A simple modal logic for belief revision

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First draft: March 2004. This version: December 2004.

Abstract

We propose a logic based on three modal operators, representing intial beliefs, information and revised beliefs. Three simple and transparent axioms are used to provide a sound and complete axiomatization of the qualitative part of Bayes' rule. Some theorems of this logic are derived concerning the interaction between current beliefs and future beliefs. Information flows and iterated revision are also discussed.

1 Introduction

The notions of static belief and of belief revision have been extensively studied in the literature. However, there is a surprising lack of uniformity in the two approaches. In the philosophy and logic literature, starting with Hintikka's [17] seminal contribution, the notion of static belief has been studied mainly within the context of modal logic. On the syntactic side a belief operator B is introduced, with the intended interpretation of $B\phi$ as "the individual believes that ϕ". Various properties of beliefs are then expressed by means of axioms, such as the positive introspection axiom $B\phi \to BB\phi$, which says that if the individual believes ϕ then she believes that she believes ϕ. On the semantic side Kripke structures (Kripke [23]) are used, consisting of a set of states (or possible worlds) Ω together with a binary relation $\mathcal{B}$ on Ω, with the interpretation of $\alpha\mathcal{B}\beta$ as "at state α the individual considers state β possible". The connection between syntax and semantics is then obtained by means of a valuation V which associates with every atomic sentence p the set of states where p is true. The pair $\langle\Omega, \mathcal{B}\rangle$ is called a frame and the addition of a valuation V to a frame yields a model. Rules are given for determining the truth of an arbitrary formula at

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*A first draft of this paper was presented at the Sixth Conference on Logic and the Foundations of Game and Decision Theory (LOFT6), Leipzig, July 2004. I am grateful to the conference participants for useful comments.

every state of a model; in particular, the formula $B\phi$ is true at state α if and only if ϕ is true at every β such that $\alpha\mathcal{B}\beta$, that is, if ϕ is true at every state that the individual considers possible at α. A property of the accessibility relation $\mathcal{B}$ is said to correspond to an axiom if every instance of the axiom is true at every state of every model based on a frame that satisfies the property and *vice versa*. For example, the positive introspection axiom $B\phi \to BB\phi$ corresponds to transitivity of the relation $\mathcal{B}$. This combined syntactic-semantic approach has turned out to be very useful. The syntax allows one to state properties of beliefs in a clear and transparent way, while the semantic approach is particularly useful in reasoning about complex issues, such as the implications of rationality in interactive situations.[1]

The theory of belief revision (known as the AGM theory due to the seminal work of Alchourron et al [1]), on the other hand, has followed a different path.[2] In this literature beliefs are modeled as sets of formulas in a given syntactic language and the problem that has been studied is how a belief set ought to be modified when new information, represented by a formula ϕ, becomes available. With a few exceptions[3], the tools of modal logic have not been explicitly employed in the analysis of belief revision.

In the economics and game theory literature, it is standard to represent beliefs by means of a probability measure over a set of states Ω and belief revision is modeled using Bayes' rule. Let P_0 be the prior probability measure representing the initial beliefs, $E \subseteq \Omega$ an event representing new information and P_1 the posterior probability measure representing the revised beliefs. Bayes' rule says that, if $P_0(E) > 0$, then, for every event A, $P_1(A) = \frac{P_0(A \cap E)}{P_0(E)}$. Bayes' rule thus implies the following, which we call the *Qualitative Bayes Rule*:

$$\text{if } \ supp(P_0) \cap E \neq \varnothing, \text{ then } \ supp(P_1) = supp(P_0) \cap E.$$

where $supp(P)$ denotes the support of the probability measure P.[4]

In this paper we propose a unifying framework for static beliefs and belief revision by bringing belief revision under the umbrella of modal logic and by providing an axiomatization of the Qualitative Bayes Rule in a simple logic based on three modal operators: B_0, B_1 and I, whose intended interpretation is as follows:

$B_0\phi$	initially (at time 0) the individual believes that ϕ
$I\phi$	(between time 0 and time 1) the individual is informed that ϕ
$B_1\phi$	at time 1 (after revising his beliefs in light of the information received) the individual believes that ϕ.

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[1] For extensive surveys of the role of beliefs and rationality in game theory see Dekel and Gul [10], Battigalli and Bonanno [4] and vand der Hoek and Pauly [19].

[2] For an extensive overview see Gärdenfors [14].

[3] For example, Battigalli and Bonanno [3], Board [6], Liau [24] and Segerberg [27].

[4] There is an ongoing debate in the philosophical literature as to whether or not Bayes' rule is a requirement of rationality: see, for example, Brown [8], Jeffrey [21], Howson and Urbach [20], Maher [25] and Teller [28].

Semantically, it is clear that the Qualitative Bayes Rule embodies the *conservativity principle* for belief revision, according to which "When changing beliefs in response to new evidence, you should continue to believe as many of the old beliefs as possible" (Harman [16], p. 46). The set of all the propositions that the individual believes corresponds to the set of states that she considers possible (in a probabilistic setting a state is considered possible if it is assigned positive probability). The conservativity principle requires that, if the individual considers a state possible and her new information does not exclude this state, then she continue to consider it possible. Furthermore, if the individual regards a particular state as impossible, then she should continue to regard it as impossible, unless her new information excludes *all* the states that she previously regarded as possible. The axiomatization we propose gives a transparent syntactic expression to the conservativity principle.

The paper is organized as follows. In Section 2 we provide a characterization of the Qualitative Bayes Rule in terms of three simple axioms. In Section 3 we provide a logic which is sound and complete with respect to the class of frames that satisfy the Qualitative Bayes Rule and prove some theorems of this logic concerning the interaction between current beliefs and future beliefs. In section 4 we discuss the relationship between our analysis and that of closely related papers in the literature. Section 5 examines the relationship between our approach and the AGM approach. In Section 6 we deal with the issue of iterated revision and Section 7 concludes.

2 Axiomatic characterization of the Qualitative Bayes Rule

We begin with the semantics. A *frame* is a quadruple $\langle \Omega, \mathcal{B}_0, \mathcal{B}_1, \mathcal{I} \rangle$ where Ω is a set of *states* and $\mathcal{B}_0$, $\mathcal{B}_1$, and $\mathcal{I}$ are binary relations on Ω, whose interpretation is as follows:

$\alpha \mathcal{B}_0 \beta$	at state α the individual initially (at time 0) considers state β possible
$\alpha \mathcal{I} \beta$	at state α, state β is compatible with the information received
$\alpha \mathcal{B}_1 \beta$	at state α the individual at time 1 (in light of the information received) considers state β possible.

Let $\mathcal{B}_0(\omega) = \{\omega' \in \Omega : \omega \mathcal{B}_0 \omega'\}$ denote the set of states that, initially, the individual considers possible at state ω. Define $\mathcal{I}(\omega)$ and $\mathcal{B}_1(\omega)$ similarly.[5] By *Qualitative Bayes Rule* (QBR) we mean the following property:

$$\forall \omega \in \Omega, \ if \ \ \mathcal{B}_0(\omega) \cap \mathcal{I}(\omega) \neq \varnothing \ \ then \ \ \mathcal{B}_1(\omega) = \mathcal{B}_0(\omega) \cap \mathcal{I}(\omega). \qquad \text{(QBR)}$$

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[5] In a probabilistic setting, if P_0 is the prior probability measure representing the initial beliefs at state ω and P_1 is the posterior probability measure representing the revised beliefs at ω then $\mathcal{B}_0(\omega) = supp(P_0)$ and $\mathcal{B}_1(\omega) = supp(P_1)$.

Thus QBR says that if at a state the information received is consistent with the initial beliefs $-$ in the sense that there are states that were considered possible initially and are compatible with the information $-$ then the states that are considered possible according to the revised beliefs are precisely those states.

On the syntactic side we consider a modal propositional logic based on three operators: B_0, B_1 and I whose intended interpretation is as explained in Section 1. The formal language is built in the usual way from a countable set S of atomic propositions, the connectives $\neg$ (for "not") and $\vee$ (for "or") and the modal operators.[6] Thus the set Φ of formulas is defined inductively as follows: $q \in \Phi$ for every atomic proposition $q \in S$, and if $\phi, \psi \in \Phi$ then all of the following belong to Φ: $\neg\phi$, $\phi \vee \psi$, $B_0\phi$, $B_1\phi$ and $I\phi$.

Remark 1 *We have allowed $I\phi$ to be a well-formed formula for every formula ϕ. As pointed out by Friedman and Halpern [12], this may be problematic. For example, it is not clear how one could be informed of a contradiction. Furthermore, one might want to restrict information to facts by not allowing $I\phi$ be a well-formed formula if ϕ contains any of the modal operators B_0, B_1 and I.[7] Without that restriction, in principle we admit situations like the following: the individual initially believes that ϕ and is later informed that he did not believe that ϕ: $B_0\phi \wedge I\neg B_0\phi$. It is not clear how such a situation could arise.[8] However, since our results remain true - whether or not we impose the restriction - we have chosen to follow the more general approach. The undesirable situations can then be eliminated by imposing suitable axioms, for example the axiom $B_0\phi \to \neg I \neg B_0\phi$, which says that if the individual initially believes that ϕ then it cannot be the case that he is informed that he did not believe that ϕ (see Section 7 for further discussion).*

The connection between syntax and semantics is given by the notion of model. Given a frame $\langle \Omega, \mathcal{B}_0, \mathcal{B}_1, \mathcal{I} \rangle$, a *model* is obtained by adding a *valuation* $V : S \to 2^{\Omega}$ (where 2^{Ω} denotes the set of subsets of Ω, usually called *events*) which associates with every atomic proposition $p \in S$ the set of states at which p is true. The truth of an arbitrary formula at a state is then defined inductively as follows ($\omega \models \phi$ denotes that formula ϕ is true at state ω; $\|\phi\|$ is the truth set of ϕ, that is, $\|\phi\| = \{\omega \in \Omega : \omega \models \phi\}$):

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[6] See, for example, Blackburn et al [5]. The connectives $\wedge$ (for "and"), $\to$ (for "if ... then ...") and $\leftrightarrow$ (for "if and only if") are defined as usual: $\phi \wedge \psi = \neg(\neg\phi \vee \neg\psi)$, $\phi \to \psi = \neg\phi \vee \psi$ and $\phi \leftrightarrow \psi = (\phi \to \psi) \wedge (\psi \to \phi)$.

[7] In an interpersonal setting, however, information that pertains to beliefs (rather than merely to facts) ought to be allowed, at least to the extent that the information received by an individual be about the beliefs of *another* individual.

[8] More examples of problematic situations are: $I(\phi \wedge \neg B_1\phi)$ (the individual is informed that ϕ and that he will not believe ϕ), $B_0\phi \wedge I\neg B_1 B_0\phi$ (the individual initially believes ϕ and is informed that he will forget that he did), etc.

if q is an atomic proposition, $\omega \models q$ if and only if $\omega \in V(q)$,
$\omega \models \neg\phi$ if and only if $\omega \not\models \phi$,
$\omega \models \phi \vee \psi$ if and only if either $\omega \models \phi$ or $\omega \models \psi$ (or both),
$\omega \models B_0\phi$ if and only if $\mathcal{B}_0(\omega) \subseteq \|\phi\|$,[9]
$\omega \models B_1\phi$ if and only if $\mathcal{B}_1(\omega) \subseteq \|\phi\|$,
$\omega \models I\phi$ if and only if $\mathcal{I}(\omega) = \|\phi\|$.

Remark 2 *Note that, while the truth conditions for* $B_0\phi$ *and* $B_1\phi$ *are the standard ones, the truth condition of* $I\phi$ *is unusual in that the requirement is* $\mathcal{I}(\omega) = \|\phi\|$ *rather than merely* $\mathcal{I}(\omega) \subseteq \|\phi\|$.[10]

We say that a formula ϕ is *valid in a model* if $\omega \models \phi$ for all $\omega \in \Omega$, that is, if ϕ is true at every state. A formula ϕ is *valid in a frame* if it is valid in every model based on that frame. Finally, we say that a property of frames is *characterized by* (or characterizes) an axiom if (1) the axiom is valid in any frame that satisfies the property and, conversely, (2) whenever the axiom is valid in a frame, then the frame satisfies the property.

We now introduce three axioms that, together, provide a characterization of the Qualitative Bayes Rule.

$$\textit{QUALIFIED ACCEPTANCE:}\;\; (I\phi \wedge \neg B_0\neg\phi) \rightarrow B_1\phi.$$

This axiom says that if the individual is informed that ϕ ($I\phi$) and he initially considered ϕ possible (that is, it is not the case that he believed its negation: $\neg B_0\neg\phi$) then he accepts ϕ in his revised beliefs. That is, information that is not surprising is believed.

The next axiom says that if the individual receives non-surprising information (i.e. information that does not contradict his initial beliefs) then he continues to believe everything that he believed before:

$$\textit{PERSISTENCE:}\;\; (I\phi \wedge \neg B_0\neg\phi) \rightarrow (B_0\psi \rightarrow B_1\psi).$$

The third axiom says that beliefs should be revised in a minimal way, in the sense that no new beliefs should be added unless they are implied by the old beliefs and the information received:

$$\textit{MINIMALITY:}\;\; (I\phi \wedge B_1\psi) \rightarrow B_0(\phi \rightarrow \psi).$$

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[9] In a probabilistic setting, where $\mathcal{B}_0(\omega)$ is the support of the probability measure representing the initial beliefs at ω, we would have that $\omega \models B_0\phi$ if and only if the individual assigns probability 1 to the event $\|\phi\|$. Similarly for $\omega \models B_1\phi$.

[10] The reason for this will become clear later. Intuitively, this allows us to distinguish between the content of information and its implications.

The Minimality axiom is not binding (that is, it is trivially satisfied) if the information is surprising: suppose that at a state, say α, the individual is informed that ϕ ($\alpha \models I\phi$) although he initially believed that ϕ was *not* the case ($\alpha \models B_0\neg\phi$). Then, for every formula ψ, the formula $(\phi \to \psi)$ is trivially true at every state that the individual initially considered possible ($\mathcal{B}_0(\alpha) \subseteq \|\phi \to \psi\|$) and therefore he initially believed it ($\alpha \models B_0(\phi \to \psi)$). Thus the axiom restricts the new beliefs only when the information received is not surprising, that is, only if $(I\phi \wedge \neg B_0\neg\phi)$ happens to be the case.

The above axioms are furthered discussed below. The following proposition gives the main result of this section.

