CAPACITIES AND PROBABILISTIC BELIEFS: A PRECARIOUS COEXISTENCE

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Klaus Nehring

Abstract

This paper raises the problem of how to define revealed probabilistic beliefs in the context of the capacity/Choquet Expected Utility model. **At** the center of the analysis is a decision-theoretically axiomatized definition of "revealed unambiguous events." The definition is shown to impose surprisingly strong restrictions on the underlying capacity and on the set of unambiguous events; in particular, the latter is always an algebra. Alternative weaker definitions violate even minimal criteria of adequacy.

Rather than finding fault with the proposed definition, we argue that our results indicate that the CEU model is epistemically restrictive, and point out that analogous problems do not arise within the Maximin Expected Utility model.

1. INTRODUCTION

Following Ellsberg's (1961) classical experiments, it has become widely accepted that the preferences of empirical decision-makers often violate the consistency conditions characteristic of classical Subjective Expected Utility theory, and in particular that they fail to reveal a well-defined subjective probability measure.

There exists by now a variety of axiomatic models designed to accommodate Ellsbergian behavior; the two most frequently studied are the Choquet and Maximin Expected Utility Models (CEU respectively MMEU) due to Schmeidler (1989) respectively Gilboa-Schmeidler (1989).

While on a heuristic and rhetorical level the episternic distinction between risk and uncertainty has been important in stimulating an interest in such non-standard models, little work has been done in determining their epistemic content, i.e. in relating preferences to appropriate notions of belief (see Epstein-Zhang (1996), Sarin-Wakker (1995), and Nehring (1994), as well as Ghirardato (1996), Mukerjee (1996), and Nehring (1991) from rather different perspectives).

This paper addresses a particular issue within this general problematics: when can one legitimately attribute to a11 agent an unambiguous probabilistic belief about an event or set of events? And, in a related vein: which conditions must preferences satisfy in order to reflect / be consistent with a set of given ("objective") probabilities?

A satisfactory answer to these basic questions seems not only essential to an adequate understanding of models of' non-probabilistic uncertainty, it also promises to have significant value in applications. By allowing to 'localize" ambiguous beliefs, it should yield models with more specific predictions and sharper comparisons to traditional "global" expected-utility models. For example, in a game-theoretic context, one may want to describe the extensive-form game itself (in particular the "moves of Nature") in standard Bayesian manner in terms of unambiguous probabilities, while allowing at the same time for ambiguity in players' beliefs about other players' strategic choices ("strategic uncertainty") .

We will conduct the analysis in the context of the CEU or "capacity" model as does most of the existing epistemic literature. The first thing to note is that, as simple and as elementary as they look, the questions raised do not have an obvious answer. Indeed, it will be seen that it is not even clear that any satisfactory answer exists within the CEU model.

The non-triviality of the issue becomes clear through the following preliminary consideration. For an agent to believe in the occurrence of some event A with subjective probability α, not only must the capacity of A, $\nu(A)$, be equal to α, but that of the complement must be equal to its probability $1-\alpha$ also. But more is required. If in addition the agent believes in the occurrence of the disjoint set B with subjective probability β, then he also believes (of conceptual necessity) that the probability of the event $A \cup B$ is equal $\alpha+\beta$, hence $\nu(A \cup B)$ must be equal to $\alpha+\beta=\nu(A)+\nu(B)$. Probability judgements have a "logical syntax" that needs to be accounted for.

In the literature, only the very recent and thorough contribution by Zhang (1997) has taken up the issue of defining revealed probabilistic beliefs explicitly in the context of an axiomatization of CEU preferences for capacities that can be represented as "inner measures".[1] Otherwise, the special case of probability one beliefs has received quite a bit of recent interest *(see* Haller (1995), Morris (1995), Sarin-Wakker (1995)); the issue has also connections with that of defining independent product capacities *(see* Hendon et al. (1995), Ghirardato (1995) and Eichberger-Kelsey (1996); cf. section 5).

The plan for the remainder of the paper is as follows.

Section **2** sets out the issue of defining "revealed unambiguous events" from a capacity, and establishes criteria for the "soundness" of any proposed definition. These criteria are violated by the simplest natural definitions (section **3).**

[1] **Sarin-Wakker (1992) define "revealed unambiguous *partitions"***

In section 4, capacities are interpreted as "rank-dependent probability assignments"; this suggests a definition of unambiguous events with a canonical look to it. It is characterized in terms of conditions on preferences whose applicability and appeal are not restricted to the CEU model. All proposed definitions are shown to coincide for the class of convex capacities.

Section 5 characterizes the surprisingly strong implications of unambiguous events for the underlying capacity, and shows that the class of unambiguous events is always an algebra. The latter implies for example that whenever a decision-maker has probabilistic beliefs about the marginal distributions of each of a collection of random variables, he has probabilistic beliefs about their joint distribution as well.

This unwelcome implication might in principle be accounted for in two ways : it may indicate that the adopted definition is too strong; alternatively, it may show that the CEU model is applicable only when an agent's probabilistic beliefs take a certain form. In the concluding section 6, we argue for the latter as the more plausible interpretation.

2. PRELIMINARIES

Let S be a finite set of states with $\#S = n$, and let A^S denote the probability-, simplex on S.

A capacity ν is a mapping from the power set 2^S of S into $[0,1]$ such that $\nu(\emptyset) = 0$, $\nu(S) = 1$, and $\nu(A) \geq \nu(B)$ whenever $A \supseteq B$. It is *convex* if for all $A, B \in 2^S$: $\nu(A) + \nu(B) \leq \nu(A \cap B) + \nu(A \cup B)$.

