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ON THE EXISTENCE OF PURE STRATEGY EQUILIBRIA IN LARGE GENERALIZED GAMES WITH ATOMIC PLAYERS

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ABSTRACT. We consider a game with a continuum of players where only a finite number of them are atomic. Objective functions and admissible strategies may depend on the actions chosen by atomic players and on aggregate information about the actions chosen by non-atomic players. Only atomic players are required to have convex sets of admissible strategies and quasi-concave objective functions.

We prove the existence of a pure strategy Nash equilibria. Thus, we extend to large generalized games with atomic players the results of equilibrium existence for non-atomic games of Schemeidler (1973) and Rath (1992). We do not obtain a pure strategy equilibrium by purification of mixed strategy equilibria. Thus, we have a direct proof of both Balder (1999, Theorem 2.1) and Balder (2002, Theorem 2.2.1), for the case where non-atomic players have a common non-empty set of strategies and integrable bounded codification of action profiles.

Our main result is readily applicable to many interesting problems in general equilibrium. As an application, we extend Aumann (1966) result on the existence of equilibrium with a continuum of traders to a standard general equilibrium model with incomplete asset markets.

KEYWORDS. Generalized games - Non-convexities - Pure-strategy Nash equilibrium.
JEL CLASSIFICATION NUMBERS. C72, C62.

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1. INTRODUCTION

In a seminal paper, Schmeidler (1973) proved that in non-convex games with a continuum of players the set of pure strategy equilibria is non-empty provided that (i) all agents are non-atomic, and (ii) objective functions depend only on their own strategy and on the average of the actions chosen by the other players. Essentially, this last assumption convexifies the game, as the integral of any correspondence is a convex set (Aumann (1965)).

In this paper, we extend Schmeidler's result to large generalized games with a finite number of atomic players. In our framework, both objective functions and admissible strategies may depend on the strategies of atomic players and on messages which aggregate information about strategies chosen by non-atomic players (i.e., not necessarily on the average of these actions). By extending the proof given by Rath (1992, Theorem 2) of Schmeidler (1973) classical result, we provide a short and direct proof of the existence of pure Nash equilibria in large generalized game, without purifying a mixed strategy equilibrium. Our theorem is related with equilibrium existence theorems in Balder (1999, Theorem 2.1) and Balder (2002, Theorem 2.2.1). However, one of the merits of our proof consist of its simplicity, as it is based only on standard fixed point arguments in compact metric spaces.

Our theorem is general enough to cope with interesting applications. A natural application of our result is to general equilibrium theory. Essentially, to prove equilibrium existence it is usual to find bounds on endogenous variables and search for an equilibrium allocation as an equilibrium in an abstract generalized game. In this type of generalized games, consumers and firms maximize their objective functions taking prices as given, and there are atomic players that determine prices, asset returns, taxes or any other endogenous variables that are taken as given by consumers or firms. Thus, for equilibrium models where agents have non-convex choice sets or their objective functions are not necessarily quasi-concave, our main result may help researchers to find an equilibrium.[1]

The rest of the paper is organized as follows: in Section 2 we present our non-convex large generalized game and we prove the existence of a pure strategy Nash equilibrium. In Section 3, we discuss the relation of our result with the existing literature. Finally, we apply our main result to prove the existence of equilibrium in an incomplete markets model with non-convex preferences.

2. PURE STRATEGY EQUILIBRIA IN LARGE NON-CONVEX GENERALIZED GAMES

Let $\mathcal{G}(T, (K_t, \Gamma_t, u_t)_{t \in T}, h)$ be a generalized game with an infinite set of players $T = T_1 \cup T_2$, where T_1 is a compact metric space and T_2 is a finite set of atomic players. Also, there is a σ-algebra of subsets of T_1, namely $\mathbb{B}(T_1)$, and a finite non-atomic measure λ such that, $(T_1, \mathbb{B}(T_1), \lambda)$ is a

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[1]Recently, Poblete-Cazenave and Torres-Martínez (2012) apply of our result to prove equilibrium existence in a general equilibrium model with limited-recourse collateralized loans.

measure space. Each player $t \in T_1$ has a compact and non-empty action space $K_t \subset \widehat{K}$, where $\widehat{K}$ is a compact metric space and $\bigcap_{t \in T_1} K_t \neq \emptyset$. On the other hand, each player $t \in T_2$ has a compact, convex and non-empty action space $K_t \subset \widehat{K}_t$, where $\widehat{K}_t$ is a compact metric space.

A *profile of actions* for players in T_1 is given by a function $f : T_1 \to \widehat{K}$ such that $f(t) \in K_t$, for any $t \in T_1$. Since T_2 is finite, a *profile of actions* for the players in T_2 is a vector $a := (a_i; i \in T_2) \in \prod_{t \in T_2} \widehat{K}_t$ such that, $a_t \in K_t$, $\forall t \in T_2$. Let $\mathcal{F}(T_i)$ be the space of all profiles of actions of agents in T_i, with $i \in \{0, 1\}$. Also, given $t \in T_2$, let $\mathcal{F}_{-t}(T_2)$ be the set of profiles of actions $a_{-t} := (a_j; j \in T_2 \setminus \{t\})$ for players in $T_2 \setminus \{t\}$.