Proposition 3 *The Qualitative Bayes Rule (QBR) is characterized by the conjunction of the three axioms Qualified Acceptance, Persistence and Minimality (that is, if a frame satisfies QBR then the three axioms are valid in it and - conversely - if the three axioms are valid in a frame then the frame satisfies QBR).*

The proof of Proposition 3 is a corollary of the following three lemmas, which characterize the three axioms individually.

Lemma 4 *The Qualified Acceptance axiom* ($(I\phi \wedge \neg B_0\neg\phi) \to B_1\phi$) *is characterized by the property:* $\forall\omega \in \Omega$, *if* $\mathcal{B}_0(\omega) \cap \mathcal{I}(\omega) \neq \varnothing$ *then* $\mathcal{B}_1(\omega) \subseteq \mathcal{I}(\omega)$.

Proof. Fix a frame where the property holds, an arbitrary model based on it, a state ω and a formula ϕ such that $\omega \models I\phi \wedge \neg B_0\neg\phi$. Then $\mathcal{I}(\omega) = \|\phi\|$. Since $\omega \models \neg B_0\neg\phi$ there exists a $\beta \in \mathcal{B}_0(\omega)$ such that $\beta \models \phi$. Thus $\mathcal{B}_0(\omega) \cap \mathcal{I}(\omega) \neq \varnothing$ and, by the property, $\mathcal{B}_1(\omega) \subseteq \mathcal{I}(\omega)$. Hence $\omega \models B_1\phi$. Conversely, fix a frame that does not satisfy the property. Then there exists a state α such that $\mathcal{B}_0(\alpha) \cap \mathcal{I}(\alpha) \neq \varnothing$ and $\mathcal{B}_1(\alpha) \nsubseteq \mathcal{I}(\alpha)$, that is, there is a $\beta \in \mathcal{B}_1(\alpha)$ such that $\beta \notin \mathcal{I}(\alpha)$. Let p be an atomic proposition and construct a model where $\|p\| = \mathcal{I}(\alpha)$. Then $\alpha \models Ip$ and, since $\mathcal{B}_0(\alpha) \cap \mathcal{I}(\alpha) \neq \varnothing$, $\alpha \models \neg B_0\neg p$. Furthermore, $\beta \not\models p$ (because $\beta \notin \mathcal{I}(\alpha)$). Thus, since $\beta \in \mathcal{B}_1(\alpha)$, $\alpha \not\models B_1 p$ and the axiom is falsified at α. ■

Note that if the truth condition for $I\phi$ were "$\omega \models I\phi$ if and only if $\mathcal{I}(\omega) \subseteq \|\phi\|$" (rather than $\mathcal{I}(\omega) = \|\phi\|$), then Lemma 4 would not be true. The implication "property violated $\Longrightarrow$ axiom not valid" would still be true (identical proof). However, the implication "property holds $\Longrightarrow$ axiom valid" would no longer be true, because it could happen that $\mathcal{I}(\omega)$ is a **proper** subset of $\|\phi\|$. For example, let $\Omega = \{\alpha, \beta, \gamma\}$, $\mathcal{B}_0(\alpha) = \{\alpha\}$, $\mathcal{B}_0(\beta) = \mathcal{B}_0(\gamma) = \{\gamma\}$, $\mathcal{I}(\alpha) = \{\alpha\}$, $\mathcal{I}(\beta) = \{\beta\}$, $\mathcal{I}(\gamma) = \{\gamma\}$, $\mathcal{B}_1(\alpha) = \mathcal{B}_1(\beta) = \{\alpha\}$ and $\mathcal{B}_1(\gamma) = \{\gamma\}$. Then the property $\forall\omega$, if $\mathcal{B}_0(\omega) \cap \mathcal{I}(\omega) \neq \varnothing$ then $\mathcal{B}_1(\omega) \subseteq \mathcal{I}(\omega)$ is satisfied (note, in particular, that $\mathcal{B}_0(\beta) \cap \mathcal{I}(\beta) = \varnothing$). Construct a model where, for some atomic proposition p, $\|p\| = \{\beta, \gamma\}$. Then, under the rule $\mathcal{I}(\beta) \subseteq \|p\|$, we would have $\beta \models Ip$ and $\beta \models \neg B_0\neg p \wedge \neg B_1 p$, so that the Qualified Acceptance axiom would be falsified at β. This frame is illustrated in Figure 1. In all the figures we represent a binary relation $R \subseteq \Omega \times \Omega$ as follows: (1) if there is an arrow from

ω to ω' then $\omega' \in R(\omega)$ (i.e. $\omega R \omega'$), (2) if a rounded rectangle encloses a set of states then, for any two states ω and ω' in that rectangle, $\omega' \in R(\omega)$ and (3) if there is an arrow from a state ω to a rounded rectangle, then for any state ω' in that rectangle, $\omega' \in R(\omega)$.

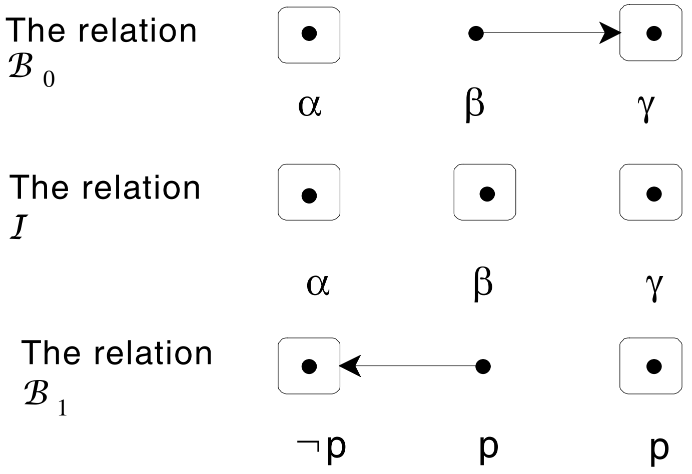

Figure 1

Lemma 5 *The Persistence axiom* $((\neg B_0 \neg \phi \wedge I\phi) \rightarrow (B_0\psi \rightarrow B_1\psi))$ *is characterized by the property:* $\forall \omega \in \Omega$, *if* $\mathcal{B}_0(\omega) \cap \mathcal{I}(\omega) \neq \varnothing$ *then* $\mathcal{B}_1(\omega) \subseteq \mathcal{B}_0(\omega)$.

Proof. Fix a frame where the property holds, an arbitrary model based on it, a state ω and formulas ϕ and ψ such that $\omega \models B_0\psi \wedge \neg B_0 \neg \phi \wedge I\phi$. Then $\mathcal{I}(\omega) = \|\phi\|$. Since $\omega \models \neg B_0 \neg \phi$, $\mathcal{B}_0(\omega) \cap \mathcal{I}(\omega) \neq \varnothing$. Then, by the property, $\mathcal{B}_1(\omega) \subseteq \mathcal{B}_0(\omega)$. Since $\omega \models B_0\psi$, $\mathcal{B}_0(\omega) \subseteq \|\psi\|$. Thus $\omega \models B_1\psi$. Conversely, fix a frame that does not satisfy the property. Then there exists a state α such that $\mathcal{B}_0(\alpha) \cap \mathcal{I}(\alpha) \neq \varnothing$ and $\mathcal{B}_1(\alpha) \nsubseteq \mathcal{B}_0(\alpha)$, that is, there exists a $\beta \in \mathcal{B}_1(\alpha)$ such that $\beta \notin \mathcal{B}_0(\alpha)$. Let p and q be atomic propositions and construct a model where $\|p\| = \mathcal{B}_0(\alpha)$ and $\|q\| = \mathcal{I}(\alpha)$. Then $\alpha \models B_0 p \wedge Iq$ and, since $\mathcal{B}_0(\alpha) \cap \mathcal{I}(\alpha) \neq \varnothing$, $\alpha \models \neg B_0 \neg q$. Since $\beta \notin \mathcal{B}_0(\alpha)$, $\beta \nvDash p$. Thus, since $\beta \in \mathcal{B}_1(\alpha)$, $\alpha \nvDash B_1 p$. Thus the instance of the axiom with $\psi = p$ and $\phi = q$ is falsified at α. ■

Note again that with the standard validation rule for the operator I, the above lemma would not be true. The implication "property violated $\Longrightarrow$ axiom not valid" would still be true (identical proof). However, the implication "property holds $\Longrightarrow$ axiom valid" would no longer be true. This can be seen in the example of Figure 1 at state β with $\phi = \psi = p$. In fact, under the rule $\beta \models Ip$ if and only if $\mathcal{I}(\beta) \subseteq \|p\|$ (rather than $\mathcal{I}(\beta) = \|p\|$) we would have $\beta \models Ip$ and $\beta \models B_0 p \wedge \neg B_0 \neg p \wedge \neg B_1 p$, so that the Persistence axiom would be falsified at β, despite the fact that the frame of Figure 1 satisfies the property that, $\forall \omega \in \Omega$, if $\mathcal{B}_0(\omega) \cap \mathcal{I}(\omega) \neq \varnothing$ then $\mathcal{B}_1(\omega) \subseteq \mathcal{B}_0(\omega)$ (notice, in particular, that $\mathcal{B}_0(\beta) \cap \mathcal{I}(\beta) = \varnothing$).

Lemma 6 *The Minimality axiom* $((I\phi \wedge B_1\psi) \rightarrow B_0(\phi \rightarrow \psi))$ *is characterized by the following property:* $\forall \omega \in \Omega$, $\mathcal{B}_0(\omega) \cap \mathcal{I}(\omega) \subseteq \mathcal{B}_1(\omega)$.

Proof. Fix a frame that satisfies the property and an arbitrary model based on it. Let α be a state and ϕ and ψ formulas such that $\alpha \models I\phi \wedge B_1\psi$. Then $\mathcal{I}(\alpha) = \|\phi\|$. By the property, $\mathcal{B}_0(\alpha) \cap \mathcal{I}(\alpha) \subseteq \mathcal{B}_1(\alpha)$. Since $\alpha \models B_1\psi$, $\mathcal{B}_1(\alpha) \subseteq \|\psi\|$. Thus, for every $\omega \in \mathcal{B}_0(\alpha) \cap \mathcal{I}(\alpha)$, $\omega \models \psi$ and therefore $\omega \models \phi \rightarrow \psi$. On the other hand, for every $\omega \in \mathcal{B}_0(\alpha)$, if $\omega \notin \mathcal{I}(\alpha)$, then $\omega \models \neg\phi$ and therefore $\omega \models \phi \rightarrow \psi$. Thus $\mathcal{B}_0(\alpha) \subseteq \|\phi \rightarrow \psi\|$, i.e. $\alpha \models B_0(\phi \rightarrow \psi)$.

Conversely, suppose the property is violated. Then there exists a state α such that $\mathcal{B}_0(\alpha) \cap \mathcal{I}(\alpha) \nsubseteq \mathcal{B}_1(\alpha)$, that is, there exists a $\beta \in \mathcal{B}_0(\alpha) \cap \mathcal{I}(\alpha)$ such that $\beta \notin \mathcal{B}_1(\alpha)$. Let p and q be atomic propositions and construct a model where $\|p\| = \mathcal{I}(\alpha)$ and $\|q\| = \mathcal{B}_1(\alpha)$. Then $\alpha \models Ip \wedge B_1q$. Since $\beta \in \mathcal{I}(\alpha)$ and $\beta \notin \mathcal{B}_1(\alpha)$, $\beta \models p \wedge \neg q$, i.e. $\beta \models \neg(p \rightarrow q)$. Thus, since $\beta \in \mathcal{B}_0(\alpha)$, $\alpha \nvDash B_0(p \rightarrow q)$. Hence the axiom is falsified at α. ■

Once again, it can be seen from Figure 1 that under the standard validation rule for I ($\omega \models I\phi$ if and only if $\mathcal{I}(\omega) \subseteq \|\phi\|$, rather than $\mathcal{I}(\omega) = \|\phi\|$) it is not true that satisfaction of the property $\forall \omega \in \Omega$, $\mathcal{B}_0(\omega) \cap \mathcal{I}(\omega) \subseteq \mathcal{B}_1(\omega)$ guarantees validity of the Minimality axiom. In fact, under the standard validation rule, Minimality would be falsified at state β with $\phi = p$ and $\psi = \neg p$.

The Qualitative Bayes Rule captures the following conservativity principle for belief revision: if the information received involves no surprises, then beliefs should be changed in a minimal way, in the sense that all the previous beliefs ought to be maintained and any new belief should be deducible from the old beliefs and the information. The extreme case of "no surprise" is the case where the individual is informed of something which he already believes. In this case the notion of minimal change would require that there be no change at all. This requirement is expressed by the following axiom:

$$\textit{NO CHANGE:}\ \ (B_0\phi \wedge I\phi) \rightarrow (B_1\psi \leftrightarrow B_0\psi).$$

Proposition 7 *Assume that initial beliefs satisfy axiom K* $(B_0\phi \wedge B_0(\phi \rightarrow \psi) \rightarrow B_0\psi)$ *and the consistency axiom D* $(B_0\phi \rightarrow \neg B_0\neg\phi)$*. Then the conjunction of Persistence and Minimality implies No Change.*

Proof. We give a syntactic proof (PL stands for 'Propositional Logic'):

1.	$B_0\phi \rightarrow \neg B_0\neg\phi$	Consistency of B_0
2.	$B_0\phi \wedge I\phi \rightarrow \neg B_0\neg\phi \wedge I\phi$	1, PL
3.	$\neg B_0\neg\phi \wedge I\phi \rightarrow (B_0\psi \rightarrow B_1\psi)$	Persistence
4.	$B_0\phi \wedge I\phi \rightarrow (B_0\psi \rightarrow B_1\psi)$	2, 3, PL
5.	$I\phi \wedge B_1\psi \rightarrow B_0(\phi \rightarrow \psi)$	Minimality
6.	$I\phi \wedge B_1\psi \wedge B_0\phi \rightarrow B_0(\phi \rightarrow \psi) \wedge B_0\phi$	5, PL
7.	$B_0(\phi \rightarrow \psi) \wedge B_0\phi \rightarrow B_0\psi$	Axiom K for B_0
8.	$I\phi \wedge B_0\phi \wedge B_1\psi \rightarrow B_0\psi$	6, 7, PL
9.	$I\phi \wedge B_0\phi \rightarrow (B_1\psi \rightarrow B_0\psi)$	8, PL
10.	$I\phi \wedge B_0\phi \rightarrow (B_0\psi \rightarrow B_1\psi) \wedge (B_1\psi \rightarrow B_0\psi)$	4, 9, PL ■

Note that without consistency of initial beliefs Proposition 7 is not true.[11] Note also that the converse of Proposition 7 does not hold: the conjunction of Persistence and Minimality cannot be derived from No Change.[12]

We conclude this section with further discussion of the axioms studied above.

