tilde{\rho}_s(\tau)\tau} V(Y_{\tau_-}^{\phi,y}) \right] d\tilde{N}_\tau^k.
\end{aligned}$$

Taking expectation and dividing by $h$ gives together with (26)

$$\begin{aligned}
(39) \qquad & \frac{1}{h} \tilde{E}_0 \left[ e^{-\tilde{\rho}_s(h)h} V(Y_h^{\phi,y}) - V(y) \right] \\
= & \tilde{E}_0 \left[ \frac{1}{h} \int_0^h e^{-\tilde{\rho}_s(\tau)\tau} \left[ \tilde{\alpha}_0^\phi \left(Y_\tau^{\phi,y}\right) \cdot V_y\left(Y_\tau^{\phi,y}\right) - \rho(s+\tau) V(Y_\tau^{\phi,y}) \right] d\tau \right] \\
& + \sum_{k=1}^d \lambda_k \tilde{E}_0 \left[ \frac{1}{h} \int_0^h e^{-\tilde{\rho}_s(\tau)\tau} \left[ V\left( Y_\tau^{\phi,y} + \tilde{\alpha}_k^\phi \left(Y_\tau^{\phi,y}\right) \right) - V(Y_\tau^{\phi,y}) \right] d\tau \right]
\end{aligned}$$

Now let $h$ tend to 0. We show that the theorem of bounded convergence can be applied to interchange limit and expectation on the right-hand side in (39). For this purpose we have to find an upper bound with finite expectation for each of the $d+1$ random variables inside the expectations. Notice that such a bound must hold uniformly over all $h$ small enough. Whereas the bound is obvious if the utility function and the coefficients are bounded, we have to do some more calculation for the more general case with linear boundedness. Herein can be seen the heart of the contribution of the present paper.

We first consider the most-left integral on the right-hand side of (39). Remember from real analysis that for any univariate piecewise continuous function $f$, $\int_x^y f(z)\, dz \leq (y-x) \max_{x \leq z \leq y} f(z)$. With this result we derive for $h \leq 1$, using the linear boundedness of $\alpha_0$, the linear boundedness of $V$ according to theorem 4.4, and the boundedness of the first derivative of $V$:

$$(40)\quad \left|\frac{1}{h}\int_0^h e^{-\tilde{\rho}_s(\tau)\tau}\left[\tilde{\alpha}_0^{\phi}\left(Y_\tau^{\phi,y}\right)\cdot V_y\left(Y_\tau^{\phi,y}\right)-\rho\left(s+\tau\right)V(Y_\tau^{\phi,y})\right]d\tau\right|$$
$$\leq\ (1+\|a_0\|_1)\|\bigtriangledown_G\|+\|\rho\|_1\|K\|_1+(\|b_0\|_1\|\bigtriangledown_G\|+m\|\rho\|_1\|A\|_1)\sup_{\tau\in[0,1]}\left\|X_{s+\tau}^{\phi,s,x}\right\|$$
$$+m\|\rho\|_1\sup_{\tau\in[0,1]}E_{s+\tau}\int_{s+\tau}^{\infty}e^{-\overline{\rho}_{s+\tau}(t)[t-(s+\tau)]}\left\|\phi^*\left(t,X_t^{\phi^*,s+\tau,X_{s+\tau}^{\phi,s,x}}\right)\right\|dt,$$

where $\|\bigtriangledown_G\|:=\sup_{y\in\tilde{\Theta}}\|G_y(y)\|<\infty$ and $\|a_0\|_1:=\sup_{\tau\in[0,1]}a_0(s+u)<\infty$, $\|b_0\|_1:=\sup_{\tau\in[0,1]}b_0(s+\tau)<\infty$, and so forth. According to lemma 4.2, $\sup_{\tau\in[0,1]}\left\|X_{s+\tau}^{\phi,s,x}\right\|$ possess finite expectation. Furthermore,