In $\mathcal{G}(T, (K_t, \Gamma_t, u_t)_{t \in T}, h)$ players do not necessarily advance the actions chosen by players in T_1. However, when making a decision, players will consider aggregate information of some characteristics of these actions. Thus, given an action profile of non-atomic players $f \in \mathcal{F}(T_1)$, each player in T will only take into account, for strategic purposes, aggregated information coded through the message $m(f) := \int_{T_1} h(t, f(t)) d\lambda$, where $h : T_1 \times \widehat{K} \to \mathbb{R}^l$ is a continuous function.

Since we want to concentrate on action profiles for which messages are well defined, we say that f is a *strategic profile* of players in T_1 if both $f \in \mathcal{F}(T_1)$ and $h(\cdot, f(\cdot))$ is a measurable function from T_1 to $\mathbb{R}^l$. Measurability restrictions are not necessary over the behavior of atomic players. For this reason, the set of strategic profiles of players in T_2 coincides with $\mathcal{F}(T_2)$.

The set of messages associated with strategic profiles of non-atomic players is given by

$$M = \left\{ \int_{T_1} h(t, f(t)) d\lambda : \quad f \in \mathcal{F}(T_1) \quad \wedge \quad h(\cdot, f(\cdot)) \text{ is measurable} \right\} \subset \mathbb{R}^l,$$

which is non-empty, because $\bigcap_{t \in T_1} K_t$ is a non-empty set and h is a continuous function. Also, since $\widehat{K}$ and T_1 are compact metric spaces, for any profile of actions $f : T_1 \to \widehat{K}$ the function $h(\cdot, f(\cdot)) : T_1 \to \mathbb{R}^l$ is bounded. As T_1 has finite measure, if $h(\cdot, f(\cdot))$ is measurable, then it is integrable. For this reason, in the definition of M we only require measurability of $h(\cdot, f(\cdot))$.

In our game, the messages about the strategic profiles of players in T_1 jointly with the strategic profiles of players in T_2 may restrict the set of admissible strategies available for a player $t \in T$. That is, given a vector $(m, a) \in M \times \mathcal{F}(T_2)$ the strategies available for a player $t \in T_1$ are given by a set $\Gamma_t(m, a) \subset K_t$, where $\Gamma_t : M \times \mathcal{F}(T_2) \twoheadrightarrow K_t$ is a continuous correspondence with non-empty and compact values. Analogously, given $(m, a_{-t}) \in M \times \mathcal{F}_{-t}(T_2)$, the set of strategies available for a player $t \in T_2$ is $\Gamma_t(m, a_{-t}) \subset K_t$, where $\Gamma_t : M \times \mathcal{F}_{-t}(T_2) \twoheadrightarrow K_t$ is a continuous correspondence with non-empty, compact and convex values. We refer to correspondences $(\Gamma_t; t \in T)$ as correspondences of admissible strategies.

Given a set A, let $\mathcal{U}(A)$ be the collection of continuous functions $u : A \to \mathbb{R}$. Assume that $\mathcal{U}(A)$ is endowed with the sup norm topology. We suppose that each player $t \in T_1$ has an objective function

4

$u_t \in \mathcal{U}(\widehat{K} \times M \times \mathcal{F}(T_2))$, and each player $t \in T_2$ has an objective function $u_t \in \mathcal{U}(M \times \mathcal{F}(T_2))$ which is quasi-concave in its own strategy. Finally, we assume that the mapping $U : T_1 \to \mathcal{U}(\widehat{K} \times M \times \mathcal{F}(T_2))$ defined by $U(t) = u_t$ is measurable.[2]

DEFINITION. *A (pure strategy) Nash equilibrium of the generalized game* $\mathcal{G}(T, (K_t, \Gamma_t, u_t)_{t \in T}, h)$ *is given by strategic profiles* (f^*, a^*) *such that,*

$$u_t(f^*(t), m(f^*), a^*) \geq u_t(f(t), m(f^*), a^*), \quad \forall f(t) \in \Gamma_t(m^*, a^*), \ \forall t \in T_1;$$

$$u_t(m(f^*), a_t^*, a_{-t}^*) \geq u_t(m(f^*), a_t, a_{-t}^*), \forall a_t \in \Gamma_t(m^*, a_{-t}^*), \ \forall t \in T_2.$$

In our definition of Nash equilibrium, every agent maximizes his objective function, while in Balder (1999, 2002) and Rath (1992) almost everyone maximizes. However, since objective functions are continuous and action spaces compact, given an equilibrium for any of the games studied in these articles, it is always possible to change the allocations associated with the set of non-atomic players that do not maximize, giving to each of them an optimal strategy, without changing the integrability of the action profile or the value of messages.

THEOREM 1. *Any generalized game* $\mathcal{G}(T, (K_t, \Gamma_t, u_t)_{t \in T}, h)$ *has a pure strategy Nash equilibrium.*

PROOF. We divide the proof into five steps.

(1) *The space of messages $M \subset \mathbb{R}^l$ is non-empty, compact and convex.*

As notices in the previous section, M is non-empty as $\bigcap_{t \in T_1} K_t \neq \emptyset$. Essentially, if we fix $k \in \bigcap_{t \in T_1} K_t$, the function $g : T_1 \to \widehat{K}$ defined by $g(t) = k$ for any $t \in T_1$ belongs to $\mathcal{F}(T_1)$ and $h(\cdot, g(\cdot))$ is trivially measurable. Therefore, $\int_{T_1} h(t, g(t)) d\lambda$ is well defined and M is non-empty.