The relatively recent literature on dynamic epistemic logic studies how actions such as public announcements lead to revision of the interactive knowledge of a group of individuals (for a survey see van der Hoek and Pauly [19] and van Ditmarsch and van der Hoek [11]). One of the issues studied in this literature is what kind of public announcements can be successful in the sense that they produce common knowledge of the announced fact. Some public announcements, although truthful, cannot be successful. For example if individual a does not know that p ($\neg K_a p$), the public announcement '$p \wedge \neg K_a p$', although truthful, "leaves a with a difficult, if not impossible task to update his knowledge; it is hard to see how to simultaneously incorporate p and $\neg K_a p$ into his knowledge" (van der Hoek and Pauly [19], p. 23). In our approach this difficulty does not arise, since we distinguish between initial beliefs (B_0) and revised beliefs (B_1). It is therefore not problematic to be told "p is true and you did not believe it before this announcement" ($p \wedge \neg B_0 p$) since this fact can be truthfully incorporated into the revised beliefs. That is, the formula $p \wedge \neg B_0 p \wedge B_1(p \wedge \neg B_0 p)$ is consistent.

If the revised beliefs satisfy positive introspection, that is, if the operator B_1 satisfies the axiom $B_1\phi \rightarrow B_1 B_1 \phi$, then the following axiom can be derived from Minimality: $I\phi \wedge B_1\phi \rightarrow B_0(\phi \rightarrow B_1\phi)$.[13] This may seem counterintuitive. However, one cannot consistently reject this principle and at the same time embrace Bayes' rule for belief revision, since the former is an implication of the latter. In fact, letting P_0 be the probability measure that represents the initial beliefs, and denoting its support by $supp(P_0)$, for every event F it is trivially rue that

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[11] As is well known (see Chellas [9]) consistency of initial beliefs is characterized by seriality of $\mathcal{B}_0$ ($\forall \omega \in \Omega$, $\mathcal{B}_0(\omega) \neq \varnothing$). If there is a state α such that $\mathcal{B}_0(\alpha) = \varnothing$ then $\alpha \models B_0\psi$ for every formula ψ.

To see that without consistency of initial beliefs Proposition 7 is not true, consider the following example. $\Omega = \{\alpha\}$, $\mathcal{B}_0(\alpha) = \varnothing$, $\mathcal{B}_1(\alpha) = \mathcal{I}(\alpha) = \{\alpha\}$. Then, for every formula ϕ, $\alpha \nvDash \neg B_0 \neg \phi$ so that Persistence is trivially valid. It is also trivially true, for every ϕ anf ψ, that $\alpha \models B_0(\phi \rightarrow \psi)$ so that Minimality is also valid. Let p be an atomic proposition such that $\alpha \models p$. Then $\alpha \models B_0 p \wedge Ip \wedge B_0 \neg p \wedge \neg B_1 \neg p$, so that the No Change axiom is falsified at α with $\phi = p$ and $\psi = \neg p$.

[12] Consider the following frame: $\Omega = \{\alpha, \beta, \gamma\}$, and for every $\omega \in \Omega$, $\mathcal{B}_0(\omega) = \{\beta, \gamma\}$, $\mathcal{I}(\omega) = \{\omega\}$ and $\mathcal{B}_1(\omega) = \{\alpha, \beta\}$. By Lemma 5, Persistence is not valid in this frame (since $\mathcal{B}_0(\beta) \cap \mathcal{I}(\beta) \neq \varnothing$ and yet $\mathcal{B}_1(\beta) \nsubseteq \mathcal{B}_0(\beta)$. By Lemma 6, also Minimality is not valid (since $\mathcal{B}_0(\gamma) \cap \mathcal{I}(\gamma) \nsubseteq \mathcal{B}_1(\gamma)$). However, No Change is trivially valid in this frame. In fact, fix an arbitrary model and an arbitrary formula ϕ. It is easy to see that, for every $\omega \in \Omega$, $\omega \nvDash B_0\phi \wedge I\phi$. For example, if $\beta \models I\phi$ then $\|\phi\| = \mathcal{I}(\beta) = \{\beta\}$, so that $\gamma \notin \|\phi\|$, implying that $\beta \nvDash B_0\phi$.

[13] Proof:

1.	$B_1\phi \rightarrow B_1 B_1 \phi$	positive inrospection axiom
2.	$I\phi \wedge B_1\phi \rightarrow I\phi \wedge B_1 B_1 \phi$	1, PL
3.	$I\phi \wedge B_1 B_1 \phi \rightarrow B_0(\phi \rightarrow B_1\phi)$	instance of Minimality with $\psi = B_1\phi$
4.	$I\phi \wedge B_1\phi \rightarrow B_0(\phi \rightarrow B_1\phi)$	2,3,PL.

$$supp(P_0) = (supp(P_0) \cap F) \cup (supp(P_0) \cap \neg F) \qquad (1)$$

(where $\neg F$ denotes the complement of F). Now, let E be an event representing new information such that $P_0(E) > 0$, that is, $supp(P_0) \cap E \neq \varnothing$. Let P_1 be the probability measure representing the revised beliefs obtained by applying Bayes' rule, so that, for every event A, $P_1(A) = \frac{P_0(A \cap E)}{P_0(E)}$. Then, as noted in Section 1,

$$supp(P_1) = supp(P_0) \cap E. \qquad (2)$$

It follows from (1) and (2) that

$$supp(P_0) \subseteq \neg E \cup supp(P_1) \qquad (3)$$

which says that for any state ω that the individual initially considers possible ($\omega \in supp(P_0)$) if event E is true at ω ($\omega \in E$) then he will later assign positive probability to ω ($\omega \in supp(P_1)$). Since, by (2), $supp(P_1) \subseteq E$, assigning prior probability 1 to the event $\neg E \cup supp(P_1)$ corresponds to the syntactic formula $B_0(\phi \to B_1\phi)$, where $\|\phi\| = E$.

3 A sound and complete logic for belief revision

We now provide a sound and complete logic for belief revision. Because of the non-standard validation rule for the information operator I, we need to add the universal or global modality A (see Blackburn et al [5], p. 415 and Goranko and Passy [15]). The interpretation of $A\phi$ is "it is globally true that ϕ". As before, a frame is a quadruple $\langle \Omega, \mathcal{B}_0, \mathcal{B}_1, \mathcal{I} \rangle$. To the validation rules discussed in Section 2 we add the following:

$$\omega \models A\phi \text{ if and only if } \|\phi\| = \Omega.$$

We denote by $\mathfrak{L}$ the logic determined by the following axioms and rules of inference.

AXIOMS:

1. All propositional tautologies.

2. Axiom K for B_0, B_1 and A (note the absence of an analogous axiom for I):

$$\begin{array}{ll} B_0\phi \wedge B_0(\phi \to \psi) \to B_0\psi & (\mathrm{K}_0) \\ B_1\phi \wedge B_1(\phi \to \psi) \to B_1\psi & (\mathrm{K}_1) \\ A\phi \wedge A(\phi \to \psi) \to A\psi & (\mathrm{K}_A) \end{array}$$

3. S5 axioms for A:

$$\begin{array}{ll} A\phi \to \phi & (\mathrm{T}_A) \\ \neg A\phi \to A\neg A\phi & (5_A) \end{array}$$

4. Inclusion axioms for B_0 and B_1 (note the absence of an analogous axiom for I):

$$\begin{array}{ll} A\phi \to B_0\phi & (\mathrm{Incl}_0) \\ A\phi \to B_1\phi & (\mathrm{Incl}_1) \end{array}$$

5. Axioms to capture the non-standard semantics for I:

$$\begin{array}{ll} (I\phi \wedge I\psi) \to A(\phi \leftrightarrow \psi) & (\mathrm{I}_1) \\ A(\phi \leftrightarrow \psi) \to (I\phi \leftrightarrow I\psi) & (\mathrm{I}_2) \end{array}$$

RULES OF INFERENCE:

1. Modus Ponens: $\frac{\phi,\ \ \phi\to\psi}{\psi}$ (MP)
2. Necessitation for A: $\frac{\phi}{A\phi}$ (Nec$_A$)

Remark 8 *Note that from (Nec*$_A$*) and (Incl*$_0$*) one obtains necessitation for* B_0 *as a derived rule of inference:* $\frac{\phi}{B_0\phi}$*. The same is true for* B_1*. On the other hand, the necessitation rule for* I *is* ***not*** *a rule of inference of logic* $\mathfrak{L}$*. Indeed necessitation for* I *is not validity preserving.*[14] *Neither is rule RK for* I*:* $\frac{\phi\to\psi}{I\phi\to I\psi}$*.*[15] *On the other hand, by Nec*$_A$ *and* I_2*, rule RE for* I*:* $\frac{\phi\leftrightarrow\psi}{I\phi\leftrightarrow I\psi}$ *is a derived rule of inference of* $\mathfrak{L}$*.*

Note that, despite the non-standard validation rule, axiom K for I, namely $I\phi \wedge I(\phi \to \psi) \to I\psi$, is trivially valid in every frame.[16] It follows from the completeness theorem proved below that axiom K for I is provable in $\mathfrak{L}$. The following proposition, however, provides a direct proof.

Proposition 9 $I\phi \wedge I(\phi \to \psi) \to I\psi$ *is a theorem of logic* $\mathfrak{L}$*.*

Proof. We give a syntactic proof ('PL' stands for 'Propositional Logic'):

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[14] If ϕ is a valid formula, then $\|\phi\| = \Omega$. Let $\alpha \in \Omega$ be a state where $\mathcal{I}(\alpha) \neq \Omega$. Then $\alpha \not\models I\phi$ and therefore $I\phi$ is not valid.

[15] Consider the following model: $\Omega = \{\alpha, \beta\}$, $\mathcal{I}(\alpha) = \{\alpha\}$, $\mathcal{I}(\beta) = \{\beta\}$, $\|p\| = \{\alpha\}$ and $\|q\| = \Omega$. Then $\|p \to q\| = \Omega$, $\|Ip\| = \{\alpha\}$, $\|Iq\| = \varnothing$ and thus $\|Ip \to Iq\| = \{\beta\} \neq \Omega$.

[16] Proof. Fix a frame, an arbitrary model and a state α. For it to be the case that $\alpha \models I(\phi \to \psi) \wedge I\phi$ we need $\mathcal{I}(\alpha) = \|\phi\|$ and $\mathcal{I}(\alpha) = \|\phi \to \psi\|$. Now, $\|\phi \to \psi\| = \|\neg\phi \vee \psi\| = \|\neg\phi\| \cup \|\psi\|$ and therefore we need the equality $\|\phi\| = \|\neg\phi\| \cup \|\psi\|$ to be satisfied. This requires $\|\phi\| = \|\psi\| = \Omega$. Thus if $\mathcal{I}(\alpha) = \|\phi\| = \|\psi\| = \Omega$, then $\alpha \models I(\phi \to \psi) \wedge I\phi \wedge I\psi$. In every other case, $\alpha \not\models I(\phi \to \psi) \wedge I\phi$ and therefore the formula $I(\phi \to \psi) \wedge I\phi \to I\psi$ is trivially true at α.

1.	$(I\phi \wedge I(\phi \to \psi)) \to A(\phi \leftrightarrow (\phi \to \psi))$	Axiom I_1
2.	$(\phi \leftrightarrow (\phi \to \psi)) \to (\phi \wedge \psi)$	Tautology
3.	$A(\phi \leftrightarrow (\phi \to \psi)) \to A(\phi \wedge \psi)$	2, inference rule RK for A (see Chellas, 1984, p. 114)
4.	$(\phi \wedge \psi) \to (\phi \leftrightarrow \psi)$	Tautology
5.	$A(\phi \wedge \psi) \to A(\phi \leftrightarrow \psi)$	4, inference rule RK for A
6.	$A(\phi \leftrightarrow \psi) \to (I\phi \leftrightarrow I\psi)$	Axiom I_2
7.	$(I\phi \wedge I(\phi \to \psi)) \to (I\phi \leftrightarrow I\psi)$	1,3,5,6, PL
8.	$(I\phi \wedge I(\phi \to \psi)) \to I\psi$	7, PL ■

Recall that a logic is *complete* with respect to a class of frames if every formula which is valid in every frame in that class is provable in the logic (that is, it is a theorem). The logic is *sound* with respect to a class of frames if every theorem of the logic is valid in every frame in that class. The following proposition is a straightforward adaptation of a result due to Goranko and Passy [15] (Theorem 6.2, p. 24). Its proof is relegated to the appendix.

Proposition 10 *Logic $\mathfrak{L}$ is sound and complete with respect to the class of all frames $\langle \Omega, \mathcal{B}_0, \mathcal{B}_1, \mathcal{I} \rangle$.*

We are interested in extensions of $\mathfrak{L}$ obtained by adding various axioms. Let $\mathfrak{R}$ ('R' stands for 'Revision') be the logic obtained by adding to $\mathfrak{L}$ the axioms discussed in the previous section:

$$\mathfrak{R} = \mathfrak{L} \ + \ \text{Qualified Acceptance} + \text{Persistence} + \text{Minimality}.$$

The following proposition is proved in the appendix.

Proposition 11 *Logic $\mathfrak{R}$ is sound and complete with respect to the class of frames $\langle \Omega, \mathcal{B}_0, \mathcal{B}_1, \mathcal{I} \rangle$ that satisfy the Qualitative Bayes Rule.*

So far we have not postulated any properties of beliefs, in particular, in the interest of generality, we have not required beliefs to satisfy the KD45 logic. In order to further explore the implications of the Qualitative Bayes Rule, we shall now consider additional axioms:

Consistency of initial beliefs	$B_0\phi \to \neg B_0 \neg \phi$	(D_0)
Positive Introspection of initial beliefs	$B_0\phi \to B_0 B_0 \phi$	(4_0)
Self Trust	$B_0(B_0\phi \to \phi)$	(ST)
Information Trust	$B_0(I\phi \to \phi)$	(IT).

Self Trust says that the individual at time 0 believes that his beliefs are correct (he believes that if he believes ϕ then ϕ is true), while Information Trust says that the individual at time 0 believes that any information he will receive will be correct (he believes that if he is informed that ϕ then ϕ is true).

