Speculative Trade under Unawareness: The Infinite Case*

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Abstract

We generalize the "No-trade" theorem for finite unawareness belief structures in Heifetz, Meier, and Schipper (2009) to the infinite case.

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JEL-Classifications: C70, C72, D53, D80, D82.

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1 Introduction

Unawareness refers to the lack of conception rather than to the lack of information. It is natural to presume that asymmetric unawareness may lead to speculative trade. Indeed, in Heifetz, Meier, and Schipper (2009) we present a simple example of speculation under unawareness in which there is common certainty of willingness to trade but agents have a strict preference to trade despite the existence of a common prior.[1] This is impossible in standard state-space structures with a common prior. In standard "No Trade" theorems, if there is common certainty of willingness to trade, then agents are necessarily indifferent to trade (Milgrom and Stokey, 1982). Somewhat surprising, in Heifetz, Meier, and Schipper (2009) we also prove a "No-trade" result according to which under a common prior there can not be common certainty of strict preference to trade. This means that arbitrary small transaction costs rule out speculation under asymmetric unawareness. The "No-trade" result in Heifetz, Meier, and Schipper (2009) has been stated for finite unawareness belief structures. In this note we generalize the result to infinite unawareness belief structures. Such a generalization is relevant since the space of underlying uncertainties may be large. Especially if it is large, agents may be unaware of some of them. Moreover, the generalization serves as a robustness check for our "No-trade" result for finite unawareness belief structures. It shows that the result in Heifetz, Meier, and Schipper (2009) is not an artefact of the finiteness assumption but holds more generally.

Recently we learned that Board and Chung (2009) present a different model of unawareness in which they also study speculative trade under what they term living in "denial" and "paranoia". They consider only finite spaces. The precise connection between our result and their result is yet to be explored.

The paper is organized as follows. The next section introduces topological unawareness belief structures. The general "No-trade" theorem is stated in Section 3. Finally, Section 4 contains the proof of the theorem.

[1] The example in Heifetz, Meier, and Schipper (2009) is a probabilistic version of the speculation example in Heifetz, Meier, and Schipper (2006). Unawareness belief structures allow us to state the common prior assumption.

2 Topological Unawareness Belief Structures

We consider an unawareness belief structure as defined in Heifetz, Meier, and Schipper (2009) but with additional topological properties.

2.1 Compact Hausdorff State-Spaces

Let $\mathcal{S} = \{S_\alpha\}_{\alpha \in \mathcal{A}}$ be a complete lattice of disjoint *state-spaces*, with the partial order $\preceq$ on $\mathcal{S}$. If S_α and S_β are such that $S_\alpha \succeq S_\beta$ we say that "S_α is more expressive than S_β – states of S_α describe situations with a richer vocabulary than states of S_β ".[2] $(\mathcal{S}, \preceq)$ is well-founded, that is, every non-empty subset $\mathcal{X} \subseteq \mathcal{S}$ contains a $\preceq$-minimal element. (That is, there is a $S' \in \mathcal{X}$ such that for all $S \in \mathcal{X}$: if $S \preceq S'$, then $S = S'$.) Each state-space $S \in \mathcal{S}$ is a non-empty compact Hausdorff space with a Borel σ-field $\mathcal{F}_S$. Denote by $\Omega = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$ the union of these spaces. Ω is endowed with the disjoint-union topology: $O \subseteq \Omega$ is open if and only if $O \cap S$ is open in S for all $S \in \mathcal{S}$.

Spaces in the lattice can be more or less "rich" in terms of facts that may or may not obtain in them. The partial order relates to the "richness" of spaces. The upmost space of the lattice may be interpreted as the "objective" state-space. Its states encompass full descriptions.

2.2 Continuous Projections

For every S and S' such that $S' \succeq S$, there is a continuous surjective projection $r_S^{S'} : S' \to S$, where r_S^S is the identity. ("$r_S^{S'}(\omega)$ is the restriction of the description ω to the more limited vocabulary of S.") Note that the cardinality of S is smaller than or equal to the cardinality of S'. We require the projections to commute: If $S'' \succeq S' \succeq S$ then $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$. If $\omega \in S'$, denote $\omega_S = r_S^{S'}(\omega)$. If $D \subseteq S'$, denote $D_S = \{\omega_S : \omega \in D\}$.

Projections "translate" states in "more expressive" spaces to states in "less expressive" spaces by "erasing" facts that can not be expressed in a lower space.