The expectation of a random-variable $f : S \rightarrow \mathbf{R}$ with respect to the capacity ν is defined as its *Choquet-integral*

$$\int f d\nu := \sum_{k=1}^{n} f(s_k) \cdot (\nu(\{s_1, ..., s_k\}) - \nu(\{s_1, ..., s_{k-1}\})),$$

with $\{s_k\}_{k=1,\ldots,n}$ chosen such that $f(s_j) \geq f(s_k)$ whenever $j \leq k$.[2]

Let C denote a set of consequences. An *act* x maps states to consequences, $x: S \to C$, or, in equivalent notation, $x \in C^S$. A preference ordering $\succeq$ on C^S has a *"Choquet Expected Utility"* (CEU) representation if there exist a capacity ν and a utility-function $u: C \to \mathbb{R}$ such that $x \succeq y$ if and only if $\int u \circ x d\nu \geq \int u \circ y d\nu$.

To simplify argument and notation, we will focus on "risk-neutral" decision-makers with $C = \mathbb{R}$ and $u = id$. As long as the "true" utility-function u is defined on a connected domain C and is continuous, this is without effective loss of generality. Under risk-neutrality, a capacity induces a unique CEU preference-ordering $\succeq_\nu$ according to the condition: $x \succeq_\nu y$ if and only if $\int x d\nu \geq \int y d\nu$.

The task is to define from a given capacity ν a collection of "revealed unambiguous" events $\mathcal{A}_\nu^{ua}$ for which the agent is understood to have probabilistic beliefs. Within the CEU-model (which is assumed throughout), this is equivalent to defining $\mathcal{A}_\nu^{ua}$ in terms of the associated preference-relation $\succeq_\nu$ due to the one-to-one relation between the two. Conceptually, a *primitive* definition of unambiguous events should be in terms of the preference relation as the primitive entity; this point of view is adopted in section 4 which attempts to provide *"the right"* definition. On the other hand, the implications of any given definition are more easily described in terms of the capacity representation; likewise, the set of *possible* definitions is more easily surveyed in terms of the representation.

To be satisfactory, $\mathcal{A}_\nu^{ua}$ should have the property that for any three events A, B, C such that the value of a probability measure on C is uniquely determined by its values on A and B, C must be in $\mathcal{A}_\nu^{ua}$ whenever both A and B are. In the measure-theoretic terminology introduced by Zhang (1997) into decision-theory, $\mathcal{A}_\nu^{ua}$ must be a λ-system.

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[2]Equivalently, this can be written as $\int f d\nu := f(s_n) + \sum_{k=1}^{n-1} \left(f(s_k) - f(s_{k+1})\right) \cdot \nu(\{s_1, \ldots, s_k\})$.

Definition 1 **A** *collection* A $\in 2^S$ *is a* A – system *if it has the following two* ***prop-*** *erties:*

i) $0, S \in$ A,

ii) $A, B \in \mathcal{A},\ A \supseteq B \Rightarrow A \backslash B \in \mathcal{A}$.

A *is an* algebra if *it satisfies in addition*

iii) $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$.

Remark: Zhang (1997, lemma 2.1) shows that a A-system defined by i) and ii) is always closed under disjoint unions:

$$A, B \in \mathcal{A}, A \cap B = \emptyset \Rightarrow A \cup B \in \mathcal{A}.$$

In general, one will not want $\mathcal{A}_{\nu}^{ua}$ to be an algebra. For instance, if S $= S_1$ x S_2, with non-singleton S_1 and S_2, then $\mathcal{A}_{\nu}^{ua} = \{$T x $S_2 \mid$ T $\subseteq S_1\}$ U $\{S_1$ x $T \mid$ T $\subseteq S_2\}$ says that an agent has "unambiguous", "probabilistic" marginal beliefs about each component of the state, but "non-probabilistic", "ambiguous" beliefs about their joint distribution. $\mathcal{A}_{\nu}^{ua}$ is a A-system but not an algebra.

Furthermore, one will want ν on $\mathcal{A}_{\nu}^{ua}$ to be "coherently interpretable" as a probability; this is captured by

Definition 2 ν *is* probabilistically coherent *on* A *if there exists a probability measure p on* 2^S *that agrees with* ν *on* A.

Note that "probabilistic coherence" implies additivity of ν on A but is not implied by it, even if A is a A-system (fact **2).**

A successful definition of "revealed unambiguous belief" makes it possible to express formally the notion that an agent's beliefs incorporate a set of "given" probabilities ("set of probabilistic constraints"). These may be thought of as *information* in the form of objective probabilities, but need not be.

Definition 3 *A* probabilistic constraint set is a pair $(\mathcal{C},\phi)$, *where* $\mathcal{C} \subseteq 2^S$ and $\phi : \mathcal{C} \to [0,1]$ *is* pmbabilistically *coherent* on C.

The capacity ν is consistent with $(\mathcal{C},\phi)$ if

i) $\nu(A) = \phi(A)$ *for all* $A \in \mathcal{C}$, and

ii) $\mathcal{A}_\nu^{ua} \supseteq \mathcal{C}$.

Example 1 *Let* $S = \{a, b, c, d\}$, $\mathcal{C} = \{\{a, b\}, \{c, d\}, \{a, d\}, \{b, c\}, \emptyset, S\}$, $\phi(\{a,b\}) = \phi(\{b,c\}) = 0.9$, $\phi(\{a,d\}) = \phi(\{c,d\}) = 0.1$, $\phi(\emptyset) = 0$, $\phi(S) = 1$, *and* $\nu(A) = \sup\{\phi(E) \mid E \in \mathcal{C}, E \subseteq A\}$[3]. *Suppose* that $\mathcal{A}_\nu^{ua} = \mathcal{C}$. Then ν is consistent with $(\mathcal{C},\phi)$, and $\mathcal{A}_\nu^{ua}$ *satisfies* all the desiderata *listed* above: it is a λ-system, and ν on $\mathcal{A}_\nu^{ua}$ *is probabilistically* coherent.

