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A fresh look on economic evolution from kinetic viewpoint

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A Fresh Look on Economic Evolution from the Kinetic Viewpoint

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Abstract:

The present paper makes a contribution to fill in a gap left open by dynamic theory and evolutionary economics as well. While the "closed loop" dynamic theory has explanation power in analyzing evolving economic systems at the price of neglecting the possible occurrence of non anticipated novelties, "open loop" evolutionary economics is subject to the converse trade-off. Basing on a general equilibrium model by T. Kehoe including production and taxes we provide a formal model of an evolving economy fully accounting for the appearance of novelty. From this emerges in a natural way the notion of an open loop evolution equilibrium. It is based on arguments from the gradual vs. bang-bang tax reform controversy and from the debate on optimal macroeconomic policy design. Existence of equilibrium is established extending an analytical result which in a different context and independently has been proved by Mas-Colell and by the author. We use the term "kinetic" to indicate that in contrast to traditional comparative statics our approach neither hinges on the uniqueness of equilibria, nor is it confined to the analysis of prescribed parameter variations.

JEL-Classification: B52, C62, D50, D58

Keywords: evolution, equilibrium, equilibrium price path, frictionless tuning of control parameters

1 Introduction

In the last years the economic profession has witnessed a growing interest in the subject of evolution. Evolutionary economists, when asked for the main difference between their approach and that of traditional dynamic theory, first and foremost point to the possible occurrence of novelties in economic systems (e.g. Witt 2003). In fact, the well-known closed loop dynamic models achieve strong results – however, at the price of disregarding the occurrence of novelties. Unfortunately, however, also the open loop evolutionary approach faces a trade-off. Not surprisingly it is reciprocal to that just mentioned: the less the possible occurrence of novelties is restricted, the less is naturally the power as an ex ante theory.

If one agrees, as we do, that the issue of novelty is essential for adequately studying evolving economic systems it appears to be reasonable to develop alternative approaches to build a bridge over the gap left by the described trade-offs. Such an alternative approach will be provided in the present paper. To be more precise, we will first design a general model of an evolving economic system accounting for novelty, and afterwards explore this modelling framework for its general properties with respect to coordinating variables like prices, production coefficients and tax rates.

This model will serve as a base for our general formalization of an evolving economy. The cornerstone of our conceptualization of evolution is to "animate" the static functional relationships of this model. To use a metaphor, our procedure is like the transition from a single frame to a movie. Unlike cinematic movies our "movie", however, consists of a continuum of frames. Speaking in terms of this metaphor any single frame corresponds to a momentary state economy, and accordingly its equilibria are momentary state equilibria. Our main analytical instrument for formalizing this idea of evolution is the intuitively appealing concept of continuous one-parametrizations (Lehmann-Waffenschmidt 1995) which in different contexts have also been used by other authors since the eighties (Mas-Colell 1985, Allen 1981).

The economic meaning of a "single frame", i.e. a momentary state in the evolution, is immediate. Since here we follow a *continuous approach* it is natural to think of all evolving economic variables, i.e consumption, production, and tax revenues, as *continuous flows* over time. Consequently, a *momentary* state of the evolving system is an infinitesimally short period of the evolution and thus is completely characterized by the *cross section sizes* of these flows. This *flow view* of demand and supply over time has also been adopted by traditional growth theory. The reader should note, however, that this flow-understanding of economic quantities over time is the *only analogy* of our present approach to conventional growth theory. This is due to the fact the growth models employ a closed loop preconception of the evolving system whereas our approach is open for novelties (see e.g. Balasko/Lang 1998).

Note that, proceeding this way, we observe and record the evolution of the economic system on the "phenomenological" level of the endogenously determined variables, but do not try to explain the evolution causally. In this respect our approach is essentially different from an approach which strives for a causal explanation of the economic evolution process. On the other side, however, it would also not be adequate to consider our approach comparative static. To see this let us recall that the comparative static method analyzes the effects of either marginal, or discrete, precisely specified parameter variations, and essentially *requires uniqueness of equilibrium.* How arduous it is to guarantee uniqueness of equilibria in economic models show the results of a research programme at the University of Bonn (e.g. Hildenbrand 1989, 1994, 1998, 1999). Our approach, in contrast, does not stick to any one of these requirements. To emphasize this very difference we use the term "kinetic" which has been picked up from physics, and is used there in the same kind to analyse evolving systems as just described.

Conceptualizing a theoretical framework of evolution taking novelty into account only means one half of our task. The second half naturally must be to develop a *notion of equilibrium* which properly accounts for novelty. Our proposal is quite simple: We just extend the idea of a momentary bookkeeping equilibrium of a single state economy to the whole evolution.

Our concept of an "open evolution equilibrium" simultaneously accounts for two aspects. On the one hand it reflects the openness of the whole approach. This means particularly that the open evolution equilibrium must not predetermine the evolving economic system in any respect. On the other hand it must link the states at different dates of the evolving economic system in some reasonable way, since it otherwise would loose its temporal characteristic. These two requirements are sufficiently well met by the idea of a *continuous string*, or say *path*, of *state equilibria* in the space $S \times [0, 1]$ (see Figure A, the intuition of this concept will be given to the reader in Section 2 below). Anyhow, as we will see later, this is even the maximally possible result in the sense that more regularity requirements on the open evolution equilibrium concept make existence go lost.

Before we will further discuss the economic achievements of our concept of an open evolution equilibrium we want to clarify what it *does not mean.* Unlike general equilibrium theory we do not pretend that the notion of equilibrium provides a description of the observed state of real economic systems. Our understanding of equilibrium is by far less ambitious. We just attribute an equilibrium the state of a reference point, or of a perfect solution, which is, nevertheless, a fundamental reference point for economic analysis. In other words: an equilibrium means a *solution* of the interdependend economic system under consideration which admits overall and simultaneous consistency of individual plans.

Let us come to further achievements of our concept of an open evolution equilibrium. To be sure this amounts to the question what is economically

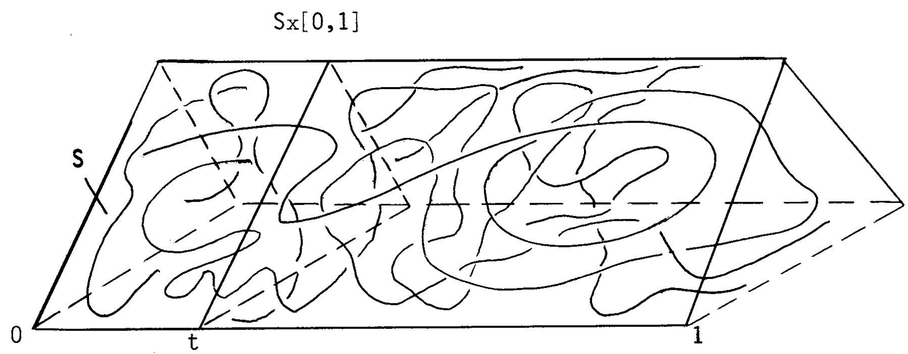

Figure A

gained by the opportunity to be capable of adjusting all variables permanently in a continuous way in an openly evolving economy – the equilibrium variables and the state determining variables, i.e., the parameters. Let us first have a closer look at the nature of the involved variables. As it has been mentioned above, the equilibrium variables are prices and tax receipts. The state control parameters are various tax (revenue) rates. As a first answer let us quote Balasko (1988, p.70, cf also Balasko 1996): "The idea that discontinuity is in itself harmful, synonymous of catastrophies, is widespread . . . from an economic point of view, a continuous evolution path is superior to any discontinuous one."

Let us substantiate this general statement starting from analyzing the advantages of the opportunity to continuously adjust, or tune, the state control parameters. The pros and cons of a *continuous, or say gradualistic, tuning* of tax parameters has intensively been discussed in the tax reform debate which originated in the late seventies (e.g. Hatta 1977) and is still discussed in the debate on optimal design of macro policy parameters (see e.g. Gandolfo/Petit 1988, Marangos 2002 and the cited literature there). In a metaphoric language the advocates of a gradualistic tuning remind their opponents of the problems which arise when one tries to put a car into reverse without stopping it. More specifically they argue that a "bang-bang" policy switch, or shock therapy, causes political, social, and administrative frictions, i.e. costs. This is due to the missing opportunity for gradual adaptation by the agents. The reason for this is the fact that any sudden policy switch disturbs the intertemporal plans and the expectations of the agents and thus contributes to uncertainty and instability of the whole system. Actually, these arguments mutatis mutandis also apply to the issue of a rule-based monetary policy as opposed to a discretionary policy.