[2]Here and in what follows, phrases within quotation marks hint at intended interpretations, but we emphasize that these interpretations are not part of the definition of the set-theoretic structure.

2.3 Events

Denote $g(S) = \{S' : S' \succeq S\}$. For $D \subseteq S$, denote $D^{\uparrow} = \bigcup_{S' \in g(S)} \left(r_S^{S'}\right)^{-1}(D)$. ("All the extensions of descriptions in D to at least as expressive vocabularies.")

An *event* is a pair (E, S), where $E = D^{\uparrow}$ with $D \subseteq S$, where $S \in \mathcal{S}$. D is called the *base* and S the *base-space* of (E, S), denoted by $S(E)$. If $E \neq \emptyset$, then S is uniquely determined by E and, abusing notation, we write E for (E, S). Otherwise, we write $\emptyset^S$ for $(\emptyset, S)$. Note that not every subset of Ω is an event.

Some fact may obtain in a subset of a space. Then this fact should be also "expressible" in "more expressive" spaces. Therefore the event contains not only the particular subset but also its inverse images in "more expressive" spaces.

Let Σ be the set of *measurable events* of Ω, i.e., $D^{\uparrow}$ such that $D \in \mathcal{F}_S$, for some state-space $S \in \mathcal{S}$. Note that unless $\mathcal{S}$ is a singleton, Σ is not an algebra because it contains distinct $\emptyset^S$ for all $S \in \mathcal{S}$.

2.4 Negation

If $(D^{\uparrow}, S)$ is an event where $D \subseteq S$, the negation $\neg(D^{\uparrow}, S)$ of $(D^{\uparrow}, S)$ is defined by $\neg(D^{\uparrow}, S) := ((S \setminus D)^{\uparrow}, S)$. Note, that by this definition, the negation of a (measurable) event is a (measurable) event. Abusing notation, we write $\neg D^{\uparrow} := (S \setminus D)^{\uparrow}$. Note that by our notational convention, we have $\neg S^{\uparrow} = \emptyset^S$ and $\neg \emptyset^S = S^{\uparrow}$, for each space $S \in \mathcal{S}$. The event $\emptyset^S$ should be interpreted as a "logical contradiction phrased with the expressive power available in S." $\neg D^{\uparrow}$ is typically a proper subset of the complement $\Omega \setminus D^{\uparrow}$. That is, $(S \setminus D)^{\uparrow} \subsetneqq \Omega \setminus D^{\uparrow}$.

Intuitively, there may be states in which the description of an event $D^{\uparrow}$ is both expressible and valid – these are the states in $D^{\uparrow}$; there may be states in which its description is expressible but invalid – these are the states in $\neg D^{\uparrow}$; and there may be states in which neither its description nor its negation are expressible – these are the states in

$$\Omega \setminus \left(D^{\uparrow} \cup \neg D^{\uparrow}\right) = \Omega \setminus S\left(D^{\uparrow}\right)^{\uparrow}.$$

2.5 Conjunction and Disjunction

If $\left\{\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right)\right\}_{\lambda \in L}$ is a finite or countable collection of events (with $D_{\lambda} \subseteq S_{\lambda}$, for $\lambda \in L$), their conjunction $\bigwedge_{\lambda \in L}\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right)$ is defined by $\bigwedge_{\lambda \in L}\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right) := \left(\left(\bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}\right), \sup_{\lambda \in L} S_{\lambda}\right)$.

Note, that since $\mathcal{S}$ is a *complete* lattice, $\sup_{\lambda\in L} S_\lambda$ exists. If $S = \sup_{\lambda\in L} S_\lambda$, then we have $\left(\bigcap_{\lambda\in L} D_\lambda^\uparrow\right) = \left(\bigcap_{\lambda\in L}\left(\left(r_{S_\lambda}^S\right)^{-1}(D_\lambda)\right)\right)^\uparrow$. Again, abusing notation, we write $\bigwedge_{\lambda\in L} D_\lambda^\uparrow := \bigcap_{\lambda\in L} D_\lambda^\uparrow$ (we will therefore use the conjunction symbol $\wedge$ and the intersection symbol $\cap$ interchangeably).

