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Revealed Unawareness

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Abstract

I develop awareness-dependent subjective expected utility by taking unawareness structures introduced in Heifetz, Meier, and Schipper (2006, 2008, 2009) as primitives in the Anscombe-Aumann approach to subjective expected utility. I observe that a decision maker is unaware of an event if and only if her choices reveal that the event is "null" and the negation of the event is "null". Moreover, I characterize "impersonal" expected utility that is behaviorally indistinguishable from awareness-dependent subject expected utility and assigns probability zero to some subsets of states that are not necessarily events. I discuss in what sense impersonal expected utility can not represent unawareness.

Keywords: Unawareness, awareness, unforeseen contingencies, null, zero probability, subjective expected utility, Anscombe-Aumann, small worlds, extensionality of acts, event exchangeability.

JEL-Classifications: C70, C72, D80, D82.

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1 Introduction

Unawareness refers to the lack of conception rather than the lack of information. There is a fundamental difference between uncertainty about which event obtains and the inability to conceive of some events. In the literature, unawareness has been defined epistemically using syntactic and semantic approaches.[1] While epistemic characterizations are conceptually insightful, the behavioral content of unawareness remains unclear. For instance, a referee of a recent report on Heifetz, Meier, and Schipper (2010a) wrote "It has become a folk wisdom among readers of this literature that unawareness is often nothing but another name for 0-probability belief. ... Is unawareness really nothing but another name for 0-probability belief? I don't know."

Heifetz, Meier, and Schipper (2006, 2008, 2009) introduced a syntax-free semantics of unawareness using state-spaces familiar to economists, decision theorists, and game theorists.[2] Instead of one state-space, it consists of a lattice of disjoint spaces, where every space in the lattice captures one particular horizon of meanings or propositions. Higher spaces capture wider horizons, in which states correspond to situations described by a richer vocabulary. In the present paper, I replace the standard state-space in the Anscombe and Aumann (1963) approach to subjective utility theory by a lattice of spaces. This is done because Dekel, Lipman, and Rustichini (1998) showed that standard state-spaces preclude unawareness while Heifetz, Meier, and Schipper (2006) showed that non-trivial unawareness obtains in a lattice of spaces. In this richer framework, I am able to characterize awareness-dependent subjective expected utility. I choose the Anscombe-Aumann approach not because I think it is the most natural one in the context of unawareness but because it is perhaps the most "standard" approach and starting point. Apart from the lattice of spaces, the setting should be entirely familiar and thus easily accessible to the reader. The message I like to convey is that unawareness structures lend themselves in a straight-forward way as primitives in subjective expected utility theory.

Acts are now defined on the union of all spaces and do not necessarily conform

[1]For a comprehensive bibliography see http://www.econ.ucdavis.edu/faculty/schipper/unaw.htm

[2]Apart from having a syntax-free semantics, Heifetz, Meier, and Schipper (2006, 2008) generalize Modica and Rustichini (1999) and a version of Fagin and Halpern (1988) to the multi-agent case. The precise connection between Fagin and Halpern (1988), Modica and Rustichini (1999), Halpern (2001) and Heifetz, Meier, and Schipper (2006) is understood from Halpern and Rêgo (2008) and Heifetz, Meier, and Schipper (2008). The connection between Heifetz, Meier, and Schipper (2006, 2008) and Galanis (2007) is explored in Galanis (2008). The relationship between Board and Chung (2009) and Heifetz, Meier, and Schipper (2006) is studied in Board, Chung, and Schipper (2009). The connections to the models of Li (2009) and Feinberg (2009) are yet to be explored.

anymore to the principle of extensionality. That is, in my approach the interpretation of the very same act depends on the awareness of the decision maker and the decision maker may evaluate acts differently depending on her awareness. For instance, consider a potential investor who considers the act "invest in firm X". Firm X is a bundle of potential opportunities and liabilities, which depend on the states of nature. Which of these opportunities and liabilities the investor has in mind is determined by her awareness of these events. An investor being aware of a potential law suit that involves the firm but unaware of a potential innovation that may enhance the value of the firm may evaluate the act differently than an investor who is unaware of the former but aware of the latter. (See Heifetz, Meier, and Schipper (2009) and Meier and Schipper (2010) for the analysis of speculative trade in such a setting.)