The counterarguments use metaphors like that of the obvious impossibility of a gradual change from driving on the left to driving on the right in a country. More substantially, the con-arguments stress the risk that a reform process may not reach its aim since agents may be time inconsistent and change their minds during the process. Moreover, a reform process of formal institutions can hardly be conceived of in a piecemeal way since this would mean an iterated change of laws and regulations.

After this brief discussion of the gains and losses of a *gradualistic policy* parameter change let us finally come to the gains and losses of the opportunity to gradually *adjust equilibrium values* throughout the evolution of the economic system. Obviously, the preceding pro arguments equally apply to this case. But there is a further specific argument: a discontinuous "catastrophic" change of commodity prices and tax revenue amounts in general will also change the economic status of the agents abruptly, i.e. their feasible activities, and particularly their consumption opportunities. And surely it is this kind of an unforeseeable discontinuous change of economic conditions which is largely disliked by real economic agents.

The paper is organized as follows. Section 2 provides further intuition on the scope of the study to the reader. In Section 3 the kinetic method is characterized. Section 4 provides the reader with analytical prerequisites, and Section 5 presents our basic model of a continuously evolving exchange economy with production and taxes. Section 6 contains the main body of our mathematical analysis, particularly the proof of existence of open evolution equilibria. Basing on the results of Section 6, Section 7 then outlines the frictionless tuning procedure. The concluding remarks in Section 8 resume the methodological status of our approach in relation to both the evolutionary and the neoclassical approach, particulary to the approach of temporary equilibrium theory (see e.g. Grandmont 1983). The Appendices A to D are found after Section 8.

2 Intuition, Scope, and Aims of the Model

Our analysis starts from the encompassing general equilibrium exchange framework with an endogenous production sphere and taxes which has been provided by T. Kehoe (1985b). In this context an equilibrium is a pair (p, r) of an n-commodity price system p and of total tax revenue r. Intuitively speaking, an equilibrium means that the whole economy is in balance in that total tax receipts equal total real-valued tax redisbursals to the agents, and all exchange markets are simultaneously cleared when the production sector is run with a suitable scale vector. The reader should bear in mind that this notion of equilibrium belongs to the *"bookkeeping category"*. This means there is no connotation of a thermodynamic end state equilibrium meaning a state of rest to which the system finally settles down.

There might be objected that the intuition of our continuous flow modelling would be sophisticated in so far as in reality prices and quantities usually change discretely. Besides resorting to the theoretical rationale of the flow conceptualization in traditional growth theory one can object against this argument the following: The continuous formalization of an evolving economy in historical time can readily be viewed as *limit case* of the periodic discrete modelling when period lengths approach zero. A great advantage of the continuous modelling is furthermore the fact that it is no longer subject to the notoriously annoying unsolvable question which period length would be the appropriate one. The problem of the "right" choice of period length particularly would apply to a model like ours which models a system of markets. To understand this it is worthwhile to recall that the economic meaning of a period is that of a planning period of the agents. Indeed, it is hard to conceive of one common period length which equally well serves as planning period for agents being active on the markets for eggs, or tissues, and for agents acting on the markets for nuclear power plants, or TV-satellites, for instance (cf. Kirzner 1990, or Bosch 1990, for instance).

To make the reader more familiar with our analytical flow conceptualization we illustrate it by the following three figures showing the evolution of the market process for commodity i. Figure B shows the evolution of the price p_i of commodity i during the time interval $[t_0, t_1]$, Figure C shows the evolution of market demand d_i of commodity i during the same time interval, and Figure D shows the evolution of resulting expenditure $p_i d_i$. The shaded areas in Figure C and D show the total amount of the demanded quantity of commodity i and total expenses during $[t_0, t_1]$, respectively.

Since the continuity requirement is the only hard assumption for our formalization of economic evolution we admit a considerably broad class of admissible model evolutions. Particularly, there is no restriction on the functional form of the model functions and their evolutions. It is in this sense that our approach accounts for novelties in the course of the evolving of the economic system. Let us emphasize that in our study we model the open evolution of an exchange economy with production and tax schemes, but do not analyse radical system transformations which change the type of the economy.

Our procedure to formally extend the static idea of a bookkeeping equilibrium to the evolution setting is the following: Think of an Euclidean space S where the equilibria of the state model may be situated, replicated as many times as there are points in the unit time interval, and then think of the geometrical space (the cylinder) A formed by stringing these copies of S along the unit interval (see Figure E below which is identical to Figure A). Mathematically this forms the product space of S and the unit interval. Now, in general the entire equilibrium set E of an evolution will be a subset shaping a more or less wild and irregular configuration in the space A. To get more familiar with this equilibrium subspace E let us now consider the equilibrium set of an arbitrarily chosen momentary state economy of the evolution, say the t-state economy. Due to the construction it is identical to the t-slice of the equilibrium set E. One knows that according to the assumptions of the basic model every t-slice will be non-empty. But, unfortunately, one knows

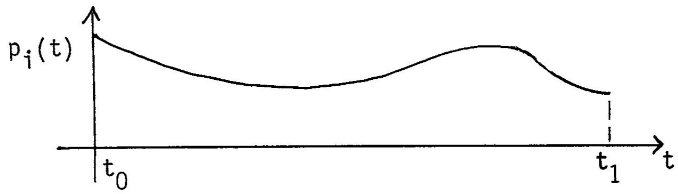

Figure B

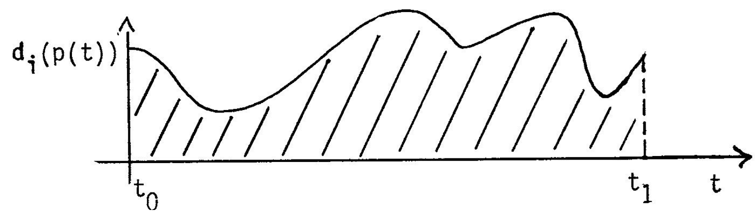

Figure C

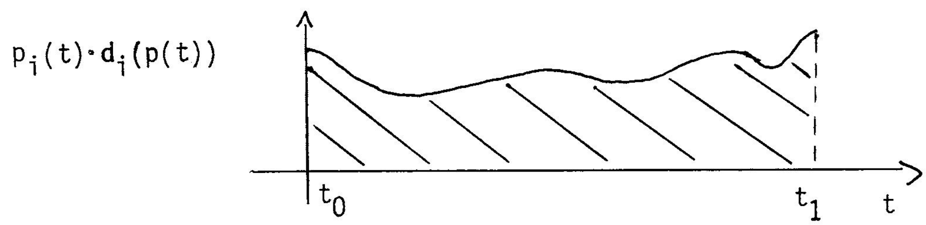

Figure D

nothing about the object which obviously deserves major interest in an evolutionary context, namely the structure of the subsequent s-state equilibrium sets for $s > t$. Actually, it is the solution of this question which our study is devoted to.

We note that uniqueness of momentary equilibria would make the proof of existence of an open evolution equilibrium in our context an easy exercise. However, uniqueness notoriously is only achieved by restrictive assumptions (cf Kehoe 1985a, b). There are more advanced results on this field making use of empirical distribution characteristics of agents by Hildenbrand and others. This approach, however, relies on a specific model framework which is completely different from ours (see e.g. Hildenbrand 1989, 1998, 1999). In contrast, our aim is to become able to cope with the multiplicity phenomenon and to establish the existence of an open evolution equilibrium. This will be accomplished by means of a certain analytical result which has been used in different contexts for the first time by the author (1983, 1985, see also 1995) and by Mas-Colell (1985).

