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# DYNAMIC UNAWARENESS AND RATIONALIZABLE BEHAVIOR\*

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February 18, 2010

## Abstract

We define generalized extensive-form games which allow for mutual unawareness of actions. We extend Pearce's (1984) notion of extensive-form (correlated) rationalizability to this setting, explore its properties and prove existence. We define also a new variant of this solution concept, prudent rationalizability, which refines the set of outcomes induced by extensive-form rationalizable strategies. We apply prudent rationalizability to the analysis of verifiable communication with unawareness. Finally, we define the normal form of a generalized extensive-form game, and characterize in it extensive-form rationalizability by iterative conditional dominance.

**Keywords:** Unawareness, extensive-form games, extensive-form rationalizability, prudent rationalizability, iterative conditional dominance.

**JEL-Classifications:** C70, C72, D80, D82.

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# 1 Introduction

In real-life dynamic interactions, unawareness of players regarding the relevant actions available to them is at least as prevalent as uncertainty regarding other players' strategies, payoffs or moves of nature. Players frequently become aware of actions they (or other players) could have taken in retrospect, when they can only re-evaluate the past actions chosen by partners or rivals who *were* aware of those actions from the start, and hence re-assess their likely future behavior. Yet, while uncertainty can be captured within the standard framework of extensive-form games with imperfect information, unawareness and mutual uncertainty regarding awareness require an extension of this framework. Such an extension is the first task of the current paper.

At first, one may wonder why the standard framework would not suffice. After all, if a player is unaware of an action which is actually available to her, then for all practical purposes she cannot choose it. Why wouldn't it be enough simply to truncate from the tree all the paths starting with such an action?

The reason is that the strategic implications of unawareness of an action are distinct from the unavailability of the same action. To see this, consider the following standard "battle-of-the-sexes" game (where Bach and Stravinsky concerts are the two available choices for each player)

		II	
		B	S
I	B	3, 1	0, 0
	S	0, 0	1, 3

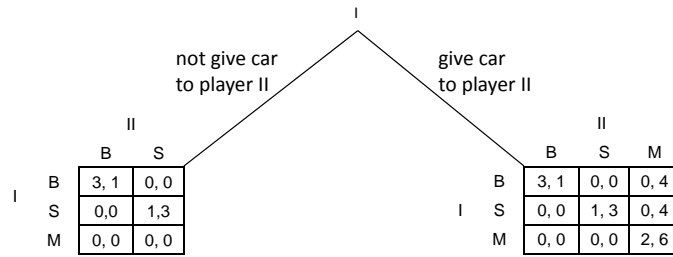
augmented by a dominant Mozart concert for player II:

		II		
		B	S	M
I	B	3, 1	0, 0	0, 4
	S	0, 0	1, 3	0, 4
	M	0, 0	0, 0	2, 6

The new game is dominance solvable, and (M,M) is the unique Nash equilibrium.

Suppose that the Mozart concert is in a distant town, and II can go there only if player I gives him her car in the first place: Here, if player I doesn't give the car to player II, player II may conclude by forward induction that player I would go to the Bach concert

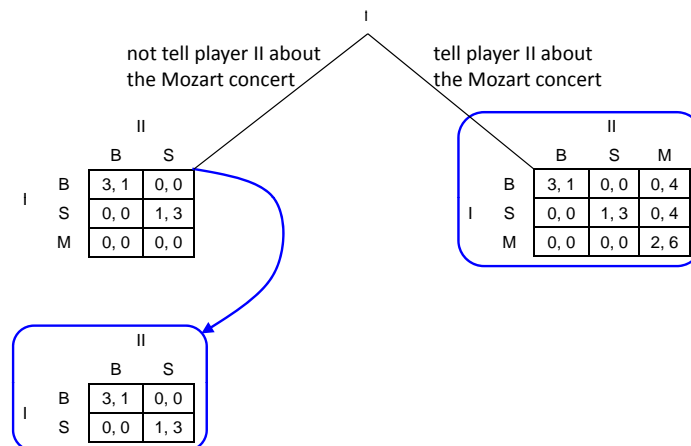
Figure 1:



with the hope of getting the payoff 3 (because by giving the car to II, player I could have achieved the payoff 2). The best reply of player II is to follow suit and attend the Bach concert as well. Hence, in the unique rationalizable outcome, player I is not to give the car to player II and to go to the Bach concert.<sup>1</sup>

But what if, instead, the Mozart concert is in town but player II is initially unaware of the Mozart concert, while player I can enable player II to go to the concert simply by telling him about it? If player II remains unaware of the Mozart concert, then neither does he conceive that player I could have told him about the Mozart concert, and in particular he cannot carry out any forward-induction calculation. For him, the game is a standard battle-of-the-sexes game, where both actions of player I are rationalizable. This strategic situation is depicted in Figure 2.

Figure 2:



<sup>1</sup>For a discussion of forward induction in battle-of-the-sexes games see van Damme (1989).

The strategic situation is not a standard extensive-form game (more on this in Section 2.6 below). If player I chooses not to tell player II about the Mozart concert, then player II's information set (depicted in blue) consists of a node in a simpler game –namely the one-shot battle-of-the-sexes with no preceding move by player I.

This is a simple example of the general novel framework that we define in Section 2 for dynamic interaction with possibly mutual unawareness of actions, generalizing standard extensive-form games. The framework will not only allow modeling of situations in which one player is certain that another player is unaware of portions of the game tree, as in the above example, but also of situations in which a player is uncertain regarding the way another player views the game tree, as well as situations in which the player is uncertain regarding the uncertainties of the other player about yet other players' views of the game tree, and so forth.

In fact, this framework allows not just for unawareness but also for other forms of misconception about the structure of the game. Section 6 specifies further properties obtaining in generalized extensive-form games where the only source of players 'misconception' is unawareness and mutual unawareness of available actions and paths in the game. Since we focus on this type of unawareness, most of the examples in the paper satisfy the further properties specified in Section 6. Nevertheless, modeling *awareness of unawareness* does require the general framework in Section 2, as explained at its end.

In this new framework, for each information set of a player her strategy specifies – from the point of view of the modeler – what the player would do if and when that information set of hers is ever reached. In this sense, a player does not necessarily 'own' her full strategy at the beginning of the game, because she might not be initially aware of all of her information sets. That's why a sensible generalization of Pearce's (1984) notion of extensive-form rationalizability is non-trivial.

In Section 3 we put forward a modified definition, prove existence, and show the sense in which it coincides with extensive-form rationalizability in standard extensive-form games.

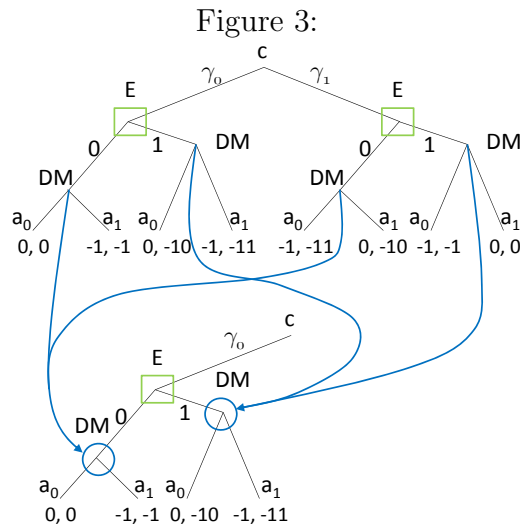
We focus here on a rationalizability solution concept rather than on some notion of equilibrium. While an equilibrium is ideally interpreted as a rest-point of some dynamic learning or adaptation process, or alternatively as a pre-meditated agreement or expectation, we find it difficult to carry over such interpretations to a setting in which every increase of awareness is by definition a shock or a surprise. Once a player's view of the game itself is challenged in the course of play, it is hard to justify the idea that a convention or an agreement for the continuation of the game are readily available.

We chose to focus on extensive-form rationalizability because it embodies forward induction reasoning. If an opponent makes a player aware of some relevant aspect of reality, it is implausible to dismiss the increased level of awareness as an unintended consequence of the opponent’s behavior. Rather, the player should try to rationalize the opponent’s choice, re-interpret the opponent’s past behavior, and try to infer from it the opponent’s future moves. Extensive-form rationalizability indeed captures a ‘best rationalization principle’ (Battigalli, 1997).

With rationalizability, generalized games are necessary for properly modeling unawareness; trying to model unawareness by having the unaware player assigning probability zero to the contingency of which she is unaware might give rise to a completely different rationalizable behavior, which does not square with unawareness in the proper sense of the word. To see this consider the following example.

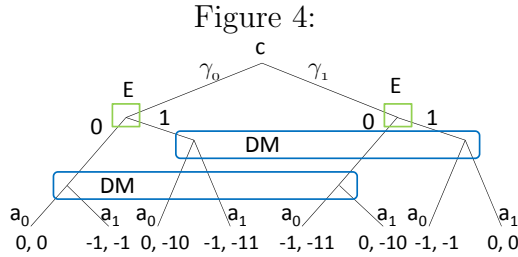
A Decision Maker (DM) has to choose between two policies,  $a_0$  and  $a_1$ . Before choosing she gets a recommendation from an expert via a narrow communication channel, through which the expert can recommend either “0” or “1”. The expert makes the recommendation after observing the state of nature, which may be either  $\gamma_0$  or  $\gamma_1$ , and which the DM does not see. The interests of the expert and the DM are completely aligned: They each bear a cost of 1 if  $a_1$  is implemented when the state of nature is  $\gamma_0$  or vice versa. The expert furthermore bears a cost of 10 from “lying”, i.e. from recommending “0” when the state of nature is  $\gamma_1$  or recommending “1” when the state of nature is  $\gamma_0$ .

Assume the DM is aware only of the state  $\gamma_0$  and unaware of  $\gamma_1$ . The dynamic interaction is hence modeled by the generalized game in Figure 3.



In this generalized game the only extensive-form rationalizable strategy of the DM is to always implement the policy  $a_0$ : she does not conceive of a contingency that would make the policy  $a_1$  superior to  $a_0$  even if she hears from the expert the recommendation “1”; in such a case she regrettably concludes that the expert behaved in an irrational way and bore the cost of “lying”.

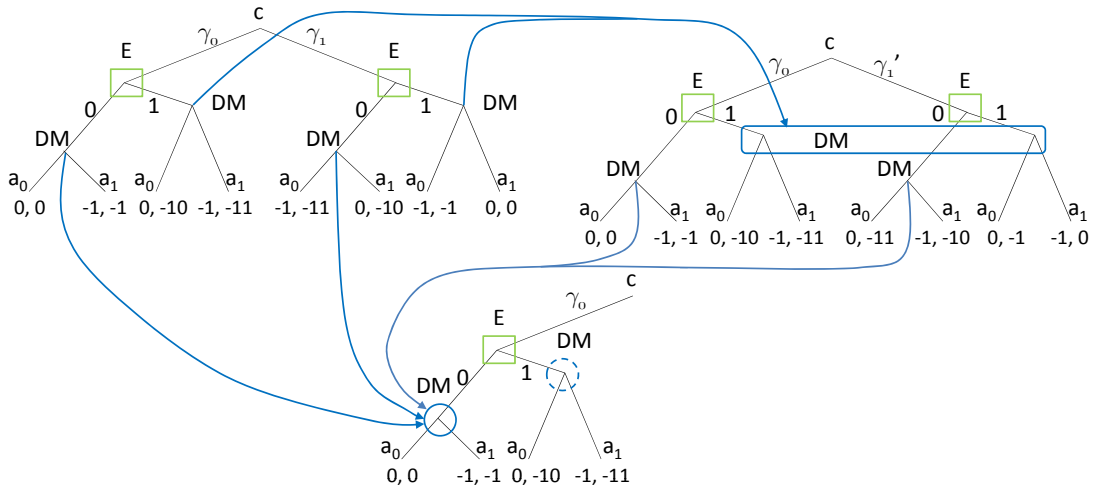
However, if we were to model the DM alternatively as being aware of  $\gamma_1$  but assigning probability zero to it, the strategic interaction would be modeled by the standard game in Figure 4.



In this game the unique extensive-form rationalizable strategy of the DM is to choose  $a_0$  upon hearing “0” from the expert, but to implement  $a_1$  upon hearing the recommendation “1”. Indeed, extensive-form rationalizability requires the DM to base her choice on a system of beliefs about the expert’s strategies with which at every information set of hers she maintains a belief that best rationalizes the choices of the expert which could have led to that information set. In particular, upon hearing the recommendation “1” from the expert, the only way for the DM to rationalize it is to assume that the state of nature is *nevertheless*  $\gamma_1$ , where recommending “1” is strictly dominant for the expert; and in  $\gamma_1$  the optimal choice for the DM is  $a_1$ .

Conceptually, upon hearing the surprising recommendation “1” both choices of the DM have their internal logic. The former gives priority to “only  $\gamma_0$  is conceivable”, the latter to the rationality of the expert. But in the latter case, if initially the DM is *genuinely* unaware of  $\gamma_1$ , there is no reason why the DM would conceive *precisely* of  $\gamma_1$  and not of some alternative description  $\gamma'_1$  of nature that would also rationalize the expert’s recommendation “1”; some such conceptualizations  $\gamma'_1$  need not necessarily induce the DM to adopt the expert’s recommendation. Generalized games lend themselves also to modeling such misconceptions that may arise upon a surprise, as demonstrated in Figure 5. Here, the DM’s rationalizable strategy is to choose  $a_0$  also upon hearing the (surprising) recommendation “1”, because the DM believes this recommendation was

Figure 5:



strictly dominant for the expert but that her interest and those of the expert are now opposed.

In Section 4 we introduce a related solution concept, prudent rationalizability, which is the direct generalization of iterated admissibility to dynamic games with unawareness. Unlike in normal-form games, this generalization is surprisingly *not* always a refinement of extensive-form rationalizability (even for standard extensive-form games). However, we prove that prudent rationalizable strategies do refine the set of *outcomes* obtainable by extensive-form rationalizable strategies. We show how prudent rationalizability is effective in ruling out less plausible rationalizable outcomes in examples due to Pearce (1984) and Ozbay (2007).

Of particular interest is the application of prudent rationalizability to the Milgrom-Roberts (1986) communication game, in which a sender sends a verifiable (and hence correct) piece of information to a receiver who makes a decision on its basis. Milgrom and Roberts (1986) showed that the unique sequential equilibrium in this game features full unraveling of information, and that at equilibrium the receiver interprets each piece of information in the most ‘skeptical’ manner. We show that the complete unraveling outcome is also the unique outcome in *prudent* strategies, and hence that it hinges on rationalizability (or, more precisely, on prudence) considerations and does not require the full power of equilibrium analysis. Nevertheless, we show that if the certified information has multiple dimensions and the receiver is *unaware* of some of them, then complete unraveling need not occur with prudent strategies. Thus, this is yet another example in



which unawareness has strategic implications which are genuinely different than those implied by asymmetric information.

In standard game theory, the extensive form has been considered as a more complete description of the strategic situation than the normal form. This has been questioned by Kohlberg and Mertens (1986) who argued that the normal form contains all strategically relevant information. For standard extensive-form games, Shimoji and Watson (1998) showed how extensive-form reasoning embodied in extensive-form rationalizability can be carried out in the normal form. Arguably generalized extensive-form games contain more "time relevant" structure than standard extensive-form games since they also formalize changes in the awareness of players. It is therefore an intriguing question whether a solution to generalized extensive-form games can be found when the analysis is carried out in the appropriately defined normal form associated to a generalized extensive-form game. In Section 5 we define the normal form associated to general extensive-form games. We extend Shimoji and Watson' characterization of extensive-form rationalizability by iterated conditional strict dominance to games with unawareness. In some applications, it may be more practical to apply iterated conditional strict dominance in the normal form rather than extensive-form rationalizability.

Our framework for dynamic interaction under unawareness seems to be simpler than the one proposed by Halpern and Rêgo (2006) and Rêgo and Halpern (2007), in which they investigated the notions of Nash and sequential equilibrium, respectively.<sup>2</sup> Feinberg (2009) defines unawareness by explicit *unbounded* sequences of mutual "views" of the game, with analogous properties both for static and for dynamic games. In his dynamic setting, he does not impose perfect recall, which might hamper the extension of known solution concepts such as sequential equilibrium or extensive-form rationalizability that rely on perfect recall; in contrast, extensive-form rationalizability and prudent rationalizability are the focal solution concepts that we extend and define and analyze in our paper, and to this effect we extend the definition of perfect recall to our setting. Li (2006) considered dynamic unawareness with perfect information, while our framework allows for both unawareness and imperfect information.

Ozbay (2007) studies sender-receiver games, in which an 'announcer' can make an unaware decision maker aware of more states of nature before the decision maker takes

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<sup>2</sup>The simplification obtained in our framework is due to the fact that our initial building block is a tree representing *physical* moves, with information sets defined only in the sub-trees which represent subjective views of the game; in contrast, Halpern and Rêgo (2006) had information sets defined already in their basic tree. As a result, not all sub-trees could be considered, and Halpern and Rêgo (2006) had to postulate additional conditions relating the information sets in sub-trees to those of the basic tree.

an action. Such games can also be naturally formulated as a particular instance of our framework. For these games Ozbay studies an equilibrium notion incorporating forward-induction reasoning. Filiz-Ozbay (2007) studies a related setting in which the aware announcer is a risk neutral insurer, while the decision maker is a risk averse or ambiguity averse insuree. At equilibrium, the insurer does not always reveal all relevant contingencies to the insuree.<sup>3</sup>

Our aim is to provide a general framework for modeling misperceptions about the availability of actions in dynamic strategic situations. Different kinds of perception biases among players in games have been a popular topic in the recent literature on behavioral game theory. For instance, in static games Eyster and Rabin (2005) analyze players with correct conjectures about opponents' actions but misperceptions about how those opponents' actions are correlated with the opponents' information. In multi-stage games with moves of nature, Jehiel (2005) studies players that bundle nodes at which other players choose into "analogy classes", correctly anticipate the average behavior for each analogy class, and thus may have misperceptions about how others' behavior is related others' information. Recently there has been a renaissance of non-equilibrium iterative solution concepts in behavioral game theory like level-k thinking and related models (e.g. Stahl and Wilson, 1995, Camerer, Hu and Chong, 2004, Crawford and Iriberri, 2007). Note that our iterative solution concepts, would-be rationalizability and prudent rationalizability, do not only provide behavioral predictions in the limit but also at every finite level of rationalization.

## 2 Generalized extensive-form games

To define a generalized extensive-form game  $\Gamma$ , consider first, as a building block, a finite perfect information game with a set of players  $I$ , a set of decision nodes  $N_0$ , active players  $I_n$  at node  $n$  with finite action sets  $A_n^i$  of player  $i \in I_n$  (for  $n \in N_0$ ), chance nodes  $C_0$ , and terminal nodes  $Z_0$  with a payoff vector  $(p_i^z)_{i \in I} \in \mathbb{R}^I$  for the players for every  $z \in Z_0$ . The nodes  $\bar{N}_0 = N_0 \cup C_0 \cup Z_0$  constitute a tree.