Preferences of the decision maker are defined on those modified acts, one preference relation for each awareness level so that the same decision maker at different awareness levels can be compared. Standard properties on preferences are imposed for each awareness level and an additional property is imposed that confines the extensionality of an act to the awareness level of the decision maker. An awareness-dependent subjective expected utility representation is then characterized in an embarrassingly straight-forward way. Indeed, the first positive main message of this paper for the applied economist may be that it is straight-forward to characterize subjective expected utility in unawareness structures. This closes an important gap in the literature as I do not know of any other choice-theoretic model that allows for non-trivial unawareness satisfying epistemic properties introduced in Fagin and Halpern (1988), Modica and Rustichini (1999) and Dekel, Lipman, and Rustichini (1998). In the literature on choice theory, non-trivial unawareness is precluded due to the use of standard state-space or it is not known whether non-trivial unawareness obtains. In contrast, unawareness is defined epistemically in the literature on unawareness but no choice-theoretic characterization has been provided. This critique applies also to our own prior work. Originally, just epistemic properties of unawareness structures have been studied in Heifetz, Meier, and Schipper (2006). Logical foundations have been provided by Halpern and Rego (2008) and Heifetz, Meier, and Schipper (2008). Unawareness structures have been applied to speculative trade in Heifetz, Meier, and Schipper (2009) and Meier and Schipper (2010), to Bayesian games in Heifetz, Meier, and Schipper (2010b), and to dynamic games and an application of verifiable communication in Heifetz, Meier, and Schipper (2010a). Yet, until now notions of utility and beliefs have been taken as primitives in those structures. The current paper shows that they can be derived from choices within unawareness structures.

The second goal is to apply the representation theorem to analyze the behavioral implications of unawareness. Consider an outside observer who wishes to know from the choices of a decision maker conforming to the Anscombe-Aumann approach whether she is unaware of an event or not. It is shown that a decision maker is unaware of the event if and only if her choices reveal that the event is "null" *and* the negation of the event is "null". This distinguishes unawareness from subjective probability zero belief, for which the event is null but its negation cannot be null. Thus unawareness does have behavioral implications different from probability zero belief. The following example illustrates the point: Consider a potential buyer of a firm. Agreements on the change of ownerships of private firms may be very complex involving many pages of legal documents. It is not inconceivable that the buyer may be miss certain important clauses and may not think about them when contemplating the transaction. In particular, the buyer may be unaware of a specific costly future law suit that the firm may or may not be involved in. Assume that the buyer can choose among two contracts. Under contract 1 the potential law suit is the buyer's responsibility. Under contract 2 the potential law suit is the seller's responsibility. Otherwise both contracts are the same in content. Being indifferent between both contracts is consistent with assigning probability zero to the event of the law suit. Assume now that a third contract is available. Under contract 3 the potential law suit is the seller's responsibility but the seller receives an additional compensation from the buyer in the event that the law suit does not obtain. Apart from this clause, the content of contract 3 is the same as the other contracts. Being indifferent between contract 3 and 2 is consistent with assigning probability zero to the event of "no law suit". Indifference between *all* three contracts is consistent with being unaware of "law suit" but not with assigning probability zero to either the the events "law suit" or "no law suit" because probability zero can not be assigned to an event and its negation. We also provide a characterization of unawareness by (an extreme form of) event exchangeability as introduced in de Finetti (1937) and further studied for instance in Chew and Sagi (2006). A decision maker is unaware of an event if and only if any pair of disjoint events with the same expressiveness are exchangeable.