Remark 12 *It is well-known that Consistency of initial beliefs corresponds to seriality of* $\mathcal{B}_0$ *(* $\mathcal{B}_0(\omega) \neq \varnothing$*, for all* $\omega \in \Omega$*) and Positive Introspection to transitivity of* $\mathcal{B}_0$ *(if* $\beta \in \mathcal{B}_0(\alpha)$ *then* $\mathcal{B}_0(\beta) \subseteq \mathcal{B}_0(\alpha)$*). It is also well-known that Self Trust is characterized by secondary reflexivity of* $\mathcal{B}_0$ *(if* $\beta \in \mathcal{B}_0(\alpha)$ *then* $\beta \in \mathcal{B}_0(\beta)$*).*[17]

Lemma 13 *Information Trust (*$B_0(I\phi \to \phi)$*) is characterized by reflexivity of* $\mathcal{I}$ *over* $\mathcal{B}_0$*:* $\forall \alpha, \beta \in \Omega$*, if* $\beta \in \mathcal{B}_0(\alpha)$ *then* $\beta \in \mathcal{I}(\beta)$*.*

Proof. Suppose the property is satisfied. Fix arbitrary α and ϕ. If $\mathcal{B}_0(\alpha) = \varnothing$ then $\alpha \models B_0\psi$ for every formula ψ, in particular for $\psi = I\phi \to \phi$. Suppose therefore that $\mathcal{B}_0(\alpha) \neq \varnothing$ and fix an arbitrary $\beta \in \mathcal{B}_0(\alpha)$. If $\mathcal{I}(\beta) \neq \|\phi\|$ then $\beta \nvDash I\phi$ and therefore $\beta \models I\phi \to \phi$. If $\mathcal{I}(\beta) = \|\phi\|$ then $\beta \models I\phi$. By the property, $\beta \in \mathcal{I}(\beta)$. Thus $\beta \models \phi$ and, therefore, $\beta \models I\phi \to \phi$. Conversely, suppose the property is violated. Then there exist α and β such that $\beta \in \mathcal{B}_0(\alpha)$ and $\beta \notin \mathcal{I}(\beta)$. Let p be an atomic proposition and construct a model where $\|p\| = \mathcal{I}(\beta)$. Then $\beta \models Ip$. Since $\beta \notin \mathcal{I}(\beta)$, $\beta \models \neg p$. Thus $\beta \nvDash Ip \to p$ and, therefore, $\alpha \nvDash B_0(Ip \to p)$. ■

Remark 14 *Since the additional axioms listed above are canonical, it follows from Proposition 11 that if* Σ *is a set of axioms from the above list, then the logic* $\mathfrak{R}+\Sigma$ *obtained by adding to* $\mathfrak{R}$ *the axioms in* Σ *is sound and complete with respect to the class of frames that satisfy the Qualitative Bayes Rule and the properties corresponding to the axioms in* Σ*. For example, the logic* $\mathfrak{R} + \{D_0, 4_0, ST\}$ *is sound and complete with respect to the class of frames that satisfy the Qualitative Bayes Rule as well as seriality, transitivity and secondary reflexivity of* $\mathcal{B}_0$*.*

By Proposition 7, No Change ($B_0\phi \wedge I\phi \to (B_1\psi \leftrightarrow B_0\psi)$) is a theorem of $\mathfrak{R} + D_0$. We now discuss some further theorems of extensions of $\mathfrak{R}$. Consider the following axiom:

$$B_0\phi \to B_0 B_1\phi$$

which says that if the individual initially believes that ϕ then she initially believes that she will continue to believe ϕ later.

Proposition 15 $B_0\phi \to B_0B_1\phi$ *is a theorem of* $\mathfrak{R} + 4_0 + ST + IT$*.*

Proof. It is shown in van der Hoek [18] (p. 183, Theorem 4.3 (c)) that axiom $B_0\phi \to B_0B_1\phi$ is characterized by the following property:

$$\forall \alpha, \beta \in \Omega, \text{ if } \beta \in \mathcal{B}_0(\alpha) \text{ then } \mathcal{B}_1(\beta) \subseteq \mathcal{B}_0(\alpha). \tag{P$_0$}$$

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[17] Furthermore, Self Trust is implied by a stronger property of beliefs, namely Negative Introspection ($\neg B_0\phi \to B_0\neg B_0\phi$), which is characterized by euclideanness of $\mathcal{B}_0$ (if $\beta \in \mathcal{B}_0(\alpha)$ then $\mathcal{B}_0(\alpha) \subseteq \mathcal{B}_0(\beta)$).

By Remark 14, the logic $\mathfrak{R} + 4_0 + ST + IT$ is sound and complete with respect to the class of frames that satisfy the Qualitative Bayes Rule as well as transitivity and secondary reflexivity of $\mathcal{B}_0$ and reflexivity of $\mathcal{I}$ over $\mathcal{B}_0$. Thus it is enough to show that this class of frames satisfies property (P_0). Fix an arbitrary frame in this class and arbitrary states α and β such that $\beta \in \mathcal{B}_0(\alpha)$. By Secondary Reflexivity of $\mathcal{B}_0$, $\beta \in \mathcal{B}_0(\beta)$. By Reflexivity of $\mathcal{I}$ over $\mathcal{B}_0$, $\beta \in \mathcal{I}(\beta)$. Thus $\mathcal{B}_0(\beta) \cap \mathcal{I}(\beta) \neq \varnothing$ and, by the Qualitative Bayes Rule, $\mathcal{B}_1(\beta) = \mathcal{B}_0(\beta) \cap \mathcal{I}(\beta)$, so that $\mathcal{B}_1(\beta) \subseteq \mathcal{B}_0(\beta)$. By transitivity of $\mathcal{B}_0$, $\mathcal{B}_0(\beta) \subseteq \mathcal{B}_0(\alpha)$. Thus $\mathcal{B}_1(\beta) \subseteq \mathcal{B}_0(\alpha)$. ■

Remark 16 *Close inspection of the proof of Proposition 15 reveals that Qualified Acceptance and Minimality play no role (since we only used the fact that* $\mathcal{B}_0(\beta) \cap \mathcal{I}(\beta) \neq \varnothing$ *implies that* $\mathcal{B}_1(\beta) \subseteq \mathcal{B}_0(\beta)$*), that is,* $B_0\phi \to B_0 B_1 \phi$ *is in fact a theorem of the logic* $\mathfrak{L}$*+Persistence* $+ 4_0 + ST + IT$.

The following frame, illustrated in Figure 2, shows that Positive Introspection of initial beliefs is crucial for Proposition 15: $\Omega = \{\alpha, \beta, \gamma\}$, $\mathcal{B}_0 = \mathcal{B}_1$, $\mathcal{B}_0(\alpha) = \{\beta\}$, $\mathcal{B}_0(\beta) = \{\beta, \gamma\}$, $\mathcal{B}_0(\gamma) = \{\gamma\}$, $\mathcal{I}(\alpha) = \mathcal{I}(\beta) = \mathcal{I}(\gamma) = \{\alpha, \beta, \gamma\}$. This frame does not validate the axiom $B_0\phi \to B_0 B_1 \phi$. In fact, let $\|p\| = \{\alpha, \beta\}$. Then $\alpha \models B_0 p$ but $\alpha \not\models B_0 B_1 p$. However, the frame satisfies the Qualitative Bayes Rule ($\forall \omega$, if $\mathcal{B}_0(\omega) \cap \mathcal{I}(\omega) \neq \varnothing$ then $\mathcal{B}_1(\omega) = \mathcal{B}_0(\omega) \cap \mathcal{I}(\omega)$) and validates Self Trust (since $\mathcal{B}_0$ is secondary reflexive) and Information Trust (since is $\mathcal{I}$ reflexive). On the other hand, Positive Introspection of Initial Beliefs does not hold, since $\mathcal{B}_0$ is not transitive (in fact, $\alpha \models B_0 p$ but $\alpha \not\models B_0 B_0 p$).

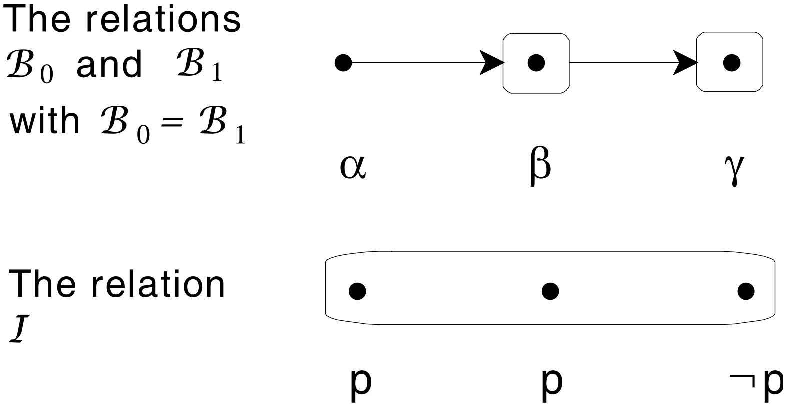

Figure 2

The next example, illustrated in Figure 3, shows that also Self Trust is crucial for Proposition 15: $\Omega = \{\alpha, \beta, \gamma, \delta, \varepsilon\}$, $\mathcal{B}_0(\alpha) = \{\beta, \gamma\}$, $\mathcal{B}_0(\beta) = \mathcal{B}_0(\gamma) = \{\gamma\}$, $\mathcal{B}_0(\delta) = \mathcal{B}_0(\varepsilon) = \{\varepsilon\}$, $\mathcal{I}(\alpha) = \{\alpha\}$, $\mathcal{I}(\beta) = \mathcal{I}(\delta) = \{\beta, \delta\}$, $\mathcal{I}(\gamma) = \{\gamma\}$, $\mathcal{I}(\varepsilon) = \{\varepsilon\}$, $\mathcal{B}_1 = \mathcal{I}$, $\|p\| = \{\beta, \gamma\}$. This frame does not validate axiom $B_0\phi \to B_0 B_1 \phi$ since $\alpha \models B_0 p$ but $\alpha \not\models B_0 B_1 p$ (since $\beta \in \mathcal{B}_0(\alpha)$ and $\beta \not\models B_1 p$ because

$\delta \in \mathcal{B}_1(\beta)$ and $\delta \nvDash p$). This frame satisfies the Qualitative Bayes Rule and validates Information Trust (since $\mathcal{I}$ is reflexive) and Positive Introspection of Initial Beliefs (since $\mathcal{B}_0$ is transitive). However Self Trust $B_0(B_0\phi \rightarrow \phi)$ is not valid, since $\mathcal{B}_0$ is not secondary reflexive (for example, let q be such that $\|q\| = \{\gamma\}$, then $\alpha \nvDash B_0(B_0 q \rightarrow q)$, since $\beta \in \mathcal{B}_0(\alpha)$ and $\beta \models B_0 q \wedge \neg q$).

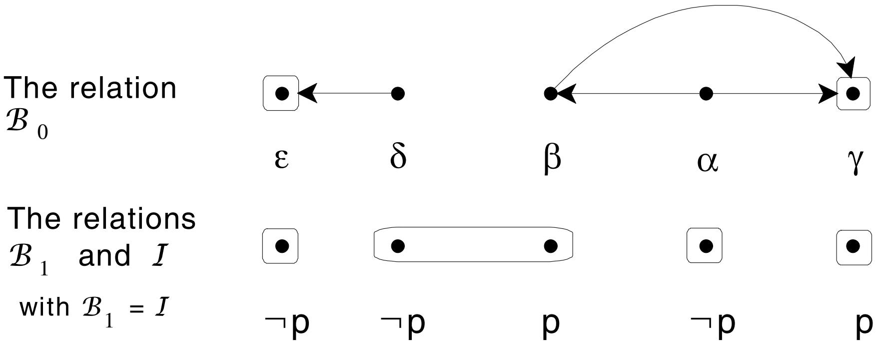

Figure 3

Similarly, it can be shown that Information Trust is necessary for Proposition 15 to be true.

Consider now the following axiom which is the converse of the previous one:

$$B_0 B_1 \phi \rightarrow B_0 \phi.$$

This axiom says that if the individual initially believes that later on she will believe ϕ then she must believe ϕ initially.

Proposition 17 *$B_0 B_1 \phi \rightarrow B_0 \phi$ is a theorem of $\mathfrak{R} + ST + IT$.*

Proof. It is shown in van der Hoek [18] (p. 183, Theorem 4.3 (e)) that axiom $B_0 B_1 \phi \rightarrow B_0 \phi$ is characterized by the following property: $\forall \alpha, \gamma \in \Omega$,

$$\text{if } \gamma \in \mathcal{B}_0(\alpha) \text{ then there exists a } \beta \in \mathcal{B}_0(\alpha) \text{ such that } \gamma \in \mathcal{B}_1(\beta). \qquad (\text{P}_1)$$

By Remark 14, the logic $\mathfrak{R} + ST + IT$ is sound and complete with respect to the class of frames that satisfy the Qualitative Bayes Rule as well as secondary reflexivity of $\mathcal{B}_0$ and reflexivity of $\mathcal{I}$ over $\mathcal{B}_0$. Thus it is enough to show that this class of frames satisfies property (P_1). Fix an arbitrary frame in this class and arbitrary states α and γ such that $\gamma \in \mathcal{B}_0(\alpha)$. By Secondary Reflexivity of $\mathcal{B}_0$, $\gamma \in \mathcal{B}_0(\gamma)$. By Reflexivity of $\mathcal{I}$ over $\mathcal{B}_0$, $\gamma \in \mathcal{I}(\gamma)$. Thus $\gamma \in \mathcal{B}_0(\gamma) \cap \mathcal{I}(\gamma)$ and, by the Qualitative Bayes Rule, $\mathcal{B}_0(\gamma) \cap \mathcal{I}(\gamma) = \mathcal{B}_1(\gamma)$, so that $\gamma \in \mathcal{B}_1(\gamma)$. Hence Property ($\text{P}_1$) is satisfied with $\beta = \gamma$. ■

Remark 18 *Close inspection of the proof of Proposition 17 reveals that Qualified Acceptance and Persistence play no role (since we only used the fact that* $\mathcal{B}_0(\gamma) \cap \mathcal{I}(\gamma) \subseteq \mathcal{B}_1(\gamma)$*), that is,* $B_0 B_1 \phi \to B_0 \phi$ *is in fact a theorem of the logic* $\mathfrak{L}$*+Minimality +* $ST + IT$ *.*

To see that Minimality is crucial, consider the following frame: $\Omega = \{\alpha, \beta\}$ and, for every $\omega \in \Omega$, $\mathcal{B}_0(\omega) = \{\beta\}$, $\mathcal{I}(\omega) = \Omega$ and $\mathcal{B}_1(\omega) = \{\alpha\}$. This frame validates Self Trust (since $\mathcal{B}_0$ is secondary reflexive) and Information Trust (since $\mathcal{I}$ is reflexive). However, it does not validate Minimality, since $\mathcal{B}_0(\alpha) \cap \mathcal{I}(\alpha) = \{\beta\} \nsubseteq \mathcal{B}_1(\alpha) = \{\alpha\}$. Let p be such that $\|p\| = \{\alpha\}$. Then $\alpha \models \mathcal{B}_0 \mathcal{B}_1 p \wedge \neg \mathcal{B}_0 p$.