We define the relation $\subseteq$ between events (E,S) and (F,S'), by $(E,S)\subseteq(F,S')$ if and only if $E\subseteq F$ as sets *and* $S'\preceq S$. If $E\neq\emptyset$, we have that $(E,S)\subseteq(F,S')$ if and only if $E\subseteq F$ as sets. Note however that for $E=\emptyset^S$ we have $(E,S)\subseteq(F,S')$ if and only if $S'\preceq S$. Hence we can write $E\subseteq F$ instead of $(E,S)\subseteq(F,S')$ as long as we keep in mind that in the case of $E=\emptyset^S$ we have $\emptyset^S\subseteq F$ if and only if $S\succeq S(F)$. It follows from these definitions that for events E and F, $E\subseteq F$ is equivalent to $\neg F\subseteq\neg E$ only when E and F have the same base, i.e., $S(E)=S(F)$.

The disjunction of $\left\{D_\lambda^\uparrow\right\}_{\lambda\in L}$ is defined by the de Morgan law $\bigvee_{\lambda\in L} D_\lambda^\uparrow = \neg\left(\bigwedge_{\lambda\in L}\neg\left(D_\lambda^\uparrow\right)\right)$. Typically $\bigvee_{\lambda\in L} D_\lambda^\uparrow \subsetneqq \bigcup_{\lambda\in L} D_\lambda^\uparrow$, and if all D_λ are nonempty we have that $\bigvee_{\lambda\in L} D_\lambda^\uparrow = \bigcup_{\lambda\in L} D_\lambda^\uparrow$ holds if and only if all the $D_\lambda^\uparrow$ have the same base-space. Note, that by these definitions, the conjunction and disjunction of (at most countably many measurable) events is a (measurable) event.

Apart from the topological conditions, the event-structure outlined so far is analogous to Heifetz, Meier, and Schipper (2006, 2008, 2009).

2.6 Regular Borel Probability Measures

Here and in what follows, the term 'events' always means measurable events in Σ unless otherwise stated.

For each $S\in\mathcal{S}$, $\Delta(S)$ is the set of *regular* Borel probability measures on $(S,\mathcal{F}_S)$. We consider this set itself as a measurable space which is endowed with the topology of weak convergence.[3]

[3] This topology is generated by the sub-basis of sets of the form

$$\{\mu\in\Delta(S):\mu(O)>r\}$$

where $O\subseteq S$ is open and $r\in\mathbb{R}$ (see e.g. Billingsley (1968), appendix III). When S is Normal (and in particular compact and/or metric), this topology coincides with the weak* topology - the weakest topology for which the mapping

$$\mu\longrightarrow\int_S f\,d\mu$$

is continuous for every continuous real-valued function f on S.

$\bigcup_{S\in\mathcal{S}} \Delta(S)$ is endowed with the disjoint-union topology: $O_\Delta \subseteq \bigcup_{S\in\mathcal{S}} \Delta(S)$ is open if and only if $O_\Delta \cap \Delta(S)$ is open in $\Delta(S)$ for all $S \in \mathcal{S}$.

Note that although each S and each $\Delta(S)$ are compact, if $\mathcal{S}$ is infinite, Ω and $\bigcup_{S\in\mathcal{S}} \Delta(S)$ are *not* compact.

2.7 Marginals

For a probability measure $\mu \in \Delta(S')$, the marginal $\mu_{|S}$ of μ on $S \preceq S'$ is defined by

$$\mu_{|S}(D) := \mu\left(\left(r_S^{S'}\right)^{-1}(D)\right), \quad D \in \mathcal{F}_S.$$

Let S_μ be the space on which μ is a probability measure. Whenever $S_\mu \succeq S(E)$ then we abuse notation slightly and write

$$\mu(E) = \mu(E \cap S_\mu).$$

If $S(E) \npreceq S_\mu$, then we say that $\mu(E)$ is undefined.

2.8 Continuous Type Mappings

Let I be a nonempty finite or countable set of individuals. For every individual, each state gives rise to a probabilistic belief over states in some space.

Definition 1 *For each individual $i \in I$ there is a continuous type mapping $t_i : \Omega \to \bigcup_{\alpha\in\mathcal{A}} \Delta(S_\alpha)$.*

We require the type mapping t_i to satisfy the following properties:

(0) Confinement: If $\omega \in S'$ then $t_i(\omega) \in \triangle(S)$ for some $S \preceq S'$.

(1) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S)$ then $t_i(\omega_{S'}) = t_i(\omega)$.

(2) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S')$ then $t_i(\omega_S) = t_i(\omega)_{|S}$.

(3) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega_{S'}) \in \triangle(S)$ then $S_{t_i(\omega)} \succeq S$.