The set M is convex because the integral of a correspondence in a non-atomic measurable space is a convex set (see Aumann (1965)): consider the correspondence $Q : T_1 \twoheadrightarrow \mathbb{R}^l$ defined by $Q(t) = h(t, K_t)$, for any $t \in T_1$. Then $M = \int_{T_1} Q(t) d\lambda$ is convex.[3]

Let $\widetilde{Q} : T_1 \twoheadrightarrow \mathbb{R}^l$ be the correspondence defined by $\widetilde{Q}(t) = h(T_1, \widehat{K})$, for any $t \in T_1$. Then $M = \int_{T_1} Q(t) d\lambda \subset \int_{T_1} \widetilde{Q}(t) d\lambda = \text{convexhull}(h(T_1, \widehat{K}))$. Therefore, since h is continuous, M is a subset of a compact set. Thus, it remains to prove that M is closed. Let $\{m_k\}_{k \in \mathbb{N}} \subset M$ be a sequence that converges to a vector $m \in \mathbb{R}^l$. Since $m_k \in M$, $m_k = \int_{T_1} h_k(t) d\lambda$, where $h_k : T_1 \to \mathbb{R}^l$ is a

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[2]Suppose that there is a finite number of types on the set of non-atomic agents, T_1. That is, there is a finite partition of T_1 into measurable sets $\{I_1, \ldots, I_r\}$ such that, two players belonging into the same element of the partition are identical. In this case, the restriction about measurability of U is trivially satisfied.

[3]This follow immediately from the definition of integral of a correspondence and the fact that we do not require action profiles to be measurable.

mesurable function and $h_k = h(\cdot, f_k(\cdot))$ for some $f_k \in \mathcal{F}(T_1)$. For each t, $\{h(t, f_k(t))\}_{k \in \mathbb{N}} \subset Q(t)$, which is a compact set. Thus, every limit point of $\{h_k(t)\}_{k \in \mathbb{N}}$ is contained in $Q(t)$. Also, since h is continuous, T_1 is compact, and $\cup_{t \in T_1} K_t \subset \widehat{K}$, it is easy to see that $\{h_k\}_{k \in \mathbb{N}}$ is uniformly bounded by an integrable function. By Aumann (1976), the limit point of $\int_{T_1} h_k(t) d\lambda$ belongs to $\int_{T_1} Q(t) d\lambda$. Therefore, the space of messages is compact.

(2) *Best-reply correspondences are closed with non-empty and compact values.*

For any $t \in T_1$, define the best-reply correspondence $B_t : M \times \mathcal{F}(T_2) \twoheadrightarrow K_t$ as

$$B_t(m, a) = \text{argmax}_{f(t) \in \Gamma_t(m, a)} u_t(f(t), m, a).$$

Analogously, for any atomic player $t \in T_2$, the best-reply correspondence $B_t : M \times \mathcal{F}_{-t}(T_2) \twoheadrightarrow K_t$ is defined by $B_t(m, a_{-t}) = \text{argmax}_{a_t \in \Gamma_t(m, a_{-t})} u_t(m, a_t, a_{-t})$. It follows that, as a consequence of Berge's Maximum Theorem, best-reply correspondences have closed graph and non-empty compact values.

(3) *For any atomic player $t \in T_2$, his best-reply correspondence has convex values.*

A direct consequence of the fact that, for players in T_2, correspondences of admissible strategies are convex valued and objective functions are quasi-concave in the own strategy.

(4) *The correspondence $\Omega : M \times \mathcal{F}(T_2) \twoheadrightarrow M$ defined by $\Omega(m, a) = \int_{T_1} h(t, B_t(m, a)) d\lambda$ is closed and has non-empty and convex values.*

Given $(m, a) \in M \times \mathcal{F}(T_2)$, by identical arguments to those made by Rath (1992, pages 430-431), there is a measurable function $f \in \mathcal{F}(T_1)$ such that $f(t) \in B_t(m, a)$ for any $t \in T_1$. Since h is continuous, $h(\cdot, f(\cdot))$ is measurable and, therefore, Ω has non-empty values. The correspondence Ω has convex values, since for any $(m, a) \in M \times \mathcal{F}(T_2)$, the set $\Omega(m, a)$ is the integral of the correspondence $t \twoheadrightarrow h(t, B_t(m, a))$.

Fix $t \in T_1$. Since B_t has closed graph, the correspondence that associate to each $(m, a) \in M \times \mathcal{F}(T_2)$ the set $h(t, B_t(m, a))$ has closed graph too (a direct consequence of the continuity of the function h and the fact that $B_t(m, a) \subset \widehat{K}$ for any $(t, m, a) \in T_1 \times M \times \mathcal{F}(T_2)$).

On the other hand, since T_1 and $\widehat{K}$ are compact and h continuous, there is a bounded function $v : T_1 \to \mathbb{R}^l$ such that $-v(t) \leq h(t, f(t)) \leq v(t)$, for any $t \in T_1$, $f \in \mathcal{F}(T_1)$ and $\int_{T_1} v(t) d\lambda$ is finite. Therefore, the correspondence that associate to each $(m, a) \in M \times \mathcal{F}(T_2)$ the integral on T_1 of the correspondence $t \twoheadrightarrow h(t, B_t(m, a))$ has closed graph (a consequence of the main result in Aumann (1976)). In other words, Ω is closed.