The third goal of this note is to analyze in what sense unawareness could be "modeled" nevertheless by probability zero. I characterize "impersonal" expected utility that is behaviorally indistinguishable from awareness-dependent expected utility. The representation delivers a probability measure on the "flattened state-space", the union of all state-spaces in the lattice, that assigns zero probability not only to null events but also to any subsets of states (that may not necessarily be events) that the decision maker does "not reason" about. I argue that such a probability zero measure can not be interpreted anymore as a "personal" or "subjective" belief but it is an artificial construct ascribed to the decision maker by the modeler. In this sense, while being behaviorally indistinguishable from unawareness, the probability zero approach misses the main goal of subjective expected utility theory, namely to ability to ascribe "personal" or "subjective" belief to a decision maker based on his choices.

I also discuss a model in which facing of a certain act may already influence the awareness of a decision maker. That is, a decision maker's awareness may depend on how fine-grained the description of an act is. Intuitively, above investor contemplating "invest in firm X" is now assumed to read all the fine-print associated with this act. In this case, revealing unawareness becomes very limited.

Awareness-dependent expected utility may be seen as a step towards analyzing Savage's (1954) "small worlds" assumption. Savage (1954, p. 82-83) used the term for the space of states of nature to indicate the "...practical necessity to confining attention, or isolating, relatively simple situations...". Savage (1954, p. 16) felt that he "was unable to formulate criteria for selecting these small worlds...". While I can not deliver such a criterion either, my approach allows the modeler to analyze the decision maker in various sets of "small worlds" which are partially ordered by their richness. The representation theorem should be interpreted either from the modeler's (bird's) point of view as contemplating a decision maker's (admittedly counterfactual) choices at various awareness levels, or from the decision maker's point of view conditional on her awareness level.[3]

The paper is organized as follows: In Section 2, I present primitives of unawareness structures. In Section 3, I develop awareness-dependent subjective expected utility with confined extensionality of acts. This is applied to the problem of revealing unawareness in Section 4. In Section 5, I characterize impersonal expected utility and discuss its relation to awareness-dependent subjective expected utility. In Section 6, I finish with a discussion of extensions and the related literature. Proofs, although mostly straightforward once the unawareness structure is in place, are collected in the appendix to show where they depart from the standard Anscombe-Aumann approach.

[3] In an extended model with states of the world (as in Heifetz, Meier, and Schipper, 2009) rather than states of nature, i.e., in which states also encode the preference and thus beliefs of the decision maker, the decision maker at a given awareness level could also reason about her own decisions at lower awareness levels.

2 Primitives of Unawareness Structures

2.1 State-Spaces

Let $\mathcal{S} = \{S_\alpha\}_{\alpha \in \mathcal{A}}$ be a finite lattice of disjoint *state-spaces*, with the partial order $\preceq$ on $\mathcal{S}$. For simplicity we assume in this paper that each S is finite. If S_α and S_β are such that $S_\alpha \succeq S_\beta$ we say that "S_α is more expressive than S_β – states of S_α describe situations with a richer vocabulary than states of S_β ".[4] Denote by $\Omega = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$ the union of these spaces.

Spaces in the lattice can be more or less "rich" in terms of facts that may or may not obtain in them. The partial order relates to the "richness" of spaces. The upmost space of the lattice may be interpreted as the "objective" state-space. Its states encompass full descriptions.

2.2 Projections

For every S and S' such that $S' \succeq S$, there is a surjective projection $r_S^{S'} : S' \to S$, where r_S^S is the identity. ("$r_S^{S'}(\omega)$ is the restriction of the description ω to the more limited vocabulary of S.") Note that the cardinality of S is smaller than or equal to the cardinality of S'. We require the projections to commute: If $S'' \succeq S' \succeq S$ then $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$. If $\omega \in S'$, denote $\omega_S = r_S^{S'}(\omega)$. If $D \subseteq S'$, denote $D_S = \{\omega_S : \omega \in D\}$.

Projections "translate" states in "more expressive" spaces to states in "less expressive" spaces by "erasing" facts that can not be expressed in a lower space.