$$E_s\sup_{\tau\in[0,1]}E_{s+\tau}\int_{s+\tau}^{\infty}e^{-\overline{\rho}_{s+\tau}(t)[t-(s+\tau)]}\left\|\phi^*\left(t,X_t^{\phi^*,s+\tau,X_{s+\tau}^{\phi,s,x}}\right)\right\|dt$$
$$\leq\ e^{\rho}E_s\int_s^{\infty}e^{-\overline{\rho}_s(t)(t-s)}\left\|\phi^*\left(t,X_t^{\phi^*,s,x}\right)\right\|dt,$$

which is finite by assumption (13). Hence, the right-hand side in (40) is an upper bound with finite expectation for the first integral on the right-hand side in (39). In analogy, for each of the remaining $k$ integrals in (39) an upper bound for all $h\leq 1$ is given by

$$\left|\frac{1}{h}\int_0^h e^{-\tilde{\rho}_s(\tau)\tau}\left[V\left(Y_\tau^{\phi,y}+\tilde{\alpha}_k^{\phi}\left(Y_\tau^{\phi,y}\right)\right)-V(Y_\tau^{\phi,y})\right]d\tau\right|$$
$$\leq\ 2\|K\|_1+m\|A\|_1\left[\|a_k\|_1+(2+\|b_k\|_1)\sup_{\tau\in[0,1]}\left\|X_{s+\tau}^{\phi,s,x}\right\|\right]$$
$$+m\sup_{\tau\in[0,1]}E_{s+\tau}\int_{s+\tau}^{\infty}e^{-\overline{\rho}_{s+\tau}(t)(t-(s+\tau))}\left\|\phi^*\left(t,X_t^{\phi,s+\tau,X_{s+\tau}^{\phi,s,x}+\alpha_k^{\phi}\left(X_{s+\tau}^{\phi,s,x}\right)}\right)\right\|dt$$
$$+m\sup_{\tau\in[0,1]}E_{s+\tau}\int_{s+\tau}^{\infty}e^{-\overline{\rho}_{s+\tau}(t)(t-(s+\tau))}\left\|\phi^*\left(X_t^{\phi,s+\tau,X_{s+\tau}^{\phi,s,x}}\right)\right\|dt.$$

Again with lemma 4.2 and assumption (13) we deduce that the expectation of this upper bound is finite. The theorem of bounded convergence can hence be applied on

(39) to interchange limit and expectation. This, finally, yields with lemma 6.2

$$\begin{aligned}
&\lim_{h\searrow 0}\frac{1}{h}\tilde{E}_0\left[e^{-\tilde{\rho}_s(h)h}V(Y_h^{\phi,y})-V(y)\right]\\
=&\ \tilde{\alpha}_0^{\phi}(y)\cdot V_y(y)-\rho(s)V(y)+\sum_{k=1}^{d}\lambda_k\left(V\left(y+\tilde{\alpha}_k^{\phi}(y)\right)-V(y)\right)\\
=&\ D^{\phi(y)}V(y)-\rho(s)V(y),
\end{aligned}$$

which is what was to be shown. ■

In the remaining part of this section we finally present the proofs of the main results from section 5.

**Proof of theorem 5.1.** Let $y\in\tilde{\Theta}$. We first prove that the optimal policy $\phi^*$ yields equality in the HJB equation (17). Take some small $h>0$. Then,

