Analytical Approximations for the Critical Stock Prices of American Options: A Performance Comparison

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Abstract

Many efficient and accurate analytical methods for pricing American options now exist. However, while they can produce accurate option prices, they often do not give accurate critical stock prices. In this paper, we propose two new analytical approximations for American options based on the quadratic approximation. We compare our methods with existing analytical methods including the quadratic approximations in Barone-Adesi and Whaley (1987) and Barone-Adesi and Elliott (1991), the lower bound approximation in Broadie and Detemple (1996), the tangent approximation in Bunch and Johnson (2000), the Laplace inversion method in Zhu (2006b), and the interpolation method in Li (2008). Both of our methods give much more accurate critical stock prices than all the existing methods above.

JEL Classification: C02; C63; G13

Keywords: American option; Analytical approximation; Critical stock price

I. Introduction

American options differ from European options in that one can exercise the option anytime before maturity. What makes them interesting is the existence of the critical stock price boundary, which separates the state space into the continuation region and the exercise region. Earliest studies on American options include McKean (1965), Samuelson (1967), and Merton (1973). Most theoretical works on American options up to now have focused on various aspects of the critical boundary. These include: van Moerbeke (1976), Kim (1990), Jacka (1991), Carr, Jarrow and Myneni (1992), Myneni (1992), Little, Pant and Hou (2002), Evans, Kuske and Keller (2002), and Zhu (2006a).

On the computational front, more than a dozen well-known methods now exist, as well as many others that have not been widely adopted. A short but relatively recent survey is Barone-Adesi (2005). Readers interested in the relative performances of different methods on pricing American options are referred to the following papers: Broadie and Detemple (1996), AitSahlia and Carr (1997), Ju and Zhong (1999), and more recently, Pressacco, Gaudenzi, Zanette and Ziani (2008). However, it is fair to say that each method has its own strengths and weaknesses, and no single method has emerged as *the* method of choice for American options under all circumstances. This is because these methods differ in many dimensions such as: accuracy in prices, critical stock prices, or greeks; speed; robustness and stability; mathematical sophistication needed; programming complexity; availability of closed-form prices, critical stock prices, or greeks; portability to other American-style derivatives; readiness to include jumps or stochastic volatility; etc. For example, while the maturity randomization method in Carr (1998) is more accurate than the quadratic approximation in Barone-Adesi and Whaley (1987), it requires a relatively high level of mathematical sophistication for full appreciation, and is also much slower than the quadratic approximation. As a second example, while the least-square Monte Carlo method of Longstaff and Schwartz (2001) might not be the most efficient method for pricing American options in the Black-Scholes model, it can be easily modified to handle American-type derivatives under general diffusion processes.

In contrast to the vast amount of attention paid by theorists to the critical stock prices, existing literature on numerical aspects of American options has focused almost exclusively on various methods' accuracy in pricing options. The only notable exception is Bunch and Johnson (2000), which positions itself as a paper focusing more on the critical stock price. While the option price understandably should be the quantity of the most interest, the critical stock price is important for at least two obvious reasons. First, an investor who holds an American option needs to know whether it is now optimal for him to exercise or not. Second, the option price can be computed from the critical stock boundary through the early exercise premium representation and many of its variants. Thus, in the current paper we take a different angle by focusing on the performance of different methods in approximating the critical stock prices, instead of the option prices.

Admittedly, the American option price and the critical stock price are two very closely-related quantities. In fact, every single method for pricing American options would have to look at the critical stock price boundary. However, it is very important for researchers to realize that although deeply related, approximating the American option price and approximating the current critical stock price are two different problems. In principle, one could have a very accurate approximation for the prices but whose performance on the current critical stock price is not as good. This is indeed the case for the EXP3 method in Ju (1998). A quick look at Figure 1 in Ju (1998) shows that EXP3 gives very inaccurate current critical stock price, by its design. On the other hand, in principle one could also have a very accurate approximation for the critical stock price that gives large errors in option prices. One simple example is an approximation in which one uses the critical stock price from very accurate methods such as a very fine binomial tree, but uses -1 as the price for the American option. This example is of course dull, but it illustrates the point. A somewhat less dull but far less obvious example is the quadratic approximation in Ju and Zhong (1999). Numerical experiments tell us that if one uses the true critical stock price in their method instead of the rough quadratic approximation, the method's accuracy actually often *decreases*.

As we will see in this paper, it is often more challenging to obtain accurate critical stock prices than the American option prices themselves. There are possibly two explanations. One skeptical view is that this is the case because most researchers have focused on the option prices rather than the critical stock prices. While there might be some truth in this view, the real explanation rests in the characteristics of the problems themselves. Finding the critical stock price of an American call option is equivalent to finding the (right-most) zero point of the early exercise premium. Now suppose American option prices (hence the early exercise premiums) are computed with some small error. This would put a narrow band around the early exercise premium curve. Graphically it is now clear that theoretically it is possible to have an extremely inaccurate critical stock price even if the early exercise premium is computed with an extremely small uniform error, because the slope of the early exercise premium with respect to current stock price is exactly zero to the left of the critical stock price. Another reason why the critical stock price problem is often more challenging is that if we rely on some root-finding algorithm such as Dekker-Brent, then we need to compute the American option prices repeatedly, making the computational cost of finding critical stock prices much higher.

To shift existing literature's focus to the critical stock price, we propose two new analytical approximations in this paper, $\texttt{QD}^+$ and $\texttt{QD}^*$. Both methods are based on the quadratic approximation of MacMillan (1986) and use more careful treatments for the high contact condition than in Barone-Adesi and Whaley (1987) and Ju and Zhong (1999). We then compare our methods with existing analytical methods on their accuracy for the critical stock prices. The methods we consider include the quadratic approximations in Barone-Adesi and Whaley (1987) and Barone-Adesi and Elliott (1991), the lower bound approximation in Broadie and Detemple (1996), the tangent approximation of the first passage probability in Bunch and Johnson (2000), the pseudo-steady state approximation of Laplace inversion method in Zhu (2006b), and the interpolation method in Li (2008). To our knowledge, this is the first paper that systematically looks at the performance of analytical approximations in terms of accuracy in the critical stock prices. We find that both of our methods give much more accurate critical stock prices than all the existing methods above.

Finally, we carry out a comparison on the accuracy of our methods in pricing options with the modified quadratic approximation in Ju and Zhong (1999). We find that both of our methods have about the same efficiency and the same accuracy in pricing American options as Ju and Zhong (1999), while producing much more accurate critical stock prices.

The current paper only considers American options under the standard Black-Scholes framework, which by now is widely regarded as being not very realistic. Thus it is important to understand why the work done in this paper is still important. The answer lies in the central role of the Black-Scholes theory in option pricing even though it is understood to be a crude model. For example, traders and academics frequently refer to the concept of implied volatility, which is "the incorrect number computed using the incorrect Black-Scholes model" as traders usually put it. Nonetheless, without such a "wrong" concept, our understanding of the financial market would be much hampered. Furthermore, to understand the pricing behavior of other more complicated models such as stochastic volatility models and jump diffusion models, researches frequently benchmark them against the Black-Scholes model. On a more pragmatic note, most option prices database provide Black-Scholes implied volatilities computed from American-style options usually by using binomial methods. A reliable and efficient method for computing American option prices (and hence implied volatilities) under the Black-Scholes model would be extremely useful.[1] Last but not the least, even if we would allow for some degree of inaccuracy for the critical stock prices in consideration of model risk, the accuracy of existing analytical methods is still quite limited, with percentage error often exceeding 1%.

Nonetheless, the above arguments should not be taken as excuses for not pursuing American option pricing under more realistic frameworks such as stochastic volatility models and jump diffusion models. On the contrary, analytical approximations for these models are much sought after. Unfortunately, increasing even just one degree of dimension from Black-Scholes to other models poses extreme difficulty. To appreciate the complexity, readers are referred to Detemple (2006), which devotes one whole chapter to multi-asset American options. Even for this special case of higher-dimensional American option pricing, the critical boundary could be quite complicated. For general multivariate American option pricing problems, as far as we know, there is no efficient and

[1] As a matter of fact, we have received quite a few phone calls from traders in Chicago asking me whether we could extend the algorithms in Li (2008, 2009) to American options.

reliable *analytical* approximations, although a large literature exists for numerical methods. The difficulty is that most one-dimensional analytical approximations rely on some approximations to the partial differential valuation equation, which cannot be readily generalized to higher dimensional case. It is very likely that researchers would have to explore other approaches. One possible approach is to construct analytical upper or lower bounds for the American option prices instead of approximations for the prices themselves.

The paper is organized as follows. Section II reviews the existing analytical approximations for the critical stock prices and then introduces two new analytical approximations based on the quadratic approximation. Section III carries out a careful numerical study on the performance of various methods on approximating the critical stock prices and finds that both of our methods give much more accurate critical stock prices than existing analytical methods. Section IV concludes.

II. Analytical Approximations for the Critical Stock Price

A. Existing Analytical Approximations for the Critical Stock Price

Given the extensive research that has been done on approximating American option prices, we will mainly focus on the issue of approximating the critical stock price under the Black-Scholes framework in this paper. What's more, we are only interested in analytical approximations of the current critical stock price. Here the word "analytical" should be interpreted as the one understood by researchers in finance. In particular, closed-form solutions are considered analytical, but so are solutions in which a root-finding algorithm such as Newton-Raphson or a simple one-dimensional numerical integration is required for just one or two times. However, methods in which a Newton-Raphson algorithm is called repeatedly are excluded. Thus, we exclude a large class of methods such as Geske and Johnson (1984), Huang, Subrahmanyam and Yu (1996), Ju (1998), Sullivan (2000), Kallast and Kivinukk (2003), Khaliqa, Vossb, and Kazmicand (2006), and Kim and Jang (2008). These methods often have their unique advantages, but they are less efficient for the purpose of computing the critical stock prices.