(5) *The generalized game $\mathcal{G}(T, (K_t, \Gamma_t, u_t)_{t \in T}, h)$ has a pure strategy Nash equilibrium.*

Define $\Gamma : M \times \mathcal{F}(T_2) \twoheadrightarrow M \times \mathcal{F}(T_2)$ by $\Gamma(m, a) = (\Omega(m, a), (B_t(m, a_{-t}))_{t \in T_2})$. Then Γ is closed and has nonempty, convex and compact values. Therefore, applying Kakutani's Fixed Point

Theorem, we conclude that Γ has a fixed point, i.e. there exists $(m^*, a^*) \in M \times \mathcal{F}(T_2)$ such that $(m^*, a^*) \in \Gamma(m^*, a^*)$. That is, for some $f^* \in \mathcal{F}(T_1)$, $m^* = \int_{T_1} h(t, f^*(t)) d\lambda$ and $f^*(t) \in B_t(m^*, a^*)$, for any $t \in T_1$. Also, for any $t \in T_2$, $a_t^* \in B_t(m^*, a_{-t}^*)$. These properties ensure that (f^*, a^*) is a pure strategy Nash equilibrium of $\mathcal{G}(T, (K_t, \Gamma_t, u_t)_{t \in T}, h)$. $\qquad\square$

3. Discussion of related literature

Rath (1992, Theorem 2) result on games with compact action spaces is an elementary proof of Schmeidler (1973) classical result. On the other hand, Balder (1999, Theorem 2.1) and Balder (2002, Theorem 2.2.1) are generalizations of Schmeidler (1973) and Rath (1992) to generalized games. Our theorem is related with Balder (1999, 2002) but still, it extends Rath (1992) on some dimensions, because we consider generalized games, where admissible strategies and objective functions may depend on atomic players actions. Different to Balder (1999), we assume that non-atomic players has a non-empty set of common strategies.

There are generalizations of our theorem that are quite straightforward but we think they would obscure the elementary nature of our proof. For example, we could avoid the assumption that action spaces of non-atomic players share a common strategy and rather use an argument along Remark 8 in Rath (1992, page 432). Similar arguments to Remark 6 in Rath's article would allow us to avoid fixing a topology over the space of objective functions. On the other hand, we could also relax substantially the hypothesis of our coding function h. In particular, as in Balder (1999, 2002), we could assume that h is a vector valued function of Carathéodory functions.

Finally, we should emphasize that one of the main differences between our theorem and Balder's results is the fact that we assume sets of strategic profiles to be integrable bounded codifications of action profiles (i.e., are integrable functions—obtained by the codification of action profiles—with respect to a finite measure space). Integrability of action profiles is something which is clearly an ungrounded hypothesis in many applications. It makes necessary to bound actions, prove equilibrium existence and arguing somehow that bounds are innocuous (for example, by constructing a sequence of equilibria for less stringent bounds and then arguing that this sequence has a convergent subsequence that is also an equilibrium).

4. Application: Equilibria in non-convex incomplete markets economies

In this section we apply our main result to extend Aumann (1966) to economies with incomplete asset markets. The main point is to show that, even in economies with incomplete asset markets, one can get rid of the convexity assumption on preferences as long as there are many traders (a continuum of traders).

Consider an economy $\mathcal{E}$ with two periods, $t \in \{0, 1\}$, and uncertainty about the realization of the state of nature in the second period. There is a finite set S of possible states of nature that can be realized at $t = 1$, and $s = 0$ denotes the only state of nature at $t = 0$. Let $S^* = S \cup \{0\}$.

There is a finite set L of perfectly divisible and perishable commodities that can be demanded for consumption in spot markets at each state of nature in the second period. We denote by $p_s \in \mathbb{R}_+^L$ the commodity price vector at $s \in S$ and by $p \in \mathbb{R}_+^{L \times S}$ the vector of commodity spot prices in the economy. There exists a finite set J of nominal assets. One unit of asset $j \in J$ delivers a payment $N_{s,j}$ when state of nature $s \in S$ is reached. We denote Assets are available in a perfectly competitive spot market in the first period. Let $q \in \mathbb{R}^J$ be the unitary asset price vector at $t = 0$ and assume all assets are in zero net supply.

There is a set of agents $H = [0, 1]$–endowed with the Lebesgue measure–that want to reallocate their income through states of nature using financial assets. Given prices (p, q), each agent $t \in [0, 1]$ maximizes his utility $U^t : \mathbb{R}_+^{L \times S} \to \mathbb{R}$ (that represent his preferences over consumption) choosing an allocation in his budget set $B^t(p, q)$, defined as the set of consumption and financial allocations $(x, z) \in \mathbb{R}_+^{L \times S} \times \mathbb{R}^J$ that satisfy,

$$\sum_{j \in J} q_j z_j \leq 0; \qquad p_s x_s \leq p_s w_s^t + \sum_{j \in J} N_{s,j} z_j, \ \ \forall s \in S.$$

where $w^t := (w_s^t; s \in S) \in \mathbb{R}_+^{L \times S}$ is the initial endowment of commodities of agent $t \in [0, 1]$.

