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Iliadi, Fotoula; Mariolis, Theodore; Soklis, George and Tsoulfidis, Lefteris

Department of Public Administration, Panteion University, Athens, Greece, Department of Economics, University of Macedonia, Thessaloniki, Greece

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Bienenfeld's Approximation of Production Prices and Eigenvalue Distribution: Some More Evidence from Five European Economies*

Fotoula Iliadi,[1] Theodore Mariolis,[1] George Soklis[1] & Lefteris Tsoulfidis[2]
[1]Department of Public Administration, Panteion University, Athens, Greece
[2]Department of Economics, University of Macedonia, Thessaloniki, Greece

Abstract
This paper tests Bienenfeld's polynomial approximation of production prices using data from ten symmetric input-output tables of five European economies. The empirical results show that the quadratic formula works extremely well and its accuracy is connected to the actual distribution of the eigenvalues of the matrices of vertically integrated technical coefficients.

Keywords: Bienenfeld's approximation, Damping ratio, Eigenvalue distribution, Empirical evidence, Production prices
JEL classifications: B51, C67, D46, D57, E11

1. Introduction

It is well known that, in a world of production of commodities by means of commodities, the pattern of the price-variations arising from a change in distribution may be complex (Sraffa, 1960, §§19-20 and 48). However, typical findings in many empirical studies of single-product systems are that (i) the production price-profit rate curves are, more often than not, monotonic; (ii) non-monotonic production price-profit rate curves are not only rare but also have no more than one extremum point; therefore, (iii) the approximation of the production prices through Bienenfeld's (1988) linear and, a fortiori, quadratic formulae works pretty well; and (iv) the so-called 'wage-profit curves' are almost linear irrespective of the numeraire chosen (i.e. the correlation coefficients between the wage and profit rates tend to be above 99%), which implies, in its turn, that there is empirical basis for searching for an 'approximate surrogate production function'.[1] As it has recently been argued, these findings could be connected to the distribution of the eigenvalues of the 'matrices of vertically integrated technical

Address for correspondence: Theodore Mariolis, Department of Public Administration, Panteion University, 136, Syngrou Ave, Athens 17671, Greece; Email: mariolis@hotmail.gr

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[1] See, for example, Petrović (1991), Shaikh (1998, 2010), Tsoulfidis and Mariolis (2007), Schefold (2008a) (and the references provided there). To our knowledge, there are only two empirical studies, based on Supply and Use Tables (SUT) and, therefore, on models of joint production, which show that the relevant actual systems do not necessarily have the usual properties of single-product systems (see Mariolis and Soklis, 2010, and Soklis, 2011). However, when the wage-profit curves are strictly decreasing, they tend to be almost linear (see Soklis, 2011, pp. 554-557).

coefficients' (Pasinetti, 1973), i.e. to the fact that the moduli of the first non-dominant eigenvalues fall quite rapidly and the rest constellate in much lower values (see Schefold, 2008b, c, and Mariolis and Tsoulfidis, 2009, 2011).

This paper tests Bienenfeld's polynomial approximation with data from the Symmetric Input-Output Tables (SIOT) of the Danish (for the years 2000 and 2004), Finnish (for the years 1995 and 2004), French (for the years 1995 and 2005), German (for the years 2000 and 2002) and Swedish (for the years 1995 and 2005) economies, and connects its accuracy to the actual eigenvalue distributions. It should be noted that we decided to use the SIOT of the above five countries mainly because (i) they include all the data required for such an investigation; (ii) the selected input-output tables are comparable to each other in terms of industry detail, but also there are cases where the length of the time span between the selected years for each country is large enough to allow for technological change to take place and give rise to possible differential results; and (iii) as far as we know, input-output data from these countries have not been used neither for testing Bienenfeld's approximation nor in other related questions. The investigation is carried out on the basis of a circulating capital model, as there are no available data for the construction of the matrices of fixed capital stocks.

The remainder of the paper is structured as follows. Section 2 presents Bienenfeld's approximation. Section 3 brings in the empirical evidence and evaluates the results. Section 4 concludes.

2. Bienenfeld's approximation

Consider a closed, linear system involving only single products, basic commodities (in the sense of Sraffa 1960, §6) and circulating capital. Furthermore, assume that (i) the input-output coefficients are fixed; (ii) the system is viable, i.e. the Perron-Frobenius (P-F hereafter) eigenvalue of the irreducible $n \times n$ matrix of input-output coefficients, $\mathbf{A}$, is less than 1;[2] (iii) the profit rate, r, is uniform; (iv) labour is not an input to the

[2] Matrices (and vectors) are delineated in boldface letters. The transpose of an $n \times 1$ vector $\mathbf{x} \equiv [x_i]$ is denoted by $\mathbf{x}^{\mathrm{T}}$, and the diagonal matrix formed from the elements of $\mathbf{x}$ is denoted by $\hat{\mathbf{x}}$. $\lambda_{\mathbf{A}1}$ denotes the P-F eigenvalue of a semi-positive $n \times n$ matrix $\mathbf{A}$ and $(\mathbf{x}_{\mathbf{A}1}, \mathbf{y}_{\mathbf{A}1}^{\mathrm{T}})$ the corresponding eigenvectors, whilst $\lambda_{\mathbf{A}k}$, $k = 2,...,n$ and $|\lambda_{\mathbf{A}2}| \geq |\lambda_{\mathbf{A}3}| \geq ... \geq |\lambda_{\mathbf{A}n}|$, denotes the non-dominant eigenvalues of $\mathbf{A}$ and $(\mathbf{x}_{\mathbf{A}k}, \mathbf{y}_{\mathbf{A}k}^{\mathrm{T}})$ the corresponding eigenvectors. Finally, $\mathbf{A}_j$ denotes the j – th column of $\mathbf{A}$, and $\mathbf{e}$ the summation vector, i.e. $\mathbf{e} \equiv [1,1,...,1]^{\mathrm{T}}$.

household sector and may be treated as homogeneous because relative wage rates are invariant (see Sraffa, 1960, §10; Kurz and Salvadori, 1995, pp. 322-325); and (v) the net product is distributed to profits and wages that are paid at the beginning of the common production period and there are no savings out of this income. Finally, the following are given: (i) the technical conditions of production, that is the pair $[\mathbf{A}, \mathbf{l}]$, where $\mathbf{l}$ $(>\mathbf{0})$ denotes the vector of direct labour coefficients; and (ii) the real wage rate, which is denoted by the (semi-) positive vector $\mathbf{b}$.