Nonetheless, ν is not "truly consistent" with $(\mathcal{C},\phi)$. In particular, $\nu(\{b\}) = 0$, while $\nu(\{a,c,d\}) = 0.1$; in terms of decision making, betting on $\{b\}$ is dispreferred to betting on its complement $S \backslash \{b\}$, i.e. $1_{\{a,c,d\}} \succ_\nu 1_{\{b\}}$, with 1_A denoting the indicator-function of the event A. Since the probability of $\{b\}$ is "objectively" at least 0.8, and thus at least four times as large as that of its complement, this seems hardly acceptable: it is materially irrational for the decision-maker to bet on the event that is unambiguously less likely in view of his information $(\mathcal{C},\phi)$.[4] It follows that on the correct definition of $\mathcal{A}_\nu^{ua}$, $\mathcal{A}_\nu^{ua}$ cannot contain C, so that ν would not be underwritten by $\mathcal{A}_\nu^{ua}$ as consistent with $(\mathcal{C},\phi)$.

The requirements on a minimally satisfactory definition of unambiguous events are summarized in the following notion of "soundness".

[3] A similar capacity is defined in example 1.1 of Zhang (1997).

[4] Note that the event $\{a,c,d\}$ is unambiguously *less likely* than the event $\{b\}$, although neither event is unambiguous in itself. Such more general forms of unambiguous probabilistic beliefs will be treated in Nehring (1997).

Definition 4 *A definition of revealed unambiguous events is a mapping $\mathcal{A}^{ua}_{\bullet} : \nu \mapsto \mathcal{A}^{ua}_{\nu}$. It is sound iff, for all capacities ν:*

a) $\mathcal{A}^{ua}_{\nu}$ is a A-system,

ii) for all $E \in 2^S$: ν is probabilistically coherent on $\mathcal{A}^{ua}_{\nu} \cup \{E\}$.

To illustrate clause ii), consider again example 1. Here ν fails to be probabilistically coherent on $\mathcal{A}^{ua}_{\nu} \cup \{b\}$, whenever $\mathcal{A}^{ua}_{\nu} \supseteq \mathcal{C}$. To be sound, ν would need to satisfy $\nu(\{b\}) \geq 0.8$ and $\nu(\{a, c, d\}) \leq 0.2$.

If $\mathcal{A}^{ua}_{\nu}$ is an algebra rather than merely a A-system, the second clause simplifies.

Fact 1 *If A is an algebra, the following two statements are equivalent:*

i) for all $E \in 2^S$: ν is probabilistically coherent on $\mathcal{A}^{ua}_{\nu} \cup \{E\}$.

ii) ν is additive on A, i.e. for all A, $B \in$ A such that $A \cap B = \emptyset$, $\nu(A) + \nu(B) = \nu(A \cup B)$.

A trivial example of a sound definition of revealed unambiguous events is the constant mapping $\nu \mapsto \{\emptyset, S\}$ for all ν. Thus "soundness" of the definition says only that the events given by $\mathcal{A}^{ua}_{\nu}$ can be thought of as "genuinely unambiguous / probabilistic"; it does not address the issue whether $\mathcal{A}^{ua}_{\nu}$ comprises *all* "genuinely probabilistic" events.

3. THE PROBLEM

A particularly simple and straightforward definition of unambiguous events is given by

$$\mathcal{A}^{3}_{\nu} := \left\{ A \in 2^S \mid \nu(A) + \nu(A^c) = 1 \right\}.$$

This however fails miserably: $\mathcal{A}^{3}_{\nu}$ is generally not closed under disjoint unions, thus failing to qualify as a A-system. Moreover, even if $\mathcal{A}^{3}_{\nu}$ happens to be an algebra, ν may fail to be additive on $\mathcal{A}^{3}_{\nu}$.

Example 2 *Let* $S=\{a,b,s\}$ *and define* ν *by*

$$\nu(\mathbf{A}) := \begin{cases} 0 & if \quad \#A \leq 1 \\ 1 & if \quad \#A \geq 2 \end{cases}$$

Here $\mathcal{A}_\nu^3 = 2^S$, *but* ν ***is not*** *a pmbability-measure.*

The example suggests that $\mathcal{A}_\nu^3$ fails to "build in" additivity with respect to events outside the partition $\{A, A^c\}$. A :natural move is to strengthen the definition to

$$\mathcal{A}_\nu^2 := \{A \in 2^S \mid \nu(A \cup B) - \nu(B) = \nu(A) \text{ for all } B \text{ such that } A \cap B = \emptyset\}.$$

$\mathcal{A}_\nu^2$ seems on the right track; for instance, it ensures additivity of ν on $\mathcal{A}_\nu^2$ whenever the latter is an algebra. $\mathcal{A}_\nu^2$ has been adopted with reservations by Zhang (1997), who gives a preference-based characterization of it and notes that it may fail to be a λ -system, violating condition ii) as for instance in example 2, where $\mathcal{A}_\nu^2 = \{A \in 2^S \mid \#A \geq 2\}$. He responds to this by simply *imposing* the second condition on $\mathcal{A}_\nu^2$; note that this is in effect a restriction on the domain of capacities to which the definition $\nu \mapsto \mathcal{A}_\nu^2$ is applied.

Yet even if this domain-restriction is accepted, $\mathcal{A}_\nu^2$ is unsound. In example 1, for instance, $\mathcal{A}_\nu^2 = \mathcal{C}$, which makes $\mathcal{A}_\nu^2$ unsound as shown above. Indeed, ν may even fail to be probabilistically coherent *on!* $\mathcal{A}_\nu^2$.

Fact 2 *There exist capacities* ν *such that* $\mathcal{A}_\nu^2$ *is a A-system and* ν *is not probabilistically coherent on* $\mathcal{A}_\nu^2$; *in particular, not every q that is additive on a A-system* $\mathcal{A}$ *can be extended to a probability-measure on* 2^S.

Proof. See appendix.