Let us finally discuss on the significance of our notion of an open evolu-

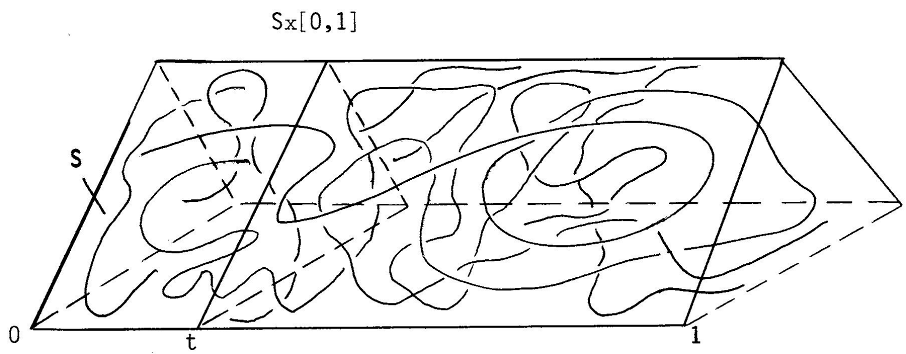

Figure E

tion equilibrium from the economic point of view. At first sight it means that one can adjust approximate equilibrium values (almost) permanently continuously while the economic system is evolving in an *open*, non-predetermined manner. The restriction "almost" is due to the weak assumptions we impose on our model of evolution, and would only be removable by additional and considerably restrictive assumptions. It means that possibly at some dates one cannot avoid to discontinuously jump when adapting the changing equilibrium values over elapsing time. To be sure, in our tax equilibrium framework the concept of an open evolution equilibrium achieves more than that. At all dates where discontinuous jumps in equilibrium values are necessary it is possible to re-tune the state parameters of the economy in a completely continuous way. Thus finally a completely "frictionless", or say continuous, adaptation of equilibrium values is possible. Of course, the latter means that one has to intervene into the open evolution of the economic system. But note that the intervention does only apply for a short time period, and, moreover, it does not involve any states of the economy which did not yet occur in the evolution up to the date of intervention. Thus, the interventions amount to repetitions of parts of the evolution of the economy, or to put it more formally, they amount to a partly backtracking in the path of states which the economy passes while it is evolving. Let us finally stress again that it is just the existence of the opportunity to continuously adjust equilibria what is shown by our result, but we do not analyze the way how this could be implemented by economic institutions in real economies.

3 The Kinetic Approach

In this Section we will briefly characterize and discuss the methodological status of the kinetic approach. In theoretical physics "kinetics" means the "description of the motion of objects without considering the forces that cause or result from the motions" (Encyclopaedia Britannica 1985). In economic literature the term kinetic has first been used in this meaning at the beginning of our century by the sociologists F.H. Giddings and F. Oppenheimer in 1911, 1916 respectively.

When speaking from kinetic analysis here we mean the following procedure. We conceptualize a formal general model of an open loop evolving economic system. Having achieved this we see that the model evolution produces a dependent "co-evolution" of momentary equilibria. It is this dependent co-evolution of equilibria which we are interested in for our further analysis of the evolution phenomenon. More specifically, we search for structure, or say regularity properties in this equilibrium co-evolution which are generally valid, i.e. which do not depend on the special shape of any specification of our general evolution model. To say it in a nutshell, the aim of kinetic analysis is not to study the "laws of evolution of the economic system", but to study the "laws of the effects on the endogenous variables", or to say it more formally, the "laws of the dependent co-evolution of evolving economic systems of equilibrium values".

It might appear natural to ask now for the relationship of the kinetic method to the conventional comparative static method. Briefly said, conventional comparative statics has a much more restricted scope than kinetics since comparative static analysis usually either analyzes the effects of a marginal, or of a discrete, change of exogenous parameters. To be sure, there are very few comparative static results in the literature which do not hinge to one of these two schemes (see for instance Arrow/Hahn 1971, Chapter 10, Theorem 5, or Quirk/Saposnik 1968, Section 6.4, Theorem 5). However, according to the traditional understanding of comparative statics also these results are restricted to well-specified parameter changes, and – what is more – require uniqueness of equilibrium. Clearly the latter severely limits the scope of the traditional comparative static approach (Kehoe 1985 a).

In contrast, our kinetic approach does not hinge to any one of these restricting prerequirements. We analyze whole evolutions of the economic system under consideration, and not only marginal, or discrete, changes of certain coefficients. Furthermore, we do not restrict the evolution of the economic system to well-specified parameter changes, but admit a maximal variety of possible evolutions which, however, must be continuous. And finally, we do not stick to the uniqueness requirement of equilibria. Actually, it is an essential feature of our approach that we are able to cope with multiplicity of the equilibrium set in that our conception and existence proof of an open evolution equilibrium do not require any kind of uniqueness.

4 Analytical Prerequisites

In this Section we will provide the reader with some analytical prerequisites which will turn out to be useful for our conceptualization and analysis of evolutions. We do not put this Section into the appendix since it provides the intuition and the precise definitions of the building blocks of the concepts used in our paper.

The most important analytical tool for our present study is the concept of a *continuous one-parametrization.* Generally, a continuous one-parametrization is a continuous mapping

$$F : X \times [0,1] \longrightarrow Y$$

where X and Y are arbitrary (Euclidean sub-) spaces. Obviously, one can also view the mapping F as a family of continuous mappings $(F_t)_{t\in[0,1]}$ where the members of the family are continuously connected, too. The space $X \times [0,1]$ is called the *homotopy space.* That the concept of a continuous one-parametrization naturally suits for formalizing evolutions makes the following Figure 1 obvious.

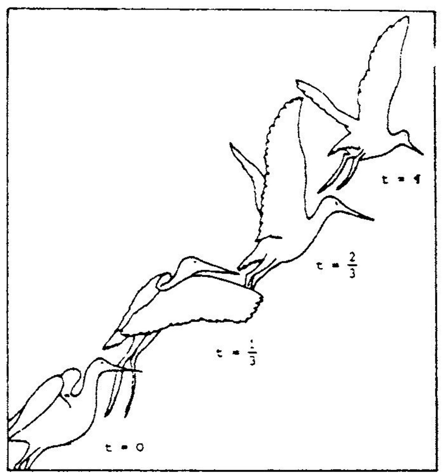

Figure 1

Let us now list some further useful notations and concepts. The symbol Δ^{n-1} means the n-1-dimensional unit simplex in $\mathbb{R}^n_+$. The symbol $\overset{\circ}{B}{}^n_\alpha\,(x)$ means the open ball in $\mathbb{R}^n$ with center x and radius α. Another term which will play a crucial role in our subsequent analysis is that of an "equilibrium-equivalent self-mapping" $g : X \longrightarrow X$. This concept refers to a given equilibrium model and means that g is a mapping whose fixed points precisely equal the set of equilibria of this equilibrium model. Clearly, many equilibrium existence proofs in the literature essentially amount to the construction of an equilibrium equivalent self-mapping.

Making later our notion of an open evolution equilibrium precise we will involve the analytical concept of *topological connectedness.* Formally, a *connected subset* of $\mathbb{R}^n$ is a set which cannot be separated into two disjoint open subsets. A subset X of $\mathbb{R}^n$ is furthermore even *path connected* if any two of its points can be linked by a continuous path lying in X. Or formally: for any two points x and y of X there is a continuous mapping $w : [0, 1] \longrightarrow X$ with $w(0) = x$ and $w(1) = y$. One has to carefully distinguish between the notion of a path w and its *arc,* i.e. its image $w[0, 1]$ in X. The *Euclidean length of a path* $w : [0, 1] \rightarrow X$ is defined as $\sup_{W_k} L(w, W_k)$ where W_k denotes a subdivision of $[0, 1]$ by $k + 1$ points $0 = t_0 < t_1 < \ldots < t_k = 1$ and

$$L(w, W_k) := \sum_{j=1}^{k} d(w(t_{j-1}), w(t_j)) = \sum_{j=1}^{k} \|w(t_j) - w(t_{j-1})\|.$$

If $\sup_{W_k} L(w, W_k)$ is finite then one says that w is of finite length, or w is *rectifiable.* It is well-known that a path w is rectifiable if and only if each of its component functions w_i, $i = 1, \ldots, k$, is of *bounded variation over* $[0, 1]$, that means

$$\sup_{W_k} \sum_{j=1}^{k} \|w_i(t_j) - w_i(t_{j-1})\| < \infty.$$

The everyday connotation of the term 'path' clearly is 'to be passable' in the intuitive geometrical sense. To be sure, this is also our intuition in this study. Unfortunately, arcs of *continuous* paths can still have unpleasant wild shapes as the following example shows: the graph of the continuous function $x \cdot \sin 1/x$ on the domain $[-1, 1]$ has *infinite length* (one estimates from below by the divergent harmonic series, see Figure 2). But even if the arc of a continuous path is of finite length, it still may *oscillate,* or *tremble,* infinitely often, as the function

$$x \mapsto \begin{cases} x^3 \sin 1/x, & x \in [-1, 0[\ \cup \ ]0, +1] \\ 0, & x = 0 \end{cases}$$

shows ('damped oscillation'). We will come back to this issue at the end of Section 6 and in Appendix C below.