On the basis of these assumptions, we can write:

$$\mathbf{p}^{\mathrm{T}} = (1+r)(w\mathbf{l}^{\mathrm{T}} + \mathbf{p}^{\mathrm{T}}\mathbf{A}) \tag{1}$$

where $\mathbf{p}$ denotes a vector of production prices and w the money wage rate. Substituting $w = \mathbf{p}^{\mathrm{aT}}\mathbf{b}$, where $\mathbf{p}^{\mathrm{a}}$ denotes the vector of the <u>actual</u> prices of production, in equation (1) yields

$$\mathbf{p}^{\mathrm{aT}} = (1+r^{\mathrm{a}})\mathbf{p}^{\mathrm{aT}}\mathbf{C} \tag{2}$$

where r^{a} denotes the actual profit rate and $\mathbf{C} \equiv \mathbf{A} + \mathbf{b}\mathbf{l}^{\mathrm{T}}$ the matrix of the 'augmented' input-output coefficients. Since a non-positive vector of commodity prices is economically insignificant, it follows that $r^{\mathrm{a}} = \lambda_{\mathbf{C}1}^{-1} - 1$ and $\mathbf{p}^{\mathrm{aT}} = \mathbf{y}_{\mathbf{C}1}^{\mathrm{T}}$.

Furthermore, equation (1) after rearrangement gives:

$$\mathbf{p}^{\mathrm{T}} = (1+r)w\mathbf{v}^{\mathrm{T}} + r\mathbf{p}^{\mathrm{T}}\mathbf{H} \tag{3}$$

where $\mathbf{H} \equiv \mathbf{A}[\mathbf{I} - \mathbf{A}]^{-1}$ $(>\mathbf{0})$ denotes the vertically integrated technical coefficients matrix, $\mathbf{I}$ the identity matrix, and $\mathbf{v}^{\mathrm{T}} \equiv \mathbf{l}^{\mathrm{T}}[\mathbf{I} - \mathbf{A}]^{-1}$ $(>\mathbf{0}^{\mathrm{T}})$ the vector of vertically integrated labour coefficients or 'labour values'. If Sraffa's (1960, ch. 4) Standard commodity is chosen as the standard of value or numeraire, i.e. $\mathbf{p}^{\mathrm{T}}[\mathbf{I} - \mathbf{A}]\mathbf{x}_{\mathbf{A}1} = 1$, with $\mathbf{l}^{\mathrm{T}}\mathbf{x}_{\mathbf{A}1} = 1$, then equation (1) implies that

$$w = (1+r)^{-1}(1 - rR^{-1}) = (1+r)^{-1}(1-\rho) \tag{4}$$

which gives a non-linear 'wage-profit curve', the result of our assumption that wages are being paid <u>ex ante</u> (see also Burmeister, 1968): $R \equiv \lambda_{\mathbf{A}1}^{-1} - 1$ $(= \lambda_{\mathbf{H}1}^{-1})$ denotes the maximum rate of profits, i.e. the rate of profits corresponding to $w = 0$ and $\mathbf{p} > \mathbf{0}$, R^{-1} equals the ratio of means of production to labour in the Standard system (which is independent of prices and distribution), and $\rho \equiv rR^{-1}$, $0 \le \rho \le 1$, denotes the 'relative rate of profits', which is no greater than the share of profits in this system (see Sraffa, 1960, §§29-32). Substituting equation (4) in equation (3) yields

$$\mathbf{p}^{\mathrm{T}} = (1-\rho)\mathbf{v}^{\mathrm{T}} + \rho\mathbf{p}^{\mathrm{T}}\mathbf{J} \tag{5}$$

where $\mathbf{J} \equiv R\mathbf{H}$, with $\lambda_{\mathbf{J}1} = R\lambda_{\mathbf{H}1} = 1$ and $|\lambda_{\mathbf{J}k}| < 1$,[3] is similar to the column stochastic matrix $\mathbf{M} \equiv \hat{\mathbf{y}}_{\mathbf{J}1}\mathbf{J}\hat{\mathbf{y}}_{\mathbf{J}1}^{-1}$ (the elements of which are independent of the choice of physical measurement units and the normalization of $\mathbf{y}_{\mathbf{A}1}$). Equation (5) implies that p_j is a convex combination of v_j and $\mathbf{p}^{\mathrm{T}}\mathbf{J}_j$, where the latter equals the ratio of means of production in the vertically integrated industry producing commodity j to means of production in the Standard system. At $\rho = 0$, we obtain

$$\mathbf{p}^{\mathrm{T}}(0) = \mathbf{v}^{\mathrm{T}} \tag{6}$$

whereas at the other extreme case, $\rho = 1$, we obtain

$$\mathbf{p}^{\mathrm{T}}(1) = \mathbf{p}^{\mathrm{T}}(1)\mathbf{J} \tag{7}$$

from which it follows that $\mathbf{p}^{\mathrm{T}}(1) = \mathbf{y}_{\mathbf{J}1}^{\mathrm{T}}$. Iff $\mathbf{l}^{\mathrm{T}}$ (and, therefore, $\mathbf{v}^{\mathrm{T}}$) is the left P-F eigenvector of $\mathbf{A}$, then $\mathbf{p}^{\mathrm{T}} = \mathbf{p}^{\mathrm{T}}(0)$, i.e. the 'pure labour theory of value' (Pasinetti, 1977, pp. 76-78) holds true. Finally, if $\rho < 1$, then from equations (5) and (6) we derive

$$\mathbf{p}^{\mathrm{T}} = (1-\rho)\mathbf{p}^{\mathrm{T}}(0)[\mathbf{I} - \rho\mathbf{J}]^{-1} = (1-\rho)\mathbf{p}^{\mathrm{T}}(0)\sum_{h=0}^{\infty}\rho^h\mathbf{J}^h \tag{8}$$

This is the so-called the 'reduction of prices to dated quantities of <u>embodied</u> labour' (Kurz and Salvadori, 1995, p. 175) in terms of $(1-\rho)\rho^h$, where (Steedman, 1999, pp. 315-316): (i) $(1-\rho)\sum_{h=0}^{\infty}\rho^h = 1$; (ii) the term $(1-\rho)\rho^h$, $h \geq 2$, takes its maximum value of $h^h(h+1)^{-(h+1)}$ ($\rightarrow 0$, i.e. tends to zero as h tends to infinity) at $\rho = h(h+1)^{-1}$, and its 'inflection value' of $2(h-1)^h(h+1)^{-(h+1)}$ ($\rightarrow 0$) at $\rho = (h-1)(h+1)^{-1}$); (iii) the ratio of the inflection value to the maximum value tends to $2e^{-1} \simeq 0.736$; (iv) the first term (the sum of the first two terms) is greater than the sum of all the remaining terms for $\rho < 2^{-1} = 0.5$ (for $\rho < 2^{-0.5} \simeq 0.707$); and, therefore, (v) only the very early terms are

[3] If $\lambda_{\mathbf{A}k}$ is positive, then $\lambda_{\mathbf{A}k} < \lambda_{\mathbf{A}1}$. If it is negative or complex, then $|\lambda_{\mathbf{A}k}| \leq \lambda_{\mathbf{A}1}$ (the equality holds iff $\mathbf{A}$ is imprimitive) and $|1-\lambda_{\mathbf{A}k}| > 1-|\lambda_{\mathbf{A}k}|$. Hence,

$$|\lambda_{\mathbf{J}k}| = R|\lambda_{\mathbf{A}k}||1-\lambda_{\mathbf{A}k}|^{-1} < R|\lambda_{\mathbf{A}k}|(1-|\lambda_{\mathbf{A}k}|)^{-1} < 1$$

holds for all k.