$t_i(\omega)$ represents individual i's belief at state ω. Properties (0) to (3) guarantee the consistent fit of beliefs and awareness at different state-spaces. *Confinement* means that at any given state $\omega \in \Omega$ an individual's belief is concentrated on states that are all described with the same "vocabulary" - the "vocabulary" available to the individual at ω. This "vocabulary" may be less expressive than the "vocabulary" used to describe statements in the state ω."

Properties (1) to (3) compare the types of an individual in a state ω and its projection to ω_S. Property (1) and (2) mean that at the projected state ω_S the individual believes everything she believes at ω given that she is aware of it at ω_S. Property (3) means that at ω an individual can not be unaware of an event that she is aware of at the projected state ω_S.

Define[4]

$$Ben_i(\omega) := \left\{ \omega' \in \Omega : t_i(\omega')_{|S_{t_i(\omega)}} = t_i(\omega) \right\}.$$

This is the set of states at which individual i's type or the marginal thereof coincides with her type at ω. Such sets are events in our structure:

Remark 1 *For any $\omega \in \Omega$, $Ben_i(\omega)$ is an $S_{t_i(\omega)}$-based event, which is not necessarily measurable.*[5]

Assumption 1 *If $Ben_i(\omega) \subseteq E$, for an event E, then $t_i(\omega)(E) = 1$.*

This assumption implies introspection (Property (va) in Proposition 9 in Heifetz, Meier, and Schipper, 2009). Note, that if $Ben_i(\omega)$ is measurable, then Assumption 1 implies $t_i(\omega)(Ben_i(\omega)) = 1$.

Definition 2 *We denote by $\underline{\Omega} := \left\langle \mathcal{S}, \left(r_{S_\beta}^{S_\alpha} \right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I} \right\rangle$ an topological unawareness belief structure.*

Topological unawareness belief structures are analogous to unawareness belief structures in Heifetz, Meier, and Schipper (2009) except for the additional topological properties.

3 A Generalized "No-Trade" Theorem

Definition 3 (Prior) *A prior for player i is a system of probability measures $P_i = \left(P_i^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ such that*

[4]The name "Ben" is chosen analogously to the "ken" in knowledge structures.

[5]Even in a standard type-space, if the σ-algebra is not countably generated, then the set of states where a player is of a certain type might not be measurable.

1. *The system is projective: If $S' \preceq S$ then the marginal of P_i^S on S' is $P_i^{S'}$. (That is, if $E \in \Sigma$ is an event whose base-space $S(E)$ is lower or equal to S', then $P_i^S(E) = P_i^{S'}(E)$.)*

2. *Each probability measure P_i^S is a convex combination of i's beliefs in S: For every event $E \in \Sigma$ such that $S(E) \preceq S$,*

$$P_i^S\left(E \cap S \cap A_i(E)\right) = \int_{S \cap A_i(E)} t_i(\cdot)(E)\, dP_i^S(\cdot). \tag{1}$$

We call any probability measure $\mu_i \in \Delta(S)$ satisfying equation (1) in place of P_i^S a prior of player i on S.

Definition 4 (Common Prior) *$P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ (resp. $P^S \in \Delta(S)$) is a common prior (resp. a common prior on S) if P (resp. P^S) is a prior (resp. a prior on S) for every player $i \in I$.*

Denote by $[t_i(\omega)] := \{\omega' \in \Omega : t_i(\omega') = t_i(\omega)\}$.

Definition 5 *A common prior $P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ (resp. a common prior P^S on S) is* positive *if and only if for all $i \in I$ and $\omega \in \Omega$: If $t_i(\omega) \in \triangle(S')$, for some S', then $P^S\left(\left([t_i(\omega)] \cap S'\right)^{\uparrow} \cap S\right) > 0$ for all $S \succeq S'$.*

Note that by Lemma 3 below, $[t_i(\omega)] \cap S' \in \mathcal{F}_{S'}$.

Recall Remark 8 in Heifetz, Meier, and Schipper (2009) according to which if $\hat{S}$ is the upmost state-space in the lattice $\mathcal{S}$, and $(P_i^S)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ is a tuple of probability measures, then $(P_i^S)_{S \in \mathcal{S}}$ is a prior for player i if and only if $P_i^{\hat{S}}$ is a prior for player i on $\hat{S}$ and P_i^S is the marginal of $P_i^{\hat{S}}$ for every $S \in \mathcal{S}$.