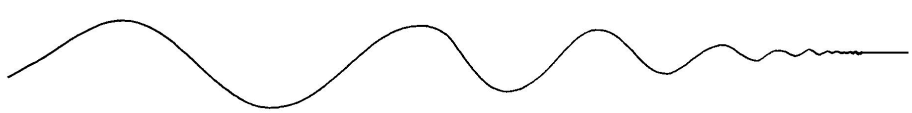

Figure 2

5 A Continuous Time Model of an Evolving Exchange Economy with Production and Taxes

In Kehoe's first version of an equilibrium model with production and taxes (1985b, pp. 318–321) an agent can be a consumer, a producer, or the government. Aggregate excess demand on the n commodity markets is given by a C^1 function

$$\begin{aligned} \zeta : \mathbb{R}^n_+ \backslash \{0^n\} \times \mathbb{R}_+ &\rightarrow \mathbb{R}^n \\ (p, r) &\mapsto \zeta(p, r) \end{aligned}$$

where r stands for the *total tax revenue.* ζ is homogeneous of degree zero, bounded from below, and satisfies the following intuitive boundary assumption$^{*)}$:
*) the function

$$\begin{aligned} k_\zeta : \mathbb{R}_+ &\rightarrow \mathbb{R}_+ \\ r &\mapsto \inf_{p \in \mathbb{R}^n_+ \backslash \{0^n\}} \{\|\zeta(p, r)\|\} \end{aligned}$$

satisfies $\lim_{r \to \infty} k_\zeta(r) = \infty$.

In words the boundary assumption$^{*)}$ just means that with the r-argument growing beyond all finite bounds the absolute value of the excess demand function $\|\zeta(p, r)\|$ also grows beyond all finite bounds *all over the price space* $\mathbb{R}^n_+ \backslash \{0^n\}$. This property of the excess demand function particularly ensures the intuitive requirement that for any real $\alpha > 0$ there is a real $\beta > 0$ such that for all $p \in \mathbb{R}^n_+ \backslash \{0\}$ one has: $r > \beta \Rightarrow \|\zeta(p, r)\| > \alpha$. Later we will see that this implies that the equilibrium set in fact is contained in a compactum which later with turn out to be essential for the construction of an equilibrium equivalent self-mapping (see Kehoe, 1985b, p. 321 first paragraph). Unfortunately, Kehoe's original condition

(A.4)

given any $p \in \mathbb{R}^n_+ \backslash \{0^n\}$

$$\lim_{r \to \infty} \|\zeta(p, r)\| = \infty$$

is too weak to ensure that. (Actually, it is not hard to find counterexamples.) However, the following additional requirement to (A.4) obviously makes it sufficient for the purposes of our later analysis: the family of *partial functions* $\zeta(p, -) : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ which is parametrized by the admissible price vectors $p \in \mathbb{R}^n_+ \backslash \{0^n\}$, is C^0-uniformly convergent on the whole domain $\mathbb{R}_+$ for varying p.

The tax payments generated by consumption and income taxation are specified by a C^1 function

$$
\begin{aligned}
t : \mathbb{R}^n_+ \backslash \{0^n\} \times \mathbb{R}_+ &\rightarrow \mathbb{R}_+ \\
(p, r) &\mapsto t(p, r)
\end{aligned}
$$

which is homogeneous of degree one, i.e. $t(\lambda p, \lambda r) = \lambda t(p, r)$ for $\lambda > 0$. Tax payments $t(p, r)$ and tax revenue r are expressed in the same units of account as expenditures $\zeta(p, r)p$. The function t furthermore satisfies an appropriately modified version of Walras' law, namely the aggregate budget constraint

$$\zeta(p, r)p + t(p, r) = r$$

for all admissible (p, r). Note that this aggregate budget constraint still does not indicate how the total tax revenue r is actually generated. The generation of r will be become clear below when also the production sphere will be introduced.

To get a better intuition on this set-up let us have a look on Kehoe's *example* (1985b, pp. 318–319) of an economy with h consumers. At price system p the j-th consumer's income is given by the value of his initial endowment bundle

$$\sum_{i=1}^{n} p_i \omega_i^j$$

plus his share of tax revenue,

$$\theta_j r.$$

Clearly, the vector of share coefficients $(\theta_1, \ldots, \theta_h)$ lies in $\overline{\Delta}^{h-1}$. For instance, $\theta_1 = \ldots = \theta_{h-1} = 0$ and $\theta_h = 1$, where agent h is the government. The endowment income $\sum_{i=1}^{h} p_i \omega_i^j$ of each consumer j is taxed at a rate $\rho_j \in [0, 1[$, and consumer j's final demand for commodity i is taxed 'ad valorem' at a rate $\tau_{ij} \in [0, 1[$ on its value. Accordingly, the utility maximization problem of consumer j is the following:

$$
\begin{aligned}
&\max \;\; u_j(x_1^j, \ldots, x_n^j) \\
&\text{so that} \;\; \sum_{i=1}^{n} p_i (1 + \tau_{ij}) x_i^j \leq (1 - \rho_j) \sum_{i=1}^{n} p_i \omega_i^j + \theta_j r \\
&x_i^j \geq 0 \text{ for all } \; i, j.
\end{aligned}
$$

u_j is a strictly concave and monotonically increasing utility function. Thus, agent j's derived excess demand function

$$\begin{array}{rcl}\zeta^j : (\mathbb{R}^n_+\backslash\{0^n\}) \times \mathbb{R}_+ & \to & \mathbb{R}^n \\ (p,r) & \mapsto & \begin{pmatrix} x^j_1(p,r) - \omega^j_1 \\ \vdots \\ x^j_n(p,r) - \omega^j_n \end{pmatrix}\end{array}$$

is continuous and the aggregate excess demand function $\sum_{j=1}^h \zeta^j$ satisfies for any $p \in \mathbb{R}^n_+\backslash\{0^n\}$ the condition $\|\zeta(p, r^m)\| \to +\infty$ as $r^m \to +\infty$. This condition means that, anything else being equal, if tax revenue becomes arbitrarily large, then the income of at least one consumer (the government for instance) becomes arbitrarily large, which in turn implies that excess demand for some good becomes arbitrarily large (cf. Kehoe, 1985b, p. 319). Accordingly, the tax payment function t is specified by

$$t(p,r) = \sum_{j=1}^h \rho_j \left(\sum_{i=1}^n p_i \omega^j_i \right) + \sum_{j=1}^h \left(\sum_{i=1}^n \tau_{ij} p_i x^j_i(p,r) \right).$$

t is C^1 and homogeneous of degree one as long as the x^j_i are. Since each individual demand function satisfies the budget constraint with equality, ζ and t satisfy the modified Walras' law. (For a further example which also allows for tax rates and revenue shares varying with income the reader is referred to Kehoe (1985b, p. 319, last paragraph). (End of example)

Now let us come back again to the general model. The production sphere is specified by an $n \times m$ activity analysis matrix $A = (a_{ij})$ with the following properties:

(1) A induces n free disposal activities, one for each commodity. Formally, this means that the last n columns of A form the negative n-dimensional unit-matrix.