The following example, illustrated in Figure 4, shows that, without Self Trust, Proposition 17 is not true: $\Omega = \{\alpha, \beta, \gamma\}$, $\mathcal{B}_0(\alpha) = \{\beta, \gamma\}$, $\mathcal{B}_0(\beta) = \mathcal{B}_0(\gamma) = \{\gamma\}$, $\mathcal{I}(\alpha) = \{\alpha\}$, $\mathcal{I}(\beta) = \mathcal{I}(\gamma) = \{\beta, \gamma\}$, $\mathcal{B}_1(\alpha) = \{\alpha\}$, $\mathcal{B}_1(\beta) = \mathcal{B}_1(\gamma) = \{\gamma\}$, $\|p\| = \{\gamma\}$. Then $\alpha \models B_0 B_1 p$ but $\alpha \nvDash B_0 p$. This frame satisfies the Qualitative Bayes Rule ($\forall \omega$, if $\mathcal{B}_0(\omega) \cap \mathcal{I}(\omega) \neq \varnothing$ then $\mathcal{B}_1(\omega) = \mathcal{B}_0(\omega) \cap \mathcal{I}(\omega)$) as well as Information Trust (since $\mathcal{I}$ is reflexive).[18].

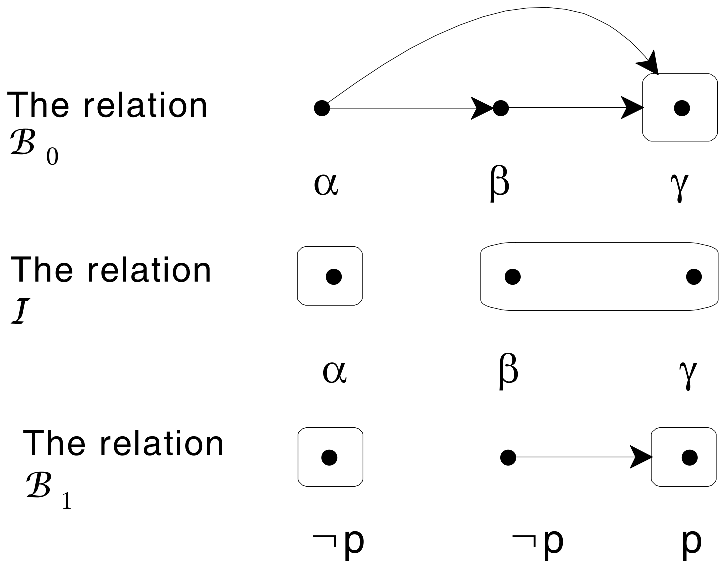

Figure 4

Putting together Propositions 15 and 17 we obtain the following corollary.

Corollary 19 $B_0 \phi \leftrightarrow B_0 B_1 \phi$ *is a theorem of* $\mathfrak{R} + 4_0 + ST + IT$*.*

Remark 20 *In the proof of Propositions 15 and 17 it was shown that axiom* $B_0 \phi \leftrightarrow B_0 B_1 \phi$ *is valid in every frame that satisfies the Qualitative Bayes Rule as well as the properties that characterize axioms* 4_0*, ST and IT (so that Corollary*

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[18] The frame also satisfies Positive Introspection of initial beliefs ($B_0 \phi \to B_0 B_0 \phi$) since $\mathcal{B}_0$ is transitive.

19 follows from the completeness theorem: see Remark 14). Note, however, that the converse is not true, that is, if a frame validates axioms 4_0, *ST*, *IT and* $B_0\phi \leftrightarrow B_0B_1\phi$ *then it does not necessarily satisfy the Qualitative Bayes Rule, as the example illustrated in Figure 5 shows.*

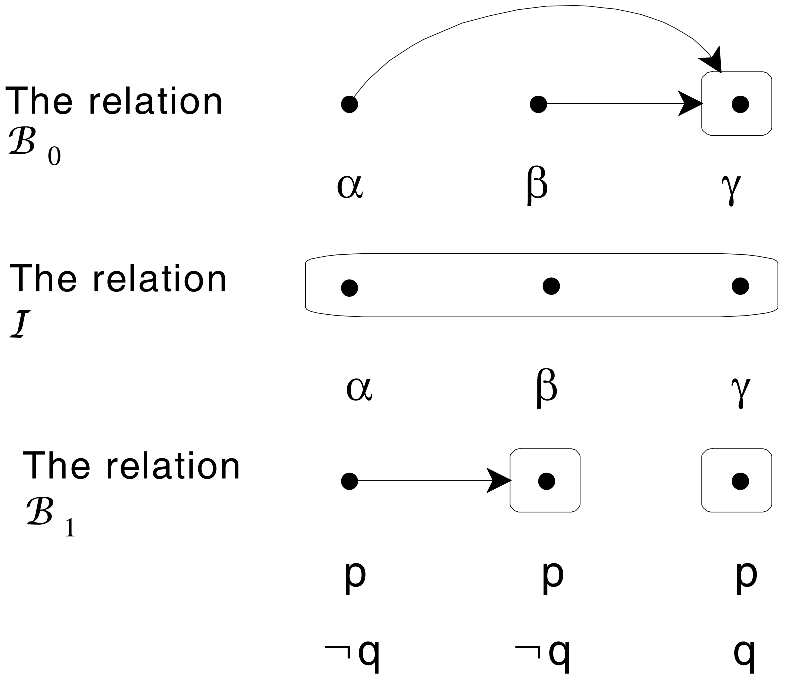

Figure 5

The frame illustrated in Figure 5 is as follows: $\Omega = \{\alpha,\beta,\gamma\}$, $\mathcal{B}_0(\alpha) = \mathcal{B}_0(\beta) = \mathcal{B}_0(\gamma) = \{\gamma\}$, $\mathcal{I}(\alpha) = \mathcal{I}(\beta) = \mathcal{I}(\gamma) = \{\alpha,\beta,\gamma\}$, $\mathcal{B}_1(\alpha) = \mathcal{B}_1(\beta) = \{\beta\}$, $\mathcal{B}_1(\gamma) = \{\gamma\}$. This frame validates Self Trust (since $\mathcal{B}_0$ is secondary reflexive) and Information Trust (since $\mathcal{I}$ is reflexive). It also validates Positive Introspection of initial beliefs (since $\mathcal{B}_0$ is transitive). Furthermore, the frame satisfies properties P_0 and P_1 (see the proofs of Propositions 15 and 17) and thus validates axiom $B_0\phi \leftrightarrow B_0B_1\phi$. However, it does not validate Persistence.[19] In fact, let $\|p\| = \Omega$ and $\|q\| = \{\gamma\}$; then $\alpha \models Ip \wedge \neg B_0\neg p \wedge B_0q$ but $\alpha \not\models B_1q$. Because of this, the Qualitative Bayes Rule is not satisfied: $\mathcal{B}_0(\alpha)\cap\mathcal{I}(\alpha) = \{\gamma\} \neq \varnothing$ and yet $\mathcal{B}_1(\alpha) \neq \{\gamma\}$.

4 Closely related literature

In this section we discuss the relationship between our approach and papers on belief revision that are closest to our analysis in that they make explicit use

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[19] Although, by Lemma 6, it does validate Minimality.

of modal logic. The relationship with the AGM literature will be discussed in Section 5.

A different axiomatization of the Qualitative Bayes Rule was provided by Battigalli and Bonanno [3] within a framework where information is *not* modeled explicitly. The logic they consider is based on four modal operators: B_0 and B_1, representing - as in this paper - initial and revised beliefs, and two knowledge operators, K_0 and K_1. Knowledge at time 1 is thought of as implicitly based on information received by the individual between time 0 and time 1 and is the basis on which beliefs are revised. The knowledge operators satisfy the S5 logic (the Truth axiom, $K_t\phi \to \phi$, and negative introspection, $\neg K_t\phi \to K_t\neg K_t\phi$), while the belief operators satisfy the KD45 logic (consistency and positive and negative introspection). Furthermore, knowledge and belief are linked by two axioms: everything that is known is believed ($K_t\phi \to B_t\phi$) and the individual knows what he believes ($B_t\phi \to K_tB_t\phi$). Within this framework Battigalli and Bonanno express the Qualitative Bayes Rule as follows: $\forall\omega \in \Omega$, if $\mathcal{K}_1(\omega)\cap\mathcal{B}_0(\omega) \neq \varnothing$ then $\mathcal{B}_1(\omega) = \mathcal{K}_1(\omega)\cap\mathcal{B}_0(\omega)$, that is, if there are states that are compatible with what the individual knows at time 1 and what he believed at time 0, then the states that he considers possible at time 1 (according to his revised beliefs) are precisely those states. The authors show that, within this knowledge-belief framework the formula $B_0\phi \leftrightarrow B_0B_1\phi$ (which says that the individual believes something at time 0 if and only if he believes that he will continue to believe it at time 1) provides an axiomatization of the Qualitative Bayes Rule. We showed in Corollary 19 that this axiom is a theorem of our logic $\mathfrak{R}$ augmented with axioms 4_0 (one of the axioms postulated by Battigalli and Bonanno), ST (implied by the negative introspection axiom for B_0, which they assume) and IT (whose counterpart in their framework, since they do not model information explicitly, is $B_0(K_1\phi \to \phi)$,which is implied by the Truth axiom of K_1). However, as pointed out above (Remark 20) in a framework where information is modeled explicitly, it is no longer true that the Qualitative Bayes Rule is characterized by axiom $B_0\phi \leftrightarrow B_0B_1\phi$. Thus moving away from the knowledge-belief framework of Battigalli and Bonanno [3] axiom $B_0\phi \leftrightarrow B_0B_1\phi$ becomes merely an implication of the Qualitative Bayes Rule under additional hypotheses.

Segerberg [27] notes the coexistence of two traditions in the literature on doxastic logic (the logic of belief), the one initiated by Hintikka [17] and the AGM approach [1], and proposes a unifying framework for belief revision. His proposal is to use dynamic logic by thinking of expansion, revision and contraction as actions. Besides the belief operator B, he introduces three operators for every (purely Boolean) formula ϕ: $[+\phi]$ for expansion, $[*\phi]$ for revision and $[-\phi]$ for contraction. Thus, for example, the intended interpretation of $[+\phi]B\chi$ is "after performing the action of expanding by ϕ the individual believes that χ". Segerberg's logic is therefore considerably more complex than ours: besides requiring the extra apparatus of dynamic logic, it involves an *infinite* number of modal operators, while our logic uses only three.

In a recent paper, Board [6] offers a syntactic analysis of belief revision. Like

Segerberg, Board makes use of an infinite number of modal operators: for every formula ϕ, an operator B^{ϕ} is introduced representing the hypothetical beliefs of the individual in the case where she learns that ϕ. Thus the interpretation of $B^{\phi}\psi$ is "upon learning that ϕ, the individual believes that ψ". The initial beliefs are represented by an operator B. On the semantic side Board considers a set of states and a collection of binary relations, one for each state, representing the plausibility ordering of the individual at that state. The truth condition for the formula $B^{\phi}\psi$ at a state expresses the idea that the individual believes that ψ on learning that ϕ if and only if ψ is true in all the most plausible worlds in which ϕ is true. The author gives a list of axioms which is sound and complete with respect to the semantics. There are important differences between our framework and his. We model information explicitly by means of a single modal operator I, while Board models it implicitly through an infinite collection of hypothetical belief operators. While we model, at any state, only the information *actually* received by the individual, Board considers all possible hypothetical pieces of information: every formula represents a possible item of information, including contradictory formulas and modal formulas. Although, in principle, we also allowed information to be about an arbitrary formula, in our approach it is possible to rule out problematic situations by imposing suitable axioms (see Remark 1 and further discussion in Section 7).

Liau [24] considers a multi-agent framework and is interested in modeling the issue of trust. He introduces modal operators B_i, I_{ij} and T_{ij} with the following intended meaning:

$B_i\psi$	Agent i believes that ψ
$I_{ij}\psi$	Agent i acquires information ψ from agent j
$T_{ij}\psi$	Agent i trusts the judgement of agent j on the truth of ψ.

On the semantic side Liau considers a set of states Ω and a collection of binary relations $\mathcal{B}_i$ and $\mathcal{I}_{ij}$ on Ω, corresponding to the operators B_i and I_{ij}. The truth conditions are the standard ones for Kripke structures ($\omega \models B_i\psi$ if and only if $\mathcal{B}_i(\omega) \subseteq \|\psi\|$ and $\omega \models I_{ij}\psi$ if and only if $\mathcal{I}_{ij}(\omega) \subseteq \|\psi\|$). Intuitively, $\mathcal{B}_i(\omega)$ is the set of states that agent i considers possible at ω according to his belief, whereas $\mathcal{I}_{ij}(\omega)$ is what agent i considers possible according to the information acquired from j. The author also introduces a relation $\mathcal{T}_{ij}$ that associates with every state $\omega \in \Omega$ a set of subsets of Ω. For any $S \subseteq \Omega$, $S \in \mathcal{T}_{ij}(\omega)$ means that agent i trusts j 's judgement on the truth of the proposition corresponding to event S. Liau considers various axioms and proves that the corresponding logics are sound and complete with respect to the semantics. One of the axioms the author discusses is $I_{ij}\psi \rightarrow B_iI_{ij}\psi$, which says that if agent i is informed that ψ by agent j then she believes that this is the case. Liau notes that, in general, this axiom does not hold, since when i receives a message from j, she may not be able to exclude the possibility that someone pretending to be j has sent the message; however, in a secure communication environment this would not happen and the axiom would hold. There are important differences between our analysis and Liau's. We don't discuss the issue of trust (although introducing an axiom such as $I\phi \rightarrow B_1\phi$ would capture the notion that information is trusted and therefore believed). On the other hand, we explicitly distinguish between beliefs held before the information is received and revised beliefs. Liau has only one belief operator and therefore does not make this distinction. Yet this distinction is very important. Suppose first that we take B to be the *initial* belief (of some agent). Then an axiom like $I\psi \rightarrow BI\psi$ would not be acceptable on conceptual grounds, even if communication is secure. For example, consider a doctor who initially in uncertain whether the patient has an infection (represented by the atomic proposition p) or not ($\neg p$). Let α be a state where p is true (the patient has an infection) and β a state where it is not. Thus the initial uncertainty can be expressed by setting $\mathcal{B}(\alpha) = \mathcal{B}(\beta) = \{\alpha, \beta\}$. The doctor orders a blood test, which, if positive, reveals that there is an infection and, if negative, reveals that there is no infection. Thus $\mathcal{I}(\alpha) = \{\alpha\}$ and $\mathcal{I}(\beta) = \{\beta\}$, so that $\alpha \models Ip$ and $\beta \models I\neg p$. Then $\alpha \models Ip$ but $\alpha \nvDash BIp$. On the other hand, if we take B to be the *revised* belief (after the information is received) then postulating the axiom $I\phi \rightarrow BI\phi$ would imply in this example that $\mathcal{B}(\alpha) = \mathcal{I}(\alpha) = \{\alpha\}$ and $\mathcal{B}(\beta) = \mathcal{I}(\beta) = \{\beta\}$, that is, that the information is necessarily believed, thus making it impossible to separate the issues of information and trust. For example, we would not be able to model a situation where the doctor receives the result of the blood test but does not trust the report because of mistakes made in the past by the same lab technician.