The reason we consider only analytical approximations is an obvious one. Without this requirement, any method that approximates the American option price can be turned into an approximation for the critical stock price. One would just compute a list of option prices with different initial stock prices and find the critical stock price such that the option should be exercised immediately. Most of the time, because of the lack of closed-form expressions for the option prices, one would have to use a numerical method such as the Dekker-Brent algorithm for the purpose of searching for the critical stock price. This will slow down the methods considerably because a bisection procedure is computationally expensive.

Although there are a lot of approximations for pricing American options, only a portion of them produce accurate critical stock prices at the same time. Still fewer are analytical methods. We take a comprehensive look at existing methods and come up with a short list of analytical approximations for the current critical stock price. They include: the initial guess of the critical stock price (`IG`) in Barone-Adesi and Whaley (1987), the quadratic approximation (`QD`) in Barone-Adesi and Whaley (1987) and Ju and Zhong (1999),[2] the refined quadratic approximation (`BE`) in Barone-Adesi and Elliott (1991), the lower bound approximation (`LB`) in Broadie and Detemple (1996),[3] the tangent approximation for first passage probability (`TA`) in Bunch and Johnson (2000), the pseudo-steady state inverse Laplace transform method (`PS`) in Zhu (2006b), and the interpolation (`IM`) method in Li (2008).

We consider only American put options. The problem of American call options can be reduced completely to that of American puts by the put-call symmetry (see McDonald and Schroder (1998)). We will use r and δ to denote the constant risk-free interest rate and the constant dividend-rate. The volatility is denoted as σ and time to maturity τ. Let the current stock price be S. We will use $p(S,\tau,K)$ to denote the price of the European put option with strike price K. That is,

$$p(S,\tau,K) = Ke^{-r\tau}\Phi(-d_2(S,\tau,K)) - Se^{-\delta\tau}\Phi(-d_1(S,\tau,K)), \tag{1}$$

where $\Phi(\cdot)$ is the cumulative normal distribution function, and

$$d_1(S,\tau,K) = \frac{\log(Se^{(r-\delta)\tau}/K)}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau}, \quad d_2(S,\tau,K) = d_1(S,\tau,K) - \sigma\sqrt{\tau}. \tag{2}$$

We will write S^*_τ for the critical stock price when the time to maturity is τ. Although simple closed-form expression for S^*_τ is not available, the asymptotic values for S^*_τ are actually known. Let $S^*_{0^+}$ be the theoretical asymptotic critical stock price immediately before maturity, and S^*_∞ the critical stock price for a perpetual American put option. Then they are given by

$$S^*_{0^+} = \min(K, rK/\delta), \tag{3}$$

and

$$S^*_\infty = \frac{K}{1-1/q_\infty}, \tag{4}$$

where

$$q_\infty = -\frac{1}{2}(N-1) - \frac{1}{2}\sqrt{(N-1)^2+4M}, \tag{5}$$

[2] Although Ju and Zhong (1999) use an improved formula for the option prices, they use exactly the same critical stock price as in MacMillan (1986), and Barone-Adesi and Whaley (1987). We will label the modified quadratic approximation in Ju and Zhong (1999) as `MQ` in our numerical study.

[3] The upper bound in Broadie and Detemple (1996) uses numerical integration and does not satisfy our definition for "analytical" approximation. Even if we allow the numerical integration in the upper bound, we would have to search the critical stock price using a slow method such as bisection because of the lack of a closed-form expression.

with

$$M = \frac{2r}{\sigma^2}, \quad N = \frac{2(r-\delta)}{\sigma^2}. \tag{6}$$

Below, we briefly discuss each of the analytical methods we are going to examine.

`IG`: (the modified initial guess in Barone-Adesi and Whaley (1987))

This is the initial guess suggested in Barone-Adesi and Whaley (1987) to feed into the Newton-Raphson algorithm for solving the critical stock price. The original form in Barone-Adesi and Whaley (1987) only applies to the case $r \geq \delta$ and would give very large errors when $r < \delta$. Our somewhat ad-hoc modification is the following:

$$S^*_\tau = S^*_{0+} + (S^*_\infty - S^*_{0+})\big(1 - e^{-\theta(\tau)}\big), \tag{7}$$

where the function $\theta(\tau)$ is chosen to be[4]

$$\theta(\tau) = \frac{K}{S^*_\infty - K}\Big((r-\delta)\tau + 2\sigma\sqrt{\tau}\Big). \tag{8}$$

`QD`: (the quadratic approximation in MacMillan (1986), Barone-Adesi and Whaley (1987))

This is a fairly popular method among practitioners due to its simplicity. To get the critical stock price S^*_τ, one solves the following equation

$$K - S^*_\tau = p(S^*_\tau, \tau, K) - \frac{S^*_\tau}{q_{\mathtt{QD}}}\Big(1 - e^{-\delta\tau}\Phi(-d_1(S^*_\tau, \tau, K)\Big), \tag{9}$$

where

$$q_{\mathtt{QD}} = -\frac{1}{2}(N-1) - \frac{1}{2}\sqrt{(N-1)^2 + 4M/h}, \tag{10}$$

with $h = 1 - e^{-r\tau}$.

`BE`: (the refined quadratic approximation in Barone-Adesi and Elliott (1991))

To determine S^*_τ, we still use equation (9) except that $q_{\mathtt{QD}}$ is now replaced by $q_{\mathtt{BE}}$ as follows:

$$q_{\mathtt{BE}} = -\frac{1}{2}(N-1) - \frac{1}{2}\sqrt{(N-1)^2 + 4(M + G(S^*_\tau, \tau))}, \tag{11}$$

with

$$G(S, \tau) = \frac{2}{\sigma^2}\frac{\Theta(S,\tau)}{K - S - p(S, \tau, K)}, \tag{12}$$

where $\Theta(S,\tau)$ is the greek Theta of the European put option, given by

$$\Theta(S,\tau) = rKe^{-r\tau}\Phi(-d_2(S,\tau,K)) - \delta S e^{-\delta\tau}\Phi(-d_1(S,\tau,K)) - \frac{\sigma S}{2\sqrt{\tau}}e^{-\delta\tau}\phi(d_1(S,\tau,K)). \tag{13}$$

The function $\phi(\cdot)$ in the above equation is the standard normal density function.

[4] Other choices have appeared in the literature. For example, Bjerksund and Stensland (1993) use the factor $S^*_{0+}/(S^*_\infty - S^*_{0+})$ instead of $K/(S^*_\infty - K)$. These choices do not make much difference, but our choice seems to produce smallest maximum absolute error.

`LB`: (the lower bound approximation in Broadie and Detemple (1996))

The lower bound approximation in Broadie and Detemple (1996) is based on approximating the American call option using a capped American call. To get S^*_τ for the American put option, we use the following put-call symmetry $S^*_\tau = K^2/L$, where L is the critical stock price of an American call option with interest rate δ and dividend rate r. Notice that the roles of r and δ have been switched. A lower bound for L can be obtained as the solution of the following equation

$$\begin{aligned} &2e^{-r\tau}(\beta-1)\Big(\Phi\big(d(L)-\sigma\sqrt{\tau}\big)-\Phi\big(d(K)-\sigma\sqrt{\tau}\big)\Big)-\frac{2e^{-\delta\tau}\beta K}{L}\Big(\Phi(d(L))-\Phi(d(K))\Big)\\ &+\left(1-\frac{2\alpha_-(L-K)}{L}\right)\Phi(-\gamma\sigma\sqrt{\tau})+\left(1-\frac{2\alpha_+(L-K)}{L}\right)\Phi(\gamma\sigma\sqrt{\tau})=0, \end{aligned} \tag{14}$$

with

$$\alpha_\pm=\frac{1}{2}(\beta\pm\gamma),\quad \beta=\frac{r-\delta}{\sigma^2}+\frac{1}{2},\quad \gamma=\frac{\sqrt{\beta^2+2\delta\sigma^2}}{\sigma^2}. \tag{15}$$

The function $d(x)$ is given by

$$d(x)=d_1(x,\tau,L)=\frac{\log(x/L)+\beta\sigma^2\tau}{\sigma\sqrt{\tau}}. \tag{16}$$

By put-call symmetry, the S^*_τ above gives an *upper* bound for the critical stock price of the American put option with interest rate r and dividend rate δ.

`TA`: (the tangent approximation for the first passage probability in Bunch and Johnson (2000))

This approximation has only been derived under the assumption $\delta=0$. The critical stock price is approximated as the solution to the following equation:

$$\log(S^*_\tau/K)=-\left(r+\frac{\sigma^2}{2}\right)\tau-g\sigma\sqrt{\tau}, \tag{17}$$

where

$$g=\pm\sqrt{2\log\left(\frac{\sqrt{\alpha}\sigma^2\exp(\alpha(r+\sigma^2/2)^2\tau/(2\sigma^2))}{2r(K/S^*_\tau)\log(K/S^*_\tau)}\right)}, \tag{18}$$

with[5]

$$\alpha=1-\frac{A}{1+\dfrac{(1+M)^2}{4}\sigma^2\tau},\quad \text{and}\quad A=\frac{1}{2}\left(\frac{M}{1+M}\right)^2. \tag{19}$$

[5]Notice that equation (27) in Bunch and Johnson (2000) contains a typo and does not agree with equation (A9) in that paper.

There is one more complexity in this approximation, that is, which sign we should use for g. Bunch and Johnson (2000) suggest that we first solve the following equation to get τ_0:

$$\tau_0 = \frac{2 \log \dfrac{2\sqrt{\alpha}}{M(1+M)\sigma^2\tau_0}}{\sigma^2(1+M)\Big(1 - \dfrac{\alpha}{4}(1+M)\Big)}. \tag{20}$$

The prescription is that if $\tau < \tau_0$, we should choose the positive root for g, and if $\tau > \tau_0$, we should choose the negative root for g. Unfortunately, this equation needs to be solved iteratively too.[6]

`PS`: (the pseudo-steady state approximation of Laplace transform in Zhu (2006b))

This method is based on the Laplace transform of the fundamental valuation partial differential equation. This method also assumes a zero dividend rate. The critical stock price is given by an inverse Laplace transform:

$$S^*_\tau = S^*_\infty + \frac{e^{-a^2\tau}K}{\pi} \int_0^\infty \frac{e^{-\tau\rho}}{a^2+\rho} e^{-f(\rho,b,\sqrt{\rho})} \sin(f(\rho,\sqrt{\rho},-b))\mathrm{d}\rho, \tag{21}$$

where for zero dividend rate, $S^*_\infty = MK/(1+M)$, and

$$f(\rho,x,y) = \frac{1}{b^2+\rho}\left(x \log\Big(\frac{\sqrt{a^2+\rho}}{M}\Big) + y \tan^{-1}\big(\sqrt{\rho}/a\big)\right) \tag{22}$$

with $a = (1+M)/2$, $b = (1-M)/2$, and $M = 2r/\sigma^2$.