$$\begin{aligned}
(41)\quad 0 =&\ \tilde{E}_0\int_0^{\infty}e^{-\tilde{\rho}_s(t)t}u^{\phi^*}\left(Y_t^{\phi^*,y}\right)dt-V(y)\\
=&\ \tilde{E}_0\int_0^{h}e^{-\tilde{\rho}_s(t)t}u^{\phi^*}\left(Y_t^{\phi^*,y}\right)dt+\tilde{E}_0\int_h^{\infty}e^{-\tilde{\rho}_s(t)t}u^{\phi^*}\left(Y_t^{\phi^*,y}\right)dt-V(y)\\
=&\ \tilde{E}_0\int_0^{h}e^{-\tilde{\rho}_s(t)t}u^{\phi^*}\left(Y_t^{\phi^*,y}\right)dt\\
&+\tilde{E}_0\left\{e^{-\tilde{\rho}_s(h)h}E\left[\int_0^{\infty}e^{-\tilde{\rho}_{s+h}(t)t}u^{\phi^*}\left(Y_{h+t}^{\phi^*,y}\right)dt\middle|\,Y_h^{\phi^*,y}\right]\right\}-V(y)\\
=&\ \tilde{E}_0\int_0^{h}e^{-\tilde{\rho}_s(t)t}u^{\phi^*}\left(Y_t^{\phi^*,y}\right)dt+\tilde{E}_0\left[e^{-\tilde{\rho}_s(h)h}V(Y_h^{\phi^*,y})-V(y)\right].
\end{aligned}$$

Dividing by $h$ and applying the limit $h\searrow 0$, this becomes

$$0=\lim_{h\searrow 0}\tilde{E}_0\frac{1}{h}\int_0^{h}e^{-\tilde{\rho}_s(t)t}u^{\phi^*}\left(Y_t^{\phi^*,y}\right)dt+\lim_{h\searrow 0}\tilde{E}_0\frac{1}{h}\left[e^{-\tilde{\rho}_s(h)h}V(Y_h^{\phi^*,y})-V(y)\right].$$

For the first term we use in analogy to appendix B the theorem of bounded convergence to interchange expectation and integral.[21] Then, we obtain with lemma 6.2, $\lim_{h\searrow 0}\tilde{E}_0\frac{1}{h}\int_0^h e^{-\tilde{\rho}_s(t)t}u^{\phi^*}\left(Y_t^{\phi^*,y}\right)dt=u^{\phi^*}(y)$. For the second term, corollary 6.3 gives the limit. Thus, altogether, $0=u^{\phi^*}(y)+D^{\phi^*(y)}V(y)-\rho(s)V(y)$, which shows that

[21] An upper bound is given by $\int_0^\infty e^{-\tilde{\rho}_s(t)t}\left|u^{\phi^*}\left(Y_t^{\phi^*,y}\right)\right|dt$, which possess finite expectation due to assumption (2).

equality in (17) is satisfied for the optimal policy.

It remains to be shown that for any $c \in \Gamma_y$, $\rho(s) V(y) \geq u(y,c) + D^c V(y)$. For this purpose we follow an argument applied by Kushner and Dupuis (1992) and Duffie (1992), in defining a policy

$$\psi_{y,h}(v) := \begin{cases} \phi(v) & \text{for } s \leq t < s+h \\ \phi^*(v) & \text{for } t \geq s+h \end{cases}, \quad v = (t,z) \in \tilde{\Phi},$$

where $\phi$ is an arbitrary admissible control with $\phi(y) = c$.[22] Since from time $h$ on the policies $\psi_{y,h}$ and $\phi^*$ equal each other, we obtain

$$W^{\psi_{y,h}}(Y_t^{\psi_{y,h},y}) = W^{\phi^*}(Y_t^{\psi_{y,h},y}) = V(Y_t^{\psi_{y,h},y}), \quad \forall t \geq h.$$

Then in analogy to (41),

$$0 \geq W^{\psi_{y,h}}(y) - V(y) = \tilde{E}_0 \int_0^h e^{-\tilde{\rho}_s(t)t} u^{\phi}(Y_t^{\phi,y})dt + \tilde{E}_0 \left[ e^{-\tilde{\rho}_s(h)h} V(Y_h^{\phi,y}) - V(y) \right]$$

Now, if we divide by $h$ and let $h$ tend toward 0, we obtain

$$0 \geq \lim_{h \searrow 0} \tilde{E}_0 \frac{1}{h} \int_0^h e^{-\tilde{\rho}_s(t)t} u^{\phi}(Y_t^{\phi,y})dt + \lim_{h \searrow 0} \frac{1}{h} \tilde{E}_0 \left[ e^{-\tilde{\rho}_s(h)h} V(Y_h^{\phi^*,y}) - V(y) \right].$$