(2) There is no output without inputs, i.e.

$$\{x \in \mathbb{R}^n \mid x = Ay, y \geq 0^n\} \cap \mathbb{R}^n_+ = \{0^n\}.$$

Production taxes are specified by an $n \times m$ matrix $A^* = (a^*_{ij})$ with $a^*_{ij} = a_{ij} - \sigma_{ij}\,|a_{ij}|$ where $\sigma_{ij} \in [0,1]$. This means, input or output of commodity i in activity j is taxed at a rate of $\sigma_{ij} \in [0,1]$. Thus,

(3) $-2|A| \leq A^* \leq A$.

Furthermore, there are no taxes at free disposal activities. Accordingly, the *revenue generated* by production taxes at prices p and at activity levels $y \in \mathbb{R}^m_+$ is

$$p'(A - A^*)y \geq 0.$$

Definition:

An *economy* is now defined as a quadruple (ζ, t, A, A^*). A *momentary equilibrium* of an economy is a pair $(p^0, r^0) \in \Delta^{n-1} \times \mathbb{R}_+$ that satisfies the following conditions,

(E.1) $p^{0'} A^* \leq 0^m$.

(E.2) there is a $y^0 \in \mathbb{R}^m_+ \backslash \{0^m\}$ such that $\zeta(p^0, r^0) = Ay^0$.

(E.3) $r^0 = t(p^0, r^0) + p^{0'}(A - A^*)y^0$

Let us briefly comment on these equilibrium conditions. From (E.2) and Walras' law follows immediately: (E.3) $\Leftrightarrow p^{0'} A^* y^0 = 0$. Actually, it is this equivalent formulation of equilibrium condition (E.3) which we will use in our later constructions.

Together with (E.1) the alternative formulation of (E.3) implies that after-tax profits are maximized at an equilibrium. (E.2) means that excess demand actually can be supplied by the producers. (E.3) expresses the fact that in equilibrium the redisbursals of the total tax revenue equal the total tax receipts $t(p^0, r^0) + p^{0'}(A - A^*)y^0$. The normalization expressed by $p^0 \in \Delta^{n-1}$ is obviously permitted by the homogeneity properties of ζ and t.

Our formalization of an *evolution of economies with production and taxes* is straightforward:

Definition:

An *evolution of economies with production and taxes* is a quadruple of four continuous one-parametrizations $(\zeta_s, t_s, (a_{ij_s}), (a^*_{ij_s}))_{s\in[0,1]}$ such that, moreover, the two component one-parametrizations $(\zeta_s)_{s\in[0,1]}$ and $(t_s)_{s\in[0,1]}$ are C^0-uniformly continuous and the one-parametrization $(a_{ij_s})_{s\in[0,1]} = (A_s)_{s\in[0,1]}$ satisfies the condition that for any $w \in \mathbb{R}^n_+$ with $\zeta_s(-,-) \geq -w$ for all $s \in [0,1]$ on the whole domain $(\mathbb{R}^n_+\backslash\{0\}) \times \mathbb{R}_+$ the set

$$\{x \in \mathbb{R}^n \mid \exists y \geq 0,\ \exists s \in [0,1] : x = A_s(y) \text{ and } \ x \geq -w\}$$

is bounded.

It is noteworthy that the last requirement in this definition is just a uniformization of assumption (2) above 'no output without inputs' on the production matrix of static economies. Nevertheless, to prove this exactly is not as easy as it seems to be at the first glance. We will provide the reader with a proof in Appendix A employing several transformation steps which turn out to be equivalences.

From our assumptions on an evolution of economies, from the definition of the equilibrium equivalent self-mapping g (see Appendix B), and from the considerations above follows directly that any admissible evolution of economies with production and taxes in fact induces a *continuous* one-parametrization of equilibrium equivalent self-mappings, as desired.

6 Existence of an Open Evolution Equilibrium

We begin this Section with giving our notion of an open evolution equilibrium a precise meaning. For this we first need a formalization of the idea of an ϵ-approximating equilibrium path:

Definition:
Let be K the domain of equilibrium values. Then for any $\epsilon > 0$ a *connecting ϵ-near-equilibrium path for the evolution* $(\zeta_s)_{s\in[0,1]}$ is a finitely piecewise linear path, or say a *polygonal path*,

$$\pi : [0,1] \longrightarrow K \times [0,1]$$

whose arc $\pi[0,1]$ lies in the ϵ-neighborhood of the equilibrium set of $(\zeta_s)_{s\in[0,1]}$ in $K \times [0,1]$ and joins the bottom $K \times \{0\}$ and the top $K \times \{1\}$ of $K \times [0,1] = \Delta^{n-1} \times [0,\beta] \times [0,1]$.

Definition:
Let any admissible evolution $(\zeta_s)_{s\in[0,1]}$ be given. Formally, an *open evolution equilibrium* for the given evolution is a *connecting* ϵ-near equilibrium path.

The reader should be well aware that an ϵ-near equilibrium price path ϵ-approximates the equilibrium set of an evolution *on the whole.* Particularly, this means that it *need not* ϵ-approximate every s-state equilibrium set when the equilibrium set of the whole evolution decomposes into several path components. Figure 3 shows an example for this. Nevertheless, we will see

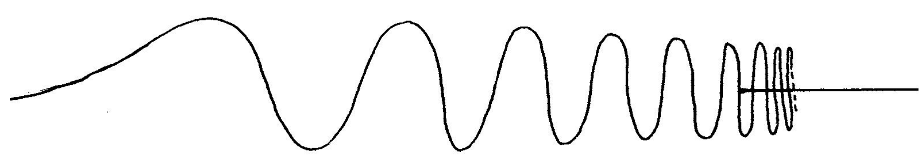

Figure 3

that for a large class of evolutions any ϵ-near equilibrium path indeed also ϵ-approximates any s-state equilibrium set (see Theorem 2 in Appendix D below). Now we are ready to state the following result which will be central for our study.

Theorem 1:
For any admissible evolution $(\zeta_s)_{s\in[0,1]}$ *there is at least one* ϵ*-near equilibrium path for any given* $\epsilon > 0$.

The reader can find the proof in Appendix C. A general categorization of equilibrium proofs in the economic literature can be found in Weintraub, Gayer (2001). In his monograph (1985, Proposition 5.8.2) Mas-Colell presents the analogue of our Theorem 1 for the basic model of an explicit finite exchange economy defined by preference relations and initial endowments. Actually, Mas-Colell's method of deriving the result is quite different from ours. Other contributions in similar directions have been given by Allen (1981), Balasko (1988), Balasko/Lang (1998) and Bonnissean and Rivea-Cayupi (1999), for instance.

To be sure the approximation of an ϵ-near-equilibrium path is necessary since in general no well-behaved path can be found in the equilibrium set. This is proven in Appendix C below.

7 Frictionless Tuning of Tax Parameters and Equilibrium Variables in an Evolving Economy

We now proceed making analytically precise the idea of a *frictionless tuning* which we have outlined intuitively in the Introduction. Note that in analytical terms the meaning of "frictionless" is "continuous".

To fix ideas let us start with a formal reformulation of an evolution. It will turn out to be more convenient for our present purposes. We represent an evolution as a composite continuous mapping

$$\begin{array}{ccccc} [0,1] & \stackrel{z}{\to} & C & \stackrel{\Phi_z}{\to} & \mathcal{E} \\ s & \mapsto & (c_{1_s},\ldots,c_{r_s}) & \mapsto & E_{(c_{1_s},\ldots,c_{r_s})} \end{array}$$

where $\mathcal{E}$ denotes the space of admissible momentary state economies and C denotes the Euclidean space of policy control parameters. The symbol z means any continuous path in the control parameter space C, and Φ_z is a mapping which may be individually chosen in dependence on the control parameter path z. More specifically, Φ_z associates an economy $E_{(c_{1_s},\ldots,c_{r_s})}$ with any admissible control parameter tuple $(c_{1_s},\ldots,c_{r_s})$ from the path z in a continuous way. Consequently, any admissible evolution is completely characterized by the evolution (the path) z of control parameters. Let us now come back to Kehoe's example.