These surjective projections may embody Savage's idea that "(i)t may be well, however, to emphasize that a state of the smaller world corresponds not to a state of the larger, but to a set of states" (Savage, 1954, p. 9).

2.3 Events

Denote $g(S) = \{S' : S' \succeq S\}$. For $D \subseteq S$, denote $D^{\uparrow} = \bigcup_{S' \in g(S)} \left(r_S^{S'}\right)^{-1}(D)$. ("All the extensions of descriptions in D to at least as expressive vocabularies.")

An *event* is a pair (E, S), where $E = D^{\uparrow}$ with $D \subseteq S$, where $S \in \mathcal{S}$. D is called the *base* and S the *base-space* of (E, S), denoted by $S(E)$. If $E \neq \emptyset$, then S is uniquely

[4] Here and in what follows, phrases within quotation marks hint at intended interpretations, but we emphasize that these interpretations are not part of the definition of the set-theoretic structure.

determined by E and, abusing notation, we write E for (E, S). Otherwise, we write $\emptyset^S$ for $(\emptyset, S)$. Note that not every subset of Ω is an event.

Some fact may obtain in a subset of a space. Then this fact should be also "expressible" in "more expressive" spaces. Therefore the event contains not only the particular subset but also its inverse images in "more expressive" spaces.

Let Σ be the set of events of Ω. Note that unless $\mathcal{S}$ is a singleton, Σ is not an algebra because it contains distinct vacuous events $\emptyset^S$ for all $S \in \mathcal{S}$. These vacuous events correspond to contradictions with differing "expressive power".

2.4 Negation

If $(D^{\uparrow}, S)$ is an event where $D \subseteq S$, the negation $\neg(D^{\uparrow}, S)$ of $(D^{\uparrow}, S)$ is defined by $\neg(D^{\uparrow}, S) := ((S \setminus D)^{\uparrow}, S)$. Note, that by this definition, the negation of a (measurable) event is a (measurable) event. Abusing notation, we write $\neg D^{\uparrow} := (S \setminus D)^{\uparrow}$. Note that by our notational convention, we have $\neg S^{\uparrow} = \emptyset^S$ and $\neg \emptyset^S = S^{\uparrow}$, for each space $S \in \mathcal{S}$. The event $\emptyset^S$ should be interpreted as a "logical contradiction phrased with the expressive power available in S." $\neg D^{\uparrow}$ is typically a proper subset of the complement $\Omega \setminus D^{\uparrow}$. That is, $(S \setminus D)^{\uparrow} \subsetneqq \Omega \setminus D^{\uparrow}$.

Intuitively, there may be states in which the description of an event $D^{\uparrow}$ is both expressible and valid – these are the states in $D^{\uparrow}$; there may be states in which its description is expressible but invalid – these are the states in $\neg D^{\uparrow}$; and there may be states in which neither its description nor its negation are expressible – these are the states in

$$\Omega \setminus \left(D^{\uparrow} \cup \neg D^{\uparrow}\right) = \Omega \setminus S\left(D^{\uparrow}\right)^{\uparrow}.$$

Thus our structure is not a standard state-space model in the sense of Dekel, Lipman, and Rustichini (1998).

2.5 Conjunction and Disjunction

If $\left\{\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right)\right\}_{\lambda \in L}$ is a collection of events (with $D_{\lambda} \subseteq S_{\lambda}$, for $\lambda \in L$), their conjunction $\bigwedge_{\lambda \in L}\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right)$ is defined by $\bigwedge_{\lambda \in L}\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right) := \left(\left(\bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}\right), \sup_{\lambda \in L} S_{\lambda}\right)$. Note, that since $\mathcal{S}$ is a *complete* lattice, $\sup_{\lambda \in L} S_{\lambda}$ exists. If $S = \sup_{\lambda \in L} S_{\lambda}$, then we have $\left(\bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}\right) = \left(\bigcap_{\lambda \in L}\left(\left(r_{S_{\lambda}}^{S}\right)^{-1}(D_{\lambda})\right)\right)^{\uparrow}$. Again, abusing notation, we write $\bigwedge_{\lambda \in L} D_{\lambda}^{\uparrow} := \bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}$ (we will therefore use the conjunction symbol $\wedge$ and the intersection symbol $\cap$

interchangeably).