Again, the limit of the first term is derived by first interchanging expectation and integral according to the theorem of bounded convergence and then by applying lemma 6.2, whereas lemma 6.3 gives the second limit. Hence, $0 \geq u(y,c) + D^c V(y) - \rho(s) V(y)$. Since $c \in \Gamma_y$ was chosen arbitrarily, the proof is completed. ■

**Proof of corollary 5.2.** Let $y \in \tilde{\Theta}$. Since according to theorem 5.1, $u^{\phi^*}(y) + D^{\phi^*(y)} V(y) \geq u(y,c) + D^c V(y)$ for all $c \in \Gamma_y$, (18) must hold as a first order condition if $\phi^*(y)$ lies in the inner of $\Gamma_y$. ■

**Proof of theorem 5.3.** We have a continuously differentiable function $J : \tilde{\Theta} \to \mathbb{R}$ that satisfies inequality (19) and, with an admissible policy $\phi^*$, equation (20). We show

(i) $J(y) \geq W^{\phi}(y)$ for any arbitrary admissible policy $\phi$ and

[22] By assumption, there exists an admissible policy $\phi$ with $\phi(y) = c$ for any $c \in \Gamma_y$.

(ii) $J(y) = W^{\phi^*}(y)$.

This implies that $\phi^*$ is an optimal policy and that $J = W^{\phi^*}$ is the value function $V$.

Step (i): Let $\phi \in \Pi$ be an arbitrary admissible policy. Then inequality (19) gives

$$(42) \quad -\rho(s) J(y) + D^{\phi(y)} J(y) \le -u^{\phi}(y), \quad \forall y \in \tilde{\Theta}.$$

Applying the change of variables formula from theorem 6.1 to the $C^1$ - function $f(v) = f(t, z) = e^{-\tilde{\rho}_s(t)t} J(v)$ and taking the expectation on both sides yields together with martingale property (26)

$$\tilde{E}_0 e^{-\tilde{\rho}_s(t)t} J(Y_t^{\phi,y}) - J(y) = \tilde{E}_0 \int_0^t e^{-\tilde{\rho}_s(\tau)\tau} \left[-\rho(s+\tau) J(Y_\tau^{\phi,y}) + D^{\phi\left(Y_\tau^{\phi,y}\right)} J(Y_\tau^{\phi,y})\right] d\tau.$$

Then, inequality (42) implies $J(y) \ge \tilde{E}_0 \int_0^t e^{-\tilde{\rho}_s(\tau)\tau} u^{\phi}(Y_\tau^{\phi,y}) d\tau + \tilde{E}_0 e^{-\tilde{\rho}_s(t)t} J(Y_t^{\phi,y})$. Letting $t$ approach infinity and applying the theorem of bounded convergence on the first term on the right-hand side gives $J(y) \ge W^{\phi}(y) + \lim_{t\to\infty} \tilde{E}_0 e^{-\tilde{\rho}_s(t)t} J(Y_t^{\phi,y})$.[23] Thus, since by assumption (22) the limit on the right-hand side is equal as 0, or greater, $J(y) \ge W^{\phi}(y)$.

Step (ii): We may rewrite (20) as $-\rho(s) J(y) + D^{\phi^*(y)} J(y) = -u^{\phi^*}(y)$. Then, in exactly the same way as in step (i), only with "=" instead of "$\le$", we can deduce $J(y) = W^{\phi^*}(y) + \lim_{t\to\infty} \tilde{E}_0 e^{-\tilde{\rho}_s(t)t} J(Y_t^{\phi^*,y})$. Since by limiting condition (21), the right-most term goes to zero, we obtain $J(y) = W^{\phi^*}(y)$, which completes the proof. ■

**Proof of corollary 5.4.** We show that the conditions of theorem 5.3 are satisfied. Then, by theorem 5.3, the result follows. At first we can derive from the nonnegativity of $u$ that the value function $V$ is nonnegative, too. Hence limiting inequality (22) holds. Thus, it remains to be shown,

$$(43) \quad u^{\phi^*}(y) + D^{\phi^*(y)} J(y) \ge u(y, c) + D^c J(y) \quad \forall c \in \Gamma_y,$$

i.e., $\phi^*(y)$ is a global maximum point of $u(y, c) + D^c J(y)$.