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<sup>3</sup>Currently we are unaware of further papers focusing directly and explicitly on dynamic games with unawareness. The literature on unawareness in general is growing fast – see e.g. <http://www.econ.ucdavis.edu/faculty/schipper/unaw.htm>

## 2.1 Partially ordered set of trees

Consider now a family  $\mathbf{T}$  of subtrees of  $\bar{N}_0$ , partially ordered ( $\preceq$ ) by inclusion. One of the trees  $T_1 \in \mathbf{T}$  is meant to represent the modeler's view of the paths of play that are *objectively* feasible; each other tree represents the feasible paths of play as *subjectively* viewed by some player at some node at one of the trees.

In each tree  $T \in \mathbf{T}$  denote by  $n_T$  the copy in  $T$  of the node  $n \in \bar{N}_0$  whenever the copy of  $n$  is part of the tree  $T$ . However, in what follows we will typically avoid the subscript  $T$  when no confusion may arise.

Denote by  $N_i^T$  the set of nodes in which player  $i \in I$  is active in the tree  $T \in \mathbf{T}$ .

We require two properties:

1. All the terminal nodes in each tree  $T \in \mathbf{T}$  are copies of nodes in  $Z_0$ .
2. If for two decision nodes  $n, n' \in N_i^T$  (i.e.  $i \in I_n \cap I_{n'}$ ) it is the case that  $A_n^i \cap A_{n'}^i \neq \emptyset$ , then  $A_n^i = A_{n'}^i$ .<sup>4</sup>

Property 1 is needed to ensure that each terminal node of each tree  $T \in \mathbf{T}$  is associated with well defined payoffs to the players. Property 2 means that  $i$ 's active nodes  $N_i^T$  are partitioned into equivalence classes, such that the actions available to player  $i$  are identical within each equivalence class and disjoint in distinct equivalence classes. It will be needed for the definition of information sets which follows shortly.<sup>5</sup>

Denote by  $N$  the union of all decision nodes in all trees  $T \in \mathbf{T}$ , by  $C$  the union of all chance nodes, by  $Z$  the union of terminal nodes, and by  $\bar{N} = N \cup C \cup Z$  (copies  $n_T$  of a given node  $n$  in different subtrees  $T$  are distinct from one another, so that  $\bar{N}$  is a disjoint union of sets of nodes). For a node  $n \in \bar{N}$  we denote by  $T_n$  the tree containing  $n$ .

## 2.2 Information sets

In standard extensive-form game, an information set  $\pi_i(n)$  of a player  $i$  is both (1) the set of nodes that the player considers as possible, and (2) the set of nodes in which the

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<sup>4</sup>Sometimes the modeler may want to impose an additional property: If in the original tree the probabilities of reaching  $\bar{n}_1, \dots, \bar{n}_k \in \bar{N}$  from the chance node  $c \in C$  are  $p_c^{\bar{n}_1} > 0, \dots, p_c^{\bar{n}_k} > 0$  but some of these nodes do not appear in the subtree, then the probabilities of reaching the remaining nodes emanating from  $c$  are renormalized so as to sum to 1 in the subtree. We do not impose this property here since it may be natural in some contexts but unnatural in others.

<sup>5</sup>The idea will be that in a given tree  $T$ , each action will correspond only to one view the player can have regarding the way the dynamic interaction has evolved that far, and will hence be available at (all the nodes of) a unique information set.

player has the same information as in the nodes which she considers as possible. In games with unawareness the two notions need not coincide, and our definition of an information set  $\pi_i(n)$  below generalizes (1) above;  $\pi_i(n)$  will be in a different tree than in the tree  $T_n$  if at  $n$  the player is unaware of some of the paths in  $T_n$ , and rather envisages the dynamic interaction as taking place in the tree containing  $\pi_i(n)$ .

Formally, for each decision node  $n \in N$ , define for each active player  $i \in I_n$  an information set  $\pi_i(n)$  with the following properties:

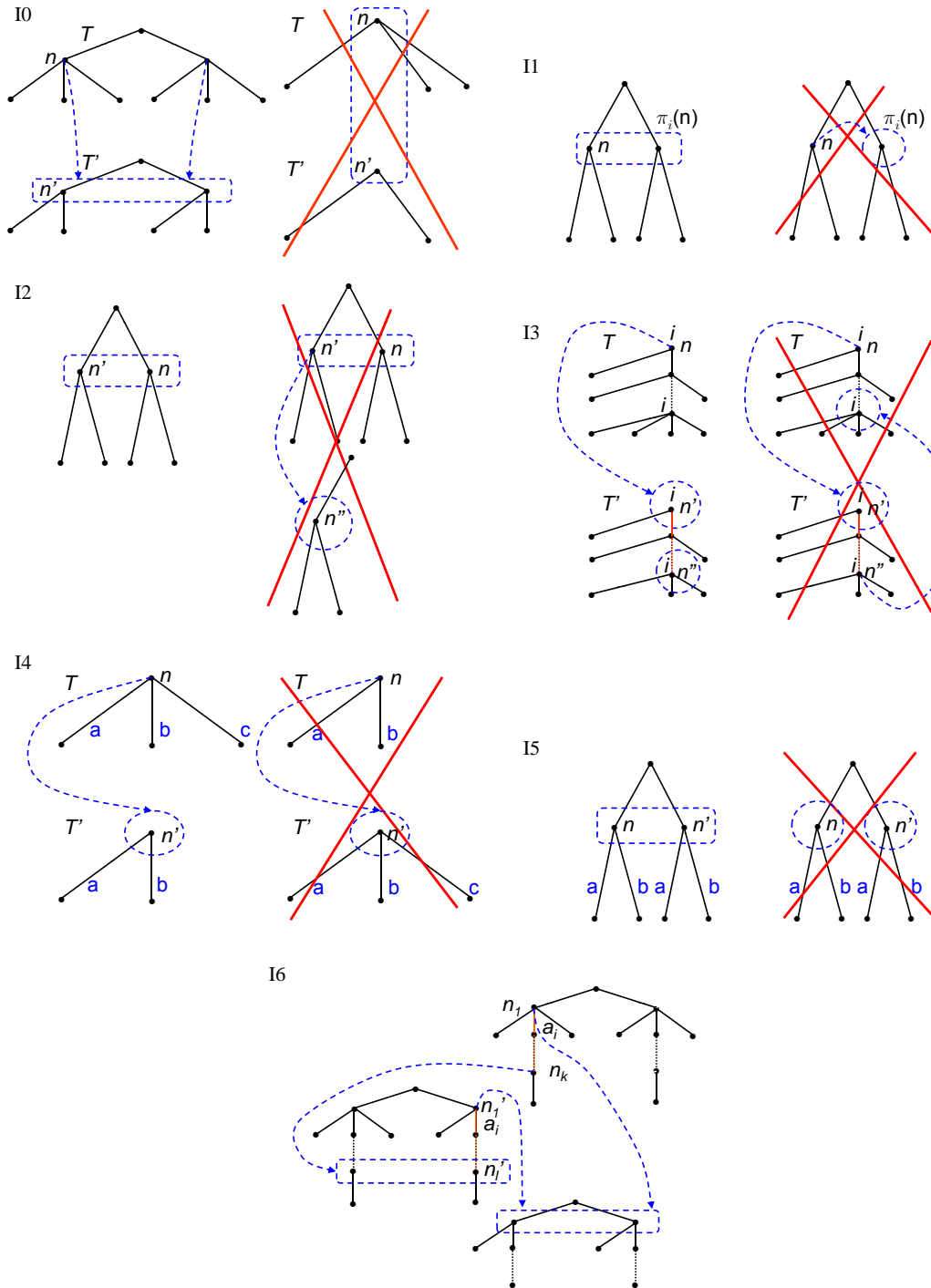
- I0 Confinement:  $\pi_i(n) \subseteq T$  for some tree  $T$ .
- I1 No delusion: If  $\pi_i(n) \subseteq T_n$  then  $n \in \pi_i(n)$ .
- I2 Introspection: If  $n' \in \pi_i(n)$  then  $\pi_i(n') = \pi_i(n)$ .
- I3 No divining of currently unimaginable paths, no expectation to forget currently conceivable paths: If  $n' \in \pi_i(n) \subseteq T'$  (where  $T' \in \mathbf{T}$  is a tree) and there is a path  $n', \dots, n'' \in T'$  such that  $i \in I_{n'} \cap I_{n''}$  then  $\pi_i(n'') \subseteq T'$ .
- I4 No imaginary actions: If  $n' \in \pi_i(n)$  then  $A_{n'}^i \subseteq A_n^i$ .
- I5 Distinct action names in disjoint information sets: For a subtree  $T$ , if  $n, n' \in T$  and  $A_n^i = A_{n'}^i$  then  $\pi_i(n') = \pi_i(n)$ .
- I6 Perfect recall: Suppose that player  $i$  is active in two distinct nodes  $n_1$  and  $n_k$ , and there is a path  $n_1, n_2, \dots, n_k$  such that at  $n_1$  player  $i$  takes the action  $a_i$ . If  $n' \in \pi_i(n_k)$ , then there exists a node  $n'_1 \neq n'$  and a path  $n'_1, n'_2, \dots, n'_\ell = n'$  such that  $\pi_i(n'_1) = \pi_i(n_1)$  and at  $n'_1$  player  $i$  takes the action  $a_i$ .

The following figures (Figure 6) illustrate properties I0 to I6.

Properties (I1), (I2), (I4), and (I5) are standard for extensive-form games, and properties (I0) and (I6) generalize other standard properties of extensive-form games to our generalized setting. The essentially new property is (I3). At each information set of a player, property (I3) confines the player's *anticipation* of her future view of the game to the view she currently holds (even if, as a matter of fact, this anticipation is about to be shuttered as the game evolves).

We denote by  $H_i$  the set of  $i$ 's information sets in all trees. For an information set  $h_i \in H_i$ , we denote by  $T_{h_i}$  the tree containing  $h_i$ . For two information sets  $h_i, h'_i$  in a given tree  $T$ , we say that  $h_i$  precedes  $h'_i$  (or that  $h'_i$  succeeds  $h_i$ ) if for every  $n' \in h'_i$  there is a path  $n, \dots, n'$  such that  $n \in h_i$ . We denote  $h_i \rightsquigarrow h'_i$ .

Figure 6: Properties I0 to I6



**Remark 1** *The following property is implied by I2 and I4: If  $n', n'' \in h_i$  where  $h_i = \pi_i(n)$  is an information set, then  $A_{n'}^i = A_{n''}^i$ .*

**Proof.** If  $n', n'' \in h_i$  where  $h_i = \pi_i(n)$  is some information set, then by introspection (I3) we must have  $\pi_i(n') = \pi_i(n'') = \pi_i(n)$ . Hence by (I4)  $A_{n'}^i \subseteq A_{n''}^i$  and  $A_{n''}^i \subseteq A_{n'}^i$ .  $\square$

**Remark 2** *Properties I0, I1, I2 and I6 imply no absent-mindedness: No information set  $h_i$  contains two distinct nodes  $n, n'$  on some path in some tree.*

**Proof.** Suppose by contradiction that there exists an information set  $h_i$  with a node  $n \in h_i$  such that some other node in  $h_i$  precedes  $n$  in the tree  $T_n$ . Denote by  $n'$  the first node on the path from the root to  $n$  that is also in  $h_i$ . Now apply I6 with  $n'_i := n'$  to get a path  $n'' = n'_1, \dots, n'_l = n'$ , with  $\pi_i(n'') = \pi_i(n_1) = \pi_i(n') = h_i$ . By I1, we have  $n'' \in h_i$  and  $n''$  is a predecessor of  $n'$ , a contradiction.  $\square$

The perfect recall property I6 and Remark 2 guarantee that with the precedence relation  $\rightsquigarrow$  player  $i$ 's information sets  $H_i$  form an *arborescence*: For every information set  $h'_i \in H_i$ , the information sets preceding it  $\{h_i \in H_i : h_i \rightsquigarrow h'_i\}$  are totally ordered by  $\rightsquigarrow$ .

For trees  $T, T' \in \mathbf{T}$  we denote  $T \succ T'$  whenever for some node  $n \in T$  and some player  $i \in I_n$  it is the case that  $\pi_i(n) \subseteq T'$ . Denote by  $\hookrightarrow$  the transitive closure of  $\succ$ . That is,  $T \hookrightarrow T''$  iff there is a sequence of trees  $T, T', \dots, T'' \in \mathbf{T}$  satisfying  $T \succ T' \succ \dots \succ T''$ .

## 2.3 Generalized games

A generalized extensive-form game  $\Gamma$  consists of a partially ordered set  $\mathbf{T}$  of subtrees of a tree  $\bar{N}_0$  satisfying properties 1-2 above, along with information sets  $\pi_i(n)$  for every  $n \in T, T \in \mathbf{T}$  and  $i \in I_n$ , satisfying properties I0-I6 above.

For every tree  $T \in \mathbf{T}$ , the  $T$ -*partial game* is the partially ordered set of trees including  $T$  and all trees  $T'$  in  $\Gamma$  satisfying  $T \hookrightarrow T'$ , with information sets as defined in  $\Gamma$ . A  $T$ -partial game is a generalized game, i.e. it satisfies all properties 1-2 and I0-I6.

We denote by  $H_i^T$  the set of  $i$ 's information sets in the  $T$ -partial game.

## 2.4 Strategies

A (pure) strategy

$$s_i \in S_i \equiv \prod_{h_i \in H_i} A_{h_i}$$

for player  $i$  specifies an action of player  $i$  at each of her information sets  $h_i \in H_i$ . Denote by

$$S = \prod_{j \in I} S_j$$

the set of strategy profiles in the generalized extensive-form game.

If  $s_i = (a_{h_i})_{h_i \in H_i} \in S_i$ , we denote by

$$s_i(h_i) = a_{h_i}$$

the player's action at the information set  $h_i$ . If player  $i$  is active at node  $n$ , we say that at node  $n$  the strategy prescribes to her the action  $s_i(\pi_i(n))$ .

In generalized extensive-form games, a strategy cannot be conceived as an ex ante plan of action. If  $h_i \subseteq T$  but  $T \not\leftrightarrow T'$ , then at  $h_i$  player  $i$  may be interpreted as being unaware of her information sets in  $H_i^{T'} \setminus H_i^T$ .

Thus, a strategy of player  $i$  should rather be viewed as a list of answers to the hypothetical questions "what would the player do if  $h_i$  were the set of nodes she considered as possible?", for  $h_i \in H_i$ . However, there is no guarantee that such a question about the information set  $h'_i \in H_i^{T'}$  would even be meaningful to the player if it were asked at a different information set  $h_i \in H_i^T$  when  $T \not\leftrightarrow T'$ . The answer should therefore be interpreted as given by the modeler, as part of the description of the situation.

For a strategy  $s_i \in S_i$  and a tree  $T \in \mathbf{T}$ , we denote by  $s_i^T$  the strategy in the  $T$ -partial game induced by  $s_i$ . If  $R_i \subseteq S_i$  is a set of strategies of player  $i$ , denote by  $R_i^T$  the set of strategies induced by  $R_i$  in the  $T$ -partial game. The set of  $i$ 's strategies in the  $T$ -partial game is thus denoted by  $S_i^T$ . Denote by  $S^T = \prod_{j \in I} S_j^T$  the set of strategy profiles in the  $T$ -partial game.

We say that a strategy profile  $s \in S$  *reaches* the information set  $h_i \in H_i$  if the players' actions and nature's moves (if there are any) in  $T_{h_i}$  lead to  $h_i$  with a positive probability. (Notice that unlike in standard games, an information set  $\pi_i(n)$  may be contained in tree  $T' \neq T_n$ . In such a case, by definition  $s_i(\pi_i(n))$  induces an action to player  $i$  also in  $n$  and not only in the nodes of  $\pi_i(n)$ .)

We say that the strategy  $s_i \in S_i$  *reaches* the information set  $h_i$  if there is a strategy profile  $s_{-i} \in S_{-i}$  of the other players such that the strategy profile  $(s_i, s_{-i})$  reaches  $h_i$ . Otherwise, we say that the information set  $h_i$  is excluded by the strategy  $s_i$ .

Similarly, we say that the strategy profile  $s_{-i} \in S_{-i}$  *reaches* the information set  $h_i$  if there exists a strategy  $s_i \in S_i$  such that the strategy profile  $(s_i, s_{-i})$  reaches  $h_i$ .

A strategy profile  $(s_j)_{j \in I}$  *reaches a node*  $n \in T$  if the players' actions  $s_j(\pi_j(n'))_{j \in I}$  and nature's moves in the nodes  $n' \in T$  lead to  $n$  with a positive probability. Since we consider only finite trees,  $(s_j)_{j \in I}$  reaches an information set  $h_i \in H_i$  if and if there is a node  $n \in h_i$  such that  $(s_j)_{j \in I}$  reaches  $n$ .

As is the case also in standard games, for every given node, a given strategy profile of the players induces a distribution over terminal nodes in each tree, and hence an expected payoff for each player in the tree.

For an information set  $h_i$ , let  $s_i \tilde{s}_i^{h_i}$  denote the strategy that is obtained by replacing actions prescribed by  $s_i$  at the information set  $h_i$  and its successors by actions prescribed by  $\tilde{s}_i$ . The strategy  $s_i / \tilde{s}_i^{h_i}$  is called an  $h_i$ -replacement of  $s_i$ .

The set of *behavioral* strategies is

$$\prod_{h_i \in H_i} \Delta(A_{h_i}).$$

## 2.5 Awareness of unawareness

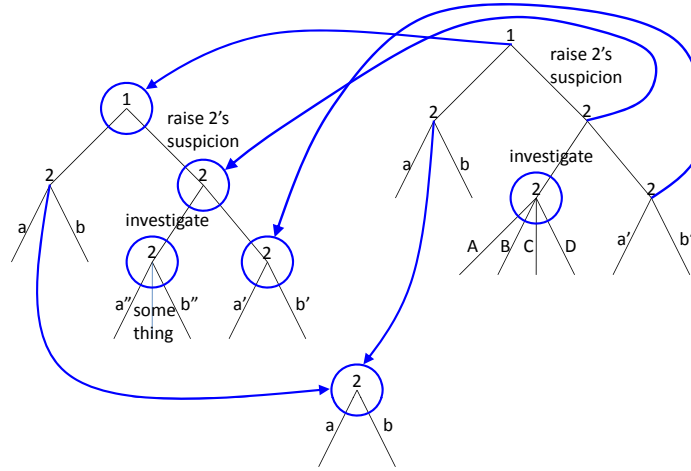
In some strategic situations a player may be aware of her unawareness in the sense that she is suspicious that *something* is amiss without being able to conceptualize this 'something'. Such a suspicion may affect her payoff evaluations for actions that she knows are available to her. More importantly, she may take actions to investigate her suspicion if such actions are physically available.