Definition 6 *Let x_1 and x_2 be real numbers and v a continuous random variable on Ω. Define the sets $E_1^{\leq x_1} := \left\{\omega \in \Omega : \int_{S_{t_1(\omega)}} v(\cdot)\, d(t_1(\omega))(\cdot) \leq x_1\right\}$ and $E_2^{\geq x_2} := \left\{\omega \in \Omega : \int_{S_{t_2(\omega)}} v(\cdot)\, d(t_2(\omega))(\cdot) \geq x_2\right\}$. We say that at ω, conditional on his information, player 1 (resp. player 2) believes that the expectation of v is weakly below x_1 (resp. weakly above x_2) if and only if $\omega \in E_1^{\leq x_1}$ (resp. $\omega \in E_1^{\geq x_2}$).*

Theorem 1 *Let $\underline{\Omega}$ be a topological unawareness belief structure and P a positive common prior. Then there is no state $\tilde{\omega} \in \Omega$ such that there are a continuous random variable*

$v : \Omega \longrightarrow \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, with the following property: at $\tilde{\omega}$ it is common certainty that conditional on her information, player 1 believes that the expectation of v is weakly below x_1 and, conditional on his information, player 2 believes that the expectation of v is weakly above x_2.

This general "No-trade" theorem implies our "No-trade" theorem for finite unawareness belief structures (Heifetz, Meier, and Schipper, 2009).

In Heifetz, Meier, and Schipper (2009) we show by example that the converse of the "No-trade" theorem does not hold.

4 Proof of Theorem 1

4.1 Preliminary Definitions and Results

For $i \in I$, $p \in [0, 1]$ and an event E, the *p -belief operator* is defined by

$$B_i^p(E) := \{\omega \in \Omega : t_i(\omega)(E) \geq p\},$$

if there is a state ω such that $t_i(\omega)(E) \geq p$, and by

$$B_i^p(E) := \emptyset^{S(E)}$$

otherwise. The mutual p-belief operator on events is defined by

$$B^p(E) = \bigcap_{i \in I} B_i^p(E).$$

The common certainty operator on events is defined by

$$CB^1(E) = \bigcap_{n=1}^{\infty} \left(B^1\right)^n (E).$$

These are standard definitions (e.g. see Monderer and Samet, 1989) adapted to our unawareness structures.

As in Heifetz, Meier, and Schipper (2009) we define for every $i \in I$ the awareness operator

$$A_i(E) := \{\omega \in \Omega : t_i(\omega) \in \Delta(S) \text{ for some } S \succeq S(E)\},$$

for every event E, if there is a state ω such that $t_i(\omega) \in \Delta(S)$ with $S \succeq S(E)$, and by

$$A_i(E) := \emptyset^{S(E)}$$

otherwise.

In Heifetz, Meier, and Schipper (2009, Proposition 1 and 2) we show that $A_i(E)$, $B_i^p(E)$, $B^p(E)$, and $CB^1(E)$ are all $S(E)$-based events. We also show in Heifetz, Meier, and Schipper (2009, Proposition 9) that standard properties of belief obtain. Moreover, in Heifetz, Meier, and Schipper (2009, Proposition 3) we show "standard" properties of awareness. One of those properties is *weak necessitation*, i.e., for any event $E \in \Sigma$, $A_i(E) = B_i^1(S(E)^{\uparrow})$. This property will be used later in the proof.

Definition 7 *An event E is evident if for each $i \in I$, $E \subseteq B_i^1(E)$.*

Proposition 1 *For every event $F \in \Sigma$:*

(i) $CB^1(F)$ is evident, that is $CB^1(F) \subseteq B_i^1(CB^1(F))$ for all $i \in I$.

(ii) There exists an evident event E such that $\omega \in E$ and $E \subseteq B_i^1(F)$ for all $i \in I$, if and only if $\omega \in CB^1(F)$.

The proof is analogous to Proposition 3 in Monderer and Samet (1989) for a standard state-space and thus omitted.

We define $G \subseteq \Omega$ to be a *measurable set* if and only if for all $S \in \mathcal{S}$, $G \cap S \in \mathcal{F}_S$. The collection of measurable sets forms a sigma-algebra on Ω.