[23] An upper bound with finite expectation is given by $\int_0^\infty e^{-\tilde{\rho}_s(\tau)\tau} \left|u^{\phi}(Y_\tau^{\phi,y})\right| d\tau$.

The first order condition for $\phi^*(y)$ to be a local maximum point is satisfied by assumption (23). From the strict concavity of $u$ and $V$ and the concavity of $\alpha_0, \ldots, \alpha_d$ we know that $u(y,c) + D^c J(y)$ is strictly concave in $c$ as well. Hence, $\phi^*(y)$ is both a local and a global maximum point. ■

**Proof of theorem 5.5.** This proof is similar to the one presented in Øksendal (2000) for controlled diffusion processes. In analogy to part (i) of the proof of theorem 5.3, we get for any admissible control $C$, $V(y) \geq W^C(y) + \lim_{t\to\infty} e^{-\tilde{\rho}_s(t)t}\tilde{E}_0 J(Y_t^{C,y})$. According to limiting inequality (25) the limit on the right-hand side is equal as 0, or greater. Thus, $V(y) \geq W^C(y)$. Since the control $C$ was chosen arbitrarily and the class of Markov controls is included in the class of generalized admissible controls (and thus $V(y) \leq V^a(y)$), the theorem follows. ■

# 7 Conclusion

In a model of optimal control where the state variable is subject to random jumps driven by one or more independent Poisson processes we have presented rigorous proofs for both the necessity and the sufficiency of the HJB equation under milder conditions than before. We especially relax the assumption of bounded utility and coefficient functions. More precisely, it could be shown that the HJB equation is still a necessary condition for optimality if these functions are linearly bounded. On the other hand, apart from a terminal condition, sufficiency could be derived even without requiring any boundedness condition.

Nevertheless, we required, at least in the necessity part, other underlying, extrinsic conditions to be satisfied, namely (i) the expected present value of the optimal control and (implicitly) the state process to be finite (see assumption (H3), (H4) and lemma 4.1) and (ii) the value function to be once continuously differentiable with bounded first derivatives. These issues are left for further research.

# A Derivation of remark 3.1 (i)

If there exist some $k$ and $t^*$ with $a_k(t^*) > 0$, the cádlág property of the boundedness coefficient $a_k$ and the fact that $Q_0(t)$ is increasing in $t$ yields $Q_0(t) > 0$ for all $t \geq t^*$. Thus, for some $T > t^*$,

$$
\begin{aligned}
B &\leq \int_0^T e^{-[\overline{\rho}_0(t) - P_0(t)]t} dt + \frac{1}{Q_0(T)} \int_T^\infty e^{-[\overline{\rho}_0(t) - P_0(t)]t} Q_0(t)\, dt \\
&\leq \int_0^T e^{-[\overline{\rho}_0(t) - P_0(t)]t} dt + \frac{A}{Q_0(T)},
\end{aligned}
$$

and hence, due to (11), $B < \infty$.