To model awareness of unawareness some of the trees may include *imaginary actions* as placeholders for actions that a player may be unaware of and terminal nodes/evaluations of payoffs that reflect her awareness of unawareness. (The approach of modeling awareness of unawareness by "imaginary moves" was proposed by Halpern and Rêgo, 2006.)

Consider the example in Figure 7. In both right and left trees, player 1 can decide whether or not to raise the suspicion of player 2. If he does not, then player 2 can decide between two actions. Since in this case player 2's information set is in the lower tree, she does not even realize that player 1 could have raised her suspicion. If player 1 raises



Figure 7: Game form with awareness of unawareness



player 2's suspicion, then player 2's information set is in the left tree. She must decide whether to investigate her suspicion or not. If she doesn't, then she can decide between two actions but this time she realizes that player 1 raised her suspicion (and could have refrained from doing so); and that she could have chosen to investigate, in which case she may have had 'something' else to do, that she cannot conceptualize in advance. Once she investigates, she becomes aware of two more actions and her information set is in the right tree. She also realizes that player 1 initially raised her suspicion without being explicitly aware of those actions of hers by himself. Note that before she decides whether or not to investigate, she is not modeled as anticipating to be in the right tree, because she cannot conceptualize the nature of the actions she reveals if and when she investigates.

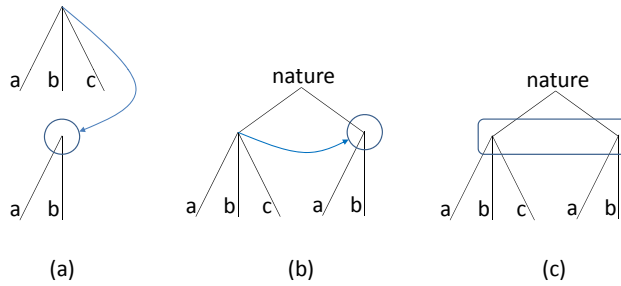
## 2.6 The connection to standard extensive-form games

Harsanyi (1967) showed how to transform games with asymmetric information into games with imperfect information about a move of nature. Can a similar idea be used to transform any generalized extensive-form game into a standard extensive-form game? Given a generalized extensive-form game  $\Gamma$  with a partially ordered set of trees  $\mathbf{T}$ , one could define the transformation of  $\Gamma$  to be the extensive-form game with an initial move of nature, in which nature chooses one of the trees in  $\mathbf{T}$ .

Notice, however, that the resulting structure would not be a standard extensive-form game. To see this, notice that every standard extensive-form game has the following property (E): the equivalence class of nodes in which a player considers as possible a

given possibility set of nodes is *identical* with that possibility set; this set is called an information set of the player, and in all of its nodes the player has the same set of available actions. In contrast, in the transformation considered above for games with misperceptions, this equivalence class may be a *strict super-set* of the possibility set. For example, when the generalized game in Figure 8(a) is transformed so as to have an initial move of nature, the possibility set for the (unique) player is the right node, while the equivalence class contains both the right and left node.

Figure 8:



Thus, if after adding the initial move of nature the information sets are defined to be synonymous with the possibility sets, the resulting game would be non-standard, because for some information set there may be additional nodes *outside it* in which the player considers it as possible (as in Figure 8(b), where in the left node the player considers only the right node as possible). If, in contrast, we choose the alternative definition, by which an information set is the equivalence class in which a player has a particular set of nodes that she considers as possible, the resulting game would again be non-standard, this time because the actions available to the player in the nodes of a given information set might not be identical across these nodes (as in Figure 8(c), where in the left node the player has more available actions than in the right node, even though both are within the same information set).

There is also another aspect that prevents the above transformation from yielding a standard extensive-form game. In a standard extensive-form game each player has a full-support prior on the moves of nature.<sup>6</sup> Using Bayes rule, the player therefore has a well-defined belief about nature at each stage of the game. In contrast, in the above transformation each player ascribes probability 1 only to one of the initial moves of nature; moreover, along the path of play the player may switch completely the move

<sup>6</sup>Moreover, in the classical definition of an extensive-form game the priors of the different players about nature are actually identical, i.e. the players have a common prior about nature.

of nature in which she confides even if nothing in the path of play itself imposed such a switch. Such a switch corresponds to a node in the generalized game in which the player is *defined* as *becoming aware* of new aspects of the dynamic interaction; such an increase of awareness may occur even when the physical path of play *per se* did not imply a surprise, and may have also been compatible with the player’s previous conception of the game. Thus, if we do add an initial move of nature to connect the trees of the generalized game, the player’s (evolving) belief about nature cannot be encapsulated within an initial probabilistic belief about nature, and must be represented explicitly by a *belief system* as part of the *definition* of the game.

Adding an initial move of nature has a further conceptual drawback. In classical extensive-form games the implicit assumption is that the players understand the entire structure of the dynamic interaction as embodied in the game tree.<sup>7</sup> Assigning probability zero to some move of nature is still compatible with realizing what could have happened if this zero-probability move were nevertheless to materialize. This is conceptually distinct from being completely *unaware* of a subset of paths in the game, and it is the latter concept that we want to model here. Moreover, as we have seen in the example of the introduction (Figures 3 and 4), it may lead to behavioral predictions different from unawareness.

Thus, standard extensive-form games are neither technically fit (without further generalization) for modeling behavior under dynamic misperceptions and unawareness, nor do they convey the appropriate conceptual apparatus for modeling such interactions, hence the need for our definition of generalized games.<sup>8</sup>

### 3 Extensive-form rationalizability

Pearce (1984) defined extensive-form (correlated) rationalizable strategies by a procedure of an iterative elimination of strategies. The idea behind the definition involves a notion of forward induction. In generic perfect-information games, rationalizable strategy pro-

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<sup>7</sup>For instance, Myerson (1991, p. 4) puts forward explicitly the tenet that game theory deals with *intelligent* players, where “a player in the game is *intelligent* if he knows everything that we know about the game and he can make any inference about the situation that we can make.”

<sup>8</sup>Even if one nevertheless prefers to model such interactions using an initial move of nature and generalizing accordingly the notions of information sets and beliefs about nature in standard extensive-form games, the properties (I0)-(I6) of our definition constitute *restrictions* on the structure of such “extended” standard games that are needed in order to guarantee e.g. that the expectations of each player about future paths are dynamically consistent (property I3) and perfect recall is well-defined (property I6).

files yield the backward induction outcome, though they need not be subgame-perfect equilibrium strategies (Reny 1992, Battigalli 1997).

In what follows we extend this definition to generalized extensive-form games.

A *belief system* of player  $i$

$$b_i = (b_i(h_i))_{h_i \in H_i} \in \prod_{h_i \in H_i} \Delta(S_{-i}^{T_{h_i}})$$

is a profile of beliefs - a belief  $b_i(h_i) \in \Delta(S_{-i}^{T_{h_i}})$  about the other players' strategies in the  $T_{h_i}$ -partial game, for each information set  $h_i \in H_i$ , with the following properties

- $b_i(h_i)$  reaches  $h_i$ , i.e.  $b_i(h_i)$  assigns probability 1 to the set of strategy profiles of the other players that reach  $h_i$ .
- If  $h_i$  precedes  $h'_i$  ( $h_i \rightsquigarrow h'_i$ ) then  $b_i(h'_i)$  is derived from  $b_i(h_i)$  by Bayes rule whenever possible.

Denote by  $B_i$  the set of player  $i$ 's belief systems.

For a belief system  $b_i \in B_i$ , a strategy  $s_i \in S_i$  and an information set  $h_i \in H_i$ , define player  $i$ 's expected payoff **at**  $h_i$  to be the expected payoff for player  $i$  in  $T_{h_i}$  given  $b_i(h_i)$ , the actions prescribed by  $s_i$  at  $h_i$  and its successors, and conditional on the fact that  $h_i$  has been reached.<sup>9</sup>

We say that with the belief system  $b_i$  and the strategy  $s_i$  player  $i$  is *rational* at the information set  $h_i \in H_i$  if either  $s_i$  doesn't reach  $h_i$  in the tree  $T_{h_i}$ , or if  $s_i$  does reach  $h_i$  in the tree  $T_{h_i}$  then there exists no  $h_i$ -replacement of  $s_i$  which yields player  $i$  a higher expected payoff in  $T_{h_i}$  given the belief  $b_i(h_i)$  on the other players' strategies  $S_{-i}^{T_{h_i}}$ .

We say that with the belief system  $b_i$  and the strategy  $s_i$  player  $i$  *would be rational* at the information set  $h_i \in H_i$  if there exists no action  $a'_{h_i} \in A_{h_i}$  such that only replacing the action  $s_i(h_i)$  by  $a'_{h_i}$  results in a new strategy  $s'_i$  which yields player  $i$  a higher expected payoff at  $h_i$  given the belief  $b_i(h_i)$  on the other players' strategies  $S_{-i}^{T_{h_i}}$ .

The difference between these two definitions is as follows. The definition of *rationality* of a strategy  $s_i$  at an information set  $h_i$  takes a global perspective. It is mute regarding information sets which the strategy  $s_i$  itself rules out. Also, at an information set  $h_i$

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<sup>9</sup>Even if this condition is counterfactual due to the fact that the strategy  $s_i$  does not reach  $h_i$ . The conditioning is thus on the event that nature's moves, if there are any, have led to the information set  $h_i$ , and assuming that player  $i$ 's past actions (in the information sets preceding  $h_i$ ) have led to  $h_i$  *even if these actions are distinct than those prescribed by  $s_i$* .

which  $s_i$  *does* reach, it considers  $h_i$ -replacements, which may alter  $s_i$  not only at  $h_i$ , but also simultaneously at  $h_i$  and/or at some of the succeeding information sets of player  $i$ .

In contrast, the second definition takes a local perspective. It takes seriously the reasoning about rationality *assuming* that  $h_i$  has been reached, whether this assumption is *realistic* (when  $h_i$  can in fact be reached with a positive probability given the actions prescribed by  $s_i$  at preceding information sets) or *counterfactual* (when  $h_i$  is ruled out by  $i$ 's own actions with the strategy  $s_i$  at preceding information sets). Moreover, it considers alternative actions  $a'_{h_i}$  only at  $h_i$  itself. This is motivated by the implicit assumption that at  $h_i$ , player  $i$  is certain that at future information sets she will be acting according to the strategy  $s_i$ , but at the same time she also realizes that at each such future information set she will have the opportunity to re-consider her action, and that at  $h_i$  she has no way to commit herself to the action she will be taking at such a future information set.

We find the second definition more appealing in the context of unawareness. With unawareness, a player does not necessarily conceive of her entire strategy. Rather, she might be aware only of a subset of her information sets. She may plan what to do if and when such an information set is reached. However, once her level of awareness gets increased along the path of play, she may suspect that a similar revelation can happen again. She may then realize that whatever she plans to do, with her current level of awareness, is in fact subject to reconsideration. That's why with unawareness, what a strategy specifies for future information sets should better be conceptualized as expressing current beliefs about one's future actions rather than as a rigid plan to which the player is bound to conform.

The following lemma describes the close connection between the two definitions when all of the information sets  $h_i$  are considered. The lemma follows from the principle of optimality in dynamic programming. The explicit proof appears in the appendix.

**Lemma 1** *With a belief system  $b_i$  of player  $i$ ,*

- (i) *if a strategy  $s_i$  of player  $i$  would be rational at all information sets  $h_i \in H_i$  then it is rational at all information sets  $h_i \in H_i$ ; and*
- (ii) *if a strategy  $s_i$  of player  $i$  is rational at all information sets  $h_i \in H_i$ , then there exists a strategy  $\hat{s}_i$  which coincides with  $s_i$  at all information sets reached by  $s_i$ , such that  $\hat{s}_i$  would be rational at all information sets  $h_i \in H_i$ .*

The connection between the two definitions described in Lemma 1 is related to the notion of a *plan of action* (Rubinstein 1991, Reny 1992). A plan of player  $i$  specifies her

action when she is called to play, and does not specify what she would do at information sets which are ruled out by that plan. Formally, a plan of action for player  $i$  is an equivalence class of strategies  $\mathcal{P}_i \subset S_i$  such that two strategies  $s_i, \hat{s}_i$  are in  $\mathcal{P}_i$  if and only if for every strategy profile  $s_{-i}$  of the other players,  $(s_i, s_{-i})$  and  $(\hat{s}_i, s_{-i})$  induce the same distribution over terminal nodes in each of the trees of the game  $\Gamma$ . If  $s_i \in \mathcal{P}_i$  we say that the strategy  $s_i$  *induces* the plan of action  $\mathcal{P}_i$ .

With this terminology, Lemma 1 implies:

**Lemma 2** *For a given belief system  $b_i$  of player  $i$ , there exists a strategy  $s_i$  which is rational at all information sets  $h_i \in H_i$  and induces the plan of action  $\mathcal{P}_i$  if and only if there exists a strategy  $\hat{s}_i$  which would be rational at all information sets  $h_i \in H_i$  and induces the plan of action  $\mathcal{P}_i$ .*

We now turn to define rationalizability in generalized extensive-form games.

**Definition 1 (Would-be rationalizable strategies)** *Define, inductively, the following sequence of belief systems and strategies of player  $i$ .*

$$B_i^1 = B_i$$

$$S_i^1 = \{s_i \in S_i: \text{there exists a belief system } b_i \in B_i^1$$

*with which for every information set  $h_i \in H_i$  player  $i$  is would-be rational at  $h_i$ \}*

$\vdots$

$B_i^k = \{b_i \in B_i^{k-1} : \text{for every information set } h_i, \text{ if there exists some profile of the other players' strategies } s_{-i} \in S_{-i}^{k-1} = \prod_{j \neq i} S_j^{k-1} \text{ such that } s_{-i} \text{ reaches } h_i \text{ in the tree } T_{h_i}, \text{ then } b_i(h_i) \text{ assigns probability 1 to } S_{-i}^{k-1, T_{h_i}}\}$

$S_i^k = \{s_i \in S_i: \text{there exists a belief system } b_i \in B_i^k \text{ with which for every information set } h_i \in H_i \text{ player } i \text{ would be rational at } h_i\}$

*The set of player  $i$ 's **would-be rationalizable strategies** is*

$$S_i^\infty = \bigcap_{k=1}^{\infty} S_i^k.$$

**Remark 3**  $S_i^k \subseteq S_i^{k-1}$  for every  $k > 1$ .

**Proof.** Consider  $s_i \in S_i^k$ . By definition,  $s_i$  would-be rational at each of player  $i$ 's information sets given some belief system  $b_i \in B_i^k$ . Since  $B_i^k \subseteq B_i^{k-1}$ ,  $s_i$  would also be

rational at each of player  $i$ 's information sets given a belief system in  $B_i^{k-1}$ , namely given  $b_i$ . Hence  $s_i \in S_i^{k-1}$ .  $\square$

The generalization of Pearce's (1984) notion of extensive-form correlated rationalizable strategies is introduced next. The inductive definition below generalizes Definition 2 in Battigalli (1997), which he proved to be equivalent to Pearce's original definition.

**Definition 2 (Extensive-form correlated rationalizable strategies)** For  $k \geq 1$  let  $\hat{B}_i^k, \hat{S}_i^k$  be defined inductively as  $B_i^k, S_i^k$  above, respectively, the only change being that the phrase "for every information set  $h_i \in H_i$  player  $i$  **would be rational** at  $h_i$ " in the definition of  $S_i^k$  is changed to "for every information set  $h_i \in H_i$  player  $i$  **is rational** at  $h_i$ " in the definition of  $\hat{S}_i^k$ . The set of player  $i$ 's **extensive-form correlated rationalizable strategies** is

$$\hat{S}_i^\infty = \bigcap_{k=1}^{\infty} \hat{S}_i^k.$$

**Remark 4**  $\hat{S}_i^k \subseteq \hat{S}_i^{k-1}$  for every  $k > 1$ .

**Proof.** Analogous to the proof of Remark 3 above.  $\square$

Lemma 2 above implies the following proposition.

**Proposition 1** The set of strategies  $S_i^k$  is contained in  $\hat{S}_i^k$ , but  $S_i^k$  induces a set of plans of action identical to the set of plans of action induced by  $\hat{S}_i^k$ . Consequently, the set of would-be rationalizable strategies is contained in the set of extensive-form correlated rationalizable strategies,

$$S_i^\infty = \bigcap_{k=1}^{\infty} S_i^k \subseteq \hat{S}_i^\infty = \bigcap_{k=1}^{\infty} \hat{S}_i^k$$

but both sets induce the same set of plans of actions.

The inclusion mentioned in the proposition may be strict. For instance, in our first example in the introduction (Figure 1), it is rationalizable for player 1 not to give the car to player 2 and to subsequently go to the Bach concert, but to have gone to the Stravinsky concert (or to the Bach concert, or to the Mozart concert) *had he given the car to 2*. In contrast, the only *would-be* rationalizable strategy of player 1 is not to give the car to player 2 and subsequently attend the Bach concert, but to have gone to the

Mozart concert had he given the car to player 2. As the proposition asserts, no difference arises between rationality and would-be rationality along the unique realized path.

**Proposition 2** *The set of would-be rationalizable strategies is non-empty. Consequently, the set of extensive-form correlated rationalizable strategies is non-empty.*

The proof is in the appendix.

What are the would-be rationalizable strategies in our battle-of-the-sexes example from the introduction (Figure 2)?

**Remark 5** *In the Bach-Stravinsky-Mozart example with unawareness from the introduction (Figure 2), no player has a unique would-be rationalizable strategy.*

**Proof.** At the first level, any strategy would-be rational for player I except all strategies that prescribe going to the Mozart concert after “don’t tell”. For player II, both the Bach concert and the Stravinsky concert would-be rational if he is unaware of the Mozart concert. If he is aware of the Mozart concert, then only this concert is rational since it is a dominant action. Thus,  $S_{II}^1 = \{(B, M), (S, M)\}$ . Not telling player II about the Mozart concert and going to the Bach concert would-be rational for player I assuming that she believes with probability at least  $\frac{1}{2}$  that player II will go to the Bach concert under such circumstances. Telling player II about the Mozart concert and going to the Mozart concert would-be rational for player I if she believes with probability at least  $\frac{1}{2}$  that player II would go to the Stravinsky concert if not told about the Mozart concert. To summarize,

$$S_I^2 = \left\{ \begin{array}{l} (\text{“don’t tell”}, B, M, B), (\text{“don’t tell”}, B, M, S), \\ (\text{“tell”}, B, M, B), (\text{“tell”}, B, M, S), (\text{“tell”}, S, M, B), (\text{“tell”}, S, M, S) \end{array} \right\}$$

where the second (resp. third) component of the strategy vector refers to player I’s choice after history “don’t tell” (resp. “tell”), and the last component denotes the action in the lower subtree. Finally, note that  $S_{II}^k = S_{II}^1$  for  $k \geq 1$  and  $S_I^k = S_I^2$  for  $k \geq 2$ .  $\square$

When we compare this example to the game in which both players are aware of the Mozart concert but player I has the option of not providing her car for going to this concert (Figure 1), we note that the strategic implications of unawareness of actions are distinct from a situation in which both players are aware of the actions but some action may not always be available. The reason is that if player I keeps player II unaware



of the Mozart concert, then player *II* can not infer the intention of player *II* to go to the Bach concert. In other words, awareness of an available action (providing the car for going to the Mozart concert) and certainty that it hasn't been taken has stronger strategic implications than unawareness of the very same action.