We assume that $(p, q) \in \triangle^S \times Q$, where $\triangle = \{p \in \mathbb{R}_+^L : \|p\| = 1\}$, $Q = \{q \in \mathbb{R}_+^J : \|q\| = 1\}$, and given $x \in \mathbb{R}^m$, $\|x\| \equiv \sum_{i=1}^m |x_i|$. As in the previous section, let $\mathcal{U}(\mathbb{R}_+^{L \times S})$ be the space of real valued continuous functions over $\mathbb{R}_+^{L \times S}$ with the sup norm topology.

DEFINITION. *A competitive equilibrium for the economy $\mathcal{E}$ is given by a vector of prices $(\overline{p}, \overline{q}) \in \triangle^S \times Q$, jointly with consumption and financial allocations $(\overline{x}^t, \overline{z}^t) \in \mathbb{R}_+^{L \times S} \times \mathbb{R}^J$ for each agent $t \in [0, 1]$, such that,*

(1) *For almost all $t \in [0, 1]$,*

$$(\overline{x}^t, \overline{z}^t) \in argmax_{(x,z) \in B^t(\overline{p}, \overline{q})} U^t(x);$$

(2) *Commodity and financial markets clear. That is,*

$$\int_{[0,1]} \overline{x}_s^t dt = \int_{[0,1]} w_s^t dt, \ \ \forall s \in S; \qquad \int_{[0,1]} \overline{z}_j^t dt = 0, \ \ \forall j \in J.$$

A standard technique for proving the existence of competitive equilibrium in convex economies with incomplete asset markets is to truncate the space of assets and commodities (see Geanakoplos and Polemarchakis (1986)), prove existence of equilibrium for this truncated economy and then, by

relaxing these bounds, show that the sequence of equilibriums of the bounded economies converge to an equilibrium of the original (unbounded) economy. We will follow the same approach on our context.

Let $K \subset \mathbb{R}_+^{S \times L} \times \mathbb{R}^J$. Given prices $(p, q) \in \triangle^S \times Q$, for each individual $t \in [0, 1]$, consider the truncated budget set $B^t(p, q; K) := B^t(p, q) \cap K$. A K-*truncated competitive equilibrium* for the economy $\mathcal{E}$ is given by prices and allocations $\left((\overline{p}, \overline{q}); (\overline{x}^t, \overline{z}^t)_{t \in [0,1]} \right)$ such that, agents maximize on the *truncated* budget constraints and commodity and financial markets clear.

The most important point of the next results is that, although utility functions are not necessarily quasi-concave, we can prove the existence of a truncated competitive equilibrium using our main result about existence of pure strategy equilibria in non-convex continuous generalized games with atomic players. Our prove is in the same spirit of Debreu (1952) prove of the existence of equilibrium in social games.

PROPOSITION. *Consider an economy $\mathcal{E}$ and let $K = [0, k_1]^{S \times L} \times [-k_0, k_0]^J$.*
Assume that the following conditions hold:

(a) *The mapping $w : [0, 1] \to \mathbb{R}_{++}^{L \times S}$ given by $w(t) = w^t$ is integrable and bounded. Thus, there is $W \in \mathbb{R}$ such that, $\|w^t\| \leq W$, $\forall t \in [0, 1]$.*

(b) *For any $t \in [0, 1]$, U^t is continuous and strictly increasing.*

(c) *The map $t \to U^t$ is measurable.*

(d) *Assets are not trivial: for any $j \in J$, $N_j := (N_{s,j}; s \in S) \in \mathbb{R}_+^S \setminus \{0\}$.*

(e) *$k_0 > W$ and $k_1 > W + k_0 \sum\limits_{(s,j) \in S \times J} N_{s,j}$.*

Then, there is a K-truncated competitive equilibrium for the economy $\mathcal{E}$.

PROOF. We divide the proof in three steps.

(A) Equilibrium existence in an abstract generalized game $\mathcal{G}_K$.

Consider a generalized game $\mathcal{G}_K = \mathcal{G}_K(T, (K_t, \Gamma_t, u_t)_{t \in T}, h)$ where the set of players is $T = [0, 1] \cup S^*$. For each player $t \in [0, 1]$, the space of actions is $K_t = K$. For the atomic player $s = 0$, let $K_0 = Q$ and for $s \in S$, let $K_s = \triangle$. We denote by (x^t, z^t) the actions of a player $t \in [0, 1]$, by q the actions of $s = 0$, and by p_s the actions of any player $s \in S$.

Let $h : K \to K$ be the identity function. Thus, the space of messages is

$$M = \left\{ \int_{[0,1]} (x^t, z^t) dt : \quad (x, z) : [0, 1] \to K \text{ measurable} \right\}.$$

The correspondence of admissible strategies of a player $t \in [0, 1]$, $\Gamma_t : M \times \triangle^S \times Q \twoheadrightarrow K$ is defined by $\Gamma_t(m, p, q) = B^t(p, q; K)$. For $s = 0$ define $\Gamma_s : M \times \triangle^S \twoheadrightarrow K_0$ by $\Gamma_s(m, p) = K_0$, and for any

$s \in S$ define $\Gamma_s : M \times \triangle^{(S-1)} \times Q \twoheadrightarrow K_s$ by $\Gamma_t(m, p_{-s}, q) = K_s$, where p_{-s} stands for the vector p with p_s deleted.