4. THE PROPOSAL

Consider a risk-neutral[5] decision-maker who has to decide between two acts x and y such that $x - y$ is $\{A, A^c\}$-measurable (i.e. constant within A and A^c) and such that $x_A > y_A$. A decision in favor of x over y can be viewed as accepting the ***incremental*** bet $x - y$ on A. If the decision-maker assigns an unambiguous subjective probability to the event A, the ***incremental*** bet has an unambiguous expectation, and it seems highly reasonable that he should accept this incremental bet if and only if its expectation is positive. Conversely, this condition yields a natural criterion for the non-ambiguity of an event.

Definition 5 The event A is $\succeq$*-unambiguous* if, for all x, y such that $x - y$ is $\{A, A^c\}$-measurable, $x \succeq y \Leftrightarrow x - y \succeq 0$.

To characterize >-,-unambiguous events directly in terms of the capacity, it proves helpful to interpret capacities as "rank dependent probability assignments".

A ***ranking*** of states is a one-to-one mapping $\rho : S \rightarrow \{1, ..., n\}$, let $\mathcal{R}$ denote the set of such rankings. The ranking, ρ is a ***neighbour*** of ρ' ("$\rho N \rho'$") iff, for at most two states $s \in S : \rho(s) \neq \rho'(s)$, and, for all $s \in S$, $|\rho(s) - \rho'(s)| \leq 1$. A mapping $\pi : \mathcal{R} \rightarrow \mathbf{A}^S$ is called a rank-dependent ***probability*** assignment (RDPA) iff for all ρ, ρ' such that $\rho N \rho'$, and all $s \in S$ such that $\rho(s) = \rho'(s) : \pi_\rho(\{s\}) = \pi_{\rho'}(\{s\})$.

For any capacity ν, define a mapping $\pi^\nu : \mathcal{R} \rightarrow \mathbf{A}^S$ by $\pi^\nu_\rho(\{t\}) = \nu(\{s \mid \rho(s) \leq \rho(t)\}) - \nu(\{s \mid \rho(s) < \rho(t)\})$. When there is no ambiguity, we will often drop the superscript in π^ν. There is a one-to-one relation between capacities and RDPAs.

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[5] **As mentioned above, this is without major loss of generality; in particular, "risk-neutrality" is an entirely standard feature of models in which consequences are defined in "probability currency", as in an Anscombe-Aumann framework. An explicit analysis along these lines will be given in Nehring (1997).**

Proposition 1 A *mapping* $\pi : \mathcal{R} \to A^S$ *is a mnk-dependent probability assignment if and only if there is a (unique) capacity* ν *such that* $\pi = \pi^{\nu}$.

Proof. The if-part is immediate from the definition of an RDPA.

For the converse, in view of the following lemma, one can set $\nu(A) = \pi_\rho(A)$ for any ρ such that $A = \{s \in S \mid \rho(s) \leq \#A\}$. This yields a capacity ν with the property that $\pi^\nu = \pi$.

Lemma 1 For all $A \in 2^S$ *and* $p, \rho' \in \mathcal{R}$ *such that* $A = \{s \in S \mid \rho(s) \leq \#A) = \{s \in S \mid \rho'(s) \leq \#A\}$: $\quad \pi_\rho(A) = \pi_{\rho'}(A)$.

Proof of lemma. Note first that the claim of the lemma is straightforward from the definition of an RDPA for all ρ, ρ' such that $\rho N \rho'$.

Now take arbitrary $\rho, \rho' \in \mathcal{R}$. It is clear that there exists a sequence of rankings $\{\rho_j\}_{j \leq k}$ such that $\rho_0 = \rho, \rho_k = \rho'$ and $\rho_j N \rho_{j+1}$ for all $j <$ k, and such that $A = \{s \in S \mid \rho_j(s) \leq \#A\}$. Since $\pi_{\rho_j}(A) = \pi_{\rho_{j+1}}(A)$ for all j from the above, one obtains $\pi_\rho(A) = \pi_{\rho'}(A)$ a desired. ■

Say that ρ is *wmonotonic* with $x \in \mathbf{R}^S$ *if*, for all s and t, $\rho(s) \geq \rho(t)$ implies $x_s \leq x_t$. It is easily verified that Choquet-integration of x amounts to ordinary integration with respect to the appropriate rank-dependent probability measure π_ρ , i.e. that $\int x dv = \int x d\pi_\rho$ for any ρ that is comonotonic to x.

An interpretation of the capacity model and of Choquet-integration along similar lines has recently been advocated by Sarin-Wakker *(1995)*. It also arises naturally from within Schmeidler's *(1989)* classic contribution, in that his Comonotonic Independence axiom is simply the Independence axiom restricted to comonotonic equivalence classes (classes of acts comonotonic to the same ranking *p)*.

On an RDPA interpretation of a capacity, ambiguity of an event is naturally associated with dependence of the assigned probability on the ranking. Correspondingly, an event is naturally defined as *unambiguous* if its rank-dependent probability does not depend on the ranking :

$$\mathcal{A}_\nu := \left\{ A \mid \pi^\nu_\rho(A) = \nu(A) \text{ for all } \rho \in \mathcal{R} \right\}$$

Note that it follows directly from the definition that $\mathcal{A}_\nu$ is a λ-system and that the definition $\nu \mapsto \mathcal{A}_\nu$ is sound.

Say that A is *connected* with respect to ρ if, for all s, s', s'' such that $\rho(s) < \rho(s') < \rho(s'')$, $A \ni s'$ whenever $A \supseteq \{s, s''\}$. Then $\mathcal{A}^2_\nu$ can be written as follows :

$\mathcal{A}^2_\nu = \{ A \mid \pi^\nu_\rho(A) = \nu(A)$ for all $\rho \in \mathcal{R}$ such that A is connected with respect to $\rho \}$.

Thus, from a rank-dependent point-of-view, $\mathcal{A}^2_\nu$ looks like an ad-hoc-restricted version of $\mathcal{A}_\nu$.