In the Bach-Stravinsky-Mozart example with unawareness from the introduction (Figure 2), the would-be rationalizable outcome is not unique. This is in contrast to the example with unavailability of actions instead, where there is a unique would-be rationalizable outcome. However, there exist also games where with unavailability of actions there are more would-be rationalizable outcomes than with unawareness of the same actions, as the example in Remark 8 in Section 5 demonstrates.

## 4 Prudent rationalizability

In normal-form games, iterated admissibility (i.e. iterative elimination of weakly dominated strategies) is a refinement of rationalizability. Van Damme (1989) and more generally Ben-Porath and Dekel (1992) showed that iterated elimination of weakly dominated strategies singles out the forward induction outcome in money-burning games.<sup>10</sup> One interpretation of iterated admissibility is that in every round of elimination, each player is prudent and hence does not exclude completely any strategy profile of the other players which has not been thus far eliminated. In this section, we use the idea of prudence to define an analogous notion of rationalizability for dynamic games:

**Definition 3 (Prudent rationalizability)** *Let*

$$\bar{S}_i^0 = S_i$$

*For  $k \geq 1$  define inductively*

$$\bar{B}_i^k = \left\{ b_i \in B_i : \begin{array}{l} \text{for every information set } h_i, \text{ if there exists some profile} \\ s_{-i} \in \bar{S}_{-i}^{k-1} = \prod_{j \neq i} \bar{S}_j^{k-1} \text{ of the other players' strategies} \\ \text{such that } s_{-i} \text{ reaches } h_i \text{ in the tree } T_{h_i}, \text{ then the support} \\ \text{of } b_i(h_i) \text{ is the set of strategy profiles } s_{-i} \in \bar{S}_{-i}^{k-1, T_{h_i}} \text{ that reach } h_i \end{array} \right\}$$

---

<sup>10</sup>A similar result was shown by Herings and Vannetelbosch (1999) who defined iterated admissibility in terms of full support beliefs and called it trembling-hand perfect rationalizability.

$$\bar{S}_i^k = \left\{ s_i \in \bar{S}_i^{k-1} : \begin{array}{l} \text{there exists } b_i \in \bar{B}_i^k \text{ such that for all } h_i \in H_i \text{ player } i \\ \text{would be rational at } h_i \end{array} \right\}$$

The set of prudent rationalizable strategies of player  $i$  is

$$\bar{S}_i^\infty = \bigcap_{k=1}^{\infty} \bar{S}_i^k$$

**Proposition 3** *The set of player  $i$ 's prudent rationalizable strategies is non-empty.*

**Proof.** First, observe that  $\bar{B}_i^k \neq \emptyset$  for every  $k \geq 1$ , because if an information set  $h_i \in H_i$  is reached by some  $s_{-i} \in \bar{S}_{-i}^{k-1}$ , then  $s_{-i}$  reaches also all of  $i$ 's information sets that precede  $h_i$  in the tree  $T_{h_i}$ .

We proceed by induction.  $\bar{S}_i^0 = S_i$  and hence non-empty. Notice also that for every  $b_i \in \bar{B}_i^1$ , a standard backward induction procedure on the arborescence of information sets  $H_i$  yields a strategy  $s_i \in \bar{S}_i^1$  with which player  $i$  would be rational  $\forall h_i \in H_i$  given  $b_i$ .

Suppose, inductively, we have already shown that  $\forall i \in I$   $\bar{S}_i^{k-1} \neq \emptyset$  (and hence that  $\bar{S}_{-i}^{k-1} \neq \emptyset$ ), and also that for every  $b_i \in \bar{B}_i^{k-1}$  there exists a strategy  $s_i \in \bar{S}_i^{k-1}$  with which player  $i$  would be rational  $\forall h_i \in H_i$  given  $b_i$ .

Let  $b_i \in \bar{B}_i^k$ . Let  $\dot{H}_i \subseteq H_i$  be the set of  $i$ 's information sets not reached by any profile  $s_{-i} \in \bar{S}_{-i}^{k-1}$  but reached by some profile  $s_{-i} \in \bar{S}_{-i}^{k-2}$ . If  $\dot{H}_i \neq \emptyset$ , for every  $h_i \in \dot{H}_i$  with no predecessor in  $\dot{H}_i$ , modify (if necessary)  $b_i(h_i)$  so as to have full support on the profiles in  $\bar{S}_{-i}^{k-2}$  that reach  $h_i$ , and in succeeding information sets modify  $b_i$  by Bayes rule whenever possible. Denote the modified belief system by  $\dot{b}_i$ . Then by construction also  $\dot{b}_i \in \bar{B}_i^k$ .

Consider a sequence of belief systems  $b_{i,n} \in \bar{B}_i^{k-1}$  such that

$$\dot{b}_i = \left( \dot{b}_i(h'_i) \right)_{h'_i \in H_i} \equiv \left( \lim_{n \rightarrow \infty} b_{i,n}(h'_i) \right)_{h'_i \in H_i}$$

and given this sequence<sup>11</sup>  $b_{i,n} \in \bar{B}_i^{k-1}$  let  $s_{i,n} \in \bar{S}_i^{k-1}$  be a corresponding sequence of strategies with the property that given  $b_{i,n}$ , it is the case that with the strategy  $s_{i,n}$  player  $i$  would be rational at every  $h_i \in H_i$ . Since player  $i$  has finitely many strategies, some strategy  $s_i$  appears infinitely often in the sequence  $s_{i,n}$ . Since expected utility is

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<sup>11</sup>To construct such a sequence  $b_{i,n} \in \bar{B}_i^{k-1}$ , for every information set  $h'_i \in H_i$  not reached by any  $s_{-i} \in \bar{S}_{-i}^{k-1}$  define  $b_{i,n}(h'_i) = \dot{b}_i(h'_i)$  for every  $n \geq 1$ ; and for every  $h'_i \in H_i$  with no predecessors but reached by some profile  $s_{-i} \in \bar{S}_{-i}^{k-1}$  define  $b_{i,n}(h'_i) \in \Delta(\bar{S}_{-i}^{k-1})$  to be any converging sequence of beliefs such that for every  $n \geq 1$  the support of  $b_{i,n}(h'_i)$  is the subset of profiles in  $\bar{S}_{-i}^{k-2}$  that reach  $h'_i$ , while  $\lim_{n \rightarrow \infty} b_{i,n}(h'_i) = \dot{b}_i(h'_i)$ . In succeeding information sets reached by some  $s_i \in \bar{S}_{-i}^{k-1}$  define  $b_{i,n}(h'_i)$  by Bayes rule whenever possible.

linear in beliefs and hence continuous, also given  $\hat{b}_i$  it is the case that with the strategy  $s_i$  player  $i$  would be rational at every  $h_i \in H_i$ . Hence  $s_i \in \bar{S}_i^k$  as well.

Now, since player  $i$ 's set of strategies  $S_i$  is finite and by definition  $\bar{S}_i^{k+1} \subseteq \bar{S}_i^k$  for every  $k \geq 1$ , for some  $\ell$  we eventually get  $\bar{S}_i^\ell = \bar{S}_i^{\ell+1} \forall i \in I$  and hence  $\bar{B}_i^{\ell+1} = \bar{B}_i^{\ell+2} \forall i \in I$ . Inductively,

$$\emptyset \neq \bar{S}_i^\ell = \bar{S}_i^{\ell+1} = \bar{S}_i^{\ell+2} = \dots$$

and therefore

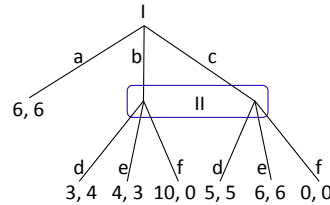
$$\bar{S}_i^\infty = \bigcap_{k=1}^{\infty} \bar{S}_i^k = \bar{S}_i^\ell \neq \emptyset$$

as required. □

#### 4.1 A first tension between rationalization and prudence: Divining the opponent's past behavior

In normal-form games, iterated admissibility is a refinement of rationalizability. Somewhat surprisingly, in extensive-form games prudent rationalizability is *not* a refinement of would-be rationalizability, as the following example (Figure 9) demonstrates.

Figure 9:



In this example, player 1 can guarantee herself the payoff 6 by choosing  $a$  and ending the game. If player 2 is called to play, should he believe that player 1 chose  $b$  or  $c$ ? If player 1 is certain that player 2 is rational, she is certain that player 2 will not choose  $f$ . Hence, if player 2 is certain that player 1 is certain that he (player 2) is rational, then at his information set player 2 is certain that player 1 chose  $c$ . The reason is that among player 1's actions leading to 2's information set,  $c$  is the only action which, assuming 2 believes  $c$  was chosen and that 2 is rational and will hence choose  $e$ , yields player 1 the payoff 6, which is just as high as the payoff she could guarantee herself with the outside option  $a$ . Hence  $(a, e)$  and  $(c, e)$  are the profiles of extensive-form (correlated) rationalizable strategies (as well as would-be rationalizable strategies) in this game.

The notion of prudence, in contrast, embodies the idea that being prudently rational, player 1 shouldn't rule out completely any of 2's possible choices, and hence that  $c$  is strictly inferior for player 1 relative to her outside option  $a$ . Hence, if 2's information set is ever reached, the only way for 2 to rationalize this is to believe that 1 chose  $b$ , based on a belief ascribing a high probability to the event that 2 will foolishly choose  $f$ . Player 2's best reply to  $b$  is  $d$ ; and player 1's best reply to  $d$  is  $a$ . Thus, the only profile of prudent rationalizable strategies in this game is  $(a, d)$ .

This example demonstrates that in dynamic interactions the notions of rationalization and prudence might involve a *tension*. Extensive-form rationalizability embodies a best-rationalization principle (Battigalli 1997, Battigalli and Siniscalchi 2002); it is driven by the assumption that in each of his information sets, a player assesses the other players' future behavior by attributing to them the 'highest' level of rationality and mutual certainty of rationality consistent with the fact that the information set has indeed been reached. But, with the additional criterion of 'prudence', what should a player believe about the behavior of his opponent if, as in the example, the opponent's only action which is compatible with common certainty of rationality is imprudent on the part of the opponent?

The definition of prudent rationalizability resolves this tension unequivocally in favor of the prudence consideration. It remains open whether and how a more balanced and elaborate definition could resolve the tension in less an extreme fashion. We plan to address this challenge in future work. However, any definition would have to cut the Gordian knot in the above example in one particular way, choosing either  $d$  or  $e$ , and indeed both potential resolutions are backed by sensible intuitions. This suggests that for dynamic interactions we need not necessarily expect one ultimate definition of rationalizability taking into account both rationalization and prudence.

**Remark 6** *The definition of prudent rationalizability employs would-be rationality. For standard extensive-form games, Brandenburger and Friedenberg (2007) studied the connection between extensive-form iteratively admissible strategies (defined on the basis of rationality rather than would-be rationality) and extensive-form rationalizability. They showed that under a "no relevant convexities" condition, extensive-form rationalizability and extensive-form iterated admissibility coincide. However, the example in Figure 9 does not satisfy this condition, and hence demonstrates that in general extensive-form iterated admissibility is not a refinement of extensive-form rationalizability.*

Nevertheless, as far as paths of play are concerned, in the above example the set of paths induced by prudent rationalizability (the path  $a$ ) is a subset of the paths induced

by (would-be) rationalizability (the paths  $a$  and  $(c, e)$ ). This is an instance of a general phenomenon:

**Proposition 4 (Prudent rationalizability refines would-be rationalizable paths)**

*The set of paths induced by profiles of prudent rationalizable strategies is a subset of the paths induced by profiles of would-be rationalizable strategies (or, equivalently, the paths induced by profiles of extensive-form correlated rationalizable strategies).*

The proof is in appendix A.

## 4.2 The refining power of prudent rationalizability

We now bring two examples demonstrating the refining power of prudent rationalizability. The first example was originally analyzed (for the full awareness case) by Milgrom and Roberts (1986) using sequential equilibrium. A second example is due to Ozbay (2007).

### 4.2.1 Milgrom-Roberts (1986) verifiable communication

Consider a merchandize whose quality  $q_i \in \{q_1, \dots, q_n\}$  is known to its seller, while a buyer knows only the prior probability distribution  $(p_1, \dots, p_n)$  of the qualities, where  $p_i > 0$  for all  $i = 1, \dots, n$ . For each quality level  $q_i$  the seller is better off the larger the quantity that she sells, while the utility of the buyer from the merchandize is strictly concave in the quantity purchases with a single peak at  $\beta(q_i)$ . Furthermore,

$$\beta(q_1) < \dots < \beta(q_n).$$

Before sale takes place, the seller has the option of providing the buyer with a certified signal about the quality of her merchandize, proving to the seller that the quality is within some range  $\{q_{\min}, \dots, q_{\max}\}$  containing the actual quality  $q_i$ .

Milgrom and Roberts (1986) proved that if the buyer's utility is strictly concave then there is a unique sequential equilibrium, in which when the quality is  $q_i$  the seller certifies to the buyer a range (possibly a singleton)  $\{q_{\min}, \dots, q_{\max}\}$  in which  $q_{\min} = q_i$ , while the buyer is skeptical and always buys  $\beta(q_{\min})$ . Thus, in this unique sequential equilibrium the quality  $q_i$  is fully revealed to the seller, who buys the optimal quantity  $\beta(q_i)$  for him.

We proceed with the caveat that the quantities which can be demanded by the buyer belong to a finite, fine grid (recall that, formally, in our formulation each player has finitely

many available actions in each information set). For simplicity, we assume further that the quantities  $\beta(q_i)$ ,  $i = 1, \dots, n$  belong to this grid. For  $1 \leq m < n$  we denote by  $[\beta(q_m), \beta(q_n)]$  the set of quantities in this grid at least as large as  $\beta(q_m)$  and no larger than  $\beta(q_n)$ .

**Proposition 5** *The strategy to buy  $\beta(q_{\min})$  when confronted with the certification that the quality is in the range  $\{q_{\min}, \dots, q_{\max}\}$  is also the unique prudent rationalizable strategy for the buyer, and certifying some range  $\{q_{\min}, \dots, q_{\max}\}$  in which  $q_{\min} = q_i$  constitute the prudent rationalizable strategies of the seller.*

Thus, any profile of prudent rationalizable strategies in this game yields the full revelation outcome indicated by Milgrom and Roberts (1986).

**Proof of Proposition 5.** When the buyer is confronted with the certificate  $\{q_n\}$ , his unique level-1 (prudent) rationalizable action is to buy  $\beta(q_n)$ , while when he is confronted with some range  $\{q_m, \dots, q_n\}$  all the quantities in the interval  $[\beta(q_m), \beta(q_n)]$  are level-1 (prudent) rationalizable (because any posterior belief of the buyer about the qualities with support  $\{q_m, \dots, q_n\}$  can be derived from a belief of the buyer that the seller provides the certificate  $\{q_m, \dots, q_n\}$  with an appropriate probability  $r_i$  when the seller knows that the quality is  $q_i \in \{q_m, \dots, q_n\}$ .) Consequently, the only level-2 prudent rationalizable strategies of the seller are those in which she provides the certificate  $\{q_n\}$  when the quality is  $q_n$  (because any other certificate that she can provide  $\{q_m, \dots, q_n\}$  will yield an expected sale strictly smaller than  $\beta(q_n)$  with a full support belief about the level-1 prudent rationalizable strategies of the buyer, that have actions in the range  $[\beta(q_m), \beta(q_n)]$ ).

Assume, inductively, that we have already proved that in all the level- $(2k - 1)$  prudent rationalizable strategies of the buyer, for every  $i = 0, \dots, k - 1$  he buys the quantity  $\beta(q_{n-i})$  when confronted with a certificate of the form  $\{q_{n-i}, \dots, q_\ell\}$ , and that in all the level- $2k$  prudent rationalizable strategies of the seller she indeed provides such a certificate when the quality is  $q_{n-i}$ . Then in all the level- $(2k + 1)$  (prudent) rationalizable strategies of the buyer he buys the quantity  $\beta(q_{n-k})$  when confronted with a certificate of the form  $\{q_{n-k}, \dots, q_\ell\}$  (because he believes that such a certificate could only be presented to him with the quality  $q_{n-k}$ , as by the induction hypothesis with each higher quality all the level- $2k$  prudent rationalizable strategies of the seller present a certificate where that higher value is the minimal value). Furthermore, when confronted with some range  $\{q_m, \dots, q_{n-k}, \dots, q_\ell\}$  all the quantities in the interval  $[\beta(q_m), \beta(q_{n-k})]$  are level- $(2k + 1)$  (prudent) rationalizable (because any posterior belief of the buyer about the

qualities with support  $\{q_m, \dots, q_{n-k}\}$  can be derived from a belief of the buyer on the level- $2k$  prudent rationalizable strategies of the seller in which the seller provides the certificate  $\{q_m, \dots, q_\ell\}$  with an appropriate probability  $r_i$  when the seller knows that the quality is  $q_i \in \{q_m, \dots, q_{n-k}\}$ .)

Consequently, in all the level- $(2k + 2)$  prudent rationalizable strategies of the seller she provides the certificate  $\{q_{n-k}, \dots, q_\ell\}$  when the quality is  $q_{n-k}$  (because any other certificate that she can provide  $\{q_m, \dots, q_{n-k}, \dots, q_\ell\}$  will yield an expected sale strictly smaller than  $\beta(q_{n-k})$  with a full support belief about the level- $(2k + 1)$  prudent rationalizable strategies of the buyer, that have actions in the range  $[\beta(q_m), \beta(q_{n-k})]$ ).

Hence, the inductive claim obtains in particular for  $k = n - 1$ , concluding what we wanted to prove.  $\square$

In fact, it is not difficult to see that the above argument does not depend on the assumption that the available certificates consist of ranges of qualities (containing the true quality). For the argument to hold it is enough to assume that for each quality level  $q_i$  one of the available certificates is the fully revealing certificate  $\{q_i\}$ .