For any $t \in [0,1]$, $u_t = U^t$. Also, objective function for players $s \in S^*$ are given by,

$$U_s(m, p, q) = q \cdot \int_{[0,1]} z_j^t dt, \quad \text{if } s = 0;$$

$$U_s(m, p, q) = p_s \cdot \int_{[0,1]} (x_s^t - w_s^t) dt, \quad \text{if } s \in S.$$

By definition, players' action spaces are non-empty and compact. Assumption (a) ensures that, for any $t \in [0,1]$, the correspondence of admissible strategies Γ_t is continuous with non-empty and compact values. Since $\triangle$ and Q are non-empty, compact and convex, it follows that, for any $s \in S^*$, the correspondence Γ_s is continuous and have non-empty, convex and compact values. Objective functions are by hypothesis continuous and, for any $s \in S^*$, U_s is linear in its own strategy and, therefore, quasi-concave on this strategy. Therefore, it follows from Theorem 1 that the generalized game $\mathcal{G}_K(T, (K_t, \Gamma_t, u_t)_{t \in T}, h)$ has a pure strategy Nash equilibria.

(B) In any Nash equilibrium of $\mathcal{G}_K$, non-atomic players have binding budget constraints.

If for some $t \in [0,1]$ the first period budget constraint is non-binding, then $\overline{z}^t = (k_0, ..., k_0)$. Thus, it follows from hypothesis (e) that, $W = W \|\overline{q}\| < \sum_{j \in J} \overline{q}_j k_0 < 0$, a contradiction. Analogously, if for some non-atomic player t the budget constraint at state of nature $s \in S$ is non-binding, then $\overline{x}_s^t = (k_1, ..., k_1)$. In this case, $k_1 = \overline{p}_s \cdot \overline{x}_s^t < \overline{p}_s \cdot w_s^t + \sum_{j \in J} N_{s,j} \overline{z}_j^t \leq \sum_{l \in L} w_{s,l}^t + \sum_{j \in J} N_{s,j} k_0 < k_1$, which is a contradiction.

(C) Any Nash equilibrium of $\mathcal{G}_K$ is a K-truncated competitive equilibrium of $\mathcal{E}$.

Let $\left((\overline{p}, \overline{q}); (\overline{x}^t, \overline{z}^t)_{t \in [0,1]}\right)$ be a pure strategy Nash equilibrium of $\mathcal{G}_K(T, (K_t, \Gamma_t, u_t)_{t \in T}, h)$. As a consequence of the previous step, it follows that,

$$q \int_{[0,1]} \overline{z}^t dt \quad \leq \quad \overline{q} \int_{[0,1]} \overline{z}^t dt = 0, \quad \forall q \in Q.$$

Evaluating inequality above in the canonical vectors of $\mathbb{R}_+^J$–which belongs to Q–we have that,

$$\int_{[0,1]} \overline{z}_j^t dt \leq 0, \quad \forall j \in J.$$

Thus, for any $s \in S$, we have that

$$\overline{p}_s \int_{[0,1]} (\overline{x}_s^t - w_s^t) dt = \sum_{j \in J} N_{s,j} \int_{[0,1]} \overline{z}_j^t dt \leq 0.$$

Therefore,

$$p_s \int_{[0,1]} (\overline{x}_s^t - w_s^t) dt \leq 0, \quad \forall (s, p_s) \in S \times \triangle,$$

which implies that, for any $(l, s) \in L \times S$,

$$\int_{[0,1]} (\overline{x}_{s,l}^t - w_{s,l}^t) dt \leq 0.$$

On the other hand, since for any $t \in [0, 1]$ the utility function U^t is strictly increasing, commodity and asset prices are strictly positive. Indeed, if there is $l \in L$ such that $\overline{p}_{s,l} = 0$, for some $s \in S$, then each player $t \in [0, 1]$ set $\overline{x}_{s,l}^t = k_1$ and, therefore, $k_1 = \int_{[0,1]} \overline{x}_{s,l}^t dt \leq \int_{[0,1]} w_{s,l}^t dt \leq W < k_1$, a contradiction. Analogously, if $\overline{q}_j = 0$ for some $j \in J$, then each agent $t \in [0, 1]$ chose $\overline{z}_j^t = k_0$ and, therefore, $k_0 = \int_{[0,1]} \overline{z}_j^t dt \leq 0$, a contradiction.

Since $(\overline{p}, \overline{q}) \gg 0$ and individuals' budget constraints are binding at each state of nature $s \in S^*$, it follows that,

$$\int_{[0,1]} (\overline{x}_s^t - w_s^t) dt = 0, \ \ \forall s \in S; \qquad\qquad \int_{[0,1]} \overline{z}_j^t dt = 0, \ \ \forall j \in J.$$

We conclude that the pure strategy equilibrium of $\mathcal{G}_K(T, (K_t, \Gamma_t, u_t)_{t \in T}, h)$ is a K-truncated competitive equilibrium for the economy $\mathcal{E}$. $\qquad\square$

Departing from the previous Proposition, we will prove the existence of a competitive equilibrium for $\mathcal{E}$ using an asymptotic argument. This is done by calculating a limit of K-truncated competitive equilibria when the size of K increases. To apply these technics, we need to ensure that financial position are uniformly bounded from below (bounded short-sales).