Recall that there are n commodities, $m > n$ production processes, and h economic agents in this example. There are *four vectors* of control parameters: the $n \cdot h$ - vector of individual ad-valorem consumption tax rates $(\tau_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,h}} \in [0,1[^{n \cdot h}$, the h-vector of individual endowment income tax rates $(\rho_j)_{j=1,\dots,h} \in [0,1[^h$, the h-vector of individual share rates of tax revenue $(\vartheta_j)_{i=1,\dots,h} \in \overline{\Delta}^{h-1}$, and the $n \cdot m$ - vector of ad-valorem production tax rates $(\sigma_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,m}} \in [0,1]^{n \cdot m}$. Clearly, all of these parameters are in principle amenable to control by some governmental authority. Accordingly, we are facing the Euclidean *control parameter space*

$$C := [0,1[^{nh} \times [0,1[^h \times \overline{\Delta}^{h-1} \times [0,1]^{n \cdot m} \subset \mathbb{R}_+^{(nh+h+h+nm)}.$$

Generally spoken "frictionless" means "without leaps in equilibrium prices (in an approximating sense)". Now let us illustrate our *frictionless tuning* procedure in this formal context. We can write the vector of control parameters $(c_{1_s}, \dots, c_{r_s})$ on any s-slice of the homotopy space $H \times [0,1]$ as it is illustrated in Figure 4.

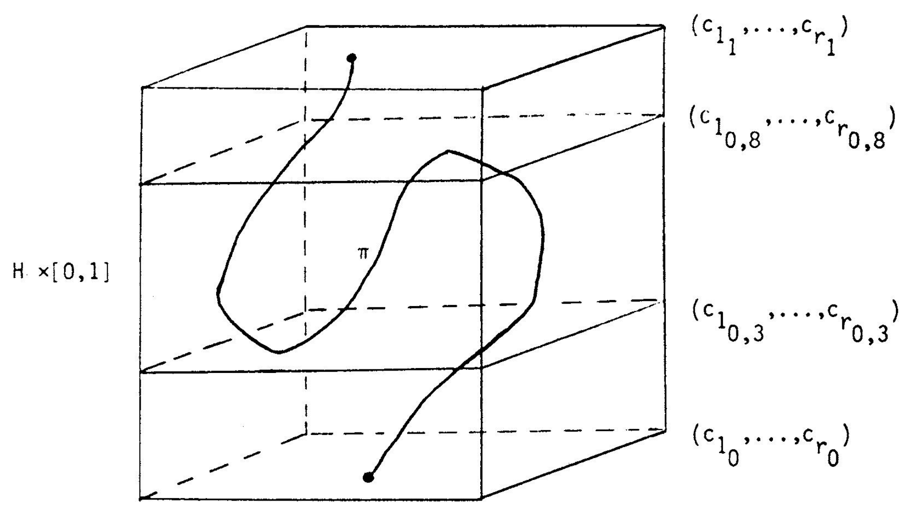

Figure 4

This means we represent the path z which is a one-parametrized subset of C by the points of the parametrizing interval. A *frictionless control* for the example of Figure 4 obviously can be achieved by following the (near-) equilibrium path π during the part from control parameter tuple $(c_{1_0}, \dots, c_{r_0})$ to $(c_{1_{0,8}}, \dots, c_{r_{0,8}})$, then running back in the control parameter path from $(c_{1_{0,8}}, \dots, c_{r_{0,8}})$ to $(c_{1_{0,3}}, \dots, c_{r_{0,3}})$, and finally running forward again from $(c_{1_{0,3}}, \dots, c_{r_{0,3}})$ to $(c_{1_1}, \dots, c_{r_1})$. (Note that $r = nh + h + h + nm$).

Extending this observation to the general case we can sum up: a frictionless control of an evolution $\Phi_z \circ z$ is possible by an appropriate continuous *re-parametrization* $\tilde{z}$ of the control parameter path z. Formally, $\tilde{z}$ is obtained by projecting the (near-) equilibrium path π on the unit interval $[0,1]$ of the homotopy space $H \times [0,1]$, i.e. $\tilde{z} = z \circ pr_2 \circ \pi : [0,1] \to z([0,1])$. Hence, both paths, the originally chosen path $\pi : [0,1] \to H \times [0,1]$ of (near-) equilibria and the appropriately re-parametrized path $\tilde{z} : [0,1] \to z([0,1]) \subset C \subset \mathbb{R}^l$, are even in the intuitive geometrical sense nicely behaved.

Obviously, one has to identify the historical time t with the evolution parameter of the re-parametrized evolution $\Phi_z \circ \tilde{z}$. Thus, the vector of control parameters at time t is $\tilde{z}(t) = z(pr_2(\pi(t)))$, and the state of the economy at time t is $\Phi_z(\tilde{z}(t)) = \Phi_z(z(pr_2(\pi(t))))$. This is completely analogous to the situation of a movie which is played with several parts backtracked whilst time is going on as usual. Viewers (economic agents) of course are not getting younger when the film (the evolution) is backtracking, but the whole movie is expanded by the length of time the repeated parts require.

Conclusions

Our findings in this Section lead to the following general conclusions. An evolution of economic states in our conceptualization is completely determined by the associated evolution of the $(nh + h + h + nm)$ tax control parameters. If a political-economic agency aims at a frictionless control in the described sense our results tell that it generally has to move back and forth appropriately in the evolution of control parameters. This is due to the general backtracking feature of (near-) equilibrium paths. In other words this means that in the present set-up in general it is inevitable for an (exogenous) political-economic agency to give the impression of somewhat being undecided – if it is purposed to ensure a frictionless control.

8 Concluding Remarks and Outlook

In our concluding remarks we want to briefly reconsider the attitude and the methodological status of our study. First of all, the reader should remember that our attitude to the notion of equilibrium is not that of traditional static equilibrium theory as a final state of rest. While the ultimate aim of traditional static equilibrium theory is to explain realized states of economic systems employing the concept of equilibrium, we, instead, merely think of equilibria as perfect momentary coordination solutions of the evolving modelled system – are they realized, or not. This particularly overcomes the awkward dilemma caused by the multiplicity of equilibria in the static context.

In a nutshell we have learned from our analysis that there are invariant regularities of the changing equilibrium solutions when the economic system evolves in a non-described continuous manner. The general structure properties we have found show that ex ante statements are well possible in our framework even when novelties, i.e. new states of the economic system, occur.

Let us now conclude with some remarks on the relationship between our approach and the neoclassical approach. At first glance our approach might seem to be closely related to the temporary equilibrium approach (see e.g. Grandmont 1983). To be sure, the two approaches have completely different scope. The main difference is due to the fact that the temporary equilibrium approach employs an intertemporal planning procedure by the agents and accordingly an intertemporal notion of equilibrium which are both closed loop. Besides that the temporary equilibrium approach is mainly interested in the money issue. In contrast, we are interested in the performance of an interdependent market system with endogenous production sphere and governmental redistribution adivities employing taxes. Actually, both our notion of equilibrium and our one-parametrizing conceptualization of economic evolution, prevents us from the reproach of trying to extend the traditional and notoriously inevolutionary approach of neoclassical economics to the evolution phenomenon (see e.g. Witt 2003). To repeat it once more, our aim is not to force the evolutionary approach into the Procrustean bed of neoclassical theory, but to contribute to overcoming the natural weakness of the open loop approach of evolutionary economics by establishing results of the kinetic type.

Appendix A

In Appendix A we are going to prove that the last requirement in the definition of an evolution of economies with production and taxes is just a uniformization of assumption (2) above 'no output without inputs' on the production matrix of static economies. The second equivalence of this chain expresses the uniformization of assumption (2).

Proposition:
The following chain of equivalences is valid.
$\forall_{w\in\mathbb{R}^n_+}\{x \in \mathbb{R}^n \mid x \in \bigcup_{s\in[0,1]} A_s(\mathbb{R}^m_+) \text{ and } x \geq -w\}$ is bounded, i.e. there is an $\alpha_w > 0$ such that $\|x\| < \alpha_w$ for all x from this set

$\Leftrightarrow \{x \in \mathbb{R}^n \mid x \in \bigcup_{s\in[0,1]} A_s(\mathbb{R}^m_+) \text{ and } x \geq (-1,\dots,-1)\}$ is bounded

$\Leftrightarrow \overline{\bigcup_{s\in[0,1]} A_s(\mathbb{R}^m_+)} \cap \mathbb{R}^n_+ = \{0^n\}$

$\Leftrightarrow$ there is a closed subspace $K \subset \mathbb{R}^n$ with $\overline{\bigcup_{s\in[0,1]} A_s(\mathbb{R}^m_+)} \subset K$ and $K \cap \mathbb{R}^n_+ = \{0^n\}$.