**Multi-dimensional certificates and unawareness.** Assume now that there are several dimensions of quality along which such certifications could be provided. To fix ideas, consider two dimensions  $L, H$  and  $^0, ^*$ . The four combinations are

$$L^0, H^0, L^*, H^*.$$

So, for instance, in the state  $L^0$  the available certificates are  $\{L, H\} \times \{^0, ^*\}$ ,  $\{L\} \times \{^0, ^*\}$ ,  $\{L, H\} \times \{^0\}$  and  $\{L\} \times \{^0\}$ .

Assume further that

$$\beta(L^*) < \beta(L^0) < \beta(H^0) < \beta(H^*)$$

Since the singleton certificates

$$\{L\} \times \{^*\}, \{L\} \times \{^0\}, \{H\} \times \{^0\}, \{H\} \times \{^*\}$$

are available, the above argument obtains and full revelation takes place in any profile of prudent rationalizable strategies of the players.

Assume, however, that the buyer is initially aware only of the  $\{L, H\}$  dimension and

is unaware of the  $\{^0, *\}$  dimension; he evaluates the merchandize as having the default quality  $L^0$  when confronted with the certificate  $\{L\}$ , and similarly, with the certificate  $\{H\}$  he evaluates the merchandize as having the default quality  $H^0$ . Assume further that the seller knows this, and that by presenting the certificates  $\{*\}$ ,  $\{^0\}$  or  $\{^0, *\}$  the seller inter alia makes the buyer aware of the  $\{^0, *\}$  dimension.

Intuitively, it is clear that the seller will want to make the buyer aware of this extra dimension when the quality is  $H^*$ , because this will lead the buyer to demand the high quantity  $\beta(H^*)$ . In contrast, when the actual quality is  $L^*$ , the seller will prefer not to present any certificate at all along the dimension  $\{^0, *\}$ : This way the buyer will remain unaware of this extra dimension, and will demand the quantity  $\beta(L^0)$  (because unraveling and full revelation will occur only along the  $\{L, H\}$  dimension); if the seller were to make the buyer aware of this extra dimension, the buyer would have demanded only  $\beta(L^*) < \beta(L^0)$ .

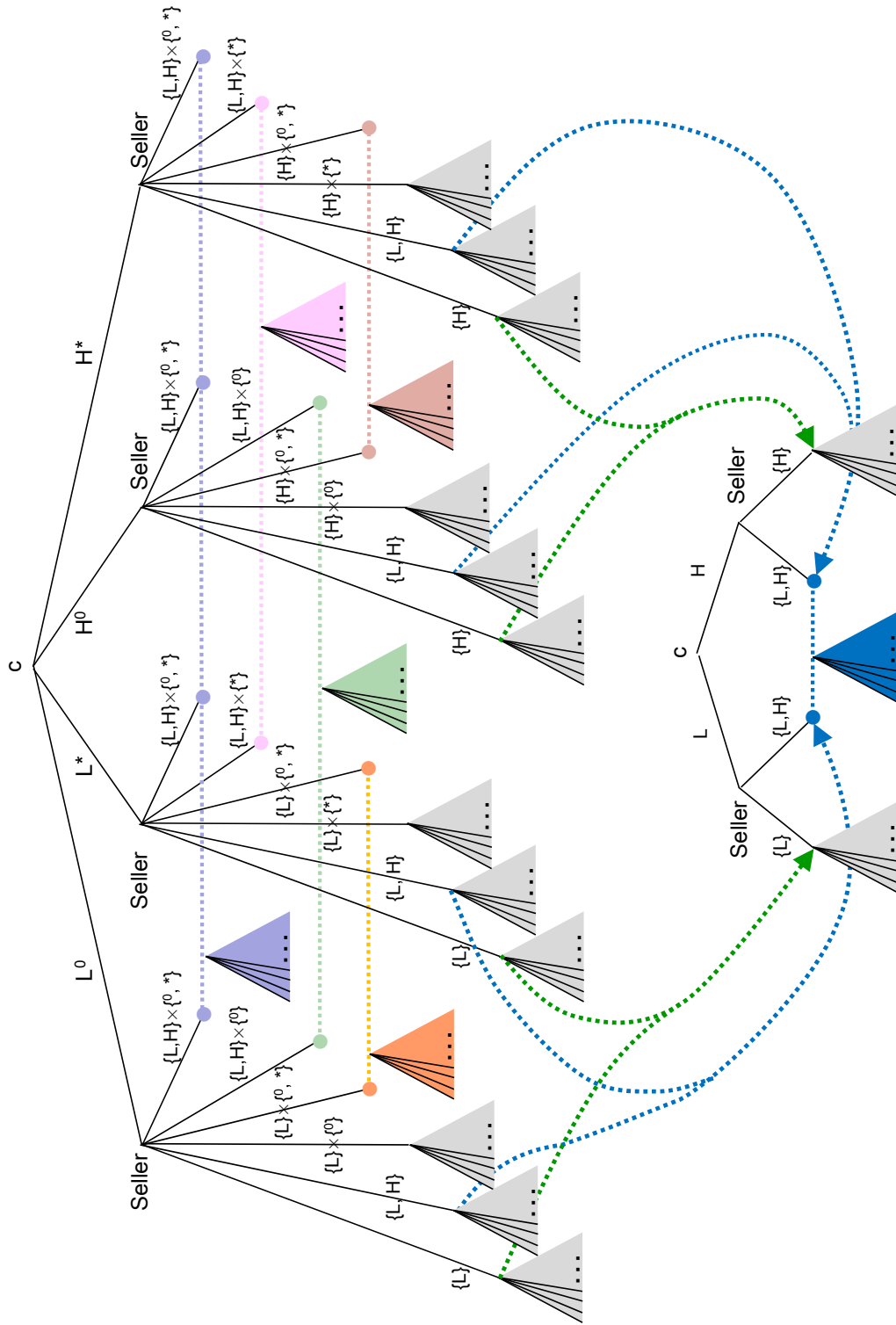
This strategic interaction is represented in the following generalized game form (Figure 10). Initially, nature,  $c$ , selects a state out of  $\{L^0, L^*, H^0, H^*\}$ . The seller observes the state of nature and chooses a certificate. Unless the seller presents a certificate involving the dimension  $\{^0, *\}$ , the buyer remains unaware of it. This is indicated by the intermitted arrows from nodes in the upper tree to nodes in the lower tree. E.g., if the seller selects the certificate  $\{L\}$ , then the buyer remains unaware of the  $\{^0, *\}$  dimension and views the game as represented by the lower tree. In particular, his information set is a singleton containing the node after nature selects  $L$  and the seller reports  $\{L\}$  in the lower tree. If the seller presents a certificate involving the  $\{^0, *\}$ -dimension, then the buyer becomes aware of it and he conceives of the entire generalized game. For instance, if the seller selects the certificate  $\{L, H\} \times \{^0, *\}$ , then the buyer's information set is given by the upmost information set drawn as an intermitted line connecting four nodes.

#### 4.2.2 An example by Ozbay (2007)

To demonstrate the extra power of prudent rationalizability, consider the following example of dynamic interaction with unawareness, which is a variant of example 3 in Ozbay (2007). There are 3 states of nature,  $\omega_1, \omega_2, \omega_3$ . A chance move chooses one out of four



Figure 10:



potential distributions over the states of nature:

$$\begin{aligned}\delta_1 &= (1, 0, 0) \\ \delta_2 &= (0, 1, 0) \\ \delta_3 &= (0, 0, 1) \\ \delta_4 &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\end{aligned}$$

An Announcer gets to know the distribution (but not the realization of the state of nature). A Decision Maker (DM) is initially aware only of the state  $\omega_1$  (and hence the DM is certain that  $\omega_1$  will be realized with certainty). However, before the DM chooses what to do, the Announcer can choose to make the DM aware of either  $\omega_2, \omega_3$ , none of them or both of them. Increased awareness makes the DM aware of the relevant marginals of the distributions. For instance, if the Announcer makes the DM aware of  $\omega_2$ , the DM becomes aware of the set of distributions

$$\begin{aligned}\delta_{1|\{\omega_1, \omega_2\}} &= (1, 0) \\ \delta_{2|\{\omega_1, \omega_2\}} &= (0, 1) \\ \delta_{4|\{\omega_1, \omega_2\}} &= \left(\frac{1}{2}, \frac{1}{2}\right)\end{aligned}$$

and also becomes certain that the Announcer knows which of these is the true distribution.<sup>12</sup>

Subsequently, the DM should choose one out of three possible actions – left, middle or right. The payoffs to the players as a function of the chosen action and the state of

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<sup>12</sup>In the spirit of Footnote 4 above, in Ozbay’s example and in what follows the DM’s beliefs about these marginal distributions will not be necessarily related to the prior probabilities with which the distributions were chosen by the chance move. That’s why we do not even bother to specify the probabilities with which the chance move chooses the different distributions.

Put differently, instead of describing this game by a partially ordered set of trees, one for each level of awareness as in Figure 11, we could have replaced each tree with an arborescence in which the initial chance move is erased. Allowing for arborescences instead of trees in the framework for dynamic unawareness of Section 2 is straightforward, but for the sake of clarity of the exposition we avoid this explicit generalization in the body of the paper.

nature appear in the following table:

	left	middle	right
$\omega_1$	3, 3	0, 0	2, 2
$\omega_2$	0, 0	5, 5	2, 2
$\omega_3$	2, 2	0, 0	2, 2

The game is thus described in Figure 11 in the following page.

It is obvious that if the Announcer announces nothing, and hence the DM is certain that  $\omega_1$  prevails, the DM will choose ‘left’.

What happens if the Announcer makes the DM aware of  $\omega_2$ ? The information set of the DM becomes

$$\left\{ \delta_{1|\{\omega_1, \omega_2\}}, \delta_{2|\{\omega_1, \omega_2\}}, \delta_{4|\{\omega_1, \omega_2\}} \right\}$$

The DM may then assign a high probability to  $\delta_{1|\{\omega_1, \omega_2\}}$ ,<sup>13</sup> and this will lead the DM to choose ‘left’. Hence, assuming such a belief by the DM, it is rationalizable for the Announcer to make the DM aware of  $\omega_2$  when the Announcer knows that the true distribution is  $\delta_1$  (i.e. when the Announcer knows that  $\omega_1$  will be realized with probability 1).

This is not very sensible, though. After all, the Announcer can ensure that the DM chooses ‘left’ by not announcing any new state. When the Announcer likes the DM to choose ‘left’, it makes no sense on the Announcer’s part to announce  $\omega_2$  and thus face the risk that the DM assigns a low probability to  $\delta_{1|\{\omega_1, \omega_2\}}$  and consequently choose ‘middle’. This idea is captured by Ozbay’s reasoning refinement to his awareness equilibrium notion<sup>14</sup>, as well as by prudent rationalizability:

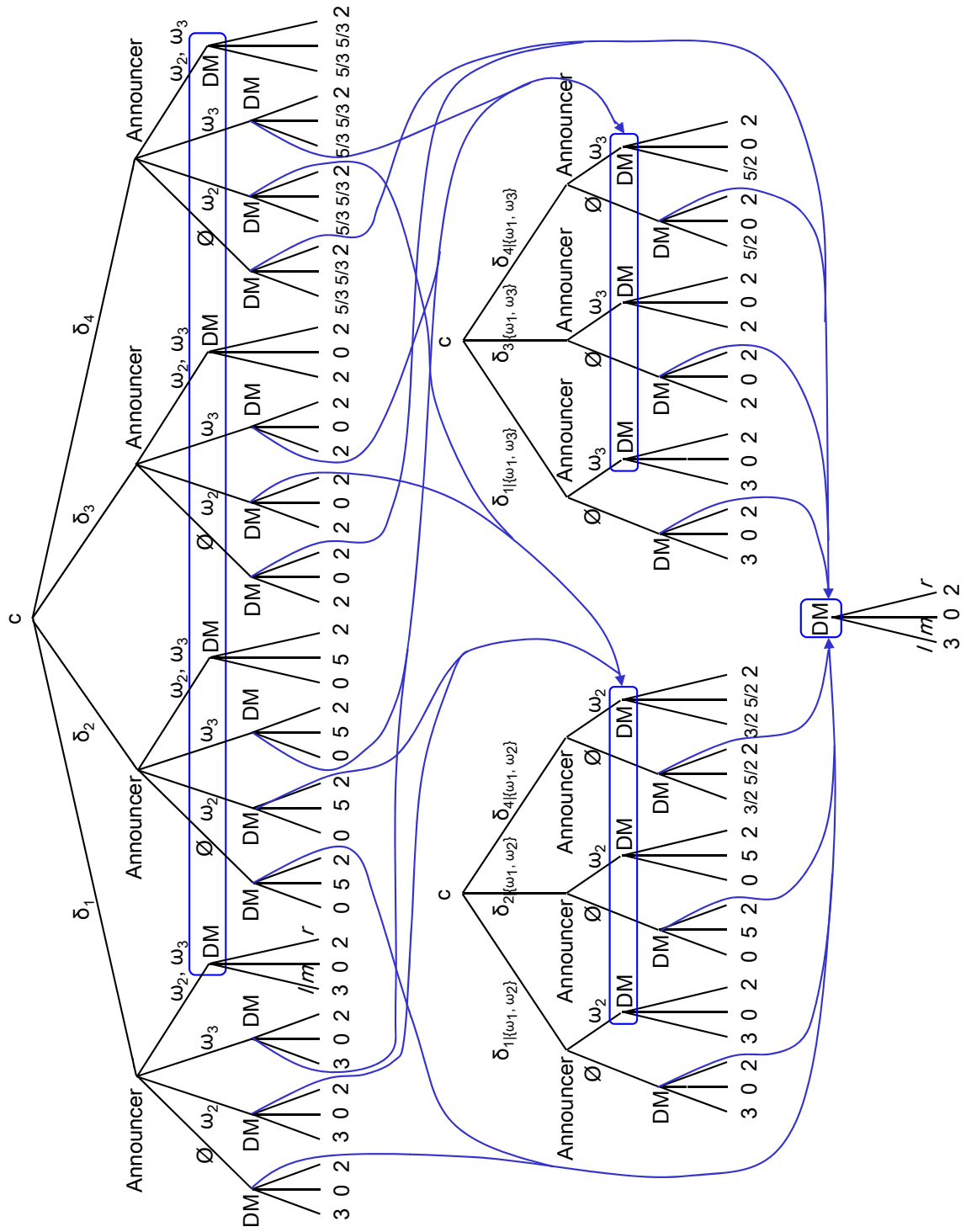
**Proposition 6** *The DM has a unique prudent rationalizable strategy. With this strategy the DM chooses ‘left’ when no new state is announced, ‘middle’ when only  $\omega_2$  is announced, ‘left’ when only  $\omega_3$  is announced, and ‘right’ when both  $\omega_2, \omega_3$  are announced.*

**Proof.**  $\bar{B}_{DM}^1$  contains belief systems in which in the information set  $\left\{ \delta_{1|\{\omega_1, \omega_2\}}, \delta_{2|\{\omega_1, \omega_2\}}, \delta_{4|\{\omega_1, \omega_2\}} \right\}$  (which follows the announcement of only  $\omega_2$  by the Announcer) the DM’s belief assigns

<sup>13</sup>That is, the DM may assign a high probability to strategies of the Announcer by which the Announcer announces  $\omega_2$  (and cause the DM’s information set to become  $\left\{ \delta_{1|\{\omega_1, \omega_2\}}, \delta_{2|\{\omega_1, \omega_2\}}, \delta_{4|\{\omega_1, \omega_2\}} \right\}$ ) when the Announcer has learned that the true distribution is  $\delta_1$ .

<sup>14</sup>As explained in the introduction, we believe that equilibrium notions are somewhat questionable in the context of unawareness, and hence our focus on rationalizability.

Figure 11:



high probabilities to  $\delta_{2|\{\omega_1, \omega_2\}}, \delta_{4|\{\omega_1, \omega_2\}}$ . The strategies in  $\bar{S}_{DM}^1$  corresponding to these belief systems prescribe ‘middle’ to the DM in the information set  $\left\{ \delta_{1|\{\omega_1, \omega_2\}}, \delta_{2|\{\omega_1, \omega_2\}}, \delta_{4|\{\omega_1, \omega_2\}} \right\}$ . The crucial point is that  $\bar{B}_{Announcer}^2$  contains *only* belief systems that assign strictly positive probabilities to these strategies of the DM. Thus, with *any* belief system in  $\bar{B}_{Announcer}^2$ , it is sub-optimal for the Announcer to announce  $\omega_2$  in the announcer’s information set  $\{\delta_1\}$ , in which the Announcer is certain of  $\omega_1$ .<sup>15</sup> Hence,  $\bar{S}_{Announcer}^2$  does not contain strategies in which the Announcer announces just  $\omega_2$  when the announcer’s information set is  $\{\delta_1\}$ . We conclude that  $\bar{B}_{DM}^3$  contains only belief systems in which the belief at the information set  $\left\{ \delta_{1|\{\omega_1, \omega_2\}}, \delta_{2|\{\omega_1, \omega_2\}}, \delta_{4|\{\omega_1, \omega_2\}} \right\}$  assigns probability zero to  $\delta_{1|\{\omega_1, \omega_2\}}$ . Hence,  $\bar{S}_{DM}^3$  contains only strategies with which the DM chooses ‘middle’ at the information set  $\left\{ \delta_{1|\{\omega_1, \omega_2\}}, \delta_{2|\{\omega_1, \omega_2\}}, \delta_{4|\{\omega_1, \omega_2\}} \right\}$ .

Furthermore, already  $\bar{S}_{DM}^1$  contains only strategies with which the DM chooses ‘left’ at the information set  $\left\{ \delta_{1|\{\omega_1, \omega_3\}}, \delta_{3|\{\omega_1, \omega_3\}}, \delta_{4|\{\omega_1, \omega_3\}} \right\}$  (i.e. when the Announcer announces just the new state  $\omega_3$ ). This is because prudent rationalizability implies that all the belief systems in  $\bar{B}_{DM}^1$  assign a positive probability to strategies of the Announcer with which the Announcer announces the new state  $\omega_3$  even when the Announcer’s information set (from the point of view of the DM!) is  $\left\{ \delta_{1|\{\omega_1, \omega_3\}} \right\}$  or  $\left\{ \delta_{4|\{\omega_1, \omega_3\}} \right\}$ .