important in determining the prices of production, provided that one is interested only in relatively low, i.e. realistic, values of ρ.[4]

For any semi-positive $\mathbf{y}^{\mathrm{T}}$, $\mathbf{y}^{\mathrm{T}}\mathbf{J}^{h}$ tends to the left P-F eigenvector of $\mathbf{J}$ as h tends to infinity, i.e.

$$\mathbf{y}^{\mathrm{T}}\mathbf{J}^{h} \to (\mathbf{y}^{\mathrm{T}}\mathbf{x}_{\mathbf{A1}})(\mathbf{p}^{\mathrm{T}}(1)\mathbf{x}_{\mathbf{A1}})^{-1}\mathbf{p}^{\mathrm{T}}(1) \tag{9}$$

(see, e.g. Seneta, 2006, pp. 9-11) and, therefore, for a sufficiently large value of t such that

$$\mathbf{p}^{\mathrm{T}}(0)\mathbf{J}^{t} \approx \mathbf{p}^{\mathrm{T}}(0)\mathbf{J}^{t+1} \approx \ldots \approx \mathbf{p}^{\mathrm{T}}(1) \tag{10}$$

equation (8) can be written as

$$\mathbf{p} \approx \mathbf{p}^{\mathrm{T}}(0) + \sum_{h=1}^{t-1} \rho^{h}(\mathbf{p}^{\mathrm{T}}(0)\mathbf{J}^{h} - \mathbf{p}^{\mathrm{T}}(0)\mathbf{J}^{h-1}) + \rho^{t}(\mathbf{p}^{\mathrm{T}}(1) - \mathbf{p}^{\mathrm{T}}(0)\mathbf{J}^{t-1}) \tag{11}$$

This is Bienenfeld's approximate formula, which is exact at the extreme, economically significant, values of ρ, and gives the correct slope of the $p_j - \rho$ curves at $\rho = 0$ (for alternative derivations, see Mariolis and Tsoulfidis, 2009, pp. 7-9, and, 2011, pp. 98-99). Setting $t = 1$, it reduces to the linear formula

$$\mathbf{p}^{\mathrm{T}} \approx \mathbf{p}^{\mathrm{T}}(0) + \rho(\mathbf{p}^{\mathrm{T}}(1) - \mathbf{p}^{\mathrm{T}}(0)) \tag{12}$$

and substituting relation (12) into

$$k_j \equiv \mathbf{p}^{\mathrm{T}}\mathbf{H}_j(p_j(0))^{-1} \tag{13}$$

which expresses the capital-intensity of the vertically integrated industry producing commodity j, yields $k_j \approx k_j(0)$, since $\mathbf{p}^{\mathrm{T}}(0)\mathbf{J} \approx \mathbf{p}^{\mathrm{T}}(1)$, i.e. the approximate $k_j - \rho$ relationships are constant. Setting $t = 2$, it reduces to the quadratic formula

$$\mathbf{p}^{\mathrm{T}} \approx \mathbf{p}^{\mathrm{T}}(0) + \rho(\mathbf{p}^{\mathrm{T}}(0)\mathbf{J} - \mathbf{p}^{\mathrm{T}}(0)) + \rho^{2}(\mathbf{p}^{\mathrm{T}}(1) - \mathbf{p}^{\mathrm{T}}(0)\mathbf{J}) \tag{14}$$

and substituting relation (14) into equation (13) yields

$$k_j \approx k_j(0) + \rho(k_j(1) - k_j(0)) \tag{15}$$

i.e. the approximate $p_j - \rho$ curves have at most one extremum point, at

$$\rho_j^{*} = 2^{-1}(p_j(0) - \mathbf{p}^{\mathrm{T}}(0)\mathbf{J}_j)(p_j(1) - \mathbf{p}^{\mathrm{T}}(0)\mathbf{J}_j)^{-1} = 2^{-1}(R^{-1} - k_j(0))(k_j(1) - k_j(0))^{-1}$$

where $0 \le \rho_j^{*} \le 1$ does not necessarily hold true, and the approximate $k_j - \rho$ relationships are linear.

[4] To our knowledge, there is no relevant empirical study where ρ is considerably greater than 0.4 (than 0.5), provided that wages are paid at the beginning (end) of the production period (see Mariolis and Tsoulfidis, 2010a, and Mariolis and Soklis, 2011, pp. 616-617).

The accuracy of relation (10) and, therefore, the accuracy of a t – th order approximation is directly related to the rate of convergence in (9), which in its turn is directly related to the magnitudes of $|\lambda_{\mathbf{J}k}|^{-1}$. In fact, the convergence is asymptotically exponential, at a rate at <u>least</u> as fast as $\log|\lambda_{\mathbf{J}2}|^{-1}$ (the number $|\lambda_{\mathbf{J}2}|^{-1}$ is known as the 'damping ratio' in population dynamics theory; see, e.g. Keyfitz and Caswell, 2005, pp. 165-166). Theoretically speaking, there are two extreme cases: (i) if $\mathbf{A}$ has rank 1, then $\lambda_{\mathbf{J}k}=0$, for all k, $\mathbf{p}^{\mathrm{T}}(0)\mathbf{J}=\mathbf{p}^{\mathrm{T}}(1)$ and, therefore, relation (12) holds exactly;[5] and (ii) if $\lambda_{\mathbf{J}k}\approx 1$, for all k, then $\mathbf{p}^{\mathrm{T}}(0)\mathbf{J}\approx\mathbf{p}^{\mathrm{T}}(0)$ and, therefore, $\mathbf{p}^{\mathrm{T}}\approx\mathbf{p}^{\mathrm{T}}(0)$ (see also Hartfiel and Meyer, 1998; Mariolis and Tsoulfidis, 2010b, Appendix, and, 2011, pp. 96-98). In more general terms, the Hopf-Ostrowski and Deutsch upper bounds (or 'coefficients of ergodicity'; Seneta, 2006, pp. 63-64) imply that

$$|\lambda_{\mathbf{J}2}|\leq 2^{-1}\max\{\sum_{f=1}^{n}\left|m_{fi}-m_{fj}\right|\}\leq(\tau-1)(\tau+1)^{-1}\leq(L-s)(L+s)^{-1}<1$$

where $\tau^2\equiv\max\{(m_{ij}m_{gl}(m_{il}m_{gj})^{-1})\}$ and $L\,(s)$ represents the largest (smallest) element of $\mathbf{M}\equiv[m_{ij}]$ (see Ostrowski, 1963, and Maitre, 1970). Thus, we may conclude that when the <u>columns</u> of $\mathbf{M}$ tend to be close to each other, a low-order approximation works pretty well.[6]

3. Results and their evaluation

The application of the previous analysis to the SIOT of the Danish (for the years 2000, $n=56$, and 2004, $n=56$), Finnish (for the years 1995, $n=57$, and 2004, $n=57$), French (for the years 1995, $n=58$, and 2005, $n=57$), German (for the years 2000, $n=57$, and 2002, $n=57$) and Swedish (for the years 1995, $n=53$, and 2005, $n=51$) economies gives the results summarized in Tables 1 to 4.[7]

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[5] In that case all the columns of $\mathbf{M}$ are equal to each other. It may also be noted that, when $\mathbf{J}$ is a random matrix, with identically and independently distributed entries, Bródy's (1997) conjecture implies that $|\lambda_{\mathbf{J}2}|$ tends to zero, with speed $1/\sqrt{n}$, when n tends to infinity (as Sun, 2008, shows, Bródy's conjecture can be proved using theorems provided by Goldberg <u>et al</u>., 2000; see also Goldberg and Neumann, 2003).