Let $\underline{\Omega}$ be an unawareness belief structure. As in Heifetz, Meier, and Schipper (2009, Section 2.13), we define the *flattened type-space* associated with the unawareness belief structure $\underline{\Omega}$ by

$$F(\underline{\Omega}) := \langle \Omega, \mathcal{F}, (t_i^F)_{i \in I} \rangle,$$

where Ω is the union of all state-spaces in the unawareness belief structure $\underline{\Omega}$, $\mathcal{F}$ is the collection of all measurable sets in $\underline{\Omega}$, and $t_i^F : \Omega \longrightarrow \Delta(\Omega, \mathcal{F})$ is defined by

$$t_i^F(\omega)(E) \quad := \quad \begin{cases} t_i(\omega)(E \cap S_{t_i(\omega)}) & \text{if } E \cap S_{t_i(\omega)} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

The definition of the belief operator as well as standard properties of belief and Proposition 1 can be extended to measurable subsets of Ω. The proofs are analogous and thus omitted.

Let $\underline{\Omega}$ be a topological unawareness belief structure and P a positive common prior. For the proof of the theorem, we have to show that there is no evident measurable set $E \in \mathcal{F}$ such that $\tilde{\omega} \in E$ and

$$\int_{\Omega} v(\cdot) d(t_1(\omega))(\cdot) \leq x_1 < x_2 \leq \int_{\Omega} v(\cdot) d(t_2(\omega))(\cdot)$$

for all $\omega \in E$.

We need the following lemmata:

Lemma 1 *Let* $\underline{\Omega}$ *be a topological unawareness belief structure,* $v : \Omega \longrightarrow \mathbb{R}$ *be a continuous random variable, and* $x \in \mathbb{R}$*. Then* $\left\{\omega \in \Omega : \int_{\Omega} v(\cdot) d(t_i(\omega))(\cdot) \geq x\right\}$ *and* $\left\{\omega \in \Omega : \int_{\Omega} v(\cdot) d(t_i(\omega))(\cdot) \leq x\right\}$ *are closed subsets of* Ω.[6]

Proof of the Lemma. Since for every $S \in \mathcal{S}$, the topology on $\Delta(S)$ coincides with the weak* topology and since in particular, $v : S \longrightarrow \mathbb{R}$ is continuous, $\left\{\mu \in \Delta(S) : \int_S v(\cdot) d\mu(\cdot) < x\right\}$ is open in $\Delta(S)$. Hence $\left\{\nu \in \bigcup_{S \in \mathcal{S}} \Delta(S) : \int_S v(\cdot) d\nu(\cdot) < x\right\}$ is open in $\bigcup_{S \in \mathcal{S}} \Delta(S)$.

By the continuity of $t_i : \Omega \longrightarrow \bigcup_{S \in \mathcal{S}} \Delta(S)$, it follows that $\left\{\omega \in \Omega : \int_{\Omega} v(\cdot) d(t_i(\omega))(\cdot) < x\right\}$ is open in Ω and hence it's relative complement with respect to Ω, $\left\{\omega \in \Omega : \int_{\Omega} v(\cdot) d(t_i(\omega))(\cdot) \geq x\right\}$ is closed in Ω. $\square$

Lemma 2 *Let* $\underline{\Omega}$ *be a topological unawareness belief structure. Let* E *be a closed subset of* Ω*. Then* $CB^1(E)$ *is a closed subset of* Ω*.*

Proof of the Lemma. The relative complement of E with respect of Ω, $\Omega \setminus E$, is open, and hence for every $S \in \mathcal{S}$, $(\Omega \setminus E) \cap S = S \setminus (E \cap S)$ is open in S. Therefore $\{\mu \in \Delta(S) : \mu(S \setminus (E \cap S)) > 0\}$ is open. It follows that $\bigcup_{S \in \mathcal{S}} \{\mu \in \Delta(S) : \mu(S \setminus (E \cap S)) > 0\}$ is open. Hence for every $i \in I$, $\left\{\omega \in \Omega : t_i(\omega) \in \bigcup_{S \in \mathcal{S}} \{\mu \in \Delta(S) : \mu(S \setminus (E \cap S)) > 0\}\right\}$ is open. It follows that it's relative complement with respect to Ω, $B_i^1(E) = \left\{\omega \in \Omega : t_i(\omega) \in \bigcup_{S \in \mathcal{S}} \{\mu \in \Delta(S) : \mu(E \cap S) = 1\}\right\}$ is closed. Since an arbitrary intersection of closed sets is closed, the Lemma follows by induction. $\square$

Lemma 3 *Let* $\underline{\Omega}$ *be a topological unawareness belief structure. Then for every* $\omega \in \Omega$*, every state-space* $S \in \mathcal{S}$ *and every player* $i \in I$*, the set* $\{\omega' \in \Omega : t_i(\omega') = t_i(\omega)\} \cap S$ *is closed in* S*.*