Also,  $\bar{B}_{DM}^1$  contains belief systems in which the DM’s belief in the information set  $\{\delta_1, \delta_2, \delta_3, \delta_4\}$  (when the Announcer announces both new states  $\omega_2, \omega_3$ ) assigns high probability to  $\delta_2$ . The strategies in  $\bar{S}_{DM}^1$  corresponding to these belief systems prescribe ‘middle’ to the DM in the information set  $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ . Hence,  $\bar{B}_{Announcer}^2$  contains *only* belief systems that assign strictly positive probabilities to these strategies of the DM. Thus, with *any* belief system in  $\bar{B}_{Announcer}^2$ , it is sub-optimal for the Announcer to announce both  $\omega_2$  and  $\omega_3$  in the announcer’s information sets  $\{\delta_1\}$  and  $\{\delta_3\}$ . Similarly,  $\bar{B}_{DM}^1$  contains belief systems in which the DM’s belief in the information set  $\{\delta_1, \delta_2, \delta_3, \delta_4\}$  assigns high probability to  $\delta_1$ . The strategies in  $\bar{S}_{DM}^1$  corresponding to these belief systems prescribe ‘left’ to the DM in the information set  $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ . Hence,  $\bar{B}_{Announcer}^2$  contains *only* belief systems that assign strictly positive probabilities to these strategies of the DM. Thus, with *any* belief system in  $\bar{B}_{Announcer}^2$ , it is sub-optimal for the Announcer to announce both  $\omega_2$  and  $\omega_3$  in the Announcer’s information sets  $\{\delta_1\}, \{\delta_2\}$  or  $\{\delta_3\}$ . We conclude that  $\bar{B}_{DM}^3$  contains only belief systems in which the belief at the information set  $\{\delta_1, \delta_2, \delta_3, \delta_4\}$  assigns probability zero to  $\delta_1, \delta_2, \delta_3$ . That is,  $\bar{B}_{DM}^3$  contains only belief systems that assign probability 1 to  $\delta_4$  at the information set  $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ . Hence,

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<sup>15</sup>Because according to every belief system in  $\bar{B}_{Announcer}^2$ , announcing just  $\omega_2$  will lead the DM with a positive probability to choose ‘middle’.

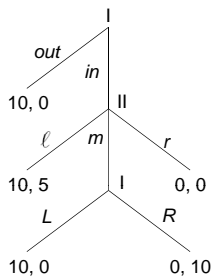
$\bar{S}_{DM}^3$  contains only strategies with which the DM chooses ‘right’ at the information set  $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ .

We thus conclude that  $\bar{S}_{DM}^3$  contains a unique strategy  $s_{DM}^*$ . This strategy prescribes the DM to choose ‘left’ in the information set  $\{\delta_1|_{\{\omega_1\}}\}$  (i.e. when the Announcer does not announce any new state), to choose ‘middle’ in the information set  $\{\delta_1|_{\{\omega_1, \omega_2\}}, \delta_2|_{\{\omega_1, \omega_2\}}, \delta_4|_{\{\omega_1, \omega_2\}}\}$  (i.e. when the Announcer announces just the new state  $\omega_2$ ), to choose ‘left’ in the information set  $\{\delta_1|_{\{\omega_1, \omega_3\}}, \delta_3|_{\{\omega_1, \omega_3\}}, \delta_4|_{\{\omega_1, \omega_3\}}\}$  (i.e. when the Announcer announces just the new state  $\omega_3$ ) and to choose ‘right’ in the information set  $\{\delta_1, \delta_2, \delta_3, \delta_4\}$  (i.e. when the Announcer announces both new states  $\omega_2, \omega_3$ ).<sup>16</sup>  $\square$

### 4.3 A second tension between rationalizability and prudence: Divining the opponent’s future behavior

In Figure 9 we demonstrated the tension between the considerations of rationalization and prudence when a player tries to divine his opponent’s past actions. A related but distinct tension arises when a player tries to deduce the opponent’s future behavior from past actions of that opponent. Consider the following example in Figure 12.

Figure 12:



In this example, *in* is imprudent for player 1 (since by going *out* she can guarantee a payoff of 10, while by moving *in* she risks getting 0 if player 2 would rather foolishly choose *r*). This means that if player 1 does move *in* and player 2 gets to play, no prudent strategy in  $\bar{S}_1^1$  reaches 2’s information set. Hence, the beliefs  $\bar{B}_2^2$  of player 2 about player 1’s future actions are not restricted. In particular, it contains beliefs by which if player

<sup>16</sup>This is also the unique strategy of the DM which is part of an awareness equilibrium satisfying reasoning refinement in Ozbay (2007).

2 chooses  $m$ , player 1 will foolishly choose  $R$  (with a high probability). That's why both  $m$  and  $\ell$  are prudent rationalizable for player 2.

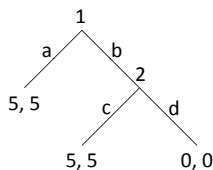
However, it is not very sensible on the part of player 2 to believe that following  $m$  player 1 may choose  $R$ . After all, when player 2 has to move, player 1 has already proved to be imprudent, but not irrational. Indeed, player 1's rationalizable (though imprudent) strategy  $(in, L)$  yields her the payoff 10 in conjunction with 2's only (would-be) rationalizable strategy  $\ell$ , as well as in conjunction with 2's prudent rationalizable strategy  $m$ ; and this payoff is the same as the payoff player 1 gets from her only prudent rationalizable strategy  $(out, L)$ .

Thus, as long as player 1 has been rational (even if imprudent) thus far, it makes more sense for player 2 to believe that player 1 will continue to be rational (though possibly imprudent) in the future. Restricting player 2's beliefs according to this logic would cross out the non-sensical choice  $m$ .

Already Pearce (1984) was well aware of this tension, which motivated his definition of **cautious extensive-form rationalizability**. That definition involves refining the set of rationalizable strategies by another round of strategy elimination with full support beliefs about the other players' surviving strategies; and then repeating this entire procedure – the standard iterative elimination process as in the definition of rationalizability, followed by one round assuming full-support beliefs – ad infinitum. In the above example, cautious extensive-form rationalizability does indeed rule out the strategy  $m$  for player 2.

However, as Pearce (1984) himself admits, the definition of cautious extensive-form rationalizability is not really satisfactory, as the following simple example of his shows.

Figure 13:



In this example, the strategy  $d$  is irrational for player 2. Once  $d$  is crossed out, both  $a$  and  $b$  are extensive-form rationalizable for player 1, and are actually also cautious extensive-form rationalizable. Notice that in contrast,  $b$  *does* get crossed out by *prudent rationalizability*, and the only prudent rationalizable strategy for player 1 is  $a$ .

To sum up, we believe it is worth exploring further a more fine-tuned *refinement*

of rationalizability which would take prudence considerations into account, one which would be more subtle than Pearce’s cautious extensive-form rationalizability. As the above examples suggest, such a definition would be involved, and would take us beyond the scope of the current paper. We plan to address this issue in future work.

#### 4.4 Strategy elimination vs. belief systems reduction

Definition 1 of would-be rationalizable strategies involves, as in Battigalli (1997), an iterative reduction procedure of belief systems (that is, by definition  $B_i^k \subseteq B_i^{k-1}$ ), and this definition implies (Remark 3) that strategies get iteratively eliminated ( $S_i^k \subseteq S_i^{k-1}$ ); and the same is true also for extensive-form correlated rationalizable strategies – by definition  $\hat{B}_i^k \subseteq \hat{B}_i^{k-1}$  and hence  $\hat{S}_i^k \subseteq \hat{S}_i^{k-1}$ . In contrast, the inductive definition of **prudent** rationalizable strategies involves an iterative elimination of strategies (that is, by definition  $\bar{S}_i^k \subseteq \bar{S}_i^{k-1}$ , in analogy with the original formulation of Pearce (1984) for extensive-form rationalizability by an iterative elimination procedure), but in the case of prudence it is **not** generally the case that  $\bar{B}_i^k \subseteq \bar{B}_i^{k-1}$ . Indeed, when  $\bar{S}_{-i}^k \subsetneq \bar{S}_{-i}^{k-1}$ :

- If the set of strategy profiles in  $\bar{S}_{-i}^k$  reaching some information set  $h_i \in H_i$  is a proper, non-empty subset of the strategy profiles in  $\bar{S}_{-i}^{k-1}$  that reach  $h_i$ , then the support of each belief  $\bar{b}_i^{k-1}(h_i)$  in each belief system  $\bar{b}_i^{k-1} \in \bar{B}_i^{k-1}$  is strictly larger than the support of any belief  $\bar{b}_i^k(h_i)$  for  $\bar{b}_i^k \in \bar{B}_i^k$ .
- For information sets  $h_i$  *not* reached by  $\bar{S}_{-i}^k$ , there is no restriction (beyond Bayes rule) on  $\bar{b}_i^k(h_i)$  for  $\bar{b}_i^k \in \bar{B}_i^k$ . No such restriction is needed, because if we define

$$m_{h_i}^k = \max \{ m < k : \text{there exists } s_{-i} \in \bar{S}_{-i}^m \text{ that reaches } h_i \}$$

then for  $s_i^k \in \bar{S}_i^k$  the restrictions on  $i$ ’s actions  $s_i^k(h_i)$  at  $h_i$  were already determined at stage  $m_{h_i}^k$ , since by definition  $s_i^k \in \bar{S}_i^k \subseteq \bar{S}_i^{m_{h_i}^k}$ .

Is it nevertheless feasible to define prudent rationalizability via a reduction process of belief systems? Asheim and Perea (2005) proposed to look at **systems of conditional lexicographic probabilities** – belief systems in which each belief at an information set is itself a **lexicographic probability system** (Blume, Brandenburger and Dekel 1991) about the other players’ strategy profiles. Using belief systems which are conditional lexicographic probabilities we could, in the spirit of Stahl (1995), put forward an equivalent definition of prudent rationalizable strategies involving an iterative reduction



procedure of belief systems rather than an iterative elimination procedure of strategies. In each round of the procedure, the surviving belief systems would be those in which at each information set, ruled-out strategy profiles of the other players (i.e. strategy profiles outside  $\bar{S}_{-i}^{m_{h_i}^k}$ ) would be deemed infinitely less likely than the surviving strategy profiles, but infinitely more likely than strategy profiles which had already been eliminated in previous rounds. We leave the precise formulation of such an equivalent definition to future work.

In their paper, Asheim and Perea (2005) proposed the notion of **quasi-perfect rationalizability**, which also involves the idea of cautious beliefs. Quasi-perfect rationalizability is distinct from our notion of prudent rationalizability. The difference is that with prudent rationalizability (as with would-be rationalizability), a player need not believe that another player's future behavior must be rationalizable to a higher order than that exhibited by that other player in the past; in contrast, with the quasi-perfect rationalizable strategies of Asheim and Perea (2005), a player should ascribe to her opponent the highest possible level of rationality in the future even if this opponent has already proved to be less rational in the past. That's why quasi-perfect rationalizability implies backward induction in generic perfect information games, while our prudent rationalizable strategies need not coincide with the backward induction strategies in such games (though they do generically lead to the backward induction path – the argument is the same as in Reny 1992 and Battigalli 1997, since in generic perfect information games prudent rationalizability coincides with extensive-form rationalizability in terms of realized paths).

## 5 Characterization by conditional dominance

### 5.1 Associated normal-form games

Consider a generalized extensive-form game  $\Gamma$  with a partially ordered set of trees  $\mathbf{T}$ . The *associated normal-form game*  $G$  is defined by  $\langle I, \langle (S_i^T)_{i \in I}, (u_i^T)_{i \in I} \rangle_{T \in \mathbf{T}} \rangle$ , where  $I$  is the set of players in  $\Gamma$  and  $S_i^T$  is player  $i$ 's set of  $T$ -partial strategies. If player  $i$  is not active in trees  $T' \in \mathbf{T}$  with  $T \prec T'$ , then  $S_i^{T'} = \emptyset$ . Recall that if player  $i$  is active at node  $n \in T$ , then at node  $n$  the strategy  $s_i \in S_i^T$  prescribes to her the action  $s_i(\pi_i(n))$ . Hence, each profile of strategies in  $S^T$  induces a distribution over terminal nodes in  $T$  (even if there is a player active in  $T$  with no information set in  $T$ ).  $u_i^T(s)$  is the expected value of the payoffs associated with the terminal nodes in  $T$  reached by  $s \in S^T$  weighted by

the probabilities associated to the moves of nature. (Note that while strategy profiles in  $S^T$  reach terminal nodes also in trees  $T' \in \mathbf{T}$ ,  $T \leftrightarrow T'$ ,  $u_i^T$  concerns payoffs in the tree  $T$  only.)

Recall that  $H_i^T$  denotes player  $i$ 's set of extensive form information sets in the  $T$ -partial game. For each  $h_i \in H_i^T$ , let  $S^T(h_i) \subseteq S^T$  be the subset of the  $T$ -partial strategy space containing  $T$ -partial strategy profiles that reach the information set  $h_i$ . Define also  $S_i^T(h_i) \subseteq S_i^T$  and  $S_{-i}^T(h_i) \subseteq S_{-i}^T$  to be the set of player  $i$ 's  $T$ -partial strategies reaching  $h_i$  and the set of profiles of the other players'  $T$ -partial strategies reaching  $h_i$  respectively. For the entire game denote by  $S(h_i) \subseteq S$  the set of strategy profiles that reach  $h_i$ . Similarly,  $S_i(h_i) \subseteq S_i$  and  $S_{-i}(h_i) \subseteq S_{-i}$  are the set of player  $i$ 's strategies reaching  $h_i$  and the set of profiles of the other players' strategies reaching  $h_i$  respectively.

Given  $\Gamma$  and its associated normal-form game  $G$ , define player  $i$ 's set of *normal-form information sets*<sup>17</sup> by

$$\mathcal{X}_i = \{S^{T_{h_i}}(h_i) : h_i \in H_i\}.$$

These are the “normal form versions” of information sets in the generalized extensive-form game.

For  $T \in \mathbf{T}$ , any set  $Y \subseteq S^T$  is called a *restriction* for player  $i$  (or an  $i$ -product set) of  $T$ -partial strategies if  $Y = Y_i \times Y_{-i}$  for some  $Y_i \subseteq S_i^T$  and  $Y_{-i} \subseteq S_{-i}^T$ . Clearly, a player's normal-form information set is a restriction. I.e., if  $S^{T_{h_i}}(h_i)$  is a normal form information set of player  $i$ , then it is a restriction for player  $i$  of  $T_{h_i}$ -partial strategy profiles.

## 5.2 Iterated conditional strict dominance and extensive-form rationalizability

We say that  $s_i \in S_i^T$  is *strictly dominated* in a restriction  $Y \subseteq S^T$  if  $s_i \in Y_i$ ,  $Y_{-i} \neq \emptyset$ , and there exists a mixed strategy  $\sigma_i \in \Delta(Y_i)$  such that  $u_i^T(\sigma_i, s_{-i}) > u_i^T(s_i, s_{-i})$  for all  $s_{-i} \in Y_{-i}$ .

Denote by  $\mathbf{S} = \bigcup_{T \in \mathbf{T}} S^T$  and  $\mathbf{S}_i = \bigcup_{T \in \mathbf{T}} S_i^T$ .

For  $T \leftrightarrow T'$  and a  $T$ -partial strategy  $s_i \in S_i^T$ , we denote the  $T'$ -partial strategy

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<sup>17</sup>We abuse here slightly existing terminology. In the literature on standard games, normal-form information sets refer more generally to subsets of the strategy space of a pure strategy reduced normal-form game for which there exists an extensive-form game with corresponding information sets (see Mailath, Samuelson and Swinkels, 1993). For our characterization, we are just interested in the normal form versions of information sets of a given generalized extensive-form game.

$s_i^{T'} \in S_i^{T'}$  induced by  $s_i$ . For  $\tilde{s}_i \in S_i^{T'}$ , define

$$[\tilde{s}_i] := \bigcup_{T \hookrightarrow T'} \{s_i \in S_i^T : s_i^{T'} = \tilde{s}_i\}.$$

That is,  $[\tilde{s}_i]$  is the set of strategies in  $\mathbf{S}_i$  which at information sets  $h_i \in H_i^{T'}$  prescribe the same actions as strategy  $\tilde{s}_i$ .

Let  $(Y^T)_{T \in \mathbf{T}}$  be a collection of  $i$ -product sets, one for each  $T \in \mathbf{T}$ . Define  $\mathbf{Y} = \bigcup_{T \in \mathbf{T}} Y^T$ . Given such a  $\mathbf{Y}$ , we say that  $s_i \in S_i^T$  is *conditionally strictly dominated on*  $(\mathcal{X}_i, \mathbf{Y})$  if (1) there exists a normal-form information set  $X \in \mathcal{X}_i$ ,  $X \subseteq S^T$  such that  $s_i$  is strictly dominated in  $X \cap Y^T$  or (2) for some  $\tilde{s}_i \in S_i^{T'}$ ,  $T \hookrightarrow T'$ ,  $s_i \in [\tilde{s}_i]$ , we have that  $\tilde{s}_i$  is strictly dominated in  $X \cap Y^{T'}$  for some normal-form information set  $X \in \mathcal{X}_i$ ,  $X \subseteq S^{T'}$ .

(Note that (2) implies (1), but the explicit distinction between (1) and (2) makes the presentation more transparent.)

For  $\mathbf{Y}$  define

$$U_i(\mathbf{Y}) = \{s_i \in \mathbf{S}_i : s_i \text{ is not conditionally strictly dominated on } (\mathcal{X}_i, \mathbf{Y})\},$$

$$U(\mathbf{Y}) = \bigcup_{T \in \mathbf{T}} \prod_{i \in I} (U_i(\mathbf{Y}) \cap S_i^T),$$

and

$$U_{-i}(\mathbf{Y}) = \bigcup_{T \in \mathbf{T}} \prod_{j \in I \setminus \{i\}} (U_j(\mathbf{Y}) \cap S_j^T).$$

Define inductively

$$U^0(\mathbf{S}) = \mathbf{S},$$

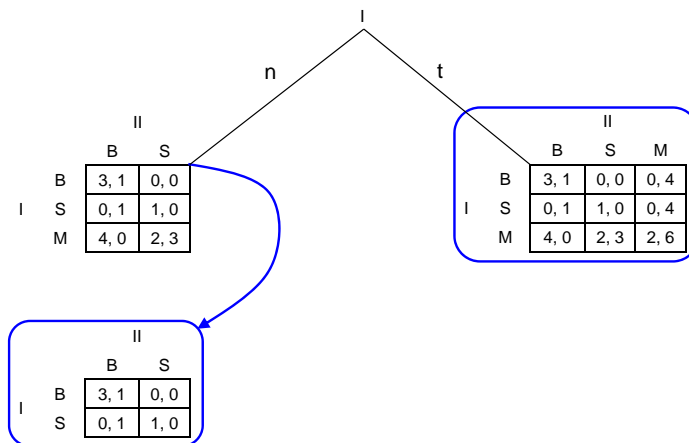
$$U^{k+1}(\mathbf{S}) = U(U^k(\mathbf{S})) \text{ for } k \geq 0,$$

$$U^\infty(\mathbf{S}) = \bigcap_{k=0}^\infty U^k(\mathbf{S}),$$

and similarly for  $U_i^k(\mathbf{S})$  and  $U_{-i}^k(\mathbf{S})$ .