That $\mathcal{A}_\nu$ is the right definition of unambiguous events is confirmed by the following theorem.

Theorem 1 *The following three statements are equivalent:*

i) $A \in \mathcal{A}_\nu$.

ii) A *is* $\succsim_\nu$*-unambiguous* .

iii) For all x, y *such that* y *is* $\{A, A^c\}$*-measurable,*

$\int (x + y)\, d\nu = \int x d\nu + \int y d\nu$.

Proof. The implications iii) $\Rightarrow$ ii) and ii) $\Rightarrow$ i) are easily verified; by contrast, the implication i) $\Rightarrow$ iii) is non-trivial.

Definition 6 *For* $A \in 2^S$*, let* $\boxed{\approx_A}$ *denote the following equivalence relation on* $\mathcal{R}$:

$\rho \approx_A \rho'$ *iff, for all* s, t *such that* $\{s, t\} \subseteq A$ *or* $\{s, t\} \subseteq A^c$:

$\rho(s) < \rho(t) \Longleftrightarrow \rho'(s) < \rho'(t)$.

Also, define for $\rho \in \mathcal{R}$ *and* $A \in 2^S$ *an associated ranking* $\boxed{\rho_A} \in \mathcal{R}$ *uniquely by the following two conditions:*

i) for all $s \in A$, $t \in A^c$: $\rho_A(s) < \rho_A(t)$ *, and*

ii) $\rho_A \approx_A \rho$.

The key to the proof is the following lemma.

Lemma 2 *If* $A \in \mathcal{A}_\nu$, *then,* for all ρ, ρ' such *that* $\rho \approx_A \rho' : \pi_\rho = \pi_{\rho'}$.

Proof of the lemma. Note first that it suffices to prove validity of the claim for neighbouring rankings ρ and ρ', since any two ρ and ρ' satisfying $\rho \approx_A \rho'$ can be connected by a chain of neighbouring rankings $\rho_1, ..., \rho_k$ satisfying $\rho_j \approx_A \rho_{j+1}$.

Assume thus $\rho N \rho'$, take any $B \in 2^S$, and let $\nu(A) = \alpha$.

The following table describes the rank-dependent probabilities for the events in $\mathcal{B} := \{A \cap B, A \cap B^c, A^c \cap B, A^c \cap B^c\}$,

E	$\pi_\rho(E)$	$\pi_{\rho'}(E)$
$A \cap B$	$\pi_\rho(A \cap B)$	$\pi_{\rho'}(A \cap B)$
$A \cap B^c$	$\alpha - \pi_\rho(A \cap B)$	$\alpha - \pi_{\rho'}(A \cap B)$
$A^c \cap B$	$\pi_\rho(A^c \cap B)$	$\pi_{\rho'}(A^c \cap B)$
$A^c \cap B^c$	$1 - \alpha - \pi_\rho(A^c \cap B)$	$1 - \alpha - \pi_{\rho'}(A^c \cap B)$

From $\rho \approx_A \rho'$ and $\rho N \rho'$, it follows that $\pi_\rho(A \cap B) = \pi_{\rho'}(A \cap B)$ or $\pi_\rho(A \cap B^c) = \pi_{\rho'}(A \cap B^c)$, as well as $\pi_\rho(A^c \cap B) = \pi_{\rho'}(A^c \cap B)$ or $\pi_\rho(A^c \cap B^c) = \pi_{\rho'}(A^c \cap B^c)$. Inspecting the table, this yields immediately $\pi_\rho(A \cap B) = \pi_{\rho'}(A \cap B)$ as well as $\pi_\rho(A^c \cap B) = \pi_{\rho'}(A^c \cap B)$, hence $\pi_\rho(13) = \pi_{\rho'}(B)$. □

Consider now $A \in \mathcal{A}_\nu$ and x, y such that y is $\{A, A^c\}$-measurable. Let ρ be any ranking that is comonotonic with x. Tnen by the $\{A, A^c\}$-measurability of y, $x + y$ is comonotonic with some ρ' such that $\rho' \approx_A \rho$. By the lemma, $\pi_\rho = \pi_{\rho'}$. Note that $\int y d\pi_{\rho'} = \int y d\nu$ since $A \in \mathcal{A}_\nu$ and y is $\{A, A^c\}$-measurable.

Thus $\int (x + y) d\nu = \int (x + y) d\pi_{\rho'} = \int x d\pi_{\rho'} + \int y d\pi_{\rho'} = \int x d\pi_\rho + \int y d\pi_{\rho'} = \int x d\nu + \int y d\nu$. ■

It is also of interest to note that the proper definition of unambiguous events is a live issue only for *non-convex* capacities: for convex capacities, all proposed definitions coincide.

Proposition 2 *For any convex ν : $\mathcal{A}_\nu = \mathcal{A}^2_\nu = \mathcal{A}^3_\nu$.*

Proof. We need only to show that $\mathcal{A}_\nu \supseteq \mathcal{A}^3_\nu$.

It is well known[6] that any convex capacity has the following representation:

$$\nu(E) = \min_{\rho \in \mathcal{R}} \pi_\rho(E) \text{ for all } E \in 2^S.$$

Suppose that $A \notin \mathcal{A}_\nu$, i.e. that for some $\rho_1, \rho_2 \in \mathcal{R}$: $\pi_{\rho_1}(A) < \pi_{\rho_2}(A)$. Since $\nu(A) \leq \pi_{\rho_1}(A)$ and $\nu(A^c) \leq 1 - \pi_{\rho_2}(A)$ by the representation, $\nu(A) + \nu(A^c) < 1$, and thus $A \notin \mathcal{A}^3_\nu$. ■

5. IMPLICATIONS

Unambiguous events turn out both to have a surprising amount of structure themselves, and entail surprisingly strong restrictions on the capacity that hosts them. The blame for these apparently excessive implications is tentatively assigned in the concluding section 6.