`IM`: (the interpolation method in Li (2008))

This method is an extension of Johnson (1983) in which the price of the American put is approximated as an interpolation of the prices of two European put options, namely, $p(S,\tau,K)$ and $p(S,\tau,Ke^{r\tau})$. To get the critical stock price, one solves

$$K - S^*_\tau = Ap(S^*_\tau,\tau,Ke^{r\tau}) + (1-A)p(S^*_\tau,\tau,K), \tag{23}$$

where

$$A = \frac{1-\Phi(-d_1(S^*_\tau,\tau,K))e^{-\delta\tau}}{[\Phi(-d_1(S^*_\tau,\tau,Ke^{r\tau})) - \Phi(-d_1(S^*_\tau,\tau,K))]e^{-\delta\tau} - q_{\mathtt{IM}}[p(S,\tau,Ke^{r\tau}) - p(S,\tau,K)]/S^*_\tau}, \tag{24}$$

with

$$q_{\mathtt{IM}} \equiv \frac{\sigma^2 - 2(r-\delta)}{2\sigma^2} - \frac{1}{2\sigma^2}\sqrt{(\sigma^2 - 2(r-\delta))^2 + 8r\sigma^2/\Psi}. \tag{25}$$

[6] Bunch and Johnson (2000) suggest an approximation for τ_0 in their equation (25). However, we will use equation (20) in order to get higher accuracy.

The parameter Ψ is obtained through empirical fitting of the following parametric form

$$\Psi = 1 - \exp(-|k_1 r\tau + k_2 \delta\tau + k_3 \sigma\sqrt{\tau} + k_4 \min(r,\delta)\tau|). \tag{26}$$

The vector k suggested in Li (2008) is given by $k = (1.3856,\ -0.1251,\ 0.005418,\ -0.2546)$.

B. Two New Approximations: $\mathtt{QD}^+$ and $\mathtt{QD}^*$

We now introduce two new analytical approximations for the critical stock price of the American put option. The methods use similar pricing formula as in the quadratic approximation of Ju and Zhong (1999), but different approximations for the critical stock price.

We first quickly review Ju and Zhong (1999). Let $h = 1 - e^{-r\tau}$ and $V(S,\tau) = P(S,\tau,K) - p(S,\tau,K)$, where $P(S,\tau,K)$ denotes the price of an American put option with strike K and time to maturity τ. The starting point is to notice that in the continuation region of the American put option, $V(S,\tau)$ satisfies the fundamental valuation partial differential equation:

$$-\frac{\partial V}{\partial \tau} + (r-\delta)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{27}$$

Ju and Zhong (1999) try the solution form $V(S,\tau) = hA(h)(1+\epsilon(S))(S/S_\tau^*)^{q_{\mathtt{QD}}}$, for some functions $A(h)$ and $\epsilon(S)$. Notice that both S_τ^* and $q_{\mathtt{QD}}$ are functions of h, and therefore τ. Plugging the trial solution into the valuation PDE results in the following ordinary differential equation for ϵ,

$$S^2\frac{\mathrm{d}^2\epsilon}{\mathrm{d}S^2} + S(2q_{\mathtt{QD}} + N)\frac{\mathrm{d}\epsilon}{\mathrm{d}S} - (1-h)M(1+\epsilon)\left(\frac{A(h)'}{A(h)} + q'_{\mathtt{QD}}(h)\log(S/S_\tau^*) - q_{\mathtt{QD}}\frac{\mathrm{d}\log S_\tau^*}{\mathrm{d}h}\right) = 0. \tag{28}$$

By trying solution of the form $\epsilon(S) = B(\log(S/S_\tau^*))^2 + C\log(S/S_\tau^*)$, Ju and Zhong (1999) obtain the following approximation for the American put option price:

$$P(S,\tau,K) = p(S,\tau,K) + \frac{K - S_\tau^* - p(S_\tau^*,\tau,K)}{1 - b(\log(S/S_\tau^*))^2 - c\log(S/S_\tau^*)}(S/S_\tau^*)^{q_{\mathtt{QD}}} \tag{29}$$

if $S > S_\tau^*$ and $P(S,\tau,K) = K - S$ otherwise. Here the coefficients b and c are given by

$$b = \frac{(1-h)Mq'_{\mathtt{QD}}(h)}{2(2q_{\mathtt{QD}} + N - 1)}, \tag{30}$$

$$c = c_0 - \frac{(1-h)M}{2q_{\mathtt{QD}} + N - 1}\left(\frac{1 - e^{-\delta\tau}\Phi(-d_1(S_\tau^*,\tau,K))}{K - S_\tau^* - p(S_\tau^*,\tau,K)} + \frac{q_{\mathtt{QD}}}{S_\tau^*}\right)\frac{\mathrm{d}\log S_\tau^*}{\mathrm{d}h}, \tag{31}$$

with

$$c_0 = -\frac{(1-h)M}{2q_{\mathtt{QD}} + N - 1}\left(\frac{1}{h} - \frac{\Theta(S_\tau^*,\tau)e^{r\tau}}{r(K - S_\tau^* - p(S_\tau^*,\tau,K))} + \frac{q'_{\mathtt{QD}}(h)}{2q_{\mathtt{QD}} + N - 1}\right), \tag{32}$$

$$q'_{\mathtt{QD}}(h) = \frac{M}{h^2\sqrt{(N-1)^2 + 4M/h}}, \tag{33}$$

and Θ given in equation (13). Ju and Zhong (1999) do not give an approximate expression for $\mathrm{d}\log S^*_\tau/\mathrm{d}h$. However, notice that c_0 is already explicitly a function of S^*_τ.

To close the system, we need to use the high contact condition, which now becomes

$$\big(1 - e^{-\delta\tau}\Phi(-d_1(S^*_\tau, \tau, K))\big)S^*_\tau + (q_{\mathtt{QD}} + c)\big(K - S^*_\tau - p(S^*_\tau, \tau, K)\big) = 0. \tag{34}$$

The prescription given in Ju and Zhong (1999) is to omit the quantity c altogether in the above equation to compute the critical stock price. One easily sees that in this case, the above equation becomes identical to equation (9) in the original quadratic approximation. As we will see in our numerical study, this results in relatively large errors in the critical stock price. Once the critical stock price is computed, equation (29) then gives the American option price. Lacking an expression for $\mathrm{d}\log S^*_\tau/\mathrm{d}h$, Ju and zhong (1999) set $c = c_0$ in equation (29) to compute the option price. This differs from Barone-Adesi and Whaley (1987) in that the latter set $c = 0$ (also $b = 0$) all the time. Therefore, Barone-Adesi and Whaley (1987) and Ju and Zhong (1999) give identical critical stock prices but different option prices.

The two new methods we propose use different prescriptions. The first method, which we label as $\mathtt{QD}^+$, sets $c = c_0$ in both equations (29) and (34). This differs from Ju and Zhong (1999) who set $c = c_0$ in equation (29) but $c = 0$ in equation (34). Intuitively, setting $c = c_0$ in both equations could be more self-consistent. As we will see in later part of our numerical study, our new prescription results in very accurate critical stock prices. However, it does not seem to give as accurate option prices as Ju and Zhong (1999), at least for the limited set of options we have tested. This behavior of the modified quadratic approximation is somewhat weird, but probably understandable given that various approximations have been made in Ju and Zhong's derivation and prescription. In fact, we have performed numerical analysis in which we feed into equation (29) the true critical stock prices with Ju and Zhong's prescription of $c = c_0$, and found that the approximating option prices become less accurate! This somewhat puzzling behavior of Ju and Zhong's approximation highlights the distinction we made in the introduction. Approximating the critical stock prices and the option prices are related, but *different* problems!

Our second method, which we label as $\mathtt{QD}^*$, keeps both equations (29) and (34) as they are. To this end, we need the quantity $\mathrm{d}\log S^*_\tau/\mathrm{d}h$. We will try to get an approximate expression for it because an exact formula does not seem readily available. Consider the high contact condition in the original quadratic approximation:

$$F(S^*_\tau, h) \equiv \big(1 - e^{-\delta\tau}\Phi(-d_1(S^*_\tau, \tau, K))\big)S^*_\tau + q_{\mathtt{QD}}\big(K - S^*_\tau - p(S^*_\tau, \tau, K)\big) = 0. \tag{35}$$

This equation defines S^*_τ as a function of h (or τ) implicitly. The quantity $\mathrm{d}\log S^*_\tau/\mathrm{d}h$ can then be computed using the implicit function theorem. The computation is straightforward albeit tedious, and the result is

$$\frac{\mathrm{d}\log S_\tau^*}{\mathrm{d}h} = -\frac{\partial F/\partial h}{\partial F/\partial S_\tau^*}, \tag{36}$$

with

$$\begin{aligned}\frac{\partial F}{\partial h} &= q_{\mathtt{QD}}\Theta(S_\tau^*,\tau)\frac{e^{r\tau}}{r} + q_{\mathtt{QD}}'(h)\Big(K - S_\tau^* - p(S_\tau^*,\tau,K)\Big) \\ &+ \frac{S_\tau^*\delta e^{-\delta\tau}\Phi(-d_1(S_\tau^*,\tau,K))}{r(1-h)} - \frac{S_*^\tau e^{-\delta\tau}n(d_1(S_\tau^*,\tau,K))}{2r\tau(1-h)}\bigg(\frac{2\log(S_\tau^*/K)}{\sigma\sqrt{\tau}} - d_1(S_\tau^*,\tau,K)\bigg),\end{aligned} \tag{37}$$

and

$$\frac{\partial F}{\partial S_\tau^*} = (1-q_{\mathtt{QD}})\Big(1 - e^{-\delta\tau}\Phi(-d_1(S_\tau^*,\tau,K))\Big) + \frac{e^{-\delta\tau}\phi(d_1(S_\tau^*,\tau,K))}{\sigma\sqrt{\tau}}. \tag{38}$$

Equation (36) gives $\mathrm{d}\log S_\tau^*/\mathrm{d}h$ (which is related to the slope of the critical boundary) explicitly in terms of S_τ^*. With the help of this equation, equations (31) now gives c explicitly in terms of S_τ^* too since c_0 is already expressed explicitly in S_τ^*.