**Example.** Consider the game below whose extensive form is identical to the Battle-of-the-Sexes game with unawareness from the introduction but whose payoffs are quite different (Figure 14). In this strategic situation, player I may deceive player II by hiding player II's dominant action M. As we will see, this example allows us to demonstrate some features of iterated conditional dominance that we couldn't have demonstrated with the introductory example.

Figure 14:



The associated normal form is given in Figure 15. The lower strategic form game

Figure 15: The Associated Normal Form Game

<i>T</i>	<i>BB</i>	<i>BS</i>	<i>SB</i>	<i>SS</i>	<i>MB</i>	<i>MS</i>
<i>nBBB</i>	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
<i>nSBB</i>	0, 1	1, 0	0, 1	1, 0	0, 1	1, 0
<i>nMBB</i>	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
<i>nBSB</i>	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
<i>nSSB</i>	0, 1	1, 0	0, 1	1, 0	1, 0	1, 0
<i>nMSB</i>	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
<i>nBMB</i>	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
<i>nSMB</i>	0, 1	1, 0	0, 1	1, 0	1, 0	1, 0
<i>nMMB</i>	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
<i>nBBS</i>	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
<i>nSBS</i>	0, 1	1, 0	0, 1	1, 0	1, 0	1, 0
<i>nMBS</i>	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
<i>nBSS</i>	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
<i>nSSS</i>	0, 1	1, 0	0, 1	1, 0	1, 0	1, 0
<i>nMSS</i>	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
<i>nBMS</i>	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
<i>nSMS</i>	0, 1	1, 0	0, 1	1, 0	1, 0	1, 0
<i>nMMS</i>	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
<i>tBBB</i>	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
<i>tSBB</i>	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
<i>tMBB</i>	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
<i>tBSB</i>	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
<i>tSSB</i>	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
<i>tMSB</i>	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
<i>tBMB</i>	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6
<i>tSMB</i>	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6
<i>tMMB</i>	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6
<i>tBBS</i>	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
<i>tSBS</i>	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
<i>tMBS</i>	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
<i>tBSS</i>	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
<i>tSSS</i>	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
<i>tMSS</i>	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
<i>tBMS</i>	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6
<i>tSMS</i>	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6
<i>tMMS</i>	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6

<i>T'</i>	<i>B</i>	<i>S</i>
<i>B</i>	3, 1	0, 0
<i>S</i>	0, 1	1, 0

is the normal form associated with the  $T'$ -partial extensive-form game and the normal

form associated with the  $T$ -partial extensive-form game is the upper strategic form game. Player I is the row player, while player II is the column player. For the row player in the upper strategic form, the first component of her strategy refers to actions at the root of the upper tree, the second to her action in the upper left subgame, the third to the upper right subgame, and the last component to the action in the lower game. For the column player, the first component of his strategy refers to the action taken in the upper information set while the second is the action taken in the lower information set.

Each boxed cell is a normal-form information set. The entire upper strategic form is the normal-form information set of player 1 (but not player 2) associated with player 1's information set at the beginning of the  $T$ -partial game (but not in the  $T'$ -partial game). We denote this information set by  $X_1(\emptyset^T)$ . The upper boxed cell in the upper strategic form is the normal-form information set of player 1 (but not of player 2) corresponding to her extensive form information set after the history  $n$  in the  $T$ -partial game (but not in the  $T'$ -partial game). We denote it by  $X_1(n)$ . The lower boxed cell in the upper strategic form game is the normal-form information set for both player 1 and 2 corresponding to the information sets after history  $t$  in the  $T$ -partial game (but not in the  $T'$ -partial game). We denote it by  $X_i(t)$ .

Finally, the lower strategic form game is a normal form information set for both player 1 and 2 both for corresponding information sets in the  $T'$ -partial normal form and in the  $T$ -partial normal form game. It is also the normal-form information set for player 2 corresponding to his information set  $\pi_2(n)$  in the  $T$ -partial game. We denote it by  $X_i(\emptyset^{T'}) = X_2(n)$ .

The definition of  $\mathbf{S}_i$  is illustrated by the example  $\mathbf{S}_2 = \{BB, BS, SB, SS, MB, MS, B, S\}$ , while the definition  $[\tilde{s}_i]$  can be illustrated by  $["S"] = \{BS, SS, MS, S\}$ . These are all the strategies of player 2 that prescribe action "S" ("Stravinsky") at the information set  $\pi_2(n)$ .

The iterated elimination of conditionally strictly dominated strategies proceeds as follows:

$$\begin{aligned}
U_i^0(\mathbf{S}) &= \mathbf{S}_i, i = 1, 2 \\
U_1^1(\mathbf{S}) &= \{nMBB, nMSB, nMMB, nMBS, nMSS, nMMS, \\
&\quad tBMB, tSMB, tMMB, tBMS, tSMS, tMMS, \\
&\quad B, S\} \\
U_2^1(\mathbf{S}) &= \{MB, B\}
\end{aligned}$$

For instance, strategy  $nSBB$  is conditionally strictly dominated by  $nMBB$  in the normal-form information set  $X_1(\emptyset^T)$  or  $X_1(n)$ . More interestingly,  $MS$  is conditionally strictly dominated on  $(\mathcal{X}_2, \mathbf{S})$  because  $MS \in [^n S^m]$  and  $S$  is strictly dominated by  $B$  in  $X_2(n)$ . So this example demonstrates that an action in the upper normal form may be deleted because of strict dominance in the lower normal form. This is one reason why we chose this game to demonstrate iterated conditional strict dominance rather than the introductory example.

Applying the definitions iteratively yields

$$\begin{aligned}
U_1^2(\mathbf{S}) &= \{nMBB, nMSB, nMMB, \\
&\quad tBMB, tSMB, tMMB, tBMS, tSMS, tMMS, B\} \\
U_2^2(\mathbf{S}) &= U_2^1(\mathbf{S}) = \{MB, B\} \\
U_1^3(\mathbf{S}) &= \{nMBB, nMSB, nMMB, B\} \\
&= U_1^k(\mathbf{S}) \text{ for } k \geq 3 \\
U_2^3(\mathbf{S}) &= U_2^2(\mathbf{S}) = \{MB, B\} \\
&= U_2^k(\mathbf{S}) \text{ for } k \geq 1
\end{aligned}$$

Note that  $U_i^\infty(\mathbf{S}) \cap S_i = \hat{S}_i^\infty$ . That is, the set of iterated elimination of conditionally strictly dominated strategies coincides with the set of extensive-form correlated rationalizable strategies, and both predict that player  $I$  will not give the car to player  $II$  and attend the Mozart concert, while player  $II$  will attend the Bach concert.

The following proposition generalizes the observation made in the example.

**Proposition 7** *For every finite generalized extensive form game,  $U_i^k(\mathbf{S}) \cap S_i = \hat{S}_i^k$ ,  $k \geq 1$ . Consequently,  $U_i^\infty(\mathbf{S}) \cap S_i = \hat{S}_i^\infty$ .*

The proof is in appendix A.

**Remark 7** *If in the definition of prudent rationalizability, would-be rationality is replaced by rationality, then prudent rationalizability can be characterized by iterated elimination of conditional weakly dominated strategies. The proof is analogous. Instead of using Lemma 3 in Pearce (1984), we would now use Lemma 4 in Pearce (1984). Moreover, iterated conditional weak dominance is equivalent to iterated admissibility in the normal-form. This is so because if a strategy weakly dominates a replacement in an*

information set, then the payoffs from the strategy and its replacement outside the information set must coincide (since otherwise it wouldn't be a replacement).

**Remark 8** Consider a game with unavailability of actions analogous to Figure 1 but with the payoffs as in the example of this section. Then the set of would-be rationalizable paths include the one in which player I gives the car to player II and they both go to the Mozart concert, as well as the paths in which player I doesn't give the car to player II and then player I goes either to the Bach or to the Mozart concert and player II goes either to the Bach or to the Stravisnky concert. In contrast with the example in the introduction, this example therefore shows that would-be rationalizability does not necessarily yield a sharper prediction under unavailability of actions than under unawareness of the same actions.

## 6 Unawareness

Generalized games can describe many types of games with subjective reasoning. In a generalized game, a player cannot imagine that she can take an action which is physically unavailable to her (property I4), but at a given information set  $\pi_i(n)$  she can nevertheless imagine that in a succeeding information set she will have an action which is actually nowhere available in the tree  $T_n$  as in the example of Figure 7. Furthermore, she can imagine that along the path of play another player will forget the history of play, i.e. that at a later information set this other player will imagine he is playing in a game tree which is completely unrelated to the game tree he imagined at an earlier stage along the path.

Since our main motivation is to analyze games with unawareness rather than games with arbitrary kinds of subjective reasoning, it is worthwhile spelling out additional properties of generalized games in which the only reason for players' misconception of the game is unawareness (and mutual unawareness) of available actions. In *extensive-form games with unawareness* the set of trees  $\mathbf{T}$  forms a join semi-lattice under the inclusion partial order relation  $\preceq$ . The maximal tree in this join semi-lattice is the modeler's objective description of feasible paths of play.

The following additional properties parallel properties of static unawareness structures in Heifetz, Meier and Schipper (2006).<sup>18</sup>

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<sup>18</sup>The number of each property corresponds to the respective property in Heifetz, Meier and Schipper (2006).

- U0 Confined awareness: If  $n \in T$  and  $i \in I_n$  then  $\pi_i(n) \subseteq T'$  with  $T' \preceq T$ .
- U1 Generalized reflexivity: If  $T' \preceq T$ ,  $n \in T$ ,  $\pi_i(n) \subseteq T'$  and  $T'$  contains a copy  $n_{T'}$  of  $n$ , then  $n_{T'} \in \pi_i(n)$ .
- U2 Introspection: If  $n' \in \pi_i(n)$  then  $\pi_i(n') = \pi_i(n)$ . (I.e. property I2.)
- U3 Subtrees preserve awareness: If  $n \in T'$ ,  $n \in \pi_i(n)$ ,  $T \preceq T'$ , and  $T$  contains a copy  $n_T$  of  $n$ , then  $n_T \in \pi_i(n_T)$ .
- U4 Subtrees preserve ignorance: If  $T \preceq T' \preceq T''$ ,  $n \in T''$ ,  $\pi_i(n) \subseteq T$  and  $T'$  contains the copy  $n_{T'}$  of  $n$ , then  $\pi_i(n_{T'}) = \pi_i(n)$ .
- U5 Subtrees preserve knowledge: If  $T \preceq T' \preceq T''$ ,  $n \in T''$ ,  $\pi_i(n) \subseteq T'$  and  $T$  contains the copy  $n_T$  of  $n$ , then  $\pi_i(n_T)$  consists of the copies that exist in  $T$  of the nodes of  $\pi_i(n)$ .

The following remark is analogous to Remark 3 in Heifetz, Meier and Schipper (2006).

**Remark 9** *U5 implies U3.*

**Proof.** If  $n \in T'$ ,  $n \in \pi_i(n)$ ,  $T \preceq T'$ , and  $T$  contains a copy  $n_T$  of  $n$ , then by U5  $\pi_i(n_T)$  must consist of the copies that exist in  $T$  of the nodes of  $\pi_i(n)$ . Since by assumption  $n \in \pi_i(n)$  and the copy  $n_T$  exists in  $T$ , we must have  $n_T \in \pi_i(n_T)$ .  $\square$

**Remark 10** *U0 implies I0. U1 implies I1.*

**Remark 11** *U0 is equivalent to I0 and  $T \succrightarrow T'$  implies  $T' \preceq T$ .*

**Proof.** I0 and  $T \succrightarrow T'$  implies  $T' \preceq T$  are equivalent to if there exists  $n \in T$  and  $i \in I_n$  such that  $\pi_i(n) \subseteq T'$  then  $T' \preceq T$ .  $\square$

All these properties are static properties in the sense that they relate nodes on one tree with copies of those nodes in another tree. One may wonder about dynamic properties of unawareness. The following property states that a player can not become unaware during the play.

DA Awareness may only increase along a path: If there is a path  $n, \dots, n'$  in some subtree  $T$  such that player  $i$  is active in  $n$  and  $n'$ , and  $\pi_i(n) \subseteq T$  while  $\pi_i(n') \subseteq T'$  then  $T' \succeq T$ .



Recall that I3 is the only completely new property imposed on information sets in generalized games.

**Remark 12** *Suppose that U0 to U2 hold. Then DA if and only if I3.*

**Proof.** More precisely, we will show first that if I1 holds, then I3 implies DA. Second, if U0 and I2 holds, then DA implies I3. This implies the result by Remark 10.

If  $n, \dots, n'$  is path in  $T$  such that  $i \in I_n \cap I_{n'}$ ,  $\pi_i(n) \subseteq T$  while  $\pi_i(n') \subseteq T'$  then by I1 we have  $n \in \pi_i(n) \subseteq T$ . Then by I3,  $\pi_i(n') \subseteq T$ , which implies DA.

If  $n' \in \pi_i(n) \subseteq T'$  and  $n', \dots, n''$  is path in  $T'$  such that  $i \in I_{n'} \cap I_{n''}$  then by I2,  $\pi_i(n') = \pi_i(n)$  and thus by DA if  $\pi_i(n'') \subseteq T''$  then  $T'' \succeq T'$ . By U0, if  $n'' \in T'$  then  $\pi_i(n'') \subseteq T''$  with  $T'' \preceq T'$ . Hence  $T'' = T'$ , which implies I3.  $\square$

## A Proofs

### A.1 Proof of Lemma 1

(i) By (I3), all information sets of player  $i$  along a path starting in  $h_i$  and ending at a terminal node are contained in  $T_{h_i}$ . Therefore, it is enough to show the claim for every subtree  $T$  in the generalized extensive-form game. Since player  $i$ 's belief system  $b_i$  satisfies updating consistency as defined by Perea (2002), the proof of Theorem 3.1 of Perea (2002) implies the claim.<sup>19</sup>

(ii) If a strategy  $s_i$  of player  $i$  is rational at all information sets  $h_i \in H_i$ , then in particular  $s_i$  would be rational in all information sets  $h_i \in H_i$  reached by  $s_i$ . Denote by  $H_i^{-s_i}$  the set of information sets not reached by  $s_i$ . By (I3), the expected payoff for player  $i$  (given the belief system  $b_i$ ) from choosing an action in  $h_i \in H_i^{-s_i}$  does not depend on her choices at information sets outside  $T_{h_i}$ .

Furthermore,  $H_i^{-s_i}$  is an arborescence with respect to the precedence relation  $\rightsquigarrow$ . Hence, a standard backward-induction procedure on  $H_i^{-s_i}$  yields an optimal action  $a_{h_i}^* \in A_{h_i}$  for player  $i$  (given  $b_i$ ) at  $h_i$  for every information set  $h_i \in H_i^{-s_i}$ . Replacing by  $a_{h_i}^*$  the action prescribed by  $s_i$  at  $h_i$  for every  $h_i \in H_i^{-s_i}$  yields a new strategy  $\hat{s}_i$  which would be rational at all information sets  $h_i \in H_i$ .  $\square$

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<sup>19</sup>Formally, theorem 3.1 in Perea (2002) refers to two-player games, but as he remarks at the top of p. 325, the argument can be extended in a straightforward manner to games with more than two players and correlated beliefs about other players' strategies.

## A.2 Proof of Proposition 2

We proceed by induction.

$B_i^1$  is non-empty. Indeed, to construct a belief system  $b_i$ , for each information set  $h_i$  with no predecessors (according to the precedence relation  $\rightsquigarrow$ ) in the arborescence of information sets  $H_i$ , assign to player  $i$  a full-support belief  $b_i(h_i)$  on the other players' strategies  $S_{-i}^{T_{h_i}}$  that reach  $h_i$ . The full-support guarantees that Bayes rule is applicable for deriving the beliefs of player  $i$  in all her remaining information sets.

Suppose, by induction, we have already shown that  $B_i^k$  is non-empty. We have to show that  $S_i^k$  is non-empty. For a typical belief system  $b_i \in B_i^k$  we have to construct a strategy with which player  $i$  would be rational at each of her information sets  $H_i$ . Since  $H_i$  is an arborescence, it is standard to construct such a strategy  $s_i$  by backward induction.

To complete the induction step, observe that  $B_i^{k+1}$  is non-empty, because by definition it singles out a non-empty subset of  $B_i^k$ .

Now, since player  $i$ 's set of strategies  $S_i$  is finite and by Remark 3  $S_i^{k+1} \subseteq S_i^k$  for every  $k \geq 1$ , for some  $\ell$  we eventually get  $S_i^\ell = S_i^{\ell+1}$  for all  $i \in I$  and hence  $B_i^{\ell+1} = B_i^{\ell+2}$  for all  $i \in I$ . Inductively,

$$\emptyset \neq S_i^\ell = S_i^{\ell+1} = S_i^{\ell+2} = \dots$$

and therefore

$$S_i^\infty = \bigcap_{k=1}^{\infty} S_i^k = S_i^\ell \neq \emptyset$$

as required. □

## A.3 Proof of Proposition 4

Denote by  $(a_i, h_i)$  the copy of the action  $a_i$  of player  $i \in I$  whenever it appears in the information set  $h_i$ . For the purpose of this proof the word ‘‘action’’ will refer to a copy  $(a_i, h_i)$  of an action *at a given information set*.

Define a *menu* of a player to be a (possibly empty) subset of (the union of) her actions in her information sets.

Define a *menu profile* to be a profile of menus, one for each player, with the following property: For each information set  $h_i$  of player  $i$ , her menu in the menu profile contains at least one action in  $h_i$  if and only if that information set is reached by a sequence of

actions of the players in the menu profile.

For a menu profile  $M$ , denote by  $M_i$  the menu of player  $i$  in  $M$ .

For a menu profile  $M$ , denote by  $P^T(M)$  all the paths from the roots to leaves in the trees of the  $T$ -partial game that one can compose from actions in  $M$  and moves of nature (if there are any). Denote also by  $P(M)$  the set of paths from roots to leaves in all the trees of the generalized games that one can compose from actions in  $M$  and moves of nature.

Now, every product of sets of strategies  $R = \prod_{i \in I} R_i$  (where  $R_i$  is a subset of  $i$ 's strategies) induces a menu profile, in which player  $i$ 's menu is defined as follows. For each information set of the player:

1) If the information set is reached by some strategy profile in the set  $R$ , the player's menu contains all the actions ascribed in that information set by  $i$ 's strategies in  $R_i$  that reach the information set.

2) If the information set is not reached by any strategy profile in  $R$ , then player  $i$ 's menu contains no action of hers in that information set.

Intuitively, player  $i$ 's menu is mute about an information set if and only if that information set is excluded by the set of strategy profiles  $R$  (case 2); otherwise (case 1) the menu contains all the actions in that information set that appear in some strategy of hers in  $R_i$  that reaches that information set.