5 Relationship to the AGM framework

The AGM theory of belief revision has been developed within the framework of belief sets. Let Φ be the set of formulas in a propositional language.[20] Given a subset $S \subseteq \Phi$, its PL-deductive closure $[S]^{PL}$ (where 'PL' stands for 'Propositional Logic') is defined as follows: $\psi \in [S]^{PL}$ if and only if there exist $\phi_1, ..., \phi_n \in S$ such that $(\phi_1 \wedge ... \wedge \phi_n) \rightarrow \psi$ is a truth-functional tautology (that is, a theorem of Propositional Logic). A *belief set* is a set $K \subseteq \Phi$ such that $K = [K]^{PL}$. A belief set K is consistent if $K \neq \Phi$ (equivalently, if there is no formula ϕ such that both ϕ and $\neg\phi$ belong to K). Given a belief set K (thought of as the initial beliefs of the individual) and a formula ϕ (thought of as a new piece of information), the *revision of K by ϕ*, denoted by K_ϕ^*, is a subset of Φ that satisfies the following conditions, known as the AGM postulates:

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[20] For simplicity we consider the simplest case where the underlying logic is classical propositional logic.

(K*1) K^*_ϕ is a belief set
(K*2) $\phi \in K^*_\phi$
(K*3) $K^*_\phi \subseteq [K \cup \{\phi\}]^{PL}$
(K*4) if $\neg\phi \notin K$, then $[K \cup \{\phi\}]^{PL} \subseteq K^*_\phi$
(K*5) $K^*_\phi = \Phi$ if and only if ϕ is a contradiction
(K*6) if $\phi \leftrightarrow \psi$ is a tautology then $K^*_\phi = K^*_\psi$
(K*7) $K^*_{\phi\wedge\psi} \subseteq \left[K^*_\phi \cup \{\psi\}\right]^{PL}$
(K*8) if $\neg\psi \notin K^*_\phi$, then $\left[K^*_\phi \cup \{\psi\}\right]^{PL} \subseteq K^*_{\phi\wedge\psi}$

(K*1) requires the revised belief set to be deductively closed. In our framework this corresponds to requiring the B_1 operator to be a normal operator, that is, to satisfy axiom K ($B_1(\phi \to \psi) \wedge B_1\phi \to B_1\psi$) and the inference rule of necessitation (from ϕ to infer $B_1\phi$).

(K*2) requires that the information be believed. In our framework, this corresponds to imposing axiom $I\phi \to B_1\phi$, which is a strengthening of Qualified Acceptance, in that it requires that if the individual is informed that ϕ then he believes that ϕ even if he previously believed that $\neg\phi$. It is straightforward to prove that this axiom is characterized by the following property: $\forall\omega \in \Omega$, $\mathcal{B}_1(\omega) \subseteq \mathcal{I}(\omega)$.

(K*3) says that beliefs should be revised minimally, in the sense that no new belief should be added unless it can be deduced from the information received and the initial beliefs. As we will show later, this requirement corresponds to our Minimality axiom $(I\phi \wedge B_1\psi) \to B_0(\phi \to \psi)$.

(K*4) says that if the information received is compatible with the initial beliefs, then any formula that can be deduced from the information and the initial beliefs should be part of the revised beliefs. As shown below, this requirement corresponds to our Persistence axiom $(I\phi \wedge \neg B_0\neg\phi) \to (B_0\psi \to B_1\psi)$.

(K*5) requires the revised beliefs to be consistent, unless the information is contradictory. As pointed out by Friedman and Halpern [12], it is not clear how information could consist of a contradiction. In our framework we can eliminate this possibility by imposing the axiom $\neg I(\phi \wedge \neg\phi)$, which is characterized by seriality of $\mathcal{I}$ ($\forall\omega \in \Omega, \mathcal{I}(\omega) \neq \varnothing$) (see Section 7). Furthermore, the requirement that revised beliefs be consistent can be captured by the consistency axiom (axiom D): $B_1\phi \to \neg B_1\neg\phi$, which is characterized by seriality of $\mathcal{B}_1$ ($\forall\omega \in \Omega$, $\mathcal{B}_1(\omega) \neq \varnothing$). Together with the axiom corresponding to (K*2), consistency of revised beliefs guarantees that information itself is consistent, that is, the conjunction of $B_1\phi \to \neg B_1\neg\phi$ and $I\phi \to B_1\phi$ implies $\neg I(\phi \wedge \neg\phi)$ (since $\mathcal{B}_1(\omega) \neq \varnothing$ and $\mathcal{B}_1(\omega) \subseteq \mathcal{I}(\omega)$ implies that $\mathcal{I}(\omega) \neq \varnothing$).

(K*6) is automatically satisfied in our framework, since if $\phi \leftrightarrow \psi$ is a tautology then $\|\phi\| = \|\psi\|$ in every model and therefore the formula $I\phi \leftrightarrow I\psi$ is valid in every frame. Hence revision based on $I\phi$ must coincide with revision based on $I\psi$.

(K*7) and (K*8) are a generalization of (K*3) and (K*4) that

> "applies to *iterated* changes of belief. The idea is that if K_ϕ^* is a revision of K and K_ϕ^* is to be changed by adding further sentences, such a change should be made by using expansions of K_ϕ^* whenever possible. More generally, the minimal change of K to include both ϕ and ψ (that is, $K_{\phi\wedge\psi}^*$) ought to be the same as the expansion of K_ϕ^* by ψ, so long as ψ does not contradict the beliefs in K_ϕ^*" (Gärdenfors [14], p. 55).[21]

We postpone a discussion of iterated revision to the next section, where we claim that the axiomatization of the Qualitative Bayes Rule that we provided can deal with iterated revision and satisfies the conceptual content of (K*7) and (K*8).

The set of postulates (K*1) through (K*6) is called the *basic set* of postulates for belief revision (Gärdenfors, [14] p. 55). The next proposition shows that our axioms imply that the basic set of postulates are satisfied.

Proposition 21 *Fix an arbitrary model and an arbitrary state* α *and let* $K = \{\psi : \alpha \models B_0\psi\}$. *Suppose that there is a formula* ϕ *such that* $\alpha \models I\phi$ *and define* $K_\phi^* = \{\psi : \alpha \models B_1\psi\}$. *If at* α *the following hypotheses are satisfied for all formulas* ψ *and* χ

$\alpha \models I\psi \to B_1\psi$	*Acceptance*
$\alpha \models (I\psi \wedge B_1\chi) \to B_0(\psi \to \chi)$	*Minimality*
$\alpha \models (I\psi \wedge \neg B_0\neg\psi) \to (B_0\chi \to B_1\chi)$	*Persistence*
$\alpha \models B_1\chi \to \neg B_1\neg\chi$	*Consistency of* B_1 *(axiom D)*

then K_ϕ^* *satisfies postulates (K*1) to (K*6).*

Proof. (K*1): we need to show that K_ϕ^* is a belief set, that is, $K_\phi^* = \left[K_\phi^*\right]^{PL}$. Clearly, $K_\phi^* \subseteq \left[K_\phi^*\right]^{PL}$, since $\psi \to \psi$ is a tautology. Thus we only need to show that $\left[K_\phi^*\right]^{PL} \subseteq K_\phi^*$. Let $\psi \in \left[K_\phi^*\right]^{PL}$, i.e. there exist $\phi_1, ..., \phi_n \in K_\phi^*$ such that $(\phi_1 \wedge ... \wedge \phi_n) \to \psi$ is a tautology. Then $\alpha \models B_1\left((\phi_1 \wedge ... \wedge \phi_n) \to \psi\right)$. By definition of K_ϕ^*, since $\phi_1, ..., \phi_n \in K_\phi^*$, $\alpha \models B_1\left(\phi_1 \wedge ... \wedge \phi_n\right)$. Thus $\alpha \models B_1\psi$, that is, $\psi \in K_\phi^*$.

(K*2): we need to show that $\phi \in K_\phi^*$, that is, $\alpha \models B_1\phi$. This is an immediate consequence of our hypotheses that $\alpha \models I\phi$ and $\alpha \models I\phi \to B_1\phi$ (by the Acceptance axiom).

(K*3): we need to show that $K_\phi^* \subseteq [K \cup \{\phi\}]^{PL}$. Let $\psi \in K_\phi^*$, i.e. $\alpha \models B_1\psi$. By hypothesis, $\alpha \models (I\phi \wedge B_1\psi) \to B_0(\phi \to \psi)$ (by Minimality) and $\alpha \models I\phi$. Thus $\alpha \models B_0(\phi \to \psi)$, that is, $(\phi \to \psi) \in K$. Hence $\{\phi, (\phi \to \psi)\} \in K \cup \{\phi\}$ and, since $(\phi \wedge (\phi \to \psi)) \to \psi$ is a tautology, $\psi \in [K \cup \{\phi\}]^{PL}$.

(K*4): we need to show that if $\neg\phi \notin K$ then $[K \cup \{\phi\}]^{PL} \subseteq K_\phi^*$. Suppose $\neg\phi \notin K$, that is, $\alpha \models \neg B_0\neg\phi$. By hypothesis, $\alpha \models I\phi$ and $\alpha \models (I\phi \wedge \neg B_0\neg\phi) \to (B_0\psi \to B_1\psi)$ (by Persistence). Thus

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[21] The expansion of K_ϕ^* by ψ is $\left[K_\phi^* \cup \{\psi\}\right]^{PL}$.

$$\alpha \models (B_0\psi \rightarrow B_1\psi), \text{ for every formula } \psi. \tag{4}$$

Let $\chi \in [K \cup \{\phi\}]^{PL}$, that is, there exist $\phi_1, ..., \phi_n \in K \cup \{\phi\}$ such that $(\phi_1 \wedge ... \wedge \phi_n) \rightarrow \chi$ is a tautology. We want to show that $\chi \in K^*_\phi$, i.e. $\alpha \models B_1\chi$. Since $(\phi_1 \wedge ... \wedge \phi_n) \rightarrow \chi$ is a tautology, $\alpha \models B_0\left((\phi_1 \wedge ... \wedge \phi_n) \rightarrow \chi\right)$. If $\phi_1, ..., \phi_n \in K$, then $\alpha \models B_0\left(\phi_1 \wedge ... \wedge \phi_n\right)$ and therefore $\alpha \models B_0\chi$. Thus, by (4), $\alpha \models B_1\chi$. If $\phi_1, ..., \phi_n \notin K$, then w.l.o.g. $\phi_1 = \phi$ and $\phi_2, ..., \phi_n \in K$. In this case we have $\alpha \models B_0\left(\phi_2 \wedge ... \wedge \phi_n\right)$ and $\alpha \models B_0\left((\phi_2 \wedge ... \wedge \phi_n) \rightarrow (\phi \rightarrow \chi)\right)$ since $(\phi_1 \wedge ... \wedge \phi_n) \rightarrow \chi$ is a tautology and it is equivalent to $(\phi_2 \wedge ... \wedge \phi_n) \rightarrow (\phi \rightarrow \chi)$. Thus $\alpha \models B_0\left(\phi \rightarrow \chi\right)$. Hence, by (4) (with $\psi = (\phi \rightarrow \chi)$), $\alpha \models B_1\left(\phi \rightarrow \chi\right)$. From the hypotheses that $\alpha \models I\phi$ and $\alpha \models I\phi \rightarrow B_1\phi$ it follows that $\alpha \models B_1\phi$. Thus $\alpha \models B_1\chi$.

(K*5): we have to show that $K^*_\phi \neq \Phi$, unless ϕ is a contradiction. As noted above, the possibility of contradictory information is ruled out by the conjunction of Consistency of revised beliefs $(B_1\psi \rightarrow \neg B_1\neg\psi)$ and Acceptance $(I\psi \rightarrow B_1\psi)$. Thus we only need to show that $K^*_\phi \neq \Phi$. By hypothesis, $B_1\psi \rightarrow \neg B_1\neg\psi$; thus if $\psi \in K^*_\phi$ then $\neg\psi \notin K^*_\phi$ and therefore $K^*_\phi \neq \Phi$.

(K*6): we have to show that if $\phi \leftrightarrow \psi$ is a tautology then $K^*_\phi = K^*_\psi$. If $\phi \leftrightarrow \psi$ is a tautology, then $\|\phi \leftrightarrow \psi\| = \Omega$, that is, $\|\phi\| = \|\psi\|$. Thus $\alpha \models I\phi$ if and only if $\alpha \models I\psi$. Hence, by definition, $K^*_\phi = K^*_\psi$. ■

6 Iterated revision

As is well known[22], the AGM postulates are not sufficient to cover iterated belief revision, that is, the case where the individual receives a sequence of pieces of information over time. Only a limited amount of iterated revision is expressed by postulates (K*7) and (K*8), which require that the minimal change of K to include both information ϕ and information ψ (that is, $K^*_{\phi\wedge\psi}$) ought to be the same as the expansion of K^*_ϕ by ψ, so long as ψ does not contradict the beliefs in K^*_ϕ.