We define the relation $\subseteq$ between events (E,S) and (F,S'), by $(E,S) \subseteq (F,S')$ if and only if $E \subseteq F$ as sets *and* $S' \preceq S$. If $E \neq \emptyset$, we have that $(E,S) \subseteq (F,S')$ if and only if $E \subseteq F$ as sets. Note however that for $E = \emptyset^S$ we have $(E,S) \subseteq (F,S')$ if and only if $S' \preceq S$. Hence we can write $E \subseteq F$ instead of $(E,S) \subseteq (F,S')$ as long as we keep in mind that in the case of $E = \emptyset^S$ we have $\emptyset^S \subseteq F$ if and only if $S \succeq S(F)$. It follows from these definitions that for events E and F, $E \subseteq F$ is equivalent to $\neg F \subseteq \neg E$ only when E and F have the same base, i.e., $S(E) = S(F)$.

The disjunction of $\left\{D_\lambda^\uparrow\right\}_{\lambda\in L}$ is defined by the de Morgan law $\bigvee_{\lambda\in L} D_\lambda^\uparrow = \neg\left(\bigwedge_{\lambda\in L} \neg\left(D_\lambda^\uparrow\right)\right)$. Typically $\bigvee_{\lambda\in L} D_\lambda^\uparrow \subsetneqq \bigcup_{\lambda\in L} D_\lambda^\uparrow$, and if all D_λ are nonempty we have that $\bigvee_{\lambda\in L} D_\lambda^\uparrow = \bigcup_{\lambda\in L} D_\lambda^\uparrow$ holds if and only if all the $D_\lambda^\uparrow$ have the same base-space. Note, that by these definitions, the conjunction and disjunction of events is a event.

2.6 Probability Measures

Let $\Delta(S)$ be the set of probability measures on S.

For a probability measure $\mu \in \Delta(S')$, the marginal $\mu_{|S}$ of μ on $S \preceq S'$ is defined by

$$\mu_{|S}(D) := \mu\left(\left(r_S^{S'}\right)^{-1}(D)\right), \quad D \subseteq S.$$

Let S_μ be the space on which μ is a probability measure. Whenever $S_\mu \succeq S(E)$ then we abuse notation slightly and write

$$\mu(E) = \mu(E \cap S_\mu).$$

If $S(E) \npreceq S_\mu$, then we say that $\mu(E)$ is undefined.

2.7 Unawareness

Definition 1 (Unawareness) *We say that a decision maker is unaware of the event E if her belief is represented by a probability measure $\mu \in \Delta(S)$ with $S \nsucceq S(E)$.*

This follows the definition of unawareness in a more sophisticated model in which states of the world rather than states of nature are considered. That is, states also capture beliefs of agents. In such a richer setting, unawareness of an agent may differ from state to state even within the same space. Unawareness operators on events can be defined and it can be shown that all properties on unawareness that have been suggested in the literature indeed obtain. See Heifetz, Meier, and Schipper (2009) for details.

Since $S(E) = S(\neg E)$ by definition, we have the following observation.

Remark 1 (Symmetry) *A decision maker is unaware of the event E if and only if she is unaware of the event $\neg E$.*

3 Subjective Expected Utility

3.1 Outcomes

Let X be an arbitrary space of outcomes or prizes. We denote by $\Delta(X)$ the set of simple probability measures on X, i.e., the set of finitely additive probability measure with finite support (see Fishburn, 1970, Section 8.2). For $p \in \Delta(X)$, we denote by *supp*(p) the support of p.

3.2 Acts

An *act* is a function $f : \Omega \longrightarrow \Delta(X)$.