Proof of the Lemma. Since $\Delta(S_{t_i(\omega)})$ is the set of regular Borel probability measures on $S_{t_i(\omega)}$ endowed with the topology of weak convergence, $\{t_i(\omega)\}$ is closed in $\Delta(S_{t_i(\omega)})$, and hence $\{t_i(\omega)\}$ is closed in $\bigcup_{S \in \mathcal{S}} \Delta(S)$. Therefore, by continuity of t_i, $t_i^{-1}(\{t_i(\omega)\}) = [t_i(\omega)]$ is closed in Ω. Hence, $[t_i(\omega)] \cap S$ is closed in S. $\square$

[6]Note that we abuse notation and write $\int_{\Omega} v(\cdot) d(t_i(\omega))(\cdot)$ instead of $\int_{S_{t_i(\omega)}} v(\cdot) d(t_i(\omega))(\cdot)$.

Lemma 4 *Let $\underline{\Omega}$ be a topological unawareness belief structure. Let P^S be a positive (common) prior on the state-space S, and let $\omega \in S$ such that $t_i(\omega) \in \Delta(S)$. Then, for every $E \in \mathcal{F}_S$, we do have $t_i(\omega)(E) = t_i(\omega)(E \cap [t_i(\omega)]) = \frac{P^S(E \cap [t_i(\omega)])}{P^S(S \cap [t_i(\omega)])}$.*

PROOF. We have $t_i(\omega)(S \cap [t_i(\omega)]) = 1$ and hence $t_i(\omega)(E) = t_i(\omega)(E \cap S \cap [t_i(\omega)]) = t_i(\omega)(E \cap [t_i(\omega)])$. Since P^S is positive, we do have $P^S(S \cap [t_i(\omega)]) > 0$.

Since $S((E \cap [t_i(\omega)])^{\uparrow}) = S$ and since $\omega' \in [t_i(\omega)]$ implies $t_i(\omega') \in \Delta(S)$, we do have $(E \cap [t_i(\omega)])^{\uparrow} \cap A_i((E \cap [t_i(\omega)])^{\uparrow}) = (E \cap [t_i(\omega)])^{\uparrow}$. We also have $(S \cap [t_i(\omega)])^{\uparrow} \subseteq A_i(S^{\uparrow}) = A_i((E \cap [t_i(\omega)])^{\uparrow})$. The last equality follows from weak necessitation. We have - by the definition of a common prior - the following (with our abuse of notation):

$$\begin{aligned} P^S(E \cap [t_i(\omega)]) &= \int_{S \cap A_i((E \cap [t_i(\omega)])^{\uparrow})} t_i(\cdot)(E \cap [t_i(\omega)]) dP^S(\cdot) \\ &= \int_{S \cap [t_i(\omega)]} t_i(\cdot)(E \cap [t_i(\omega)]) dP^S(\cdot) \\ &\quad + \int_{(S \cap A_i(S^{\uparrow})) \setminus (S \cap [t_i(\omega)])} t_i(\cdot)(E \cap [t_i(\omega)]) dP^S(\cdot) \end{aligned}$$

But if $\omega' \in (S \cap A_i((E \cap [t_i(\omega)])^{\uparrow})) \setminus (S \cap [t_i(\omega)])$, then $t_i(\omega')(E \cap [t_i(\omega)]) = 0$, and hence, we have

$$\begin{aligned} P^S(E \cap [t_i(\omega)]) &= \int_{S \cap [t_i(\omega)]} t_i(\cdot)(E \cap [t_i(\omega)]) dP^S(\cdot) \\ &= t_i(\omega)(E \cap [t_i(\omega)]) \int_{S \cap [t_i(\omega)]} 1 dP^S(\cdot) \\ &= t_i(\omega)(E \cap [t_i(\omega)]) P^S(S \cap [t_i(\omega)]). \end{aligned}$$

Since $P^S(S \cap [t_i(\omega)]) > 0$, it follows that $t_i(\omega)(E \cap [t_i(\omega)]) = \frac{P^S(E \cap [t_i(\omega)])}{P^S(S \cap [t_i(\omega)])}$. □

4.2 Proof of the Theorem

Suppose by contradiction, that there are $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and a continuous random variable $v : \Omega \longrightarrow \mathbb{R}$ such that $CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \neq \emptyset$, where

$$\begin{aligned} E_1^{\leq x_1} &:= \left\{ \omega \in \Omega : \int_{S_{t_1(\omega)}} v(\cdot) d(t_1(\omega))(\cdot) \leq x_1 \right\}, \text{ and} \\ E_2^{\geq x_2} &:= \left\{ \omega \in \Omega : \int_{S_{t_2(\omega)}} v(\cdot) d(t_2(\omega))(\cdot) \geq x_2 \right\}. \end{aligned}$$

Let S be a $\preceq$-minimal state-space with the property that $S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \neq \emptyset$.