Combining, equations (31), (34) and (36) together give a one-dimensional algebraic equation for S_τ^*, which can be solved numerically. The derivative of c_0 with respect to S_τ^* can be computed easily with software such as Mathematica in order to use the Newton-Raphson algorithm. Alternatively, one could use a quasi-Newton-Raphson algorithm in which the derivative is approximated numerically without much sacrifice in efficiency. The IMSL numerical library includes this algorithm. After S_τ^* is computed, equation (31) gives the value for c. The value of c when plugged into equation (29) in turn gives the approximated price for the American put option.

One might ask whether it is possible to obtain a more accurate approximation for $\mathrm{d}\log S_\tau^*/\mathrm{d}h$ by using equation (34) rather than its approximation equation (35). Unfortunately, this would make the algebra quite complicated and unmanageable. Furthermore, as various different types of approximations are made consecutively along the way, it is not necessary true that a more accurate $\mathrm{d}\log S_\tau^*/\mathrm{d}h$ would lead to a more accurate critical stock price.

It might appear that our approximations have a much higher computational cost than Ju and Zhong (1999), especially in $\mathtt{QD}^*$ because of the extra computation for $\mathrm{d}\log S_\tau^*/\mathrm{d}h$. This is actually not the case. The only two expensive computation in $\mathrm{d}\log S_\tau^*/\mathrm{d}h$ are $\Phi(-d_1)$ and $\Phi(-d_2)$. The equation that Ju and Zhong (1999) uses to determine the critical stock price, that is, equation (35), also needs to compute $\Phi(-d_1)$ and $\Phi(-d_2)$. When implementing our method, it is important to save these values so that they can be retrieved without recomputing. Thus, our methods have essentially the same efficiency as Ju and Zhong (1999).[7]

[7] The situation is exactly the same for Ju and Zhong (1999) and Barone-Adesi and Whaley (1987). The method in Ju and Zhong (1999) seems to be more complex than the original quadratic approximation in Barone-Adesi and Whaley (1987), but the two methods have essentially the same efficiency as the exhibits in Ju and Zhong (1999) show.

The QD* approximation has one additional unique advantage in that equation (36) gives an approximation for the rate of change of S^*_τ as the maturity shortens, which might be of interest to traders of American options:

$$\frac{\mathrm{d}S^*_\tau}{\mathrm{d}\tau} = -\frac{e^{r\tau}}{rS^*_\tau}\frac{\partial F/\partial h}{\partial F/\partial S^*_\tau}. \tag{39}$$

Our numerical study (not reported in this paper) shows that this approximation for the rate of change of the critical stock price is a fairly good one, although its accuracy is not as good as our approximations for the option prices and the critical stock prices.

One shortcoming shared by all variants of the quadratic approximation is that the errors cannot be quantified. There is no theoretical bounds on the errors available. Also, the method is not convergent. It cannot be made arbitrarily accurate by including "more and more" terms. In particular, although intuitively the two new quadratic algorithms seem to have been more careful, it is not necessarily true that it will outperform the original quadratic approximation. Therefore, the accuracy and robustness of all these quadratic approximations (as well as other analytical approximations) would have to be studied numerically.

III. Numerical Study

We are now going to perform a numerical study on the analytical approximations for American options discussed in the previous section. We will mainly be interested in their accuracy in the critical stock prices. For comparison purposes, we also compute the critical stock prices from the binomial tree method of Cox, Ross and Rubinstein (1979). A bisection method is used to search for the critical stock prices. Two numbers of time steps are used, namely, 15000 and 150. We label them $\mathtt{BT}_{15\mathrm{K}}$ and $\mathtt{BT}_{150}$. The results from $\mathtt{BT}_{15\mathrm{K}}$ are used as the true critical stock prices. Computing the critical stock prices accurately is much more challenging than computing the option prices accurately. For example, the difference in option prices from a binomial tree with 10000 and 20000 time steps are usually very small. However, the percentage difference in the critical stock price could be as large as in the order of 0.1%.[8] Also, for two instances we had to switch to a tree with 30000 time steps. In these two instances, the results from $\mathtt{BT}_{150}$ violate the lower bound approximation of Broadie and Detemple (1996). In general, the true critical stock prices for the options in the tables of this paper most likely lie within a ± 0.02 range of the $\mathtt{BT}_{15\mathrm{K}}$ values.

We did not provide the derivative information usually needed in the Newton-Raphson algorithm in the last Section for the analytical methods. This is because many of them are quite cumbersome and lengthy. Software such as Mathematica or Maple can produce these derivatives with ease. Also,

[8] While we do not have evidence, we suspect that one reason almost all previous research has focused on the option prices alone might be the difficulty in computing the benchmark critical stock prices. It took many days to compute the critical stock prices in the tables of this paper from $\mathtt{BT}_{15\mathrm{K}}$ with three computers running.

one could employ quasi-Newton-Raphson algorithm in which the derivative is computed numerically without losing much efficiency at all.

We first briefly discuss the efficiency of various analytical methods we consider. Except for the the pseudo-steady state inverse Laplace transform method (`PS`) in Zhu (2006b), all methods require a computation time smaller than 10^{-3} second per option when implemented using MATLAB. However, the `TA` method on average requires twice the computation time as the quadratic approximation because one extra Newton-Raphson algorithm needs to be called to compute τ_0 in equation (20). The `PS` approximation requires about 0.02 second in Mathematica for the computation of one current critical stock price because of the one-dimensional numerical integration.[9]

We consider only American put options under the Black-Scholes setup. Four different maturities are considered: $\tau = 0.1$, 0.5, 1, and 3 years, representing near-term, short-term, medium-term and long-term options, respectively. For each maturity, we consider three different volatility values: $\sigma = 0.2$, 0.4, and 0.6. The interest rate r and dividend rate δ are chosen to be either 0.02, 0.04, or 0.08. The results are presented in Tables 1 to 4. The `TA` and `PS` methods are not included because they only apply to the zero dividend case. For each parameter combination $(\tau, \sigma, r, \delta)$, we compute the critical stock prices from each of the methods considered. The average absolute error (AAE) and average percentage error (APE) are presented together with the maximum absolute error (MAE) and maximum percentage error (MPE). For each measure, the number from the best performing method is highlighted in boldface.

Tables 1 to 4 show that the accuracy of $\mathtt{BT}_{150}$ decreases with time to maturity significantly. The AAE's for near-term, short-term, medium-term and long-term options are 0.28, 0.47, 0.58, and 0.82, respectively. This is understandable as a tree with 150 time steps is much coarser for a long maturity than for a short maturity. The accuracy of the initial guess (`IG`) in Barone-Adesi and Whaley (1987) is not very good, with APE's around 7% for all four maturities. Nevertheless, it provides good initial values to feed into the Newton-Raphson algorithm for other methods. The APE's for the quadratic approximation `QD` are around 1% for the near-term options, and around 2% for all other maturities. The `BE` method improves over `QD` somewhat for all maturities except $\tau = 3$ years. The `LB` method is considerably more accurate than the quadratic approximation. The AAE's for `LB` are about half of those for `QD` for all four maturities. One particularly nice feature of the `LB` approximation is that it provides upper bounds for the critical stock prices for the put option, which can be easily verified in all four tables. The last two columns present the results from the two new methods introduced in this paper. The performance of these two methods are very similar to each other and is better than all the other analytical approximations. The APE's for these two

[9] One-dimensional integration is usually faster than this. However, for the integral in Zhu (2006b), the integrand is usually fairly small while the integration range is quite wide. In particular, we find that the built-in integration functions `quad` and `quadl` in MATLAB are not able to handle this integral and give implausible results.

methods are usually around 0.5%.

Table 5 considers the case of zero dividend rate. Panel A presents the results for near-term and short-term options, while Panel B presents the results for medium-term and long-term options. The `TA` and `PS` methods which only work for zero dividend rate are now added to the table. The performance of the previous seven methods remains roughly the same. The `TA` method in Bunch and Johnson (2000) gives fairly accurate critical stock prices for options in Panel A, but its accuracy decreases significantly for medium-term and long-term options.[10] Somewhat disappointingly, the `PS` method in Zhu (2006b) gives very inaccurate results, the APE's being about 8% and 9% in the two panels, respectively.[11] Overall, the two new methods ($\texttt{QD}^+$ and $\texttt{QD}^*$) still give the most accurate approximations for the critical stock prices.

Tables 1 to 5 only aggregate the options along the time to maturity dimension. Table 6 presents the summary performance of various analytical methods in terms of AAE's along other dimensions. Although not the focus of this paper, the performance of various methods is often monotone when one of the parameters changes. We are particularly interested in the last two columns. One might be tempted to recommend one of the two methods over the other one by looking at this table. For example, it seems that when τ is small, we should choose $\texttt{QD}^+$ over $\texttt{QD}^*$. We want to give a fair warning that although we have tried to sample the parameter space uniformly, we have used only a finite number of options. A somewhat safe conclusion to draw from this table is that the performance of the $\texttt{QD}^+$ and $\texttt{QD}^*$ methods is quite consistent along different dimensions.