Note that different from Anscombe-Aumann acts, f is not defined on just one state-space but on the union of spaces Ω. This is interpreted as follows: Let's say an individual investing in a firm (e.g., the act f) perceives a lottery of outcomes. Which lottery obtains depends on which event obtains. She may be unaware of some events but not of others. If the state $\omega \in S$ obtains and her awareness level is given by space $S' \prec S$, then the lottery perceived is not $f(\omega)$ but $f(\omega_{S'})$. An act at a certain state may mean different things to different agents depending on their awareness level. We aim to capture the awareness level of the decision maker by her preferences only and not by the acts she is facing. That's why acts are labels whose interpretation depends on the awareness of the decision maker. Alternatively, we could assume that acts faced by the decision maker may influence her awareness. This alternative assumption will be discussed in Section 6.2.

For any event E and acts f and g, define a *composite act* $f_E g$ by

$$f_E g(\omega) = \begin{cases} f(\omega) & \text{if } \omega \in E \\ g(\omega) & \text{otherwise.} \end{cases}$$

Note that different from composite acts in the Anscombe-Aumann approach, g is not only prescribed on the negation of E but also on all states that are neither in E nor in $\neg E$.

For any collection of pairwise disjoint events $E_1, E_2, ..., E_n \subset \Sigma$ and acts $f^1, f^2, ..., f^n, g \in \mathcal{A}$, let $f^1_{E_1} f^2_{E_2} ... f^n_{E_n} g$ denote the composite act that yields $f^i(\omega)$ if $\omega \in E_i$ for $i = 1, ..., n$, and $g(\omega)$ otherwise.

If f and g are acts and $\alpha \in [0,1]$ then $\alpha f + (1-\alpha)g$ is an act defined pointwise by $(\alpha f + (1-\alpha)g)(\omega) = \alpha f(\omega) + (1-\alpha)g(\omega)$ for all $\omega \in \Omega$. Let $\mathcal{A}$ denote the set of all acts.

Remark 2 *$\mathcal{A}$ is a* mixture space*. I.e., for all $f, g \in \mathcal{A}$ and all $\alpha, \beta \in [0,1]$, (i) $1f + 0g = f$, (ii) $\alpha f + (1-\alpha)g = (1-\alpha)g + \alpha f$, and (iii) $\alpha[\beta f + (1-\beta)g] + (1-\alpha)g = \alpha\beta f + (1-\alpha\beta)g$.*

3.3 Preferences

The decision maker's choices are represented by a collection of preferences, $\{\succsim_S\}_{S \in \mathcal{S}}$, one for each space $S \in \mathcal{S}$ with each $\succsim_S$ defined on $\mathcal{A}$.

For each $S \in \mathcal{S}$, strict preference, $\succ_S$, is defined on $\mathcal{A}$ by $\succsim_S$ and not $\precsim_S$. Indifference, $\sim_S$, is defined on $\mathcal{A}$ by $\succsim_S$ and $\precsim_S$.

Preferences are allowed to vary with state-spaces. The idea is that an act f may be preferred over the act g at a certain awareness level but g may be preferred over f at a different awareness level. E.g., suppose to you prefer onions over any other food. Yet, if you were aware that Dr. Weissbarth of Stockton University suspects onions to cause the fatal disease cuppacuppitis then you may rank onions below some other vegetable.

3.4 Assumptions on Preferences

The following five well known properties are standard in the Anscombe-Aumann approach, but adapted here to the lattice of state-spaces.