If  $M$  is the menu profile induced by  $R$ , then every strategy in  $R_i$  together with a belief about  $R_{-i}$  induce a belief  $\beta^T$  about the paths of actions in  $P^T(M)$  for every tree  $T$  of the generalized game.

Next, denote by  $M^k$  the menu profile induced by  $S^k = \prod_{i \in I} S_i^k$ , the set of level  $k$  would-be rationalizable strategy profiles; and denote by  $\bar{M}^k$  the menu profile induced by  $\bar{S}^k = \prod_{i \in I} \bar{S}_i^k$ , the set of level  $k$  prudent rationalizable strategy profiles.

Proposition 4 is implied by the following lemma:

**Lemma 3** *For all  $\ell \geq 0$ ,  $\bar{M}^\ell \subseteq M^\ell$ . In particular  $\bar{M}^\infty \subseteq M^\infty$ .*

**Proof.** The proof is by induction.

For  $\ell = 0$  we have  $M^0 = \bar{M}^0$ , the menu profile which includes all actions at all the information sets of all the players.

Suppose the claim holds for  $\ell \leq k$ .

By the induction hypothesis  $P(\bar{M}^\ell) \subseteq P(M^\ell)$  for every  $\ell \leq k$ .

We will now prove the claim for  $\ell = k + 1$ , i.e. that  $\bar{M}_i^{k+1} \subseteq M_i^{k+1}$  for every player  $i \in I$ .

To this end we have to show that for every player  $i \in I$ , every  $\bar{s}_i^{k+1} \in \bar{S}_i^{k+1}$ , every information set  $h_i \in H_i$  which is reached both by  $\bar{s}_i^{k+1}$  and by some strategy profile in  $\bar{S}_{-i}^{k+1}$  (meaning that  $\bar{s}_i^{k+1}(h_i) \in \bar{M}_i^{k+1}$ ), it is the case that

- a)  $h_i$  is also reached by  $S^{k+1}$ , and
- b)  $\bar{s}_i^{k+1}(h_i) \in M_i^{k+1}$  as well.

In fact, it is enough to show that b) holds. To see this, proceed inductively along each feasible path of the generalized game (in each of its trees). If player  $i$  is the first to play in this path (apart from nature, if there are nature moves in the path), and if  $h_i$  is the information set in which she makes this initial move, then condition a) automatically obtains for  $h_i$ , and we only need to prove b). Inductively, if we reach a node in the path which is not in  $P(\bar{M}^{k+1})$ , we have nothing to prove for this node's information set when considering this path.<sup>20</sup> If all the nodes  $n_1 \dots n_m$  in an initial segment of the path are on a path in  $P(\bar{M}^{k+1})$  and we have already proved conditions a) and b) for all the information sets of these nodes, then it already follows that a) holds for the information set of the next node  $n_{m+1}$  in the path [because b) holds for the previous node  $n_m$  for the player (or players) active in  $n_m$ ]. It thus remains to show b) for such an information set.

So we now proceed to prove b).

Suppose  $h_i$  is reached by  $\bar{S}_{-i}^{k+1}$  and by  $\bar{s}_i^{k+1} \in \bar{S}_i^{k+1}$ . Since by definition  $\bar{S}_{-i}^{k+1} \subseteq \bar{S}_{-i}^k$ , we have  $\bar{s}_i^{k+1} \in \bar{S}_i^k$  and hence  $m_i^{k+1}(h_i) = k$ . Consider a belief system  $b_i \in \bar{B}_i^{k+1}$  with a full-support belief  $b_i(h_i)$  on the strategy profiles  $\bar{S}_{-i}^k$  that reach  $h_i$ , and with which  $\bar{s}_i^{k+1}$  would be rational at  $h_i$  (i.e. player  $i$  cannot improve her expected payoff by changing  $\bar{s}_i^{k+1}$  only at  $h_i$ , from  $\bar{s}_i^{k+1}(h_i)$  to some other action  $a'_{h_i}$  available there).

The strategy  $\bar{s}_i^{k+1}$  together with the belief  $b_i(h_i)$  on the other players' strategies induce a full support belief  $\beta$  on the paths of actions in  $P(\bar{M}^k)$  reaching  $h_i$  and along which player  $i$  uses the strategy  $\bar{s}_i^{k+1}$ . Since by the induction hypothesis  $P(\bar{M}^k) \subseteq P(M^k)$ , it follows that  $\beta$  is a belief on the paths of actions in  $P(M^k)$  reaching  $h_i$  and along which player  $i$  uses the strategy  $\bar{s}_i^{k+1}$ .

Denote by  $\bar{s}_i^{k+1}|a'_{h_i}$  the strategy one gets from  $\bar{s}_i^{k+1}$  by altering the action at the information set  $h_i$  from  $\bar{s}_i^{k+1}(h_i)$  to  $a'_{h_i}$ . The altered strategy  $\bar{s}_i^{k+1}|a'_{h_i}$  together with the belief  $b_i(h_i)$  on the other players' strategies induce a full support belief  $\beta'$  on the paths

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<sup>20</sup>We may have to consider this information set again when we analyze another path passing through it.

of actions in  $P(\bar{M}^k)$  reaching  $h_i$  and along which player  $i$  uses the strategy  $\bar{s}_i^{k+1}|a'_{h_i}$ .

The fact that  $\bar{s}_i^{k+1}$  would-be rational given the belief system  $b_i$  means that in particular at the information set  $h_i$ , with the belief  $b_i(h_i)$  on the other players' strategies, the expected payoff to player  $i$  given  $\beta$  is not smaller than the expected payoff to player  $i$  given  $\beta'$ .

This yields the conclusion b) that we wanted, namely that  $\bar{s}_i^{k+1}(h_i) \in M_i^{k+1}$ .  $\square$

## A.4 Proof of Proposition 7

A *general belief system* of player  $i$

$$\tilde{b}_i = (\tilde{b}_i(h_i))_{h_i \in H_i} \in \prod_{h_i \in H_i} \Delta(S_{-i}^{T_{h_i}})$$

is a profile of beliefs – a belief  $\tilde{b}_i(h_i) \in \Delta(S_{-i}^{T_{h_i}})$  about the other players' strategies in the  $T_{h_i}$ -partial extensive-form game, for each information set  $h_i \in H_i$ , such that  $\tilde{b}_i(h_i)$  reaches  $h_i$ , i.e.,  $\tilde{b}_i(h_i)$  assigns probability 1 to the set of strategy profiles of the other players that reach  $h_i$ . The difference between a belief system and a general belief system is that in the latter *we do not impose Bayes rule*.

For  $k \geq 1$  let  $\tilde{B}_i^k$  and  $\tilde{S}_i^k$  be defined inductively like  $\hat{B}_i^k, \hat{S}_i^k$  in Definition 2, respectively, the only change being that belief systems are replaced by generalized belief systems.

**Lemma 4**  $U_i^k(\mathbf{S}) \cap S_i = \tilde{S}_i^k$  for  $k \geq 1$ . Consequently,  $U_i^\infty(\mathbf{S}) \cap S_i = \tilde{S}_i^\infty$ .

**Proof of the Lemma.** We proceed by induction. The case  $k = 0$  is straight-forward since  $U_i^0(\mathbf{S}) \cap S_i = S_i = \tilde{S}_i^0$  for all  $i \in I$ .

Suppose now that we have shown  $U_i^k(\mathbf{S}) \cap S_i = \tilde{S}_i^k$  for all  $i \in I$ . We want to show that  $U_i^{k+1}(\mathbf{S}) \cap S_i = \tilde{S}_i^{k+1}$  for all  $i \in I$ .

“ $\subseteq$ ”: First we show, if  $s_i \in U_i^{k+1}(\mathbf{S}) \cap S_i$  then  $s_i \in \tilde{S}_i^{k+1}$ .

$s_i \in U_i^{k+1}(\mathbf{S}) \cap S_i$  if  $s_i \in S_i$  is not conditionally strictly dominated on  $(\mathcal{X}_i, U^k(\mathbf{S}))$ .

$s_i \in S_i$  is not conditionally strictly dominated on  $(\mathcal{X}_i, U^k(\mathbf{S}))$  if for all  $T' \in \mathbf{T}$  with  $T_1 \hookrightarrow T'$  and all  $\tilde{s}_i \in S_i^{T'}$  such that  $s_i \in [\tilde{s}_i]$ , we have that there does not exist a normal-form information set  $X \in \mathcal{X}_i$  with  $X \subseteq S^{T'}$  such that  $\tilde{s}_i$  is strictly dominated in  $X \cap U^k(\mathbf{S})$ .

For any information set  $h_i \in H_i$ , if  $\tilde{s}_i \in S_i^{T_{h_i}}$  is not strictly dominated in  $S^{T_{h_i}}(h_i) \cap U^k(\mathbf{S})$ , then

- (i) either  $\tilde{s}_i$  does not reach  $h_i$ , in which case  $\tilde{s}_i$  is trivially rational at  $h_i$ ; or
- (ii) by Lemma 3 in Pearce (1984) there exists a belief  $\tilde{b}_i(h_i) \in \Delta(S_{-i}^{T_{h_i}}(h_i) \cap U_{-i}^k(\mathbf{S}))$  for which  $\tilde{s}_i$  is rational at  $h_i$ . Since by the induction hypothesis  $U^k(\mathbf{S}) \cap S = \tilde{S}^k$ , we have in this case that there exists a belief at  $h_i$  with  $\tilde{b}_i(h_i)(\tilde{S}_{-i}^{k, T_{h_i}}) = 1$  for which  $\tilde{s}_i$  is rational at  $h_i$ .

By definitions of  $[\tilde{s}_i]$  and “reach”, if  $\tilde{s}_i$  reaches  $h_i$  in the tree  $T_{h_i}$  and  $s_i \in [\tilde{s}_i]$ , then  $s_i$  reaches  $h_i$  in the tree  $T_{h_i}$ . Hence, if  $\tilde{s}_i \in S_i^{T_{h_i}}$  is rational at  $h_i$  given  $\tilde{b}_i(h_i)$ , then  $s_i \in [\tilde{s}_i]$  is rational at  $h_i$  given  $\tilde{b}_i(h_i)$ .

We need to show that beliefs in (ii) define a generalized belief system in  $\tilde{B}_i^{k+1}$ . Consider any  $\tilde{b}'_i = (\tilde{b}'_i(h_i))_{h_i \in H_i} \in \tilde{B}_i^{k+1}$ . For all  $h_i \in H_i$  for which there exists a profile of player  $i$ 's opponents' strategies  $s_{-i} \in \tilde{S}_{-i}^k$  that reach  $h_i$ , replace  $\tilde{b}'_i(h_i)$  by  $\tilde{b}_i(h_i)$  as defined in (ii). Call the new belief system  $\tilde{b}_i$ . Then this is a generalized belief system. Moreover,  $\tilde{b}_i \in \tilde{B}_i^{k+1}$ .

Hence, if  $s_i$  is not conditionally strictly dominated on  $(\mathcal{X}_i, U^k(\mathbf{S}))$  then there exists a generalized belief system  $\tilde{b}_i \in \tilde{B}_i^{k+1}$  for which  $s_i$  is rational at every  $h_i \in H_i$ . Thus  $s_i \in \tilde{S}_i^{k+1}$ .

“ $\supseteq$ ”: We show next, if  $s_i \in \tilde{S}_i^{k+1}$  then  $s_i \in U_i^{k+1}(\mathbf{S}) \cap S_i$ .

If  $s_i \in \tilde{S}_i^{k+1}$  then there exists a generalized belief system  $\tilde{b}_i \in \tilde{B}_i^{k+1}$  such that for all  $h_i \in H_i$  the strategy  $s_i$  is rational given  $\tilde{b}_i(h_i)$ . That is, either

- (I)  $s_i$  does not reach  $h_i$ , or
- (II)  $s_i$  reaches  $h_i$  and there does not exist an  $h_i$ -replacement of  $s_i$  which yields a higher expected payoff in  $T_{h_i}$  given  $\tilde{b}_i(h_i)$  that assigns probability 1 to  $T_{h_i}$ -partial strategies of player  $i$ 's opponents in  $\tilde{S}_{-i}^{k, T_{h_i}}$  that reach  $h_i$  in  $T_{h_i}$ . By the induction hypothesis,  $\tilde{S}_{-i}^k = U_{-i}^k(\mathbf{S}) \cap S_{-i}^{T_{h_i}}$ . Hence  $\tilde{b}_i(h_i) \in \Delta(U_{-i}^k(\mathbf{S}) \cap S_{-i}^{T_{h_i}}(h_i))$ .

If  $s_i \in [\tilde{s}_i]$  with  $\tilde{s}_i \in S_i^{T_{h_i}}$  and  $s_i$  reaches  $h_i$  in the tree  $T_{h_i}$ , then  $\tilde{s}_i$  reaches  $h_i$  in the tree  $T_{h_i}$ . Hence, if  $s_i \in [\tilde{s}_i]$  with  $\tilde{s}_i \in S_i^{T_{h_i}}$  is rational at  $h_i$  given  $\tilde{b}_i(h_i)$ , then  $\tilde{s}_i$  is rational at  $h_i$  given  $\tilde{b}_i(h_i)$ .

Thus, if  $s_i$  is rational at  $h_i$  given  $\tilde{b}_i(h_i)$ , then  $\tilde{s}_i \in S_i^{T_{h_i}}$  with  $s_i \in [\tilde{s}_i]$  is not strictly dominated in  $U_{-i}^k(\mathbf{S}) \cap S_{-i}^{T_{h_i}}(h_i)$  either because  $s_i$  does not reach  $h_i$  (case (I)), or because of Lemma 3 in Pearce (1984) (in case (II)).

It then follows that if the strategy  $s_i$  is rational at all  $h_i \in H_i$  given  $\tilde{b}_i$  then  $s_i$  is not conditionally strictly dominated on  $(\mathcal{X}_i, U^k(\mathbf{S}))$ . Hence  $s_i \in U_i^{k+1}(\mathbf{S}) \cap S_i$ .  $\square$

**Lemma 5**  $\tilde{S}_i^k = \hat{S}_i^k$  for  $k \geq 1$ . Consequently,  $\tilde{S}_i^\infty = \hat{S}_i^\infty$ .

**Proof of the Lemma.**  $\hat{S}_i^k \subseteq \tilde{S}_i^k$  for  $k \geq 1$  since if  $s_i$  is rational at each information set  $h_i \in H_i$  given the belief system  $b_i \in B_i$  then there exists a generalized belief system  $\tilde{b}_i \in \tilde{B}_i^k$ , namely  $\tilde{b}_i = b_i$ , such that  $s_i$  is rational at each information set  $h_i \in H_i$  given  $\tilde{b}_i$ .

We need to show the reverse inclusion,  $\tilde{S}_i^k \subseteq \hat{S}_i^k$  for  $k \geq 1$ . The first step is to show how to construct a (consistent) belief system from a generalized belief system. Let  $s_i$  be rational given  $\tilde{b}_i \in \tilde{B}_i^1$ , i.e.  $s_i \in \tilde{S}_i^1$ . Consider an information set  $h_i^0 \in H_i$  such that in  $T_{h_i}$  there does not exist an information set  $h_i$  that precedes  $h_i^0$ . Define  $b_i(h_i^0) \equiv \tilde{b}_i(h_i^0)$ .

Assume, inductively, that we have already defined  $b_i$  for a subset of information sets  $H'_i \subseteq H_i$  such that for each  $h'_i \in H'_i$  all the predecessors of  $h'_i$  are also in  $H'_i$ . For each successor information set  $h''_i$  of each information set  $h'_i \in H'_i$  such that  $h''_i \notin H'_i$  define  $b_i(h''_i)$  as follows:

- If  $b_i(h'_i)$  reaches  $h''_i$  define  $b_i(h''_i)$  by using Bayes rule, i.e. if  $s_{-i}^{T_{h'_i}} \in S_{-i}(h''_i)$

$$b_i(h''_i)(s_{-i}^{T_{h'_i}}) = \frac{b_i(h'_i)(s_{-i}^{T_{h'_i}})}{\sum_{\tilde{s}_{-i}^{T_{h'_i}} \in S_{-i}(h''_i)} b_i(h'_i)(\tilde{s}_{-i}^{T_{h'_i}})}$$

and  $b_i(h''_i)(s_{-i}^{T_{h'_i}}) = 0$  else.

- If  $b_i(h'_i)$  does not reach  $h''_i$  let  $b_i(h''_i) \equiv \tilde{b}_i(h''_i)$ .

Since there are finitely many information sets in  $H_i$ , this inductive definition will be concluded in a finite number of steps.

Next, assuming that  $s_i$  is rational at each information set  $h_i \in H_i$  with the generalized belief system  $\tilde{b}_i$ , we will show that  $s_i$  is also rational at each information set  $h_i \in H_i$  according to the belief system  $b_i$ .

Consider again  $h_i^0 \in H_i$  with no predecessors in  $T_{h_i^0}$ . Since  $b_i(h_i^0) = \tilde{b}_i(h_i^0)$  and  $s_i$  is rational at  $h_i^0$  given  $\tilde{b}_i(h_i^0)$ ,  $s_i$  is also rational at  $h_i^0$  given  $b_i(h_i^0)$ .

Assume, inductively, that we have already shown the claim for a subset of information sets  $H'_i \subseteq H_i$  such that for each  $h'_i \in H'_i$  all the predecessors of  $h'_i$  are also in  $H'_i$ . Consider

a successor information set  $h_i''$  of an information set  $h_i' \in H_i'$  such that  $h_i'' \notin H_i'$ . Notice that each  $h_i''$ -replacement is also an  $h_i'$ -replacement. Therefore,

- If  $b_i(h_i')$  reaches  $h_i''$ ,  $b_i(h_i'')$  is derived from  $b_i(h_i')$  by Bayes rule, and hence any  $h_i''$ -replacement improving player  $i$ 's expected payoff according to  $b_i(h_i'')$  would improve player  $i$ 's payoff also according to  $b_i(h_i')$ , contradicting the induction hypothesis. Hence  $s_i$  is rational at  $h_i''$  given  $b_i(h_i'')$ .
- If  $b_i(h_i')$  does not reach  $h_i''$ , then  $b_i(h_i'') = \tilde{b}_i(h_i'')$ . Hence,  $s_i$  is rational at  $h_i''$  also given  $b_i(h_i'')$ .

Applying the same argument inductively yields  $\tilde{S}_i^k = \hat{S}_i^k \forall k \geq 1$ . This concludes the proof of the lemma.  $\square$

Lemmata 4 and 5 together yield  $U_i^k(\mathbf{S}) \cap S_i = \hat{S}_i^k$  for  $k \geq 1$ . Since it applies for all  $k \geq 1$  and  $i \in I$ , this completes the proof of the proposition.  $\square$

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