In models with nominal assets and finitely many agents, short-sales can be endogenously bounded, using second period budget constraints and monotonicity of preferences. Indeed, optimal portfolios are the solution of a linear system which depends of prices and consumption allocations, parameters that are in compact sets when there is a finite number of agents.

However, the presence of a continuum of agents in our model, implies that equilibrium consumption allocations are not necessarily (ex-ante) uniformly bounded. For these reason, we require an additional condition over the financial structure, a kind of non-arbitrage requirement: independently of financial prices, agents can not have access to unbounded resources at first period without promise unbounded payments at second period (see the statement of Theorem 2).

REMARK. As will become clear in the proof of Theorem 2, equilibrium existence can be ensured without any non-arbitrage requirement over the financial structure, provided that exogenous short-sale constraints be imposed. In any case, in our model we can allow for assets whose payments continuously depends on prices (i.e. real assets or derivatives). However, for simplicity of notation, we restrict our atention to nominal securities.

THEOREM 2. *Under Assumptions (a)-(e), assume that asset payments $N = (N_{s,j})_{(s,j) \in S \times J}$ avoids arbitrage opportunities, in the sense that,*

$$\forall a > 0 \, \exists b > 0 : Nz \geq -a(1, \ldots, 1) \implies \min_{q \in Q} qz \geq -b.$$

Then, there exists a competitive equilibrium of $\mathcal{E}$.

PROOF. Given $n \in \mathbb{N}$, let $K_n = [0, k_n]^{S \times L} \times [-n, n]^J$, where $k_n = 2 \left(W + n \sum_{s,j} N_{s,j} \right)$. It follows from Proposition 1 that, for any $n > W$ there is a K_n-truncated equilibrium $\left((p^n, q^n), (x_n^t, z_n^t)_{t \in [0,1]} \right)$.

On the one hand, there is $\hat{z} > 0$ such that, $z_{n,j}^t > -\hat{z}$, $\forall (t, j) \in [0, 1] \times J$, $\forall n > W$.[4] Thus, the sequence of integrable functions $\{ g_n : [0, 1] \to \mathbb{R}^{L \times S} \times \mathbb{R}^J \}_{n > W}$ defined by $g_n(t) = (x_n^t, z_n^t)$ is uniformly bounded from below by an integrable function. On the other hand, since $\Delta^S \times Q$ is compact, there is a subsequence $\{ (p^{n_k}, q^{n_k}) \}_{k \in \mathbb{N}}$ of $\{ (p^n, q^n) \}_{n > W}$ which converges to a vector of prices $(\tilde{p}, \tilde{q}) \in \Delta^S \times Q$.

Given a subset of an Euclidean space, $A \in \mathbb{R}^m$, let $\mathrm{CL}(A)$ be the set of cluster points of A. It follows from considerations above that, applying the weak version of multidimensional Fatou's Lemma (see Hildenbrand (1974, page 69)), we can find allocations $(\tilde{x}^t, \tilde{z}^t)_{t \in [0,1]} \subset \mathbb{R}_+^{L \times S} \times \mathbb{R}^J$ such that,

(i) For any $(l, j) \in L \times J$,

$$\left(\int_{[0,1]} \tilde{x}_l^t dt, \int_{[0,1]} \tilde{z}_j^t dt \right) \leq \lim_{k \to +\infty} \left(\int_{[0,1]} x_{n_k,l}^t dt, \int_{[0,1]} z_{n_k,j}^t dt \right) = \left(\int_{[0,1]} w_l^t dt, 0 \right).$$

(ii) There is a full measure set $H_1 \subseteq [0, 1]$ such that, for any $t \in H_1$,

$$(\tilde{x}^t, \tilde{z}^t) \in \mathrm{CL}(\{ (x_{n_k}^t, z_{n_k}^t) \}_{k \in \mathbb{N}}).$$

CLAIM A. *For any $t \in H_1$, the allocation $(\tilde{x}^t, \tilde{z}^t)$ belongs on the budget set $B^t(\tilde{p}, \tilde{q})$ and it is an optimal choice at prices $(\tilde{p}, \tilde{q})$.*

Proof. Fix $t \in H_1$. The allocation $(\tilde{x}^t, \tilde{z}^t) \in B^t(\tilde{p}, \tilde{q})$, a consequence of item (ii) above and the closed graph property of B^t. Let $N_t \subset \mathbb{N}$ be a set of indexes such that $\{ (x_{n_k}^t, z_{n_k}^t) \}_{k \in N_t}$ converges to $(\tilde{x}^t, \tilde{z}^t)$. Given any allocation $(x^t, z^t) \in B^t(\tilde{p}, \tilde{q})$, the lower hemicontinuity of B^t ensures that there is a sequence $\{ (\dot{x}_{n_k}^t, \dot{z}_{n_k}^t) \}_{k \in N_t}$ which converges to (x^t, z^t) such that, for any $k \in N_t$, $(\dot{x}_{n_k}^t, \dot{z}_{n_k}^t) \in B^t(p^{n_k}, q^{n_k})$. Thus, for any $k \in N_t$, $U^t(\dot{x}_{n_k}^t) \leq U^t(x_{n_k}^t)$. Taking the limit as k goes to