Property 1 (Weak Order) *For all $S \in \mathcal{S}$, $\succ_S$ is complete and transitive.*

Property 2 (Archimedean Continuity) *For all $S \in \mathcal{S}$ and $f, g, h \in \mathcal{A}$, if $f \succ_S g \succ_S h$, then there exists $\alpha, \beta \in (0,1)$ such that $\alpha f + (1-\alpha)h \succ_S g \succ_S \beta f + (1-\beta)h$.*

Property 3 (Independence) *For all $S \in \mathcal{S}$, $f, g, h \in \mathcal{A}$ and $\alpha \in (0,1)$, if $f \succ_S g$ then $\alpha f + (1-\alpha)h \succ_S \alpha g + (1-\alpha)h$.*

Definition 2 (Null Event) *An event E is S-*null *if $S(E) \preceq S$ and $f_E g \sim_S h_E g$ for all $f, g, h \in \mathcal{A}$. A state ω is S-null if $\{\omega\}^{\uparrow}$ is S-null. An event E is S-nonnull if $S(E) \preceq S$ and $f_E g \succ_S h_E g$ for some $f, g, h \in \mathcal{A}$.*

This definition generalizes Savage's notion of null-event to our structure. We will show that it captures "events conceived but assigned probability zero" rather than "events not conceived of". We think that indeed this is in the spirit of Savage's notion of null-event because in Savage "events not conceived of" are simply not considered in the decision maker's small world.

Remark 3 *For each $S \in \mathcal{S}$:*

(i) For any event F with $S(F) \npreceq S$, F is neither S-null nor S-nonnull.

(ii) $\emptyset^{S'}$ is S-null if and only if $S' \preceq S$.

Property 4 (Nondegeneracy) *For all $S \in \mathcal{S}$ there exist $f, g \in \mathcal{A}$ such that $f \succ_S g$.*

Property 5 (State Independence) *If $f \in \mathcal{A}$, $p, q \in \Delta(X)$ are such that $p_{\{\omega\}^\uparrow} f \succ_S q_{\{\omega\}^\uparrow} f$ for some ω, then for all S-nonnull ω' we have $p_{\{\omega'\}^\uparrow} f \succ_S q_{\{\omega'\}^\uparrow} f$*

If the decision maker has preference $\succsim_S$, then the following property suggests the interpretation that she has *"awareness level" S*. This property is trivially satisfied in standard state-space models. Yet, it is key in the current approach.

Property 6 (Confined Extensionality) *For any $S \in \mathcal{S}$, if $f, g \in \mathcal{A}$ are such that $f(\omega) = g(\omega)$ for all $\omega \in S$, then $f \sim_S g$.*

The examples in Figures 1 and 2 illustrate Property 6. There are only two spaces, S_1 and S_2. Different shades represent different outcomes. For instance, in Figure 1, the left composite act yields "grey" in state ω_1 but "white" in states ω_2 and ω_3. If the decision maker's awareness level is given by the lower space S_2, then she does not care what happens in the upper space because she is unaware of those events. Figure 2 illustrates that if the decision maker's awareness level is given by the upper space S_1, then she cares only about states in S_1. She neglects whatever happens in lower spaces presumably because she fully understands that she is aware.

The proofs of the following three remarks can be found in the appendix.

Remark 4 *Property 6 implies: For all events E, if $S(E) \npreceq S$, then*

(i) $f_E g \sim_S g$ for all $f, g \in \mathcal{A}$.

Figure 1: Illustration of Property 6

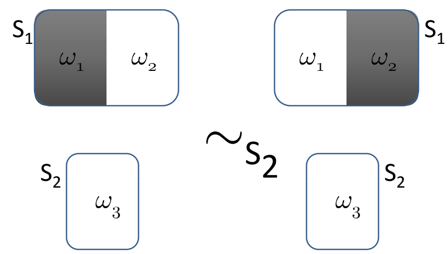
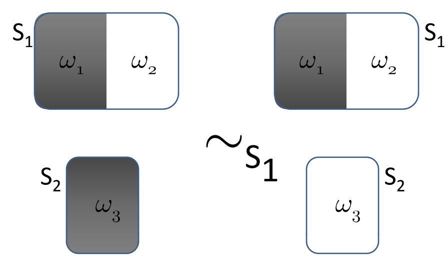


Figure 2: Illustration of Property 6



(ii) $f_E g \sim_S h_E g$ *and* $f_{\neg E} g \sim_S h_{\neg E} g$ *for all* $f, g, h \in \mathcal{A}$.