[6] For an alternative, but rather different approximation formula (the 'spectral approximation'), which is also exact at the extreme values of ρ, and its accuracy is also directly related to the magnitudes of $|\lambda_{\mathbf{J}k}|^{-1}$, see Mariolis and Tsoulfidis (2011, pp. 99-100 and 112-115).

[7] The SIOT and the corresponding levels of sectoral employment are provided via the Eurostat website (http://ec.europa.eu/eurostat). The degree of disaggregation is such that 59 product/industry groups are

Table 1 reports (i) the maximum, actual and relative profit rates (estimated from equations (1), with $w=0$, and (2)); and (ii) the deviation between the vector of the actual production prices, $\mathbf{p}^{\mathrm{aT}}$ (see equation (2)), and the vector of the approximate production prices, $\mathbf{p}_{\mathrm{B}}^{\mathrm{a\,T}}$, which is estimated from Bienenfeld's quadratic approximation (see relation (14)) and $\rho = \rho^{\mathrm{a}}$. This deviation is measured by the 'd– distance' (Steedman and Tomkins, 1998; see also Mariolis and Soklis, 2011), which is independent of the choice of numeraire and physical measurement units, and defined as $d^{\mathrm{a}} \equiv \sqrt{2(1-\cos\theta^{\mathrm{a}})}$, where θ^{a} denotes the Euclidean angle between the vectors $\mathbf{p}^{\mathrm{aT}}(\hat{\mathbf{p}}_{\mathrm{B}}^{\mathrm{a}})^{-1}$ and $\mathbf{e}$. Since, however, the SIOT have different dimensions and the theoretically minimum value of $\cos\theta^{\mathrm{a}}$ equals $1/\sqrt{n}$, Table 1 also reports the 'normalized d- distance' (Mariolis and Soklis, 2010, p. 94), defined as $d_{\mathrm{n}}^{\mathrm{a}} \equiv d^{\mathrm{a}} D^{-1}$, where $D \equiv \sqrt{2[1-(1/\sqrt{n})]}$ denotes the theoretically maximum value of the 'd- distance'.

Table 1. The actual profit rates and the deviation between the actual production prices and their quadratic approximation

	Denmark		Finland		France		Germany		Sweden	
	2000	2004	1995	2004	1995	2005	2000	2002	1995	2005
R	0.920	0.867	0.699	0.645	0.899	0.855	1.000	1.052	0.859	0.807
r^{a}	0.344	0.326	0.323	0.325	0.322	0.308	0.342	0.362	0.336	0.297
ρ^{a}	0.374	0.376	0.462	0.504	0.358	0.360	0.342	0.344	0.392	0.368
d^{a}	0.325	0.372	0.012	0.057	0.007	0.028	0.034	0.033	0.247	0.436
$d_{\mathrm{n}}^{\mathrm{a}}$	0.247	0.283	0.009	0.043	0.005	0.021	0.026	0.025	0.188	0.333

Table 2 reports the Euclidean angles (measured in degrees), which depend on the choice of physical measurement units, and the 'd– distances' between $\mathbf{p}^{\mathrm{T}}(1)$ and $\mathbf{l}^{\mathrm{T}}$,

identified. However, there are cases in which all the elements or only the labour inputs associated with certain industries equal zero. Therefore, we remove them from our analysis or we make the appropriate aggregations, respectively (see also Soklis, 2009, Appendix 1). Finally, for the construction of the relevant variables ($\mathbf{A}$, $\mathbf{l}$ and $\mathbf{b}$) we follow the usual procedure (see, e.g. Ochoa, 1989, Appendix). Mathematica 7.0 is used in the calculations, whilst the precision in internal calculations is set to 16 digits. The analytical results are available on request from the authors.

and between $\mathbf{p}^{\mathrm{T}}(1)$ and $\mathbf{p}^{\mathrm{T}}(0)\mathbf{J}^{t}$, $t = 0,1,2,...,5$ (see relation (10)): the angles (distances) are denoted by ϕ_{I} (d_{I}) and ϕ_t (d_t), respectively.

Table 2. Indicators of the accuracy of Bienenfeld's approximation

	Denmark		Finland		France		Germany		Sweden	
	2000	2004	1995	2004	1995	2005	2000	2002	1995	2005
ϕ_{I}	47.96°	51.73°	54.83°	61.23°	46.99°	51.72°	49.67°	49.59°	46.03°	48.07°
ϕ_0	28.50°	33.06°	36.34°	47.96°	28.82°	31.22°	30.87°	31.14°	27.11°	27.01°
ϕ_1	8.33°	13.67°	13.81°	35.77°	9.27°	11.51°	9.65°	9.99°	6.71°	5.35°
ϕ_2	3.75°	7.64°	7.04°	31.12°	3.90°	6.46°	4.80°	5.31°	3.01°	2.13°
ϕ_3	1.86°	4.48°	3.84°	27.32°	1.64°	3.83°	2.51°	3.03°	1.46°	0.91°
ϕ_4	0.98°	2.72°	2.18°	23.89°	0.72°	2.33°	1.34°	1.76°	0.72°	0.39°
ϕ_5	0.52°	1.68°	1.26°	20.79°	0.35°	1.41°	0.72°	1.03°	0.36°	0.17°
d_{I}	0.686	0.792	0.838	0.915	0.782	0.801	0.731	0.729	0.765	0.923
d_0	0.419	0.502	0.557	0.729	0.478	0.483	0.472	0.485	0.408	0.404
d_1	0.151	0.230	0.218	0.438	0.177	0.186	0.186	0.201	0.120	0.101
d_2	0.066	0.122	0.099	0.307	0.072	0.087	0.084	0.093	0.049	0.034
d_3	0.030	0.067	0.052	0.241	0.027	0.042	0.044	0.051	0.022	0.013
d_4	0.014	0.038	0.029	0.198	0.010	0.023	0.024	0.030	0.011	0.005
d_5	0.007	0.023	0.017	0.166	0.004	0.013	0.013	0.018	0.005	0.002