# B Interchanging expectation and integral in (29)

If we define the process $H_\tau := a_0(\tau) + b_0(\tau) Z_\tau + \sum_{k=1}^d \lambda_k [a_k(\tau) + b_k(\tau) Z_\tau]$, (29) reads $E_s Z_t^{s,x} = \|x\| + E_s \int_s^t H_\tau d\tau$. We may express the integral as a limit of Rieman sums by $\int_s^t H_\tau d\tau = \lim_{\Delta \to 0} \Delta \sum_{T=0}^{n_\Delta - 1} H_{s+T}$, where $\Delta$ is the length of the subintervals for an equidistant partition of the interval $[s, t]$ and $n_\Delta$ the number of these subintervals, i.e., $\Delta \cdot n_\Delta = t - s$. Now the problem of interchanging expectation and integral has been converted into a problem of interchanging expectation and limit. Here the theorem of bounded convergence comes into play. We have to find an upper bound with finite expectation for the absolute value of $\Delta \sum_{T=0}^{n_\Delta - 1} H_{s+T}$ that holds uniformly for all $\Delta$ small enough. Since the boundedness coefficients $a_0, \ldots, a_d$ and $b_0, \ldots, b_d$ are nonnegative, $Z^{s,x}$ is nonnegative, too, and has increasing paths. Therefore,

$$
\begin{aligned}
\left\| \Delta \sum_{T=0}^{n_\Delta - 1} H_{s+T} \right\| &= \Delta \sum_{T=0}^{n_\Delta - 1} H_{s+T} \\
&\leq (t - s) \left[ \|a_0\|_{s,t} + \sum_{k=1}^d \lambda_k \|a_k\|_{s,t} + \left( \|b_0\|_{s,t} + \sum_{k=1}^d \lambda_k \|b_k\|_{s,t} \right) Z_t^{s,x} \right],
\end{aligned}
$$

where, for $k = 0, \ldots, d$, $\|a_k\|_{s,t} := \max_{s \leq \tau \leq t} |a_k(\tau)|$ and $\|b_k\|_{s,t} := \max_{s \leq \tau \leq t} |b_k(\tau)|$. Thus, since the right-hand side has clearly finite expectation, the theorem of bounded convergence allows to interchange expectation and limit, and we obtain

$$\begin{aligned}E_s Z_t^{s,x} &= \|x\| + E_s \lim_{\Delta\to 0} \Delta \sum_{T=0}^{n_\Delta - 1} H_{s+T} \\ &= \|x\| + \lim_{\Delta\to 0} \Delta \sum_{T=0}^{n_\Delta - 1} E_s H_{s+T} = \|x\| + \int_s^t E_s H_\tau d\tau.\end{aligned}$$

# C Deriving (31) in the proof of corollary 4.3

We show how the theorems of monotone and bounded convergence can be used to interchange expectation and integral in $E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt$. At first, we consider the expectation of the finite horizon integral $\int_s^T e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt$. Here, in analogy to appendix B and with the upper bound $(T-s)\, Z_T^{s,x}$, the theorem of bounded convergence yields together with (30)

$$\begin{aligned}(44) \quad E_s \int_s^T e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt &= \int_s^T e^{-\overline{\rho}_s(t)(t-s)} E_s Z_t^{s,x} dt \\ &= \int_s^T e^{-[\overline{\rho}_s(t) - P_s(t)](t-s)} \left[\|x\| + Q_s(t)\right] dt.\end{aligned}$$

In the next step, we write $\int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt = \lim_{T\to\infty} \int_s^T e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt$. Since $\int_s^T e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt$ is increasing in $T$ and since according to (44),

$$\begin{aligned}\sup_{T\geq s} E_s \int_s^T e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt &= \int_s^\infty e^{-[\overline{\rho}_s(t) - P_s(t)](t-s)} \left[\|x\| + Q_s(t)\right] dt \\ &= A(s) + B(s)\|x\| < \infty,\end{aligned}$$

the theorem of monotone convergence tells us that $\int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt$ possess finite expectation and that

$$E_s \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} Z_t^{s,x} dt = \int_s^\infty e^{-\overline{\rho}_s(t)(t-s)} E_s Z_t^{s,x} dt = A(s) + B(s)\|x\|.$$

This, together with $\left\|X^{\phi,s,x}\right\| \leq Z^{s,x}$, yields inequality (31).

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