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[4] For any $n > W$ and $t \in [0, 1]$, it follows from second period budget constraints that, $-(p^n w_s^t; s \in S) \leq Nz_n^t$. Therefore, Assumption (a) guarantees that $-W(1, \ldots, 1) \leq Nz_n^t$. Thus, as for any $z \in \mathbb{R}^J$, $\min_{q \in Q} qz = \min_{j \in J} z_j$, the non-arbitrage condition imposed in the statement of Theorem 2 guarantees that there exists $\hat{z} > 0$ such that $z_{n,j}^t \geq -\hat{z}$. Since this lower bound is independent of (n, t), $\{ z_n^t \}_{t \in [0,1]}$ is a bounded from below subset of $\mathbb{R}^J$.

infinity, we obtain that $U^t(x^t) \leq U^t(\tilde{x}^t)$. That is, $(\tilde{x}^t, \tilde{z}^t)$ is an optimal choice at prices $(\tilde{p}, \tilde{q})$. $\quad\square$

It follows from Claim A and the monotonicity of preferences (Assumption (b)) that $(\tilde{p}, \tilde{q}) \gg 0$. Thus, there exists $\epsilon > 0$, $m^* \in \mathbb{N}$, such that, $(\tilde{p}, \tilde{q}) \gg \epsilon(1, \ldots, 1)$ and, for any $n_k > m^*$, we have $\|(\tilde{p}, \tilde{q}) - (p^{n_k}, q^{n_k})\| < \epsilon$. Particularly, for any $n_k > m^*$, $(p^{n_k}, q^{n_k}) \gg 0$.

Fix $t \in [0, 1]$. It follows from the first period budget constraint that,

$$-\hat{z} < z_{n_k, j}^t \leq \frac{\hat{z}(\#J - 1)}{q_j^{n_k}} < Z := \max_{j \in J} \frac{\hat{z}(\#J - 1)}{\tilde{q}_j - \epsilon}, \quad \forall n_k > m^*, \forall j \in J.$$

Also, second period budget constraints ensure that, for any $s \in S$,

$$0 \leq x_{n_k, s, l}^t \leq \frac{W + \sum_{j \in J} N_{s,j} Z}{p_l^{n_k}} < \frac{W + \sum_{j \in J} N_{s,j} Z}{\tilde{p}_l - \epsilon}, \quad \forall n_k > m^*, \forall l \in L.$$

It follows that $\{g_{n_k}(t)\}_{n_k > m^*}$ is bounded. Moreover, since the upper bound of $\{g_{n_k}(t)\}_{n_k > m^*}$ does not depend on the identity of $t \in [0, 1]$, we conclude that $\{g_{n_k}\}_{n_k > m^*}$ are uniformly integrable functions (see Hildenbrand (1974, page 52)). Therefore, applying the strong version of Fatou's Lemma (see Hildenbrand (1974, page 69)), we obtain allocations $(\overline{x}^t, \overline{z}^t)_{t \in [0,1]} \subset \mathbb{R}_+^{L \times S} \times \mathbb{R}^J$ such that,

(iii) For any $(l, j) \in L \times J$,

$$\left(\int_{[0,1]} \overline{x}_l^t dt, \int_{[0,1]} \overline{z}_j^t dt \right) = \lim_{k \to +\infty} \left(\int_{[0,1]} x_{n_k, l}^t dt, \int_{[0,1]} z_{n_k, j}^t dt \right) = \left(\int_{[0,1]} w_l^t dt, 0 \right).$$

(iv) There is a full measure set $H_2 \subseteq [0, 1]$ such that, for any $t \in H_2$,

$$(\overline{x}^t, \overline{z}^t) \in \mathrm{CL}(\{(x_{n_k}^t, z_{n_k}^t)\}_{k \in \mathbb{N}}).$$

It follows from item (iii) that allocations $(\overline{x}^t, \overline{z}^t)_{t \in [0,1]} \subset \mathbb{R}_+^{L \times S} \times \mathbb{R}^J$ satisfy market clear conditions of equilibrium definition. Moreover, analogous arguments to those made in Claim A ensure that, for any agent $t \in H_2$, $(\overline{x}^t, \overline{z}^t)$ belongs on the budget set $B^t(\tilde{p}, \tilde{q})$ and it is an optimal choice at prices $(\tilde{p}, \tilde{q})$. That is, $\left((\tilde{p}, \tilde{q}), (\overline{x}^t, \overline{z}^t)_{t \in [0,1]} \right)$ is an equilibrium of $\mathcal{E}$. $\quad\square$

5. Concluding remarks

We analyzed the existence of pure strategy Nash equilibria in large non-convex generalized games. Inspired by Rath (1992), we used simple arguments of mathematical analysis to obtain our result. Pure strategy Nash equilibria appear as fixed points of a convex valued correspondence, rather than by purification of mixed strategy equilibria, as in Balder (1999, 2002).

Our result is a tool to prove general equilibrium existence in non-convex economies. To illustrate this possibility, we applied our main result to prove existence of equilibria in a non-convex incomplete markets economy. Recently, Poblete-Cazenave and Torres-Martínez (2012) also applied Theorem 1 to analyze the existence of equilibrium in economies with limited-recourse collateralized loans.

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