Theorem 2 *For any capacity ν, $\mathcal{A}_\nu$ is an algebra.*

Proof . We need to show that $\mathcal{A}_\nu$ is intersection-closed. Thus, take $A, B \in \mathcal{A}_\nu$ and let $\mathcal{B} := \{A \cap B, A \cap B^c, A^c \cap B, A^c \cap B^c\}$.

Since we know that, for all $\rho \in \mathcal{R}$, $\pi_\rho(A) = \nu(A)$ and $\pi_\rho(B) = \nu(B)$, we have

$$\begin{aligned} \pi_\rho(A \cap B^c) &= \nu(A) - \pi_\rho(A \cap B), \\ \pi_\rho(A^c \cap B) &= \nu(B) - \pi_\rho(A \cap B), \\ \pi_\rho(A^c \cap B^c) &= 1 + \pi_\rho(A \cap B) - \nu(A) - \nu(B). \end{aligned} \tag{1}$$

We need to show that $\pi_\rho(A \cap B)$ is independent of ρ.

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[6]See for example Chateauneuf-Jaffray (1989).

Consider ρ, ρ' such that ρ is a neighbour of ρ'. From the definitional property of an RDPA it follows that $\pi_\rho(E) = \pi_{\rho'}(E)$ for at least two $E \in B$. However, in view of (1), this implies that the rank-dependent probability of all four events in B stays the same, and in particular, that $\pi_\rho(A \cap B) = \pi_{\rho'}(A \cap B)$.

Now take arbitrary $\rho, \rho' \in \mathcal{R}$. It is clear that there always exist a sequence of rankings $\{\rho_j\}_{j \leq k}$ such that $\rho_0 = \rho, \rho_k = \rho'$ and $\rho_j N \rho_{j+1}$ for all $j <$ k. Since $\pi_{\rho_j}(A \cap B) = \pi_{\rho_{j+1}}(A \cap B)$ for all j from the above, one obtains $\pi_\rho(A \cap B) = \pi_{\rho'}(A \cap B)$ a desired. ■

This is not all; in addition, a capacity is always "additively separable" across its unambiguous events.

For a capacity ν, define the set of its "separating events"

$$\mathcal{A}_\nu^4 := \left\{A \in 2^S \mid \nu(B) = \nu(B \cap A) + \nu(B \cap A^c)\right\}.$$

Theorem 3 *For any capacity ν, $\mathcal{A}_\nu = \mathcal{A}_\nu^4$.*

Proof.

$\underline{\mathcal{A}_\nu \subseteq \mathcal{A}_\nu^4}$: Take any $A \in \mathcal{A}_\nu$ and $B \in 2^S$. Let ρ be any ranking such that, for all $s_1 \in A \cap B$, $s_2 \in A^c \cap B$ and $s_3 \in B^c$: $\rho(s_1) < \rho(s_2) < \rho(s_3)$.

By construction,

$$\pi_\rho(B) = \nu(B).$$

Since $\rho \approx_A \rho_A$ by definition, one obtains from lemma 2,

$$\pi_\rho(B) = \pi_{\rho_A}(B).$$

From the interdefinition of π and ν and the definition of ρ_A, one obtains

$$\pi_{\rho_A}(B) = \nu(A \cap B) + [\nu(A \cup (A^c \cap B)) - \nu(A)].$$

Finally, since $\mathcal{A}_\nu \subseteq \mathcal{A}_\nu^2$,

$$\nu(A \cup (A^c \cup B)) - \nu(A) = \nu(A^c \cap B).$$

These four equalities imply $\nu(\mathrm{B}) = \nu(\mathrm{A} \cap \mathrm{B}) + \nu(A^c \cap \mathrm{B})$, as desired. $\square$

$\underline{\mathcal{A}_\nu \supseteq \mathcal{A}_\nu^4}$:

Take any $\mathrm{A} \in \mathcal{A}_\nu^4$, and arbitrary $\rho, \rho' \in \mathcal{R}$; we have to show that $\pi_\rho(\mathrm{A}) = \pi_{\rho'}(\mathrm{A})$.

The key is the following lemma.

Lemma 3 *If $A \in \mathcal{A}_\nu^4$, then,* **for** *all $p \in \mathcal{R}$: $\pi_\rho = \pi_{\rho_A}$*

Proof of lemma.

For any $j \leq n$, let $S_j^\rho := \{s \in S \mid \rho(s) \leq j\}$.

Fix any j. By definition, $\pi_\rho\left(S_j^\rho\right) = \nu\left(S_j^\rho\right)$.

Since $A \in$ A",,

$$\nu\left(S_j^\rho\right) = \nu\left(S_j^\rho \cap A\right) + \nu\left(S_j^\rho \cap A^c\right),$$

as well as

$$\nu\left(S_j^\rho \cap A^c\right) = \nu\left(\left(S_j^\rho \cap A^c\right) \cup \mathrm{A}\right) - \nu(A),$$

and thus

$$\nu\left(S_j^\rho\right) = \nu\left(S_j^\rho \cap A\right) + \nu\left(\left(S_j^\rho \cap A^c\right) \cup A\right) - \nu(A).$$

In turn, the right-hand side of this equation is easily verified to be equal to $\pi_{(\rho_A)}\left(S_j^\rho\right)$. We thus have n-, $\left(S_j^\rho\right) = \pi_{\rho_A}(\mathrm{S})$ for all $j \leq n$, and therefore also n-, $= \pi_{\rho_A}$. $\square$

The claim of the theorem is now easily established.