Proof. The first and the last equivalence are trivial, whereas the crucial middle one is not.
"$\Rightarrow$:" Let us abbreviate $N := \bigcup_{s\in[0,1]} A_s(\mathbb{R}^m_+)$ and assume that there is an

$x \in \overline{N} \cap \mathbb{R}^n_+$ with $x \neq 0$. Then there is a sequence x^k in N with $x^k \to x$. Without loss of generality we may assume that $\|x^k\| = \|x\| = 1$ and $x^k_i \geq -1$ for all k and $1 \leq i \leq n$. Define

$$m^k := \max\{\left|x^k_i\right| \mid x^k_i < 0\}.$$

Note that the right set is non-empty since $N \cap \mathbb{R}^n_+ = \{0^n\}$. The latter is due to the assumption 'no output without inputs'. Since $x^k \to x \in \mathbb{R}^n_+$, the sequence m^k converges to zero. Put $y^k := \frac{x^k}{m^k}$. Clearly $y^k \in N$ and $\|y^k\| \to \infty$ for $k \to \infty$. If we can show that $y^k \geq (-1, \ldots, -1)$ for all k, then the presumption that $\{x \in N \mid x \geq (-1, \ldots, -1)\}$ is bounded contradicts $\|y^k\| \to \infty$. Consequently, the assumption that there is an $x \in \overline{N} \cap \mathbb{R}^n_+$ with $x \neq 0$ is wrong. Now let us choose any $j \in \{1, \ldots, n\}$. Clearly, $y^k_j < 0$. From $\left|x^k_j\right| \leq m^k$ follows $\left|y^k_j\right| = \frac{\left|x^k_j\right|}{m^k} \leq 1$. But this means that $y^k_j \geq -1$, and we are done.

"$\Leftarrow$": We begin with the observation that $K := \overline{N} \cap S^{n-1}$ is compact.

Define:

$$\begin{array}{rcl} \lambda : K & \to & \mathbb{R}_+ \\ x & \mapsto & \min\limits_{\substack{\text{i with} \\ x_i < 0}} \dfrac{1}{|x_i|} = \dfrac{1}{\left[\max\limits_{\substack{\text{i with} \\ x_i < 0}} |x_i|\right]} \end{array} .$$

Actually, λ is well-defined since $K \cap \mathbb{R}^n_+ = \emptyset$ by presumption. Moreover, λ is continuous. Consequently, there is a $\lambda_0 > 0$ with

$$\forall_{x \in K}\ \lambda(x) \leq \lambda_0.$$

Choose now any $x \in \overline{N}$ with $0^n \neq x$ and $x \geq (-1, \ldots, -1)$. Define $z := \frac{x}{\|x\|} \in K$. There is an $\tilde{i} \in \{1, \ldots, n\}$ such that $z_{\tilde{i}} < 0$ and $\lambda(z) = \frac{1}{\left|z_{\tilde{i}}\right|}$. Clearly, $\frac{1}{\left|z_{\tilde{i}}\right|} = \frac{\|x\|}{\left|x_{\tilde{i}}\right|} \leq \lambda_0$, and from $-1 \leq x_{\tilde{i}} \leq 0$ follows

$$\|x\| \leq \lambda_0 \left|x_{\tilde{i}}\right| \leq \lambda_0.$$

This means that any $x \in \overline{N}$ with $x \geq (-1, \ldots, -1)$ lies in the n-ball $B^n_{\lambda_0}(0^n)$, and this completes the proof. □

Appendix B

Now we are going to provide an equilibrium equivalent self-mapping for the presented model. Before reporting on Kehoe's construction we have to do a last preparatory step (cf. Kehoe (1985b), p. 321, first paragraph). We have to ensure that in equilibrium tax revenues cannot exceed some fixed upper bound $\beta > 0$. This implies that all candidates for equilibria lie in the compact convex set $\Delta^{n-1} \times [0, \beta]$ which will be crucial for our constructions. The existence of such a β can be seen in the following way: the boundedness from below of ζ, say by $-w,\ w \in \mathbb{R}^n_+$, and assumption (2) on the production sphere clearly imply that the *production possibility set* $P := \{x \in \mathbb{R}^n \mid x \geq -w,\ x = Ay$ for some $y \geq 0^m\}$ is *bounded,* i.e. there is a real $\alpha > 0$ such that $P \subset \overset{\circ}{B}{}^n_\alpha$. Furthermore, due to boundary assumption* in Section 5 there is clearly a real

$\beta > 0$ so that $\|\zeta(p, r)\| \geq \alpha$ for any pair $(p, r) \in \Delta^{n-1} \times [\beta, \infty[$. But this implies that all equilibria already must lie in $\Delta^{n-1} \times [0, \beta]$.

Kehoe proposes the following mapping which is used in the present study as an equilibrium equivalent self-mapping (1985b, pp. 321–322):

$$\begin{aligned} g : \Delta^{n-1} \times [0, \beta] &\rightarrow \Delta^{n-1} \times [0, \beta] \\ (p, r) &\mapsto (x, y) \end{aligned}$$

where (x, y) solves the following program:
$\min 1/2[(x - p - \zeta(p, r))(x - p - \zeta(p, r)) + (y - t(p, r))^2]$
so that

(i) $(x, y) \in \Delta^{n-1} \times [0, \beta]$

(ii) $x'A - (1 + y - r)p'(A - A^*) \leq 0^m$.

We have to verify four issues:

(1) The constraint set is non-empty. This follows directly from assumption (2) on the production sphere.

(2) The constraint set is a subset of $\Delta^{n-1} \times [0, \beta]$. This follows from the assumption that there are no taxes on free disposal activities.

(3) $g(p, r)$ is continuous.

This follows from the facts that for any pair of arguments (p, r) the constraint set obviously is closed and convex and varies continuously as a point-to-set mapping, and the objective function of the program is strictly convex. The latter follows from the positive definiteness of the Hesse matrix of the objective function (recall that (p, r) is fixed): its gradient is

$$1/2 \begin{pmatrix} 2x_1 - 2(p_1 + \zeta_1(p,r)) \\ \vdots \\ 2x_n - 2(p_n + \zeta_n(p,r)) \\ 2y - 2t(p,r) \end{pmatrix},$$

and consequently the $(n+1) \times (n+1)$ Hesse matrix becomes

$$1/2 \begin{pmatrix} 2 & & 0 \\ & \ddots & \\ 0 & & 2 \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

(4) (p^0, r^0) is an equilibrium of (ζ, t, A, A^*) if and only if it is a fixed point of the associated mapping g. This is shown in the proof of Theorem 1 by Kehoe (1985b, p. 322).

Appendix C

The following proof of Theorem 1 will not only provide the reader with the logical chain of mathematical arguments establishing the statement of the theorem, but also with a detailed discussion on the meaning of every single step. Parts of the proof are adopted from Lehmann-Waffenschmidt (1995).

Fortunately, the major part of work has already been done. Actually, from Section 5, last paragraph, and Appendix B follows that the given evolution $(\zeta_s)_{s\in[0,1]}$ induces a *continuous* one-parametrization of equilibrium equivalent self-mappings

$$(g_s)_{s\in[0,1]} : K \times [0,1] \longrightarrow K$$

with $K = \Delta^{n-1} \times [0, \beta]$. Let us now look at the properties of K. K is compact, and, particularly, it is a Euclidean neighborhood retract. Since K is furthermore contractible, it is also acyclic. Hence, any self-mapping of K has Lefschetz number +1 (see Brown 1971, II.c).