In our framework we model, at very state, only the information that is actually received by the individual and do not model how the individual would have modified his beliefs if he had received a different piece of information. Thus we cannot compare the revised beliefs the individual holds after receiving information ϕ with the beliefs he would have had if he had been informed of both ϕ and ψ. On the other hand, it is possible in our framework to model the effect of receiving first information ϕ and then information ψ. Indeed, any sequence of pieces of information can be easily modeled. In order to do this, we need to add a time index to the belief and information operators. Thus, for $t \in \mathbb{N}$ (where $\mathbb{N}$ denotes the set of natural numbers), we have a belief operator B_t representing the individual's beliefs at time t. In order to avoid confusion, we attach a double index $(t, t+1)$ to the an information operator, so that $I_{t,t+1}$

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[22] See, for example, Rott [9] (p. 170).

represents the information received by the individual between time t and time $t+1$. Thus the intended interpretation is as follows:

$B_t\phi$	at time t the individual believes that ϕ
$I_{t,t+1}\phi$	between time t and time $t+1$ the individual is informed that ϕ
$B_{t+1}\phi$	at time $t+1$ (in light of the information received between t and $t+1$) the individual believes that ϕ.

Let $\mathcal{B}_t$ and $\mathcal{I}_{t,t+1}$ be the associated binary relations. The iterated version of the qualitative Bayes rule then is the following simple extension of QBR: $\forall\omega \in \Omega, \forall t \in \mathbb{N}$,

$$\textit{if } \mathcal{B}_t(\omega) \cap \mathcal{I}_{t,t+1}(\omega) \neq \varnothing \textit{ then } \mathcal{B}_{t+1}(\omega) = \mathcal{B}_t(\omega) \cap \mathcal{I}_{t,t+1}(\omega). \qquad \text{(IQBR)}$$

The iterated Bayes rule plays an important role in game theory, since it is the main building block of two widely used solution concepts for dynamic (or extensive) games, namely Perfect Bayesian Equilibrium [23] and Sequential Equilibrium (Kreps and Wilson [22]). The idea behind these solution concepts is that, during the play of the game, a player should revise his beliefs by using Bayes' rule "as long as possible". Thus if an information set has been reached that had positive prior probability, then beliefs at that information set are obtained by using Bayes' rule (with the information being represented by the set of nodes in the information set under consideration). If an information set is reached that had zero prior probability, then new beliefs are formed more or less arbitrarily, but from that point onwards these new beliefs must be used in conjunction with Bayes' rule, unless further information is received that is inconsistent with those revised beliefs. This is precisely what IQBR requires.

Within this more general framework, a simple adaptation of Propositions 3 and 11 yields the following result:

Proposition 22 *(1) The Iterated Qualitative Bayes Rule (IQBR) is characterized by the conjunction of the following three axioms:*

Iterated Qualified Acceptance:	$(\neg B_t\neg\phi \wedge I_{t,t+1}\phi) \rightarrow B_{t+1}\phi$
Iterated Persistence:	$(\neg B_t\neg\phi \wedge I_{t,t+1}\phi) \rightarrow (B_t\psi \rightarrow B_{t+1}\psi)$
Iterated Minimality	$(I_{t,t+1}\phi \wedge B_{t+1}\psi) \rightarrow B_t(\phi \rightarrow \psi)$.

(2) The logic obtained by adding the above three axioms to the straightforward adaptation of logic $\mathfrak{L}$ to a multi-period framework is sound and complete with respect to the class of frames that satisfy the Iterated Qualitative Bayes Rule.

7 Conclusion

The simple modal language proposed in this paper has two advantages: (1) information is modeled directly by means of a modal operator I, so that (2)

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[23] See, for example, Battigalli [2], Bonanno [7], Fudenberg and Tirole [13].

three operators are sufficient to axiomatize the qualitative version of Bayes' rule. Previous modal axiomatizations of belief revisions required an infinite number of modal operators and captured information only indirectly through this infinite collection. We also showed that a multi-period extension of our framework allows one to deal with information flows and iterated belief revision.

While the belief operators B_0 and B_1 are normal modal operators, the information operator I is not normal in that the inference rule "from $\phi \to \psi$ to infer $I\phi \to I\psi$" does not hold.[24] This is a consequence of using a non-standard rule for the truth of $I\phi$ ($\omega \models I\phi$ if and only if $\mathcal{I}(\omega) = \|\phi\|$, whereas the standard rule would simply require $\mathcal{I}(\omega) \subseteq \|\phi\|$). However, the addition of the global or universal modality allowed us to obtain a logic of belief revision which is sound and complete with respect to the class of frames that satisfy the Qualitative Bayes Rule.

As pointed out in Remark 1, one might want to impose restrictions on the type of formulas that can constitute information (that is, on what formulas ϕ can be under the scope of the operator I). This is best done by imposing suitable axioms, rather than by restricting the syntax itself. For example, contradictory information is ruled out by imposing axiom $\neg I(\phi \wedge \neg\phi)$, which is characterized by seriality of $\mathcal{I}$ ($\forall\omega, \mathcal{I}(\omega) \neq \varnothing$).[25] Other axioms one might want to impose are: $B_0\phi \to \neg I\neg B_0\phi$ (if you initially believed that ϕ then you cannot be informed that you did not believe that ϕ)[26], $\neg I(\phi \wedge \neg B_1\phi)$ (you cannot be informed that ϕ and that you will not believe that ϕ), etc. In this paper we have focused on

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[24] Furthermore, no formula of the type $I\phi$ or its negation is universally valid. Recall, however, that I trivially satisfies axiom K: $I(\phi \to \psi) \wedge I\phi \to I\psi$.

[25] Proof. Suppose $\mathcal{I}$ is serial and $\neg I(\phi \wedge \neg\phi)$ is not valid, that is, there is a state α and a formula ϕ such that $\alpha \models I(\phi \wedge \neg\phi)$. Then $\mathcal{I}(\alpha) = \|\phi \wedge \neg\phi\|$. But $\|\phi \wedge \neg\phi\| = \varnothing$, while by seriality $\mathcal{I}(\alpha) \neq \varnothing$. Conversely, suppose that $\mathcal{I}$ is not serial. Then there exists a state α such that $\mathcal{I}(\alpha) = \varnothing$. Since, for every formula ϕ, $\|\phi \wedge \neg\phi\| = \varnothing$, it follows that $\alpha \models I(\phi \wedge \neg\phi)$ so that $\neg I(\phi \wedge \neg\phi)$ is not valid.

Note that, given the non-standard validation rule for $I\phi$, the equivalence of axiom D ($I\phi \to \neg I\neg\phi$) and seriality breaks down. It is still true that if $\mathcal{I}$ is serial then the axiom $I\phi \to \neg I\neg\phi$ is valid, but the converse is not true. Proof of the first part: assume seriality and suppose that the axiom is not valid, i.e. there is a formula ϕ such that $\alpha \models I\phi \wedge I\neg\phi$. Then $\mathcal{I}(\alpha) = \|\phi\|$ and $\mathcal{I}(\alpha) = \|\neg\phi\|$. By seriality, there exists a $\beta \in \mathcal{I}(\alpha)$. Then $\beta \models \phi \wedge \neg\phi$, which is impossible. Now, to see that the converse is not true, first note that the truth condition for $I\phi$ is equivalent to

$$\forall\beta,\text{ if } \beta \in \mathcal{I}(\alpha) \text{ then } \beta \models \phi,\ \text{ and } \forall\gamma,\text{ if } \gamma \models \phi \text{ then } \gamma \in \mathcal{I}(\alpha).$$

Thus $\alpha \models \neg I\neg\phi$ iff $\alpha \not\models I\neg\phi$ iff $\mathit{not}(\forall\beta, \beta \in \mathcal{I}(\alpha) \Longrightarrow \beta \models \neg\phi \ \mathit{and}\ \forall\gamma, \gamma \models \neg\phi \Longrightarrow \gamma \in \mathcal{I}(\alpha))$ which is equivalent to

$$\text{either } \exists\beta \in \mathcal{I}(\alpha) \text{ such that } \beta \models \phi \text{ or } \exists\gamma \text{ such that } \gamma \models \neg\phi \text{ and } \gamma \notin \mathcal{I}(\alpha).$$

Now, suppose that $\mathcal{I}(\alpha) = \varnothing$. Then, for every formula ϕ either $\|\phi\| \neq \varnothing$, in which case $\alpha \not\models I\phi$ and therefore $\alpha \models I\phi \to \psi$ for every formula ψ (in particular for $\psi = \neg I\neg\phi$) or $\|\phi\| = \varnothing$, in which case $\alpha \models I\phi$ and, since $\alpha \models \neg\phi$ and $\alpha \notin \mathcal{I}(\alpha)$, $\alpha \models \neg I\neg\phi$. Thus validity of $I\phi \to \neg I\neg\phi$ does not guarantee seriality of $\mathcal{I}$ (let $\mathcal{I}$ be empty everywhere, then the axiom is valid!).

[26] Indeed, one might want to go further and impose memory axioms: $B_0\phi \to B_1B_0\phi$ (if in the past you believed ϕ then later on you remember this) and $\neg B_0\phi \to B_1\neg B_0\phi$ (at a later time you remember what you did *not* believe in the past).

characterization and completeness results and we leave the study of desirable refinements of the proposed logic for future work.

A APPENDIX

In this appendix we prove Propositions 10 and 11. First some preliminaries.

Let $\mathbb{M}$ be the set of maximally consistent sets (MCS) of formulas of $\mathfrak{L}$. Define the following binary relation $\mathcal{A} \subseteq \mathbb{M} \times \mathbb{M}$: $\alpha\mathcal{A}\beta$ if and only if $\{\phi : A\phi \in \alpha\} \subseteq \beta$. Such a relation is well defined (see Chellas, 1984, Theorem 4.30(1), p. 158) and is an equivalence relation because of axioms T_A and 5_A (Chellas, 1984, Theorem 5.13 (2) and (5), p. 175).

Lemma 23 *Let $\alpha, \beta \in \mathbb{M}$ be such that $\alpha\mathcal{A}\beta$ and let ϕ be a formula such that $I\phi \in \alpha$ and $\phi \in \beta$. Then, for every formula ψ, if $I\psi \in \alpha$ then $\psi \in \beta$, that is, $\{\psi : I\psi \in \alpha\} \subseteq \beta$.*

Proof. Suppose that $\alpha\mathcal{A}\beta$, $I\phi \in \alpha$ and $\phi \in \beta$. Fix an arbitrary ψ such that $I\psi \in \alpha$. Then $I\phi \wedge I\psi \in \alpha$. Since $(I\phi \wedge I\psi) \to A(\phi \leftrightarrow \psi)$ is a theorem, it belongs to every MCS, in particular to α. Hence $A(\phi \leftrightarrow \psi) \in \alpha$. Then, since $\alpha\mathcal{A}\beta$, $\phi \leftrightarrow \psi \in \beta$. Since $\phi \in \beta$, it follows that $\psi \in \beta$. ■

Similarly to the definition of $\mathcal{A}$, let the binary relations $\mathcal{B}_0$ and $\mathcal{B}_1$ on $\mathbb{M}$ be defined as follows: $\alpha\mathcal{B}_0\beta$ if and only if $\{\phi : B_0\phi \in \alpha\} \subseteq \beta$ and $\alpha\mathcal{B}_1\beta$ if and only if $\{\phi : B_1\phi \in \alpha\} \subseteq \beta$. It is straightforward to show that, because of axioms Incl_0 and Incl_1, both $\mathcal{B}_0$ and $\mathcal{B}_1$ are subrelations of $\mathcal{A}$, that is, $\alpha\mathcal{B}_0\beta$ implies $\alpha\mathcal{A}\beta$ and $\alpha\mathcal{B}_1\beta$ implies $\alpha\mathcal{A}\beta$.

Let ω_0 be an arbitrary object such that $\omega_0 \notin \mathbb{M}$, that is, ω_0 can be anything but a MCS. Define the following relation $\mathcal{I}$ on $\mathbb{M} \cup \{\omega_0\}$: $\alpha\mathcal{I}\beta$ if and only if

> *either* $\alpha\mathcal{A}\beta$ (thus $\alpha, \beta \in \mathbb{M}$) and, for some ϕ, $I\phi \in \alpha$ and $\phi \in \beta$
> *or* $\alpha \in \mathbb{M}$, $\beta = \omega_0$ and, for all ϕ, $I\phi \notin \alpha$.

Definition 24 *An augmented frame is a quintuple $\langle\Omega, \mathcal{B}_0, \mathcal{B}_1, \mathcal{I}, \mathcal{A}\rangle$ obtained by adding an equivalence relation $\mathcal{A}$ to a regular frame $\langle\Omega, \mathcal{B}_0, \mathcal{B}_1, \mathcal{I}\rangle$ with the additional requirements that $\mathcal{B}_0 \subseteq \mathcal{A}$ and $\mathcal{B}_1 \subseteq \mathcal{A}$.*

The structure $\langle\mathbb{M} \cup \{\omega_0\}, \mathcal{B}_0, \mathcal{B}_1, \mathcal{I}, \mathcal{A}\rangle$ defined above is an augmented frame. For every $\alpha \in \mathbb{M}$, let $\mathcal{A}(\alpha) = \{\omega \in \mathbb{M} : \alpha\mathcal{A}\omega\}$. Consider the canonical model based on this frame defined by $\|p\| = \{\omega \in \mathbb{M} : p \in \omega\}$, for every atomic proposition p. For every formula ϕ define $\|\phi\|$ according to the semantic rules given in Section 2, with the following modified truth conditions for the operators I and A: $\alpha \models I\phi$ if and only if $\mathcal{I}(\alpha) = \|\phi\| \cap \mathcal{A}(\alpha)$ and $\alpha \models A\phi$ if and only if $\mathcal{A}(\alpha) \subseteq \|\phi\|$. The proof of the following lemma is along the lines of Goranko and Passy [15] (p. 25).

Lemma 25 *For every formula ϕ, $\|\phi\| = \{\omega \in \mathbb{M} : \phi \in \omega\}$.*

Proof. The proof is by induction on the complexity of ϕ. For the non-modal formulas and for the cases where ϕ is either $B_0\psi$ or $B_1\psi$ or $A\psi$, for some ψ, the proof is standard (see Chellas, 1984, Theorem 5.7, p. 172). That proof makes use of rule of inference RK for the modal operators. Since this rule of inference does not hold for I (see Remark 8), we need a different proof for the case where $\phi = I\psi$ for some ψ. By the induction hypothesis, $\|\psi\| = \{\omega \in \mathbb{M} : \psi \in \omega\}$. We need to show that $\|I\psi\| = \{\omega \in \mathbb{M} : I\psi \in \omega\}$, that is, that

(1) if $\alpha \models I\psi$ (i.e. $\mathcal{I}(\alpha) = \|\psi\| \cap \mathcal{A}(\alpha)$) then $I\psi \in \alpha$, and

(2) if $I\psi \in \alpha$ then $\mathcal{I}(\alpha) = \|\psi\| \cap \mathcal{A}(\alpha)$ (i.e. $\alpha \models I\psi$).