So far we have focused on the accuracy of various analytical approximations on the critical stock prices. Our conclusion is that the two new methods are more accurate than the existing analytical approximations. It is natural to ask whether these two methods also give accurate approximations for the option prices. Tables 7 and 8 examine this issue. We compare the prices from our methods with those from the `MQ` method in Ju and Zhong (1999). Table 7 uses the same 27 near-term and short-term options considered in Geske and Johnson (1984), Huang, Subramanyam, and Yu (1996), and Ju and Zhong (1999). The current stock price S is fixed at 40, with interest rate $r = 0.0488$ and dividend rate $\delta = 0$. Table 7 shows that for near-term or short-term options, the two new methods $\texttt{QD}^+$ and $\texttt{QD}^*$ give much more accurate critical stock prices, especially for $\texttt{QD}^*$. In terms of pricing, the APE's of the two new methods are 0.3% and 0.1%, respectively, while the APE for `MQ` is about 0.2%. In Table 8, the comparison is done using the same 20 long-term options considered in Barone-Adesi and Whaley (1987), and Ju and Zhong (1999). For all 20 options, the strike price K

[10] Another shortcoming of the `TA` method is that equations (17) to (20) do not always have a solution. In fact, for the last parameter combination (3.0, 0.6, 0.08) in Panel B, we are not able to obtain a solution for these equations. The number 36.02 is obtained by minimizing the difference between the two sides of equation (17).

[11] We believe that the `PS` method could be of some value if one can show that the critical stock price it provides is a lower bound for the true critical stock price. This seems to be true in Table 5 and from the way the pseudo-steady state approximation works. However, Zhu (2006b) does not make this claim.

is fixed at 100, with interest rate $r = 0.08$, volatility $\sigma = 0.2$, and time to maturity $\tau = 3$ years. Again, we see that the two new methods give much more accurate critical stock prices (especially for $\texttt{QD}^+$). In terms of option prices, all three methods have very similar AAE's. These two tables show that our methods give accurate option prices as well as accurate critical stock prices.

IV. Conclusion

Existing analytical approximations for American options have focused on their accuracy on option prices. In this paper, we examine their performance on approximating the critical stock prices. We propose two new analytical approximations, $\texttt{QD}^+$ and $\texttt{QD}^*$, based on the quadratic approximation. Both methods use more careful treatments for the high contact condition than Barone-Adesi and Whaley (1987) and Ju and Zhong (1999).

We then compare our methods with seven existing analytical methods on their accuracy for the critical stock prices. We find that both of our methods give much more accurate critical stock prices than all the existing methods. We also carry out a comparison on the accuracy of our methods in pricing options with the modified quadratic approximation in Ju and Zhong (1999). We find that both of our methods have about the same accuracy as Ju and Zhong (1999), while producing much more accurate critical stock prices.

Finally, while the study of American options is one of the oldest problems faced by researchers in finance, we do not think the saga will end soon. For example, it might be very fruitful to develop analytical approximations for American options written on multiple assets.

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Table 1
Critical Stock Prices of Near-term American Puts ($\tau = 0.1$)

This table presents the critical stock prices as a function of volatility σ, interest rate r, and dividend rate δ for near-term American puts. Results from the following methods are presented: the binomial tree method with 15000 time steps ($\mathtt{BT}_{15K}$, treated as the true values) and 150 time steps ($\mathtt{BT}_{150}$), the initial guess (`IG`) and the quadratic approximation (`QD`) in Barone-Adesi and Whaley (1987), the refined quadratic approximation (`BE`) in Barone-Adesi and Elliott (1991), the lower bound (`LB`) in Broadie and Detemple (1996), the interpolation method (`IM`) in Li (2008), and the two methods ($\mathtt{QD}^+$ and $\mathtt{QD}^*$) introduced in this paper. The average absolute error (AAE) and average percentage error (APE) are also presented together with the maximum absolute error (MAE) and maximum percentage error (MPE).

(σ, r, δ)	$\mathtt{BT}_{15K}$	$\mathtt{BT}_{150}$	IG	QD	BE	LB	IM	$\mathtt{QD}^+$	$\mathtt{QD}^*$
0.2, 0.02, 0.02	83.80	84.02	88.56	84.49	83.65	84.23	84.38	83.99	83.46
0.2, 0.02, 0.04	48.06	48.17	46.56	47.82	47.84	48.08	47.76	48.08	48.18
0.2, 0.02, 0.08	24.06	24.11	24.11	23.91	23.94	24.07	24.03	24.07	24.13
0.2, 0.04, 0.02	88.11	88.32	89.25	88.81	87.92	88.49	88.69	88.30	87.93
0.2, 0.04, 0.04	84.92	85.12	88.82	85.56	84.76	85.32	85.51	85.09	84.67
0.2, 0.04, 0.08	48.09	48.20	47.41	47.82	47.86	48.11	47.85	48.12	48.22
0.2, 0.08, 0.02	90.72	90.91	90.56	91.25	90.48	90.98	91.20	90.85	90.66
0.2, 0.08, 0.04	89.79	89.98	90.07	90.37	89.57	90.09	90.33	89.94	89.70
0.2, 0.08, 0.08	86.12	86.31	89.20	86.68	85.94	86.47	86.67	86.27	85.95
0.4, 0.02, 0.02	70.24	70.60	78.20	71.40	69.99	70.96	71.01	70.55	69.65
0.4, 0.02, 0.04	46.17	46.37	40.92	45.73	45.75	46.22	45.29	46.21	46.38
0.4, 0.02, 0.08	23.11	23.21	21.62	22.86	22.90	23.13	22.89	23.14	23.23
0.4, 0.04, 0.02	75.47	75.79	78.90	76.74	75.17	76.20	76.39	75.82	75.03
0.4, 0.04, 0.04	72.13	72.46	78.61	73.21	71.86	72.80	73.00	72.42	71.69
0.4, 0.04, 0.08	46.20	46.40	42.13	45.73	45.77	46.24	45.62	46.24	46.42
0.4, 0.08, 0.02	79.42	79.78	80.25	80.53	79.05	80.01	80.25	79.71	79.20
0.4, 0.08, 0.04	78.14	78.47	79.89	79.27	77.79	78.76	79.05	78.44	77.87
0.4, 0.08, 0.08	74.17	74.51	79.24	75.14	73.86	74.77	75.05	74.43	73.87
0.6, 0.02, 0.02	58.88	59.34	68.98	60.35	58.58	59.80	59.63	59.27	58.14
0.6, 0.02, 0.04	44.32	44.62	35.84	43.72	43.72	44.39	42.66	44.36	44.60
0.6, 0.02, 0.08	22.20	22.35	18.98	21.87	21.90	22.24	21.66	22.23	22.36
0.6, 0.04, 0.02	64.28	64.77	69.66	65.92	63.92	65.23	65.32	64.73	63.65
0.6, 0.04, 0.04	61.28	61.70	69.45	62.66	60.93	62.14	62.25	61.65	60.72
0.6, 0.04, 0.08	44.37	44.66	37.01	43.73	43.76	44.43	43.33	44.42	44.66
0.6, 0.08, 0.02	68.90	69.36	70.98	70.43	68.45	69.74	69.93	69.31	68.53
0.6, 0.08, 0.04	67.55	67.98	70.71	69.06	67.13	68.40	68.65	67.96	67.15
0.6, 0.08, 0.08	63.89	64.33	70.22	65.14	63.50	64.67	64.95	64.23	63.51
AAE		0.28	3.93	0.84	0.31	0.43	0.74	**0.20**	0.30
APE		0.005	0.070	0.013	0.006	0.006	0.013	**0.003**	0.005
MAE		0.50	10.10	1.64	0.61	0.96	1.66	**0.46**	0.74
MPE		0.008	0.191	0.026	0.014	0.016	0.037	**0.007**	0.013

Table 2
Critical Stock Prices of Short-term American Puts ($\tau = 0.5$)

This table presents the critical stock prices as a function of volatility σ, interest rate r, and dividend rate δ for short-term American puts. Results from the following methods are presented: the binomial tree method with 15000 time steps ($\mathtt{BT}_{15K}$, treated as the true values) and 150 time steps ($\mathtt{BT}_{150}$), the initial guess (IG) and the quadratic approximation (QD) in Barone-Adesi and Whaley (1987), the refined quadratic approximation (BE) in Barone-Adesi and Elliott (1991), the lower bound (LB) in Broadie and Detemple (1996), the interpolation method (IM) in Li (2008), and the two methods ($\mathtt{QD}^+$ and $\mathtt{QD}^*$) introduced in this paper. The average absolute error (AAE) and average percentage error (APE) are also presented together with the maximum absolute error (MAE) and maximum percentage error (MPE).