Table 3 reports $|\lambda_{\mathbf{J}2}|^{-1}$, $|\lambda_{\mathbf{J}3}|^{-1}$, $|\lambda_{\mathbf{J}n}|^{-1}$ (see also Figure 1, which displays the location of all the eigenvalues in the complex plane) and the following measures of the distribution of the moduli of the non-dominant eigenvalues of $\mathbf{J}$: (i) the arithmetic mean, *AM*, that gives equal weight to all moduli; (ii) the standard deviation, *SD*; (iii) the coefficient of variation, $CV \equiv SD(AM)^{-1}$; (iv) the geometric mean, *GM*, which in our case can be written as $|\det \mathbf{J}|^{(n-1)^{-1}}$ and assigns more weight to lower moduli, and, therefore, is more appropriate for detecting the central tendency of an exponential set of numbers; (v) the so-called spectral flatness, $SF \equiv GM(AM)^{-1}$; (vi) $\pi_2 \equiv \max\{\pi_k \equiv |\lambda_{\mathbf{J}k}|(\sum_{k=2}^{n}|\lambda_{\mathbf{J}k}|)^{-1}\}$, where π_k represents a set of relative frequencies; (vii) the relative (or normalized) entropy, *RE*, defined as the ratio of the 'information content or Shannon entropy', *E*, to its maximum possible value, i.e. $RE \equiv E(E_{\max})^{-1}$, where $E \equiv -\sum_{k=2}^{n} \pi_k \log\pi_k$ and $E_{\max} \equiv \log(n-1)$ is the maximum value of *E* corresponding to $\pi_k = (n-1)^{-1}$ for all *k*; and (viii) the relative 'equivalent number', *REN* ≡ EN$(n-1)^{-1}$, where *EN* denotes the so-called equivalent number, which is determined by the equation $\log EN = E$ and represents the number of eigenvalues with equal moduli that would result in the same amount of entropy. *SF* and *RE* are known to be alternative, but different, measures of similarity (or closeness) of the moduli and take on values from near zero to one: when all $|\lambda_{\mathbf{J}k}|$ are equal to each other, then *AM* = *GM*, $\pi_k = (n-1)^{-1}$ and, therefore, *SF* = *RE* = *REN* = 1.[8] However, a low *SF* rather reflects the presence of a much lower than the average π_n, whereas a low *RE* rather reflects the presence of a much higher than the average π_2.

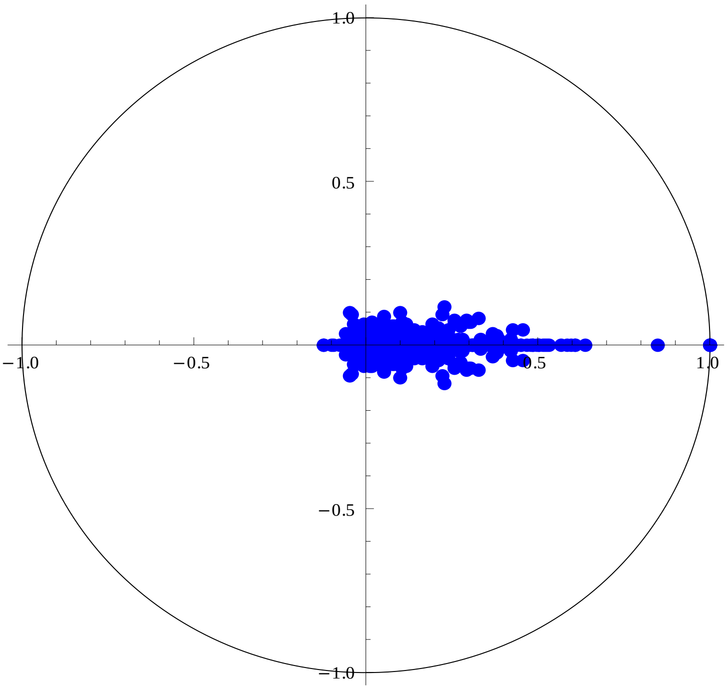


Figure 1. The location of all the eigenvalues in the complex plane

[8] For a connection between *SF* and entropy expressions, see Mariolis and Tsoulfidis (2011, p. 104, footnote 24).

Table 3. The distribution of the moduli of the non-dominant eigenvalues

	Denmark		**Finland**		**France**		**Germany**		**Sweden**	
	2000	**2004**	**1995**	**2004**	**1995**	**2005**	**2000**	**2002**	**1995**	**2005**
$\lvert\lambda_{\mathbf{J}2}\rvert^{-1}$	1.914	1.568	1.676	1.177	1.636	1.702	1.753	1.641	1.881	2.369
$\lvert\lambda_{\mathbf{J}3}\rvert^{-1}$	2.057	1.990	2.308	1.990	1.889	2.208	2.013	1.939	2.302	2.563
$\lvert\lambda_{\mathbf{J}n}\rvert^{-1}$	1541.750	815.370	735.900	2517.050	29324.400	817.635	729.625	158.114	299.881	1092.060
AM	0.118	0.108	0.100	0.103	0.131	0.128	0.178	0.178	0.098	0.099
SD	0.014	0.013	0.015	0.020	0.021	0.015	0.023	0.023	0.014	0.010
CV	0.119	0.120	0.150	0.194	0.160	0.117	0.129	0.129	0.143	0.101
GM	0.069	0.065	0.047	0.047	0.059	0.076	0.106	0.111	0.050	0.052
SF	0.585	0.602	0.470	0.456	0.450	0.594	0.596	0.624	0.510	0.525
π_2 (%)	8	11	11	15	8	8	6	6	10	9
RE	0.870	0.863	0.825	0.821	0.856	0.880	0.900	0.900	0.828	0.849
R EN (%)	60	58	50	48	57	63	66	66	50	56

Finally, in all cases the moduli of the first non-dominant eigenvalues fall quite rapidly and the rest constellate in much lower values forming a 'long tail'. In plotting these data for each of the countries and years we found, after various experimentations, that a single exponential functional form fits all the data quite well, as this can be judged by the high R – squared, R^2, and the fact that all the estimated coefficients are statistically significant, with zero probability values. The equation that captures this configuration of eigenvalues is of the following form:

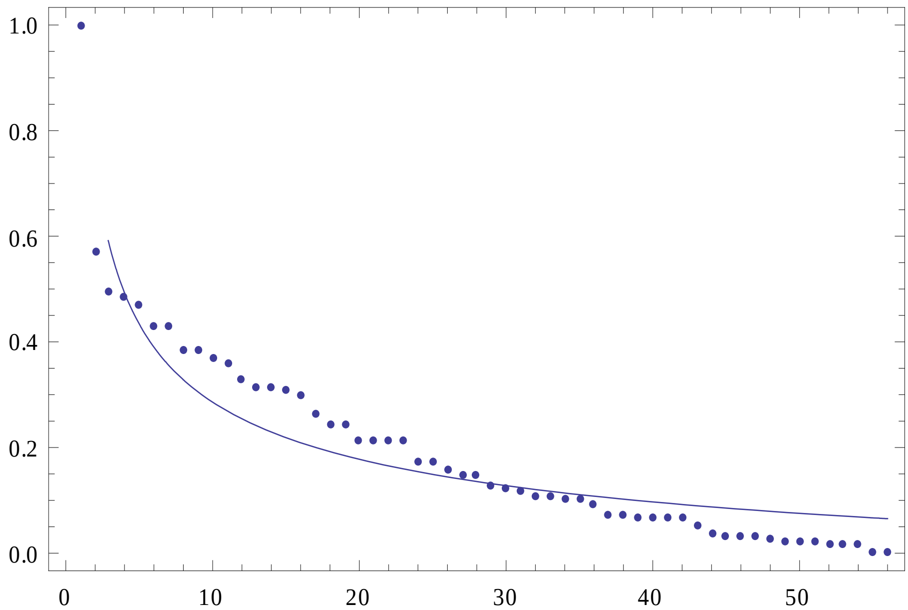
$$y = c + b\exp(x^a)$$

where a = – 0.3, 0.630 (Sweden, 2005) $\leq b \leq$ 0.754 (France, 1995), – 1.012 (Finland, 2004) $\leq c \leq$ – 0.857 (Sweden, 2005), and 0.959 (Germany, 2000) $\leq R^2 \leq$ 0.994 (Finland, 1995) (see Table 4, which also reports the values of the parameter a that approximately maximize R^2, as well as the relevant values of *b*, *c*, and R^2, and Figure 2). It is expected, therefore, that the *SF* would be relatively low and that the opposite would hold true regarding *RE*. In fact, the results (see Table 3) show that (i) the *SF* is in the range of 0.450 (France, 1995)-0.624 (Germany, 2002); (ii) the *RE* is in the range of 0.821 (Finland, 2004)-0.900 (Germany, 2000 and 2002), and the relevant π_2's are 15% and 6%, respectively; and (iii) the linear regressions between *SF* and *RE*, and between

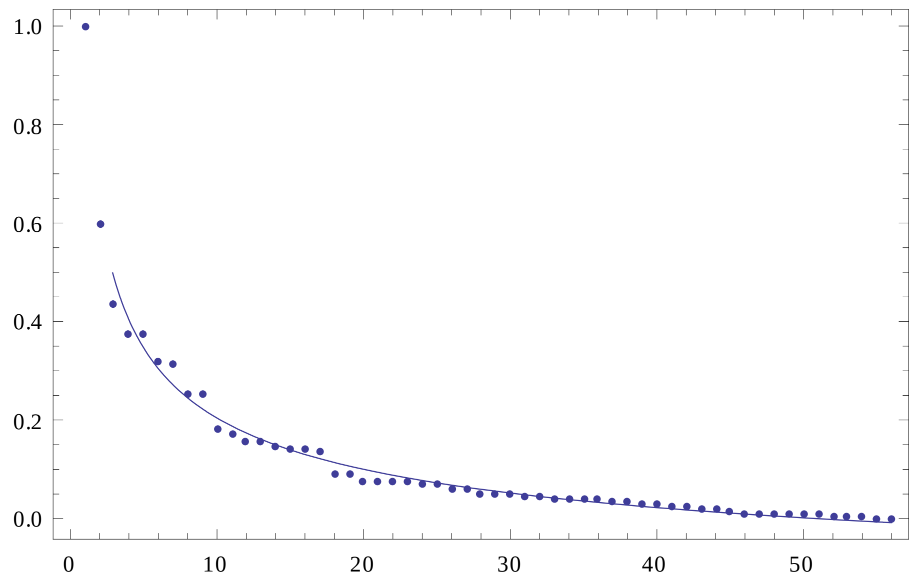
GM and *RE* (both not reported), give statistically significant R^2 values of 0.692 and 0.883, respectively. Thus, it could be concluded that these measures, both separate and combined, give a quite good description of the central tendency and also the skewness of the distribution of the moduli.

Table 4. Estimates of the exponential fit of the moduli of the eigenvalues

	Denmark		Finland		France		Germany		Sweden	
	2000	**2004**	**1995**	**2004**	**1995**	**2005**	**2000**	**2002**	**1995**	**2005**
a	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3
b	0.679	0.675	0.702	0.742	0.754	0.680	0.730	0.737	0.682	0.630
c	-0.902	-0.906	-0.956	-1.012	-1.000	-0.892	-0.918	-0.930	-0.934	-0.857
R^2	0.991	0.987	0.994	0.977	0.987	0.987	0.959	0.964	0.992	0.975
a^*	-0.3	-0.4	-0.3	-0.4	-0.2	-0.3	-0.1	-0.1	-0.4	-0.5
b^*	–	0.653	–	0.717	0.845	–	1.168	1.177	0.656	0.601
c^*	–	-0.781	–	-0.873	-1.323	–	-2.254	-2.273	-0.801	-0.668
$(R^2)^*$	–	0.993	–	0.979	0.994	–	0.982	0.985	0.995	0.983



(a)



(b)

Figure 2. Exponential fit ($a = -0.3$) of the moduli of the eigenvalues; (a) Germany, 2000; and (b) Finland, 1995

From these tables and figures, the associated numerical results and the hitherto analysis we arrive at the following conclusions:

(i). Although ρ^{a} is relatively low, i.e. in the range of 0.342-0.504, there are cases (Denmark and Sweden) where the deviation between the actual production prices and their quadratic approximation is considerably high (see Table 1). It need hardly be reminded, however, that Bienenfeld's approximation is certainly exact only at the extreme values of ρ.

(ii). In all cases, $\mathbf{p}^{\text{T}}(1)$ deviates <u>considerably</u> from $\mathbf{l}^{\text{T}}$ (see Table 2). However, setting aside the Finnish economy for the year 2004, $\mathbf{p}^{\text{T}}(0)\mathbf{J}^{t}$ tends rather quickly to $\mathbf{p}^{\text{T}}(1)$: ϕ_5 is in the range of 0.17$^{\text{o}}$ (Sweden, 2005)-1.68$^{\text{o}}$ (Denmark, 2004), d_5 is in the range of 0.002 (Sweden, 2005)-0.023 (Denmark, 2004) and, as it is easily checked, the average percentage decrease of d_t, i.e. $\bar{\hat{d}} \equiv 5^{-1}\sum_{t=0}^{4} 1 - d_{t+1} d_t^{-1}$, is in the range of 45.8% (Denmark, 2004)-64.9% (Sweden, 2005), whilst for the Finnish economy, 2004, d_5 is almost 0.166 and $\bar{\hat{d}}$ is almost 25.1%. Thus, it is expected that low-order Bienenfeld's approximations would be adequate. Finally, it should be noted that there is a direct relationship between

$|\lambda_{J2}|^{-1}$ and $\bar{\hat{d}}$: Spearman's coefficient is almost 0.721 and the regression $y = bx^a$ gives an R^2 value of 0.989 and statistically significant coefficients ($a \simeq 0.990$ and $b \simeq 0.296$).