By standard properties of beliefs, we have $CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \subseteq B_i^1(CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))$ for $i = 1, 2$. This implies that for each $\omega \in S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ and $i = 1, 2$, we have $t_i(\omega)(CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) = 1$, which by the minimality of S implies that $t_i(\omega) \in \Delta(S)$ and $t_i(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) = 1$.

By Lemma 2, $S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))$ is closed in S. Therefore it is easy to verify that if flattened, $F(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))$, that is $S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ with the induced structure, is a standard topological type-space (as in Heifetz, 2006), since for each $\omega \in S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$, we have $t_i(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) = 1$ for $i = 1, 2$.

Since P^S is a positive prior on S, we have that $P^S(S \cap [t_i(\omega)]) > 0$, for each $\omega \in S$.

For $\omega \in S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ we also have $t_i(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap [t_i(\omega)]) = 1$, and by Lemma 4, we have $t_i(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap [t_i(\omega)]) = \frac{P^S(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap [t_i(\omega)])}{P^S(S \cap [t_i(\omega)])}$.

Hence, since $P^S(S \cap [t_i(\omega)]) > 0$, it follows that $P^S(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap [t_i(\omega)]) = P^S(S \cap [t_i(\omega)]) > 0$. It follows that $P^S(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) > 0$. Therefore it is easy to check that $\frac{P^S(\cdot)}{P^S(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))}$ is a common prior on $F(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))$.

Claim: Let $\omega \in CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap S$. Then $\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_1(\omega))(\cdot) \leq x_1$ and $\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_2(\omega))(\cdot) \geq x_2$.

We prove the second inequality, the first is analogous to the second one. We know already that $t_2(\omega) \in \Delta(S)$. By the definitions $\omega \in S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ implies $\omega \in S \cap B_2^1(E_2^{\geq x_2})$, and therefore $t_2(\omega)([t_2(\omega)] \cap E_2^{\geq x_2} \cap S) = 1$. It follows that $[t_2(\omega)] \cap E_2^{\geq x_2} \cap S$ is non-empty. Let $\omega' \in [t_2(\omega)] \cap E_2^{\geq x_2} \cap S$. Then we have $\int_S v(\cdot) d(t_2(\omega'))(\cdot) \geq x_2$. But we have $t_2(\omega) = t_2(\omega')$ and therefore $\int_S v(\cdot) d(t_2(\omega))(\cdot) \geq x_2$.

Since S is compact and $v : S \longrightarrow \mathbb{R}$ is continuous, there is a $\bar{v} \in \mathbb{R}$ such that $|v(\tilde{\omega})| \leq \bar{v}$ for all $\tilde{\omega} \in S$.

Since $t_2(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) = 1$, we have

$$\begin{aligned}
\left| \int_{S \setminus (S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))} v(\cdot) d(t_2(\omega))(\cdot) \right| &\leq \bar{v} \int_{S \setminus (S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))} 1 d(t_2(\omega))(\cdot) \\
&= \bar{v} \quad t_2(\omega)(S \setminus (S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))) \\
&= 0.
\end{aligned}$$

Hence, we have

$$\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_2(\omega))(\cdot) = \int_S v(\cdot) d(t_2(\omega))(\cdot) \geq x_2$$

and this finishes the proof of the claim.

It follows that we have found a standard topological type-space $S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ in the sense of Heifetz (2006) with a common prior and a continuous random variable $v : S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \longrightarrow \mathbb{R}$ such that

$$\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_1(\omega))(\cdot) \leq x_1 < x_2 \leq \int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_2(\omega))(\cdot).$$

Note that if we replace $v(\cdot)$ by $v(\cdot) - \frac{x_1 + x_2}{2}$, we get

$$\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) - \frac{x_1 + x_2}{2} d(t_1(\omega))(\cdot) < 0 < \int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) - \frac{x_1 + x_2}{2} d(t_2(\omega))(\cdot).$$

But this is a contradiction to Feinberg's (2000) Theorem (Proposition 1 in Heifetz, 2006). Hence this completes the proof of the theorem. □

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