(σ, r, δ)	$\mathtt{BT}_{15K}$	$\mathtt{BT}_{150}$	IG	QD	BE	LB	IM	$\mathtt{QD}^+$	$\mathtt{QD}^*$
0.2, 0.02, 0.02	72.34	72.71	77.30	73.34	72.00	72.97	73.27	72.61	72.07
0.2, 0.02, 0.04	45.85	46.08	42.98	45.26	45.37	45.89	45.37	45.90	46.12
0.2, 0.02, 0.08	23.06	23.17	23.11	22.62	22.80	23.07	23.09	23.10	23.27
0.2, 0.04, 0.02	79.78	80.15	80.02	80.75	79.34	80.28	80.70	80.03	79.68
0.2, 0.04, 0.04	74.77	75.12	78.40	75.59	74.38	75.30	75.63	75.00	74.64
0.2, 0.04, 0.08	45.99	46.22	44.71	45.27	45.50	46.03	45.49	46.05	46.32
0.2, 0.08, 0.02	85.23	85.63	84.48	85.86	84.70	85.48	86.08	85.35	85.22
0.2, 0.08, 0.04	83.39	83.78	82.93	84.09	82.89	83.72	84.26	83.55	83.38
0.2, 0.08, 0.08	77.37	77.76	79.88	77.98	76.93	77.79	78.11	77.56	77.36
0.4, 0.02, 0.02	52.40	52.93	59.01	53.84	51.91	53.30	53.55	52.79	52.01
0.4, 0.02, 0.04	41.06	41.51	32.75	40.50	40.28	41.22	40.38	41.08	41.34
0.4, 0.02, 0.08	21.09	21.30	18.48	20.51	20.65	21.13	21.04	21.14	21.35
0.4, 0.04, 0.02	60.06	60.65	61.62	61.62	59.45	60.92	61.25	60.48	59.84
0.4, 0.04, 0.04	55.96	56.50	60.60	57.19	55.38	56.75	57.13	56.31	55.77
0.4, 0.04, 0.08	41.62	42.04	35.18	40.79	40.80	41.73	40.97	41.66	42.01
0.4, 0.08, 0.02	67.27	67.89	66.26	68.54	66.49	67.89	68.38	67.58	67.22
0.4, 0.08, 0.04	65.19	65.83	65.10	66.43	64.46	65.85	66.34	65.51	65.13
0.4, 0.08, 0.08	59.92	60.52	62.94	60.85	59.25	60.57	60.99	60.21	59.90
0.6, 0.02, 0.02	38.05	38.63	44.86	39.60	37.53	39.01	39.07	38.47	37.65
0.6, 0.02, 0.04	32.52	33.07	24.72	32.86	31.83	32.99	32.53	32.62	32.47
0.6, 0.02, 0.08	19.27	19.56	14.16	18.58	18.69	19.33	19.01	19.32	19.57
0.6, 0.04, 0.02	44.93	45.56	47.20	46.64	44.27	45.91	46.01	45.39	44.66
0.6, 0.04, 0.04	41.98	42.59	46.53	43.33	41.34	42.85	43.12	42.36	41.78
0.6, 0.04, 0.08	34.45	34.99	27.04	34.38	33.65	34.81	34.45	34.52	34.59
0.6, 0.08, 0.02	52.28	53.03	51.47	53.82	51.42	53.10	53.32	52.68	52.20
0.6, 0.08, 0.04	50.50	51.21	50.68	51.94	49.69	51.32	51.59	50.90	50.43
0.6, 0.08, 0.08	46.49	47.18	49.19	47.54	45.72	47.23	47.61	46.82	46.48
AAE		0.47	3.28	0.96	0.60	0.50	0.78	0.23	**0.18**
APE		0.010	0.081	0.020	0.013	0.010	0.015	**0.004**	0.004
MAE		0.76	8.31	1.71	0.86	0.98	1.18	0.47	**0.41**
MPE		0.017	0.265	0.041	0.030	0.025	0.027	**0.011**	0.015

Table 3
Critical Stock Prices of Medium-term American Puts ($\tau = 1$)

This table presents the critical stock prices as a function of volatility σ, interest rate r, and dividend rate δ for medium-term American puts. Results from the following methods are presented: the binomial tree method with 15000 time steps ($\mathtt{BT}_{15\mathrm{K}}$, treated as the true values) and 150 time steps ($\mathtt{BT}_{150}$), the initial guess (IG) and the quadratic approximation (QD) in Barone-Adesi and Whaley (1987), the refined quadratic approximation (BE) in Barone-Adesi and Elliott (1991), the lower bound (LB) in Broadie and Detemple (1996), the interpolation method (IM) in Li (2008), and the two methods ($\mathtt{QD}^+$ and $\mathtt{QD}^*$) introduced in this paper. The average absolute error (AAE) and average percentage error (APE) are also presented together with the maximum absolute error (MAE) and maximum percentage error (MPE).

(σ, r, δ)	$\mathtt{BT}_{15\mathrm{K}}$	$\mathtt{BT}_{150}$	IG	QD	BE	LB	IM	$\mathtt{QD}^+$	$\mathtt{QD}^*$
0.2, 0.02, 0.02	66.30	66.75	70.55	67.33	65.81	66.96	67.34	66.59	66.14
0.2, 0.02, 0.04	44.32	44.63	40.72	43.46	43.66	44.37	43.69	44.39	44.70
0.2, 0.02, 0.08	22.41	22.57	22.44	21.72	22.05	22.43	22.45	22.47	22.74
0.2, 0.04, 0.02	75.47	75.96	75.13	76.44	74.84	75.96	76.51	75.71	75.43
0.2, 0.04, 0.04	69.59	70.08	72.47	70.35	69.03	70.12	70.51	69.83	69.57
0.2, 0.04, 0.08	44.59	44.91	43.02	43.50	43.92	44.63	43.85	44.68	45.08
0.2, 0.08, 0.02	82.71	83.26	81.98	83.28	82.01	82.90	83.80	82.78	82.69
0.2, 0.08, 0.04	80.34	80.87	79.71	80.97	79.66	80.62	81.37	80.47	80.35
0.2, 0.08, 0.08	73.15	73.64	74.97	73.57	72.52	73.54	73.86	73.33	73.24
0.4, 0.02, 0.02	44.09	44.70	48.70	45.44	43.46	44.96	45.25	44.47	43.89
0.4, 0.02, 0.04	35.79	36.30	28.30	35.60	34.97	36.11	35.69	35.84	35.96
0.4, 0.02, 0.08	19.77	20.05	16.73	18.95	19.20	19.81	19.82	19.83	20.12
0.4, 0.04, 0.02	52.77	53.46	52.92	54.20	51.96	53.57	53.89	53.16	52.70
0.4, 0.04, 0.04	48.55	49.23	51.34	49.59	47.79	49.28	49.68	48.88	48.54
0.4, 0.04, 0.08	37.41	37.95	31.50	36.70	36.48	37.62	37.06	37.46	37.77
0.4, 0.08, 0.02	61.70	62.53	60.07	62.78	60.68	62.23	62.83	61.96	61.73
0.4, 0.08, 0.04	59.29	60.09	58.35	60.30	58.32	59.86	60.39	59.57	59.34
0.4, 0.08, 0.08	53.63	54.34	55.13	54.22	52.72	54.18	54.59	53.88	53.76
0.6, 0.02, 0.02	29.51	30.12	33.41	30.80	28.89	30.34	30.42	29.87	29.33
0.6, 0.02, 0.04	25.47	26.01	19.41	25.91	24.78	25.98	25.83	25.62	25.47
0.6, 0.02, 0.08	17.06	17.43	11.84	16.32	16.38	17.15	17.11	17.08	17.35
0.6, 0.04, 0.02	36.74	37.55	36.99	38.15	35.94	37.58	37.56	37.14	36.69
0.6, 0.04, 0.04	34.05	34.76	36.03	35.09	33.27	34.79	35.03	34.38	34.05
0.6, 0.04, 0.08	28.23	28.82	22.42	28.20	27.39	28.62	28.49	28.34	28.41
0.6, 0.08, 0.02	45.23	46.17	43.34	46.42	44.16	45.90	46.01	45.56	45.30
0.6, 0.08, 0.04	43.38	44.25	42.23	44.42	42.36	44.04	44.19	43.71	43.47
0.6, 0.08, 0.08	39.48	40.28	40.14	40.08	38.51	40.06	40.37	39.74	39.63
AAE		0.58	2.62	0.85	0.75	0.47	0.75	0.21	**0.15**
APE		0.014	0.076	0.021	0.019	0.011	0.015	0.005	**0.004**
MAE		0.94	7.49	1.43	1.07	0.86	1.16	**0.40**	0.49
MPE		0.022	0.306	0.044	0.040	0.028	0.031	**0.012**	0.017

Table 4
Critical Stock Prices of Long-term American Puts ($\tau = 3$)

This table presents the critical stock prices as a function of volatility σ, interest rate r, and dividend rate δ for long-term American puts. Results from the following methods are presented: the binomial tree method with 15000 time steps ($\texttt{BT}_{15K}$, treated as the true values) and 150 time steps ($\texttt{BT}_{150}$), the initial guess (`IG`) and the quadratic approximation (`QD`) in Barone-Adesi and Whaley (1987), the refined quadratic approximation (`BE`) in Barone-Adesi and Elliott (1991), the lower bound (`LB`) in Broadie and Detemple (1996), the interpolation method (`IM`) in Li (2008), and the two methods ($\texttt{QD}^{+}$ and $\texttt{QD}^{*}$) introduced in this paper. The average absolute error (AAE) and average percentage error (APE) are also presented together with the maximum absolute error (MAE) and maximum percentage error (MPE).