For (1) we prove the contrapositive, namely that if $\alpha \in \mathbb{M}$ and $I\psi \notin \alpha$ then $\mathcal{I}(\alpha) \neq \|\psi\| \cap \mathcal{A}(\alpha)$. Suppose that $\alpha \in \mathbb{M}$ and $I\psi \notin \alpha$. Two cases are possible: (1.a) $I\chi \notin \alpha$ for every formula χ, or (1.b) $I\chi \in \alpha$ for some χ. In case (1.a), by definition of $\mathcal{I}$, $\mathcal{I}(\alpha) = \{\omega_0\}$. Since $\omega_0 \notin \mathbb{M}$ (and $\mathcal{A}(\alpha) \subseteq \mathbb{M}$) it follows that $\mathcal{I}(\alpha) \neq \|\psi\| \cap \mathcal{A}(\alpha)$. In case (1.b) it must be that $(I\chi \rightarrow I\psi) \notin \alpha$ (since $I\psi \notin \alpha$). By axiom I_2, $A(\chi \leftrightarrow \psi) \rightarrow (I\chi \rightarrow I\psi) \in \alpha$. Thus $A(\chi \leftrightarrow \psi) \notin \alpha$. Since α is a MCS, $\neg A(\chi \leftrightarrow \psi) \in \alpha$. Now, $\neg A(\chi \leftrightarrow \psi)$ is propositionally equivalent to $\neg A \neg\neg(\chi \leftrightarrow \psi)$, which in turn is equivalent to $\neg A \neg \left((\chi \wedge \neg\psi) \vee (\psi \wedge \neg\chi)\right)$. Thus this formula belongs to α. Hence there is a β such that $\alpha\mathcal{A}\beta$ and either (1.b.1) $(\chi \wedge \neg\psi) \in \beta$ or (1.b.2) $(\psi \wedge \neg\chi) \in \beta$. In case (1.b.1), $\chi \in \beta$ and $\psi \notin \beta$. By definition of $\mathcal{I}$, since $\alpha\mathcal{A}\beta$ and $I\chi \in \alpha$ and $\chi \in \beta$, we have that $\beta \in \mathcal{I}(\alpha)$ while $\beta \notin \|\psi\|$, since $\psi \notin \beta$ and, by the induction hypothesis, $\|\psi\| = \{\omega \in \mathbb{M} : \psi \in \omega\}$. Thus $\mathcal{I}(\alpha) \neq \|\psi\| \cap \mathcal{A}(\alpha)$. In case (1.b.2), $\chi \notin \beta$ and $\psi \in \beta$, so that, by the induction hypothesis, $\beta \in \|\psi\|$; furthermore, $\beta \in \mathcal{A}(\alpha)$. We want to show that $\beta \notin \mathcal{I}(\alpha)$, so that $\mathcal{I}(\alpha) \neq \|\psi\| \cap \mathcal{A}(\alpha)$. To see this, suppose by contradiction that $\beta \in \mathcal{I}(\alpha)$. Then by definition of $\mathcal{I}$, there is some ζ such that $I\zeta \in \alpha$ and $\zeta \in \beta$. By Lemma 23 $\{\theta : I\theta \in \alpha\} \subseteq \beta$, implying that $\chi \in \beta$, since, by hypothesis, $I\chi \in \alpha$. But this contradicts $\chi \notin \beta$. This completes the proof of (1).

Next we prove (2). Suppose that $I\psi \in \alpha$. First we show that $\|\psi\| \cap \mathcal{A}(\alpha) \subseteq \mathcal{I}(\alpha)$. Fix an arbitrary $\beta \in \|\psi\| \cap \mathcal{A}(\alpha)$. Since $\beta \in \|\psi\|$, by the induction hypothesis, $\psi \in \beta$ and, therefore, by definition of $\mathcal{I}$, $\beta \in \mathcal{I}(\alpha)$. Next we show that $\mathcal{I}(\alpha) \subseteq \|\psi\| \cap \mathcal{A}(\alpha)$. Fix an arbitrary $\beta \in \mathcal{I}(\alpha)$. By definition of $\mathcal{I}$, $\beta \in \mathcal{A}(\alpha)$ and there exists a χ such that $I\chi \in \alpha$ and $\chi \in \beta$. By Lemma 23, $\{\theta : I\theta \in \alpha\} \subseteq \beta$ and therefore, since $I\psi \in \alpha$, $\psi \in \beta$. By the induction hypothesis, $\|\psi\| = \{\omega \in \mathbb{M} : \psi \in \omega\}$. Thus $\beta \in \|\psi\| \cap \mathcal{A}(\alpha)$. ■

Proposition 26 *Logic $\mathfrak{L}$ is sound and complete with respect to the class of augmented frames $\langle \Omega, \mathcal{B}_0, \mathcal{B}_1, \mathcal{I}, \mathcal{A} \rangle$ under the semantic rules given in Section 2, with the following modified truth conditions for the operators I and A: $\alpha \models I\phi$ if and only if $\mathcal{I}(\alpha) = \|\phi\| \cap \mathcal{A}(\alpha)$ and $\alpha \models A\phi$ if and only if $\mathcal{A}(\alpha) \subseteq \|\phi\|$, where $\mathcal{A}(\alpha) = \{\omega \in \Omega : \alpha\mathcal{A}\omega\}$.*

Proof. (A) SOUNDNESS. It is straightforward to show that the inference rules MP and NEC_A are validity preserving and axioms K_0, K_1, K_A, T_A, 5_A, Incl_0 and Incl_1, are valid in all augmented frames. Thus we only show that axioms I_1 and I_2 are valid in all augmented frames.

1. Validity of axiom I_1: $I\phi \wedge I\psi \rightarrow A(\phi \leftrightarrow \psi)$. Fix an arbitrary model, and suppose that $\alpha \models I\phi \wedge I\psi$. Then $\mathcal{I}(\alpha) = \|\phi\| \cap \mathcal{A}(\alpha)$ and $\mathcal{I}(\alpha) = \|\psi\| \cap \mathcal{A}(\alpha)$. Thus $\|\phi\| \cap \mathcal{A}(\alpha) = \|\psi\| \cap \mathcal{A}(\alpha)$ and hence $\mathcal{A}(\alpha) \subseteq \|\phi \leftrightarrow \psi\|$, yielding $\alpha \models A(\phi \leftrightarrow \psi)$.

2. Validity of axiom I_1: $A(\phi \leftrightarrow \psi) \rightarrow (I\phi \leftrightarrow I\psi)$. Fix an arbitrary model and suppose that $\alpha \models A(\phi \leftrightarrow \psi)$. Then $\mathcal{A}(\alpha) \subseteq \|\phi \leftrightarrow \psi\|$ and, therefore, $\|\phi\| \cap \mathcal{A}(\alpha) = \|\psi\| \cap \mathcal{A}(\alpha)$. Thus, $\alpha \models I\phi$ if and only if $\mathcal{I}(\alpha) = \|\phi\| \cap \mathcal{A}(\alpha)$ if and only if $\mathcal{I}(\alpha) = \|\psi\| \cap \mathcal{A}(\alpha)$, if and only if $\alpha \models I\psi$. Hence $\alpha \models I\phi \leftrightarrow I\psi$.

(B) COMPLETENESS. Let ϕ be a formula that is valid in all augmented frames. Then ϕ is valid in the canonical structure $\langle \mathbb{M} \cup \{\omega_0\}, \mathcal{B}_0, \mathcal{B}_1, \mathcal{I}, \mathcal{A} \rangle$ defined above, which is an augmented frame. Thus ϕ is valid in the canonical model based on this frame. By Lemma 25, for every formula ψ, $\|\psi\| = \{\omega \in \mathbb{M} : \psi \in \omega\}$. Thus ϕ belongs to every MCS and therefore is a theorem of $\mathfrak{L}$ (Chellas, 1984, Theorem 2.20, p. 57). ■

To prove Proposition 10, namely that logic $\mathfrak{L}$ is sound and complete with respect to the class of frames $\langle \Omega, \mathcal{B}_0, \mathcal{B}_1, \mathcal{I} \rangle$, we only need to invoke the result (Chellas, 1984, Theorem 3.12, p. 97) that soundness and completeness with respect to the class of augmented frames (where $\mathcal{A}$ is an equivalence relation) implies soundness and completeness with respect to the generated sub-frames (where $\mathcal{A}$ is the universal relation). The latter are precisely what we called frames. In a frame where the relation $\mathcal{A}$ is the universal relation the semantic rule $\alpha \models I\phi$ if and only if $\mathcal{I}(\alpha) = \|\phi\| \cap \mathcal{A}(\alpha)$ becomes $\alpha \models I\phi$ if and only if $\mathcal{I}(\alpha) = \|\phi\|$ and the semantic rule $\alpha \models A\phi$ if and only if $\mathcal{A}(\alpha) \subseteq \|\phi\|$ becomes $\alpha \models A\phi$ if and only if $\|\phi\| = \Omega$, since $\mathcal{A}(\alpha) = \Omega$.

Next we turn to the proof of Proposition 11, namely that logic $\mathfrak{R}$ is sound and complete with respect to the class of frames $\langle \Omega, \mathcal{B}_0, \mathcal{B}_1, \mathcal{I} \rangle$ that satisfy the Qualitative Bayes Rule (QBR).

Proof. (A) SOUNDNESS. This follows from Propositions 3 and 10.

(B) COMPLETENESS. By Proposition 10 we only need to show that the frame associated with the canonical model is a QBR frame. First we show that

$$\forall \omega \in \mathbb{M}, \quad if \;\; \mathcal{B}_0(\omega) \cap \mathcal{I}(\omega) \neq \varnothing \;\; then \;\; \mathcal{B}_1(\omega) \subseteq \mathcal{I}(\omega). \tag{5}$$

Let $\beta \in \mathcal{B}_0(\alpha) \cap \mathcal{I}(\alpha)$. Since $\mathcal{B}_0(\omega) \subseteq \mathbb{M}$, $\beta \in \mathbb{M}$ and therefore, by definition of $\mathcal{I}$, there exists a formula ϕ such that $I\phi \in \alpha$ and $\phi \in \beta$. Since $\beta \in \mathcal{B}_0(\alpha)$, $\neg B_0 \neg \phi \in \alpha$ (Chellas, 1984, Theorem 5.6, p. 172). Thus $(I\phi \wedge \neg B_0 \neg \phi) \in \alpha$. Since Qualified Acceptance is a theorem, $(I\phi \wedge \neg B_0 \neg \phi) \rightarrow B_1\phi \in \alpha$. Thus $B_1\phi \in \alpha$. We want to show that $\mathcal{B}_1(\alpha) \subseteq \mathcal{I}(\alpha)$. Fix an arbitrary $\gamma \in \mathcal{B}_1(\alpha)$. Then, by definition of $\mathcal{B}_1$, $\{\psi : B_1\psi \in \alpha\} \subseteq \gamma$. In particular, since $B_1\phi \in \alpha$, $\phi \in \gamma$. By definition of $\mathcal{I}$, since $I\phi \in \alpha$ and $\phi \in \gamma$, $\gamma \in \mathcal{I}(\alpha)$.

Next we show that

$$\forall \omega \in \mathbb{M}, \quad if \;\; \mathcal{B}_0(\omega) \cap \mathcal{I}(\omega) \neq \varnothing \;\; then \;\; \mathcal{B}_1(\omega) \subseteq \mathcal{B}_0(\omega). \tag{6}$$

Let $\beta \in \mathcal{B}_0(\alpha) \cap \mathcal{I}(\alpha)$. As shown above, there exists a ϕ such that $I\phi \in \alpha$, $\phi \in \beta$ and $\neg B_0 \neg\phi \in \alpha$. By Persistence, for every formula ψ, $(I\phi \wedge \neg B_0 \neg\phi) \rightarrow (B_0\psi \rightarrow B_1\psi) \in \alpha$. Thus

$$(B_0\psi \rightarrow B_1\psi) \in \alpha. \tag{7}$$

Fix an arbitrary $\gamma \in \mathcal{B}_1(\alpha)$. Then, by definition of $\mathcal{B}_1$, $\{\psi : B_1\psi \in \alpha\} \subseteq \gamma$. We want to show that $\gamma \in \mathcal{B}_0(\alpha)$, that is, that $\{\psi : B_0\psi \in \alpha\} \subseteq \gamma$. Let ψ be such that $B_0\psi \in \alpha$. By (7) $B_1\psi \in \alpha$ and therefore $\psi \in \gamma$.

Finally we show that

$$\forall\omega \in \mathbb{M}, \quad \mathcal{B}_0(\omega) \cap \mathcal{I}(\omega) \subseteq \mathcal{B}_1(\omega). \tag{8}$$

Fix an arbitrary $\alpha \in \mathbb{M}$. If $\mathcal{B}_0(\alpha) \cap \mathcal{I}(\alpha) = \varnothing$, there is nothing to prove. Suppose therefore that $\beta \in \mathcal{B}_0(\alpha) \cap \mathcal{I}(\alpha)$ for some β. Then there exists a ϕ such that $I\phi \in \alpha$ and $\phi \in \beta$. Fix an arbitrary $\gamma \in \mathcal{B}_0(\alpha) \cap \mathcal{I}(\alpha)$. We want to show that $\gamma \in \mathcal{B}_1(\alpha)$, that is, that $\{\psi : B_1\psi \in \alpha\} \subseteq \gamma$. Let ψ be an arbitrary formula such that $B_1\psi \in \alpha$. Then $(I\phi \wedge B_1\psi) \in \alpha$. By Minimality, $(I\phi \wedge B_1\psi) \rightarrow B_0(\phi \rightarrow \psi) \in \alpha$. Thus $B_0(\phi \rightarrow \psi) \in \alpha$. Since $\gamma \in \mathcal{B}_0(\alpha)$, $(\phi \rightarrow \psi) \in \gamma$. Since $I\phi \in \alpha$, $\mathcal{I}(\alpha) = \|\phi\|$. Thus, since $\gamma \in \mathcal{I}(\alpha)$, $\gamma \models \phi$ and, by Lemma 25, $\phi \in \gamma$. It follows from this and $(\phi \rightarrow \psi) \in \gamma$ that $\psi \in \gamma$. ■

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