We have $\pi_\rho(\mathrm{A}) = \pi_{\rho_A}(A)$ (by lemma 3),

$= \nu(\mathrm{A})$ (by definition),

$= \pi_{\rho'_A}(\mathrm{A})$ (by definition),

$= \pi_{\rho'}(\mathrm{A})$ (by lemma 3 again). $\blacksquare$

Remark: Zhang (1997) shows that

$\mathcal{A}_\nu^4 = \mathcal{A}_\nu^5 := \left\{A \in 2^S \mid \nu(A_1 \cup B) = \nu(A_1) + \nu(B) \text{ for all } A_1 \subseteq A \text{ and } B \subseteq A^c\right\}$,

considers (and rejects) $\mathcal{A}_\nu^5$ as a possible ***definition*** of unambiguous events, and gives a decision-theoretic (almost-) characterization. The intuitive content of $\mathcal{A}_\nu^4$

or $\mathcal{A}_\nu^5$ as capturing the events to which the agent assigns an unambiguous subjective probability is however not clear. And indeed, as pointed out in section 6, the decision-theoretic definitions of unambiguous events underlying A, and & diverge outside the CEU model.

Theorems **2** and **3** yield as a corollary a characterization of the class of capacities consistent with a given set of **probabilistic** constraints.

For $C \in 2^S$, let C* denote the algebra generated by $\mathcal{C}$, $C^* := \cap\{\mathcal{B} \supseteq C \mid \mathcal{B}$ is an algebra), and let $\mathcal{F}^*$ denote the minimal non-empty ele nents of that algebra which form a partition of S.

Corollary 1 ν is consistent *with the* constmints (C, ϕ) if and only if

i) for all $A \in \mathcal{C}$, $\nu(A) = \phi(A)$, and

ii) for all $A \in 2^S : \nu(A) = \sum_{F \in \mathcal{F}^*} \nu(A \cap F)$

Proof. "If": By theorem **3** and ii), $\mathcal{A}_\nu \supseteq C^* \supseteq C$; hence ν is consistent with (C, ϕ) by i).

"Only if": i) is obvious.

ii) Let $\mathcal{F}^* = \{F_i\}_{i \leq k}$ and define $B_j = \bigcup_{i \geq j} F_i$. By theorem 2, $\mathcal{A}_\nu \supseteq \mathcal{F}^*$. Since $B_{j+1} = B_j \cap F_j^c$, it follows from theorem **3**

that $\nu(A \cap B_j) = \nu(A \cap F_j) + \nu(A \cap B_{j+1})$ for all $j : 1 \leq j \leq k-1$.

Repeated substitutions yield immediately $\nu(A) = \nu(A \cap B_1) = \sum_{j \leq k} \nu(A \cap F_j)$.

■

Corollary 1 suggests a natural definition of the independent product of a capacity and a probability measure, for what it is worth[7]. Suppose that $S = S_1 \times S_2$, $\mathcal{A}_2 = \left\{S_1 \times A \mid A \in 2^{S_2}\right\}$. Let a probability ϕ_2 on $\mathcal{A}_2$ be given, as well as a "marginal capacity" ν_1 on $\mathcal{A}_1$ analogously defined.

[7] **In view of the epistemic restrictedness of the capacity-framework suggested in section 6.**

Proposition 3 Them *exists* a unique product capacity ν ($=: \nu_1 \otimes \phi_2$) such that

i) ν is *consistent* with $(\mathcal{A}_2, \phi_2)$, and

ii) for all $A \in S_1, B \in S_2 : \nu(A \times B) = \nu(A \times S_2) \cdot \phi(S_1 \times B)$.

Proof. Uniqueness: For $s \in S_2$, let $E_s = \{t \in S_1 \mid (t,s) \in E\}$. By corollary 1 and i) $\nu(E) = \sum_{s \in S_2} \nu(E_s \times \{s\})$, hence by ii), $\nu(E)$ is uniquely determined by

$$\nu(E) = \sum_{s \in S_2} \nu_1(E_s \times S_s) \cdot \phi_2(S_1 \times \{s\}). \tag{2}$$

Existence: Y defined by (2) clearly satisfies i) and ii). ■

The charm of proposition 3 lies in the fact that the consistency requirement i) uniquely singles out the product capacity $\nu_1 \otimes \phi_2$ which has been considered (and compared to alternative definitions:)by Hendon et al. (1995) and Ghirardato (1995), and also appears in Eichberger-Kelsey (1996).

6. DISCUSSION

The results of section 5 indicate that a capacity-representation of preferences and probabilistic constraints on beliefs do not live together very harmoniously; in many situations, one will have to give. Which of the two will depend on one's judgement about which is more fundamental. To us, it seems evident that probabilistic constraints are the more fundamental notion; indeed, it seems hard to even imagine what kind of argument might be adduced that could render probabilistic constraints defeasible.

This judgment is confirmed by the fact that it takes very little to obtain consistency with probabilistic constraints on preferences and beliefs in a satisfactory way. In particular, consistency can be achieved in the MMEU model in which capacities are replaced by closed convex sets of probabilities Π, and Choquet integration by "maximin integration" $\int x d\Pi := \min_{\pi \in \Pi} \int x d\pi$.

In the MMEU-model, an event A is naturally defined as Π-unambiguous if $\pi(A) = \pi'(A)$ for all a, a' $\in \Pi$; note that this definition coincides with the one given for capacities whenever the two integration-functionals coincide (i.e. for convex capacities ν **and** their core, cf. proposition 2). Under this definition, it can be shown that the preference-based characterization of unambiguous events in the manner of theorem 1 is preserved, while none of the adverse consequences are entailed.

The latter is demonstrated by considering the following example (cf. Zhang (1997), example 1.1).

Example 3 Let S = T x T, with T = {a, b) and define the probabilistic *constraints* by (C, ϕ), *with* C = {$\emptyset$, S,{a) x *T*,*{b)* x T,T x {a), T x {b)) and ϕ({a) x T) = ϕ(T x {a)) $= \frac{1}{2}$. Note that C *defines* a A-system, not an algebra.