(σ, r, δ)	$\texttt{BT}_{15K}$	$\texttt{BT}_{150}$	IG	QD	BE	LB	IM	$\texttt{QD}^{+}$	$\texttt{QD}^{*}$
0.2, 0.02, 0.02	56.20	56.89	58.34	56.96	55.39	56.80	57.22	56.47	56.27
0.2, 0.02, 0.04	40.37	40.87	36.43	39.10	39.41	40.48	39.60	40.44	40.91
0.2, 0.02, 0.08	21.15	21.42	21.10	19.71	20.56	21.15	21.16	21.23	21.67
0.2, 0.04, 0.02	68.49	69.28	67.57	69.16	67.50	68.84	69.65	68.65	68.54
0.2, 0.04, 0.04	61.32	62.03	62.51	61.57	60.39	61.72	62.09	61.52	61.50
0.2, 0.04, 0.08	41.51	42.03	39.91	39.53	40.48	41.55	40.20	41.61	42.20
0.2, 0.08, 0.02	79.22	80.19	79.08	79.56	78.28	79.25	80.84	79.16	79.13
0.2, 0.08, 0.04	75.94	76.84	75.57	76.19	74.95	76.04	77.22	75.94	75.91
0.2, 0.08, 0.08	66.92	67.73	67.58	66.58	65.91	67.12	67.29	67.02	67.11
0.4, 0.02, 0.02	31.96	32.74	32.71	32.73	31.09	32.59	32.81	32.25	32.06
0.4, 0.02, 0.04	26.34	26.97	21.34	26.13	25.45	26.69	26.57	26.45	26.57
0.4, 0.02, 0.08	16.86	17.27	13.98	15.65	16.08	16.90	17.17	16.88	17.22
0.4, 0.04, 0.02	42.05	43.02	40.45	42.75	40.89	42.57	42.68	42.30	42.19
0.4, 0.04, 0.04	37.95	38.84	37.82	38.18	36.86	38.41	38.66	38.17	38.17
0.4, 0.04, 0.08	29.71	30.42	26.00	28.70	28.63	29.91	29.46	29.78	30.08
0.4, 0.08, 0.02	53.90	55.18	52.37	54.19	52.48	54.13	54.86	53.98	53.95
0.4, 0.08, 0.04	51.07	52.29	49.74	51.17	49.69	51.32	51.81	51.17	51.18
0.4, 0.08, 0.08	45.09	46.18	44.76	44.58	43.78	45.34	45.50	45.20	45.32
0.6, 0.02, 0.02	18.59	19.25	18.38	19.13	17.89	19.07	19.01	18.80	18.69
0.6, 0.02, 0.04	16.20	16.78	12.47	16.25	15.50	16.53	16.51	16.32	16.34
0.6, 0.02, 0.08	12.13	12.58	8.84	11.49	11.47	12.24	12.72	12.14	12.32
0.6, 0.04, 0.02	26.04	26.97	24.23	26.47	25.03	26.48	26.02	26.25	26.19
0.6, 0.04, 0.04	23.91	24.77	22.91	24.02	22.95	24.29	24.26	24.08	24.10
0.6, 0.04, 0.08	19.98	20.71	16.66	19.42	19.06	20.20	20.08	20.05	20.20
0.6, 0.08, 0.02	36.10	37.41	34.03	36.15	34.71	36.36	36.03	36.21	36.22
0.6, 0.08, 0.04	34.29	35.55	32.51	34.15	32.96	34.55	34.22	34.40	34.44
0.6, 0.08, 0.08	30.78	31.90	29.69	30.19	29.52	30.99	30.91	30.86	30.97
AAE		0.82	1.72	0.57	1.01	0.28	0.57	**0.13**	0.21
APE		0.024	0.066	0.019	0.030	0.009	0.015	**0.004**	0.007
MAE		1.31	5.01	1.99	1.42	0.63	1.62	**0.29**	0.69
MPE		0.037	0.271	0.071	0.055	0.026	0.048	**0.012**	0.025

Table 5
Critical Stock Prices of American Puts with Zero Dividend Rate

This table presents the critical stock prices as a function of time to maturity τ, volatility σ and interest rate r. The dividend rate is fixed at 0 for all the options. Results from the following methods are presented: the binomial tree method with 15000 time steps ($\mathtt{BT}_{15K}$, treated as the true values) and 150 time steps ($\mathtt{BT}_{150}$), the initial guess (IG) and the quadratic approximation (QD) in Barone-Adesi and Whaley (1987), the refined quadratic approximation (BE) in Barone-Adesi and Elliott (1991), the lower bound (LB) in Broadie and Detemple (1996), the tangent approximation in Bunch and Johnson (2000), the pseudo-steady state approximation in Zhu (2006), the interpolation method (IM) in Li (2008), and the two methods ($\mathtt{QD}^{+}$ and $\mathtt{QD}^{*}$) introduced in this paper. The average absolute error (AAE) and average percentage error (APE) are also presented together with the maximum absolute error (MAE) and maximum percentage error (MPE).

Panel A: Near-term and Short-term Options

(τ, σ, r)	$\mathtt{BT}_{15K}$	$\mathtt{BT}_{150}$	IG	QD	BE	LB	TA	PS	IM	$\mathtt{QD}^{+}$	$\mathtt{QD}^{*}$
0.1, 0.2, 0.02	87.90	88.09	88.98	88.64	87.71	88.31	88.12	85.10	88.38	88.11	87.68
0.1, 0.2, 0.04	89.60	89.78	89.75	90.22	89.38	89.92	89.75	87.88	90.05	89.76	89.48
0.1, 0.2, 0.08	91.44	91.63	91.06	91.92	91.19	91.66	91.53	90.56	91.87	91.55	91.40
0.1, 0.4, 0.02	74.80	75.15	78.46	76.19	74.51	75.59	75.25	67.79	75.54	75.19	74.23
0.1, 0.4, 0.04	77.48	77.83	79.22	78.74	77.14	78.17	77.74	72.54	78.25	77.82	77.12
0.1, 0.4, 0.08	80.47	80.80	80.63	81.55	80.08	81.02	80.54	77.37	81.22	80.75	80.29
0.1, 0.6, 0.02	63.11	63.54	69.17	64.93	62.78	64.17	63.84	52.51	63.91	63.63	62.27
0.1, 0.6, 0.04	66.37	66.86	69.89	68.07	65.97	67.32	66.87	58.37	67.29	66.84	65.79
0.1, 0.6, 0.08	70.06	70.50	71.25	71.59	69.58	70.88	70.26	64.55	71.00	70.47	69.72
0.5, 0.2, 0.02	79.04	79.44	78.97	80.17	78.60	79.61	79.05	75.95	79.84	79.32	78.87
0.5, 0.2, 0.04	82.71	83.12	81.82	83.57	82.21	83.10	82.49	81.15	83.50	82.91	82.66
0.5, 0.2, 0.08	86.69	87.12	85.94	87.25	86.15	86.88	86.15	86.12	87.53	86.77	86.67
0.5, 0.4, 0.02	58.10	58.64	59.94	59.98	57.54	59.12	58.49	50.50	59.13	58.60	57.68
0.5, 0.4, 0.04	63.20	63.79	62.74	64.82	62.51	64.03	63.10	58.29	64.23	63.61	63.00
0.5, 0.4, 0.08	69.09	69.75	67.47	70.34	68.26	69.67	68.40	66.47	70.15	69.37	69.04
0.5, 0.6, 0.02	42.14	42.76	45.45	44.21	41.58	43.30	42.93	31.89	43.02	42.70	41.61
0.5, 0.6, 0.04	47.48	48.14	47.90	49.40	46.77	48.51	47.81	40.01	48.43	47.99	47.20
0.5, 0.6, 0.08	53.92	54.68	52.30	55.53	53.02	54.73	53.55	49.21	54.90	54.33	53.84
AAE		0.45	1.61	1.31	0.48	0.69	0.34	4.85	0.81	0.34	**0.28**
APE		0.007	0.025	0.021	0.007	0.011	0.005	0.079	0.012	0.005	**0.004**
MAE		0.76	6.05	2.07	0.90	1.15	0.79	10.60	1.06	**0.56**	0.85
MPE		0.015	0.096	0.049	0.017	0.027	0.019	0.243	0.021	**0.013**	0.013

Panel B: Medium-term and Long-term Options

$(\tau,\, \sigma,\, r)$	$\texttt{BT}_{15\mathrm{K}}$	$\texttt{BT}_{150}$	IG	QD	BE	LB	TA	PS	IM	$\texttt{QD}^{+}$	$\texttt{QD}^{*}$
1.0, 0.2, 0.02	74.24	74.76	73.38	75.49	73.64	74.85	74.00	71.30	75.19	74.55	74.13
1.0, 0.2, 0.04	79.24	79.78	77.99	80.12	78.55	79.60	78.58	77.91	80.22	79.41	79.22
1.0, 0.2, 0.08	84.60	85.19	84.04	85.12	83.91	84.73	83.39	84.21	85.76	84.64	84.57
1.0, 0.4, 0.02	49.90	50.55	50.18	51.79	49.18	50.90	50.15	42.77	50.88	50.39	49.63
1.0, 0.4, 0.04	56.31	57.04	54.65	57.88	55.41	57.09	55.86	51.92	57.31	56.70	56.23
1.0, 0.4, 0.08	63.88	64.73	61.85	64.98	62.81	64.36	62.51	61.75	65.03	64.11	63.90
1.0, 0.6, 0.02	33.10	33.82	34.26	34.95	32.45	34.13	33.82	24.23	33.73	33.60	32.82
1.0, 0.6, 0.04	39.27	40.04	38.00	40.93	38.40	40.16	39.40	32.88	39.93	39.71	39.18
1.0, 0.6, 0.08	47.01	47.94	44.50	48.30	45.88	47.66	46.17	43.16	47.75	47.33	47.05
3.0, 0.2, 0.02	66.02	66.80	64.10	67.21	65.06	66.55	64.97	63.68	67.14	66.28	66.02
3.0, 0.2, 0.04	73.77	74.64	72.64	74.47	72.73	73.99	71.80	72.95	75.19	73.85	73.75
3.0, 0.2, 0.08	81.86	82.88	82.08	82.28	81.04	81.87	78.81	81.70	83.79	81.78	81.75
3.0, 0.4, 0.02	37.34	38.19	35.26	38.77	36.37	38.11	37.22	31.94	37.83	37.72	37.35
3.0, 0.4, 0.04	45.95	47.02	43.32	46.93	44.68	46.46	44.63	42.90	46.58	46.20	46.03
3.0, 0.4, 0.08	56.58	57.95	55.06	57.01	55.12	56.77	53.55	55.36	57.80	56.63	56.58
3.0, 0.6, 0.02	21.14	21.92	19.56	22.16	20.39	21.75	21.60	15.55	20.95	21.44	21.19
3.0, 0.6, 0.04	28.23	29.23	25.65	28.96	27.16	28.72	27.89	24.28	27.98	28.48	28.36
3.0, 0.6, 0.08	37.92	39.27	35.63	38.14	36.48	38.18	36.02	35.75	37.90	38.02	38.00
AAE		0.86	1.53	1.06	0.95	0.53	1.06	3.45	0.86	0.26	**0.08**
APE		0.019	0.035	0.023	0.020	0.012	0.020	0.087	0.015	0.006	**0.002**
MAE		1.37	2.63	1.89	1.46	1.02	3.05	8.87	1.94	0.50	**0.29**
MPE		0.037	0.091	0.056	0.038	0.031	0.054	0.268	0.024	0.015	**0.009**

Table 6
Summary Performance of Analytical Approximations for S^*_τ

This table presents the average absolute errors (AAE) for the critical stock prices along various dimensions (Column 1). The options used are those in Tables 1 to 5. Results from the following methods are presented: the binomial tree method with 15000 time steps ($\mathtt{BT}_{15\mathrm{K}}$, treated as the true values) and 150 time steps ($\mathtt{BT}_{150}$), the initial guess (`IG`) and the quadratic approximation (`QD`) in Barone-Adesi and Whaley (1987), the refined quadratic approximation (`BE`) in Barone-Adesi and Elliott (1991), the lower bound (`LB`) in Broadie and Detemple (1996), the tangent approximation in Bunch and Johnson (2000), the pseudo-steady state approximation in Zhu (2006b), the interpolation method (`IM`) in Li (2008), and the two methods ($\mathtt{QD}^+$ and $\mathtt{QD}^*$) introduced in this paper. The last row reports the standard deviations of errors in approximating the critical stock price when all the options are used.