This means that we have posed ourselves in a situation to which the following result from one-parametrized algebraic topological fixed point theory applies:

Proposition:

Let K be a compact subset of $\mathbb{R}^n$ and a neighbourhood retract. Let $(g_s)_{s\in[0,1]} : K \times [0,1] \longrightarrow K$ be a continuous family of maps, and let F be the union of the fixed-points of the mappings g_s, i.e.,

$$F := \bigcup_{s\in[0,1]} Fix\ (g_s) \subset K \times [0,1].$$

Then the fixed-point index λ of g_s equals the Lefschetz number of g_s, and is independent of s. If $\lambda \neq 0$, then F has a connected component C which meets bottom $K \times \{0\}$ and top $K \times \{1\}$ of the homotopy space.

Definition:

We will call a connected component of the equilibrium set of an evolution of economies which meets bottom and top of the homotopy space a *joining equilibrium component of the evolution.*

The Proposition has been proven by Puppe (1979, Corollary 5.6). Apparently, the existence of a joining equilibrium component C for the evolution $(\zeta_s)_{s\in[0,1]}$ brings us more closely to our goal. However, such a connected joining equilibrium component may still display some geometrically bad features. Let us come back to this after the proof will be finished.

Now, let us do the last step of our proof of Theorem 1 by demonstrating that there is a near-equilibrium path in any relatively open ϵ-neighborhood

$$\left[\bigcup_{x\in C} \overset{\circ}{B}{}_{\epsilon}^{n+1}(x)\right] \cap [K \times [0,1]] =: \bigcup_{x\in C} \overset{\circ}{B}{}_{\epsilon_r}^{n+1}(x) =: C_\epsilon$$

of any joining component C of the equilibrium set of the given evolution $(\zeta_s)_{s\in[0,1]}$ (note that by definition $\overset{\circ}{B}{}_{\epsilon_r}^{n+1}(x) = \overset{\circ}{B}{}_{\epsilon}^{n+1}(x) \cap (K \times [0,1])$).

As C is compact, finitely many relatively open ϵ_r-balls $\overset{\circ}{B}{}_{\epsilon_r}^{n+1}(x_1)$, $\ldots, \overset{\circ}{B}{}_{\epsilon_r}^{n+1}(x_k)$ of the ϵ-neighbourhood C_ϵ are sufficient to cover C. Denote their union by C_ϵ^f.

Now consider all pairs (x_i, x_j), $i \neq j$, of centers of the relative ϵ_r-balls $\overset{\circ}{B}{}_{\epsilon_r}^{n+1}(x_i)$, and consider the graph g'_C consisting of all segments $\overline{x_i x_j}$ which *are contained in* C_ϵ^f. If one adds all segments to g'_C which are orthogonal to $\mathbb{R}_+^n \times \{0\}$ and connect a center $x_i \in \{x_1, \ldots, x_k\}$ with $\mathbb{R}_+^n \times \{0\}$ or with $\mathbb{R}_+^n \times \{1\}$ *within* $\overset{\circ}{B}{}_{\epsilon_r}^{n+1}(x_i)$, one obtains a *finitely polygonal graph* g_C in C_ϵ^f which contains a near-equilibrium price path as desired. □

Appendix D

We now proceed by pointing out the reasons why it is generally necessary to approximate a joining equilibrium component by near-equilibrium paths in order to get a *nicely behaved* path in the homotopy space. Let us use the notion of a 'nicely behaved path' for the moment in the intuitive geometric sense which means that a particle moving along a nicely behaved path moves in a highly regular manner. Particularly, there should not occur any complicated movements like oscillations for instance. Thus, a finitely piecewise linear near-equilibrium path is a *prototype* of a nicely behaved path.

Let us now look at the behavior of joining equilibrium components. First and foremost a joining equilibrium component *need not be path connected.* In other words, it may for instance contain parts like the closure of the graph of $\sin 1/x$. We will give an example of an evolution producing this below. However, even if a joining equilibrium component is path connected, it may well happen that some of its points can only be connected by paths with infinitely long arcs caused by infinitely many oscillations.

Another example of a path whose arc is of finite length though it undergoes infinitely many oscillations is given by a "saw tooth path". It consists of infinitely many segments whose lengths can be estimated from above by the terms of a sequence which generates a convergent series (Figure 5).

Unfortunately, any of these complications actually *can occur* in the equilibrium set of an evolution. They even *cannot be removed by additional differentiability conditions* on the evolution. The following example makes this intuitive.

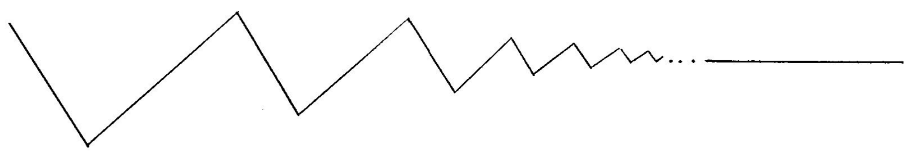

Figure 5

ζ_0 is a smooth function with a linear part over $[y, z]$ (use the function $x \mapsto \begin{cases} 0, & x \leq 0 \\ e^{-1/x^2}, & x > 0 \end{cases}$ at the bends $\zeta_0(y)$ and $\zeta_0(z)$). Actually, the following movement of ζ_0 yields a *smooth* one-parametrization $(\zeta_s)_{s\in[0,1]} : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}$; ζ_s is linear over $[y, z]$ for any s, and $\zeta_s(y)$ performs a damped oscillation whose time path looks like $x \cdot \sin \frac{1}{x}$. If $\zeta_s(z)$ correspondingly oscillates in counter-rhythm, this results in a *smoothly oscillating movement* $(\zeta_s)_{s\in[0,1]}$ with final state ζ_1 as in Figure 6. Thus, the trace $(G_s)_{s\in[0,1]}$ of the oscillating unique zero in the homotopy space $\mathbb{R}_+ \times [0, 1]$ *looks like the closure of the graph of* $\sin \frac{1}{x}$.

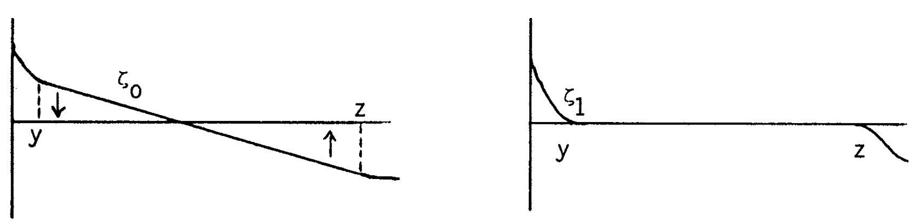

Figure 6

From the construction of the last part of the proof of Theorem 1 in Appendix C follows immediately

Theorem 2:

If at least one joining equilibrium component of an evolution $(\zeta_s)_{s\in[0,1]}$ *is even path connected, then for any* $\epsilon > 0$ *there is an* ϵ*-near equilibrium price path for* $(\zeta_s)_{s\in[0,1]}$ *which particularly also* ϵ*-approximates every* s*-state equilibrium set.*

Proof. Just *exclude* from the construction of the finitely polygonal graph g_C in the final part of the proof of Theorem 1 all segments $\overline{x_i x_j}$ with the following property: the endpoints x_i and x_j cannot be connected by a path which lies *in* C *and in* $\overset{\circ}{B}{}^{n+1}_{\epsilon_r}(x_i) \cup \overset{\circ}{B}{}^{n+1}_{\epsilon_r}(x_j)$. □

Figure 7 shows an example of a segment which will be excluded.

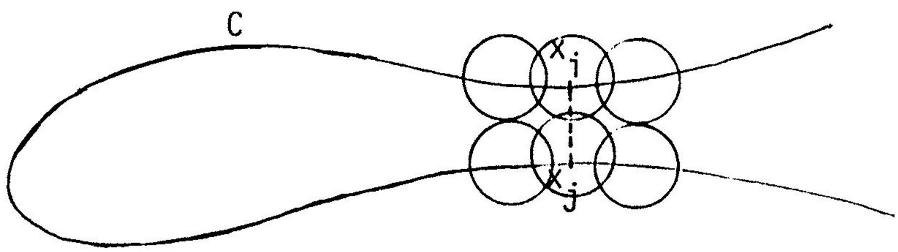

Figure 7

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