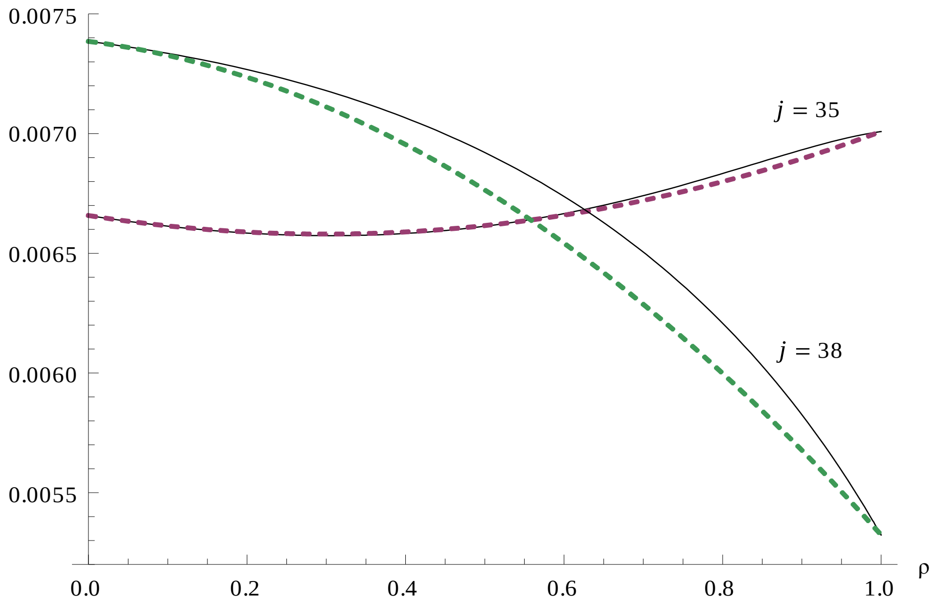
(iii). Non-monotonic price-profit rate curves could not only be considered as rare but also have no more than one extremum point and, therefore, Bienenfeld's quadratic approximation track down accurately enough the trajectories of the actual prices of production. More specifically, there are 105 cases of non-monotonic price movement (i.e. 105/559 $\simeq$ 18.8% of the tested cases) and the arithmetic mean of the mean of the relative errors, $\overline{MRE}$, between the actual, $p_j(\rho)$ (see equation (8)), and the approximate, $p_{Bj}(\rho)$ (see relation (14)), curves, i.e.

$$\overline{MRE} \equiv n^{-1}\sum_{j=1}^{n} MRE_j \text{ , where } MRE_j \equiv \int_0^1 \left|(p_j(\rho) - p_{Bj}(\rho))(p_j(\rho))^{-1}\right| d\rho ,$$

is in the range of 0.267% (Sweden, 2005)-7.069% (Finland, 2004) (see Table 5, which reports the percentage of non-monotonic curves, indicated by n.-m., $\min\{MRE_j\}$, $\max\{MRE_j\}$ and $\overline{MRE}$). For reasons of clarity of presentation and economy of space, in Figure 3 we display only a set of three graphs associated with the Danish, 2004, Finnish, 2004, and Swedish, 2005, economies, respectively, and some of the actual (depicted by solid lines) and the approximate (depicted by dotted lines) curves. Finally, it should be noted that there is an inverse relationship between $|\lambda_{J2}|^{-1}$ (or $\bar{\hat{d}}$) and $\overline{MRE}$: Spearman's coefficient is almost – 0.770 (or – 0.976) and the regression $y = bx^a$ gives an R^2 value of 0.993 (or 0.995) and statistically significant coefficients ($a \simeq -5.870$ (or – 2.985) and $b \simeq 18.370$ (or 0.114); see Figure 4).

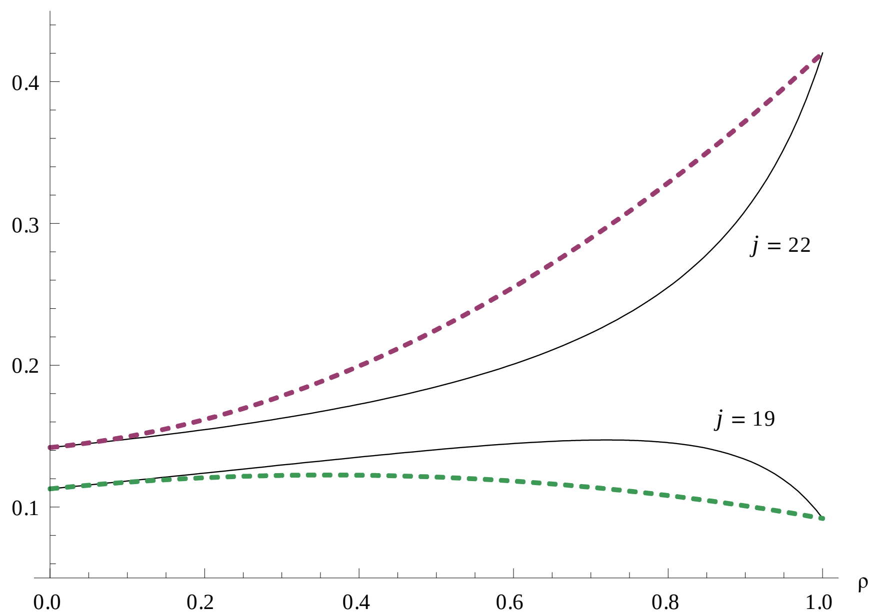
Table 5. The percentage of non-monotonic price-profit rate curves and the accuracy of Bienenfeld's quadratic approximation

	Denmark		**Finland**		**France**		**Germany**		**Sweden**	
	2000	**2004**	**1995**	**2004**	**1995**	**2005**	**2000**	**2002**	**1995**	**2005**
n.-m. (%)	23.2	32.1	15.8	22.8	18.9	14.0	15.8	12.3	20.8	11.8
min $\{MRE_j\}$ (%)	0.042	0.042	0.023	2.152	0.015	0.012	0.042	0.003	0.020	0.000
max $\{MRE_j\}$ (%)	2.243	4.431	2.667	17.199	3.096	6.947	3.926	5.264	1.897	1.498
$\overline{MRE}$ (%)	0.651	1.394	1.208	7.069	0.614	0.712	0.757	0.818	0.517	0.267



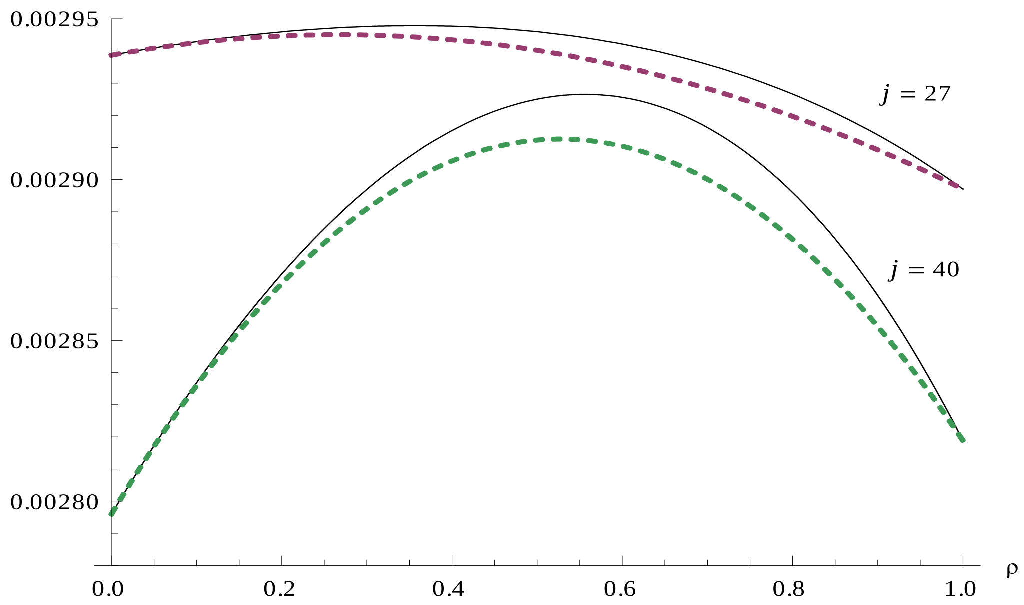
$MRE_{35} \simeq 0.123\%$, $MRE_{38} \simeq 1.728\%$