Dimension	$\mathtt{BT}_{150}$	IG	QD	BE	LB	TA	PS	IM	$\mathtt{QD}^+$	$\mathtt{QD}^*$
$\tau = 0.1$	0.29	3.45	0.93	0.31	0.49	0.30	4.95	0.73	**0.23**	0.31
$\tau = 0.5$	0.50	2.77	1.08	0.61	0.56	0.38	4.75	0.82	0.26	**0.18**
$\tau = 1.0$	0.61	2.29	0.97	0.77	0.51	0.65	4.16	0.79	0.24	**0.14**
$\tau = 3.0$	0.87	1.74	0.63	1.03	0.31	1.47	2.74	0.63	**0.14**	0.17
$\sigma = 0.2$	0.42	1.43	0.72	0.54	0.28	0.79	1.55	0.77	**0.14**	0.16
$\sigma = 0.4$	0.61	2.83	0.97	0.74	0.51	0.71	4.46	0.80	0.24	**0.21**
$\sigma = 0.6$	0.67	3.43	1.01	0.75	0.61	0.61	6.45	0.66	0.28	**0.23**
$r = 0.02$	0.44	3.58	0.93	0.56	0.44	0.45	6.14	0.59	**0.20**	0.27
$r = 0.04$	0.56	2.65	0.97	0.69	0.51	0.54	4.04	0.77	0.24	**0.22**
$r = 0.08$	0.70	1.44	0.81	0.79	0.45	1.11	2.28	0.87	0.21	**0.11**
$\delta = 0$	0.65	1.57	1.18	0.71	0.61	0.70	4.15	0.84	0.30	**0.18**
$\delta = 0.02$	0.62	2.49	1.02	0.66	0.61	—	—	0.89	0.28	**0.20**
$\delta = 0.04$	0.55	3.26	0.71	0.69	0.43	—	—	0.77	0.19	**0.18**
$\delta = 0.08$	0.44	2.92	0.69	0.65	0.22	—	—	0.47	**0.10**	0.24
All	0.57	2.56	0.90	0.68	0.47	0.70	4.15	0.74	0.22	**0.20**
	(0.28)	(2.27)	(0.49)	(0.31)	(0.31)	(0.75)	(2.84)	(0.39)	(0.15)	(0.17)

Table 7
Pricing Performance of Quadratic Approximations on Near or Short-term Puts

This table compares the pricing performance of three modifications of the quadratic approximation in Barone-Adesi and Whaley (1987), namely, the modified quadratic approximation (MQ) in Ju and Zhong (1999), and the two modifications ($\mathtt{QD}^+$ and $\mathtt{QD}^*$) introduced in this paper. The comparison is done using the 27 near-term and short-term options considered in Geske and Johnson (1984), Huang, Subramanyam, and Yu (1996), and Ju and Zhong (1999). The current stock price S is fixed at 40, with interest rate $r = 0.0488$ and dividend rate $\delta = 0$.

	Critical stock price				Option price			
(K, σ, τ)	$\mathtt{BT}_{15K}$	MQ	$\mathtt{QD}^+$	$\mathtt{QD}^*$	$\mathtt{BT}_{15K}$	MQ	$\mathtt{QD}^+$	$\mathtt{QD}^*$
35, 0.2, 0.0833	31.75	31.94	31.80	31.71	0.006	0.006	0.006	0.006
35, 0.2, 0.3333	29.95	30.21	30.01	29.93	0.200	0.201	0.200	0.201
35, 0.2, 0.5833	29.10	29.38	29.16	29.09	0.433	0.433	0.432	0.433
40, 0.2, 0.0833	36.28	36.51	36.34	36.25	0.852	0.851	0.849	0.851
40, 0.2, 0.3333	34.22	34.52	34.29	34.21	1.580	1.576	1.571	1.576
40, 0.2, 0.5833	33.26	33.57	33.32	33.25	1.991	1.984	1.979	1.984
45, 0.2, 0.0833	40.82	41.07	40.88	40.78	5.000	5.000	5.000	5.000
45, 0.2, 0.3333	38.50	38.84	38.58	38.48	5.088	5.084	5.080	5.087
45, 0.2, 0.5833	37.42	37.77	37.49	37.41	5.267	5.260	5.253	5.263
35, 0.3, 0.0833	29.79	30.10	29.87	29.72	0.077	0.077	0.077	0.077
35, 0.3, 0.3333	26.87	27.28	26.97	26.83	0.698	0.697	0.695	0.697
35, 0.3, 0.5833	25.50	25.93	25.61	25.48	1.220	1.218	1.215	1.217
40, 0.3, 0.0833	34.04	34.40	34.14	33.97	1.310	1.309	1.306	1.309
40, 0.3, 0.3333	30.71	31.18	30.83	30.67	2.483	2.477	2.471	2.477
40, 0.3, 0.5833	29.15	29.64	29.26	29.12	3.170	3.161	3.153	3.160
45, 0.3, 0.0833	38.30	38.70	38.41	38.21	5.060	5.059	5.055	5.062
45, 0.3, 0.3333	34.55	35.08	34.68	34.50	5.706	5.699	5.687	5.701
45, 0.3, 0.5833	32.79	33.34	32.92	32.76	6.244	6.231	6.218	6.232
35, 0.4, 0.0833	27.86	28.27	27.97	27.75	0.247	0.247	0.246	0.246
35, 0.4, 0.3333	23.97	24.49	24.10	23.90	1.346	1.344	1.342	1.344
35, 0.4, 0.5833	22.18	22.72	22.31	22.14	2.155	2.150	2.146	2.150
40, 0.4, 0.0833	31.84	32.31	31.97	31.72	1.768	1.767	1.764	1.767
40, 0.4, 0.3333	27.39	27.99	27.55	27.32	3.387	3.381	3.374	3.381
40, 0.4, 0.5833	25.35	25.96	25.50	25.30	4.353	4.342	4.333	4.341
45, 0.4, 0.0833	35.82	36.34	35.96	35.68	5.287	5.288	5.279	5.289
45, 0.4, 0.3333	30.82	31.49	30.99	30.73	6.510	6.501	6.487	6.502
45, 0.4, 0.5833	28.52	29.21	28.69	28.47	7.383	7.367	7.352	7.368
AAE		0.43	0.11	**0.05**		0.004	0.009	**0.004**
APE		0.014	0.004	**0.002**		0.002	0.003	**0.001**
MAE		0.69	0.17	**0.14**		0.016	0.031	**0.015**
MPE		0.014	0.004	**0.002**		0.002	0.003	**0.001**

Table 8
Pricing Performance of Quadratic Approximations on Long-term Puts

This table compares the pricing performance of three modifications of the quadratic approximation in Barone-Adesi and Whaley (1987), namely, the modified quadratic approximation (MQ) in Ju and Zhong (1999), and the two modifications ($\mathtt{QD}^+$ and $\mathtt{QD}^*$) introduced in this paper. The comparison is done using the 20 long-term options considered in Barone-Adesi and Whaley (1987), and Ju and Zhong (1999). For all 20 options, the strike price K is fixed at 100, with interest rate $r = 0.08$, volatility $\sigma = 0.2$, and time to maturity $\tau = 3$ years.

	Critical stock price				Option price			
(S, δ)	$\mathtt{BT}_{15K}$	MQ	$\mathtt{QD}^+$	$\mathtt{QD}^*$	$\mathtt{BT}_{15K}$	MQ	$\mathtt{QD}^+$	$\mathtt{QD}^*$
80, 0.12	54.52	52.45	54.59	55.09	25.658	25.725	25.753	25.715
90, 0.12					20.083	20.185	20.202	20.176
100, 0.12					15.498	15.608	15.618	15.601
110, 0.12					11.803	11.905	11.912	11.900
120, 0.12					8.886	8.974	8.978	8.970
80, 0.08	66.92	66.58	67.02	67.11	22.205	22.148	22.162	22.149
90, 0.08					16.207	16.170	16.181	16.170
100, 0.08					11.704	11.700	11.708	11.700
110, 0.08					8.390	8.390	8.395	8.390
120, 0.08					5.930	5.968	5.971	5.967
80, 0.04	75.94	76.19	75.94	75.91	20.350	20.336	20.334	20.337
90, 0.04					13.497	13.471	13.465	13.471
100, 0.04					8.944	8.931	8.926	8.931
110, 0.04					5.912	5.920	5.916	5.920
120, 0.04					3.8974	3.922	3.920	3.922
80, 0.00	81.90	82.28	81.78	81.75	20.000	20.000	20.000	20.000
90, 0.00					11.698	11.705	11.703	11.709
100, 0.00					6.932	6.956	6.950	6.958
110, 0.00					4.155	4.190	4.185	4.191
120, 0.00					2.510	2.551	2.548	2.551
AAE		0.76	**0.08**	0.23		0.040	0.042	**0.038**
APE		0.013	**0.001**	0.004		0.004	0.004	**0.004**
MAE		2.07	**0.12**	0.57		0.110	0.120	**0.103**
MPE		0.038	**0.002**	0.010		0.016	**0.015**	0.016