In the MMEU model (but not in the CEU model!), these constraints are consistent with "complete ignorance" about the joint distribution of the first and second component, i.e.. with setting $\int 1_{\{(a,a),(b,b)\}} d\Pi = \int 1_{\{(a,b),(b,a)\}} d\Pi = 0$. This is uniquely achieved by the set of priors $\Pi^* = \left\{\pi \in A^S \mid a(\{a) \times T) = a(T \times \{a)) = \frac{1}{2}\right\}$ which is not the core of a convex capacity.

It is easily verified that the set of Π^*-unambiguous events is exactly the A-system C. Note also that the analogue to the problematic separability condition for unambiguous events as in theorem **3** is not entailed; for instance, for $A = \{a\}$ x T and B = T x {a}, we have $\int 1_B d\Pi^* = \frac{1}{2} \neq 0 = \int 1_{B\cap A} d\Pi^* + \int 1_{B \cap A^c} d\Pi^*$, while A is Π^*-unambiguous.

If ***A,*** is accepted as the correct definition of unambiguous events in the CEU model (for instance on the basis of its equivalence with the class of &unambiguous events), theorems 2 and **3** are naturally read as describing epistemic pmsuppositions of the CEU model. In particular, for the CEU-model to be applicable, the decision maker's probabilistic beliefs must range over an algebra.

– It may seem hard to imagine how capacities could possibly be episternically restrictive, since their definition.seems to involve only trivial assumptions (essentially monotonicity). Such an intuition forgets, however, that capacities acquire decision-theoretic meaning only as parameters of Choquet integrals $x \mapsto \int x d\nu$, a point argued extensively in Sarin-Wakker (1995). The class of Choquet integrals, as well as the class of preference orders it serves to represent, is characterized by non-trivial properties which a priori might well be restrictive. –

The analysis of this paper has been special in two dimensions: it has focused on the **CEU** model, and it has been concerned with unconditional probabilistic beliefs. An analysis more general in both respects will be pursued in future work (Nehring 1997); it will entail the proposed definition A, as a special case.

APPENDIX

Proof of Fact 2.

By complexifying example 1, this can be shown with the help of the following lemma.

Lemma 4 *Suppose $\mathcal{A} \in 2^S$ has the following three properties:*

i) $\emptyset \in \mathcal{A}$,

ii) $A \in \mathcal{A}$ *implies* $A^c \in \mathcal{A}$,

iii) $A, B \in \mathcal{A} \setminus \{\emptyset\}$ *and* $A \cap B = \emptyset$ *imply* $B = A^c$.

Suppose also that $q : \mathcal{A} \to [0,1]$ *satisfies, for* all $A \in \mathcal{A}$:

i) $q(0) = 0$

ii) $q(A) > 0$ *if* $A \neq \emptyset$, *and*

iii) $q(A) + q(A^c) = 1$.

Then $\mathcal{A}$ *is a* λ*-system, and* q *can be extended to a capacity* ν *such that* $\mathcal{A}^2_\nu = \mathcal{A}$.

Proof of lemma.

It is straightforward to verify that $\mathcal{A}$ is a A-system. Define ν on 2^S by $\nu(A) = \sup\{q(E) \mid E \in \mathcal{A}, E \subseteq A)$; following Zhang (1997), ν may be called the "inner measure" of q. The set-function ν is evidently a well-defined capacity; it has the following two properties:

i) $A \in \mathcal{A}$ and $B \subset A^c$ (strictly) implies $\nu(B) = 0$.

ii) $A \in \mathcal{A}$ and $A \subsetneq B \subset S$ imply $\nu(B) = q(A)$.

Verification: i) The assumptions imply $A^c \in \mathcal{A}$, hence, for no $E \subseteq B$, $E \in \mathcal{A}$.

ii) Similarly, the assumptions imply: if $E \subsetneq B$ and $E \in \mathcal{A}$ then $E = A$.

Consider $A \in \mathcal{A}$ and B disjoint from A.

If $B = A^c$, then $\nu(A \cup B) = \nu(A) + \nu(B)$ by assumption ii) on q.

If $B \subset A^c$, then $\nu(A \cup B) = \nu(A) = \nu(A) + \nu(B)$ by properties i) and ii) of ν.

This shows that $\mathcal{A} \subseteq \mathcal{A}^2_\nu$.

Consider now $A \notin \mathcal{A}$. By the assumptions on $\mathcal{A}$, at most one of $\{A, A^c)$ contains some E $\in$ A.

Hence by properties i) and ii) of v, and assumption ii) on q : $\nu(A) + \nu(A^c) < 1$, which shows that $\mathcal{A}_\nu^2 \subseteq$ A. $\square$

Consider now $\boldsymbol{A}$ and q given by the following table, letting $S = T \text{ x } T$ with $T = \{a, b, c)$.

$A \in \mathcal{A}$	$q(A)$
$\{a, b\} \times T$	α
$\{c\} \times T$	$1 - \alpha$
$T \times \{a, b\}$	β
$T \text{ x } \{c\}$	$1 - \beta$
$\{b, c\} \times \{b, c\}$	7
$(\{a) \times T) \cup (T \times \{a))$	$1 - \gamma$
$\emptyset$	0
$T \times T$	1

A is easily checked to satisfy the assumptions of the lemma; q satisfies the assumptions as well whenever $a, \beta, \gamma \in (0, 1)$. Let ν denote the inner measure induced by A and q. Then ν is probabilistically *incoherent* on $\mathcal{A}_\nu^2 =$ A whenever $a + \beta + \gamma < 1$.

This is seen as follows. Suppose q (= ν on A) has an additive extension p on 2^S.

Then $p(\{(c, c))) \geq 1 - p(\{a, b\} \text{ x } T) - p(T \text{ x } \{a, b\}) = 1 - \alpha - \beta$, but also $p(\{(c, c))) \leq \gamma$, which implies $1 \leq a + \beta + \gamma$. $\blacksquare$

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