(a)



$MRE_{19} \simeq 13.373\%$, $MRE_{22} \simeq 17.199\%$

(b)



$\text{MRE}_{27} \simeq 0.139$ %, $\text{MRE}_{40} \simeq 0.307$ %

(c)

Figure 3. Actual and approximate price-profit rate curves: (a) Denmark, 2004; (b) Finland, 2004; and (c) Sweden, 2005

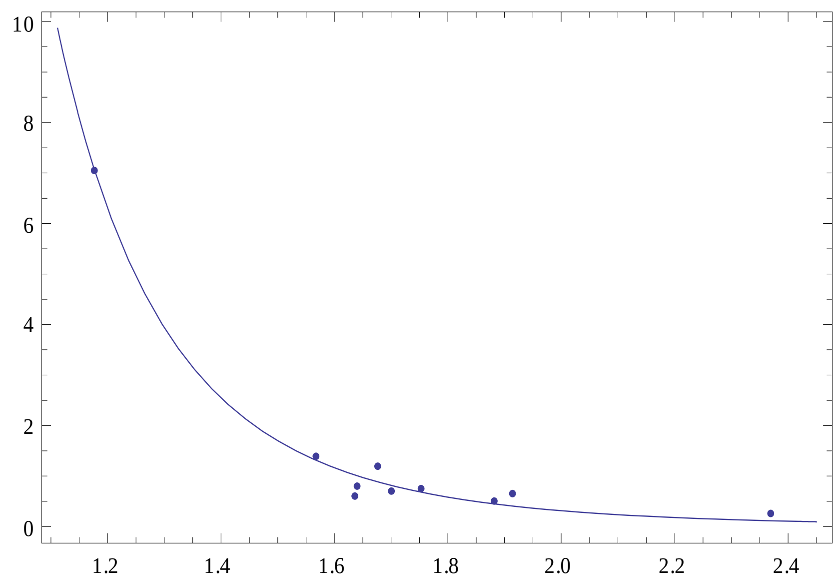


Figure 4. Mean error of Bienenfeld's quadratic approximation vs. damping ratio

(iv). Since Bienenfeld's quadratic approximation works extremely well, the $k_j - \rho$ curves are almost linear (see relation (15)). Figure 5 is representative and displays the capital-intensities of the French economy, $k_{\mathrm{y}} \equiv \mathbf{p}^{\mathrm{T}}\mathbf{Hy}(\mathbf{v}^{\mathrm{T}}\mathbf{y})^{-1}$, where $\mathbf{y}$ denotes the net output vector of the economy, as functions of ρ: they are strictly <u>increasing</u> functions (the same holds true for the German economy, whilst the functions associated with all the other economies are strictly decreasing) and the mean of the relative errors between the actual and the linear curves are in the range of 0.129% (1995)-0.548% (2005).

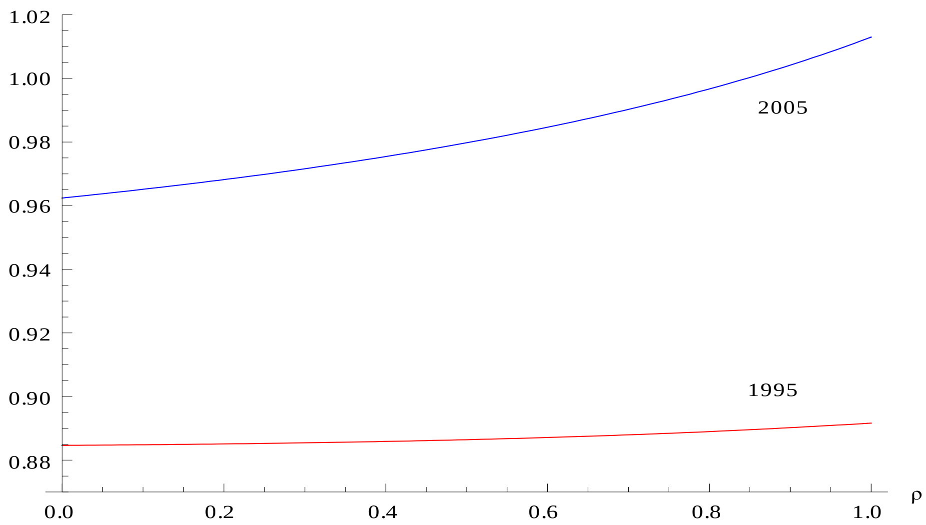


Figure 5. The capital-intensities of the French economy as functions of the relative profit rate

(v). Our results on the distribution of the moduli of the eigenvalues are in absolute accordance, both qualitatively and quantitatively, with those on many diverse economies (i.e. Canada, China, Greece, Japan, Korea, UK, USA, where $19 \leq n \leq 100$; see Mariolis and Tsoulfidis, 2011, pp. 101-109; Schefold, 2008c, pp. 34-36). Thus, it is reasonable to expect that there is a strong tendency towards uniformity in the eigenvalue distribution across countries and over time. Furthermore, moving from the flow to (the more realistic) stock input-output data, it has been found that the damping ratios rise even more abruptly, whilst the third or fourth eigenvalues become 'indistinguishable' from the rest (see Steenge and Thissen, 2005; Mariolis and Tsoulfidis, 2011, pp. 109-111), lending support to the idea of approximating the actual price curves linearly (see also Shaikh, 1998, 2010).

4. Concluding remarks

Using data from ten symmetric input-output tables of five European economies, it has been found that Bienenfeld's quadratic formula track down accurately enough the actual prices of production as functions of the profit rate. More specifically, (i) non-monotonic functions are observed in about 19% of the tested cases; (ii) there is no function with more than one extremum point; and (iii) the arithmetic mean of the mean of the relative errors between the actual and the approximate functions is in the range of 0.3%-7.1%. These findings have been connected to the distribution of the eigenvalues of the matrices of vertically integrated technical coefficients, and, in fact, it has been detected statistically significant relationships between the damping ratio and indicators of the accuracy of Bienenfeld's approximation.

Since there is no little evidence that actual economies exhibit the said attributes and since the production price-profit rate curves associated with an n – sector system may admit up to $2n-4$ extremum points, it seems that there is basis for hypothesizing that the effective dimensions of actual economies are between 2 and 3. In that case, although the 'neoclassical parable relations' do not necessarily hold, there are implications for the empirical counterparts of some capital theory propositions.

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