Remark 5 *Properties 1 and 6 imply if* $S' \preceq S$, *then* $f_{S'\uparrow} g \succsim_S h_{S'\uparrow} g$ *if and only if* $f \succsim_S h$.

Remark 6 *Properties 1, 4, and 6 imply that for each* $S \in \mathcal{S}$ *there exists a state* $\omega \in S$ *that is* S*-nonnull.*

3.5 Awareness-Dependent Subjective Expected Utility

Definition 3 (ASEU) *We say that* $\{\succsim_S\}_{S \in \mathcal{S}}$ *admits an* awareness-dependent subjective expected utility *(ASEU) representation if there exists a collection of nonconstant von Neumann-Morgenstern utility functions* $\{u_S : X \longrightarrow \mathbb{R}\}_{S \in \mathcal{S}}$ *and a collection of probability measures* $\{\mu_S \in \Delta(S)\}_{S \in \mathcal{S}}$ *such that for all* $S \in \mathcal{S}$ *and* $f, g \in \mathcal{A}$,

$$f \succ_S g \text{ if and only if } \int_S u_S \circ f d\mu_S > \int_S u_S \circ g d\mu_S,$$

and

$$\mu_S(\{\omega\}) = 0 \text{ if and only if } \omega \text{ is } S\text{-null.}$$

Moreover, if there exists another collection of von Neumann-Morgenstern utility functions $\{v_S : X \longrightarrow \mathbb{R}\}_{S\in\mathcal{S}}$ and a collection of probability measures $\{\nu_S \in \Delta(S)\}_{S\in\mathcal{S}}$, then for any $S \in \mathcal{S}$ there are constants $a_S > 0$ and b_S such that $v_S(x) = a_S u_S(x) + b_S$ and $\nu_S = \mu_S$.

The specification outlined so far allows me to apply the Anscombe and Aumann (1963) approach to each $S \in \mathcal{S}$ separately to prove in the appendix the following result.

Theorem 1 (Representation) *$\{\succsim_S\}_{S\in\mathcal{S}}$ admits an awareness-dependent subjective expected utility representation if and only if it satisfies Properties 1 to 6.*

Definition 4 *An awareness-dependent subjective expected utility representation has* awareness-independent utilities *if or all $S, S' \in \mathcal{S}$ there exist constants $a_{S'S} > 0$ and $b_{S'S}$ such that $u_S = a_{S'S} u_{S'} + b_{S'S}$.*

If an awareness-dependent subjective expected utility representation has awareness-independent utilities, then the utility function u_S at awareness level S is also a utility function for any awareness level $S' \in \mathcal{S}$ because conditional on each awareness level, utilities are unique up to affine transformations. I believe that in reality this may not be satisfied except in rather special cases.

Property 7 (Awareness-Independent Ranking) *For $p, q \in \Delta(X)$, $p \succ_S q$ if and only if $p \succ_{S'} q$ for all $S', S \in \mathcal{S}$.*

Proposition 1 *$\{\succsim_S\}_{S\in\mathcal{S}}$ admits an awareness-dependent subjective expected utility representation with awareness-independent utilities if and only if it satisfies Properties 1 to 7.*

4 Revealed Unawareness

Suppose an outside observer wishes to infer from choices of a decision maker whether she is unaware of an event E or not. The outside observer does not know the preferences of the decision maker nor does he know which preference relation is related to which awareness level (the mapping from state-spaces to binary relations over acts). All he knows is that the choices of the decision maker are summarized by one preference relation in $\{\succsim_S\}_{S\in\mathcal{S}}$

satisfying Properties 1 to 6. We denote by $\precsim$ the observed choices and define $\prec$ and $\sim$ as usual.

The following behavioral implications of unawareness are proved in the appendix.

Proposition 2 (Revealed Unawareness) *Let $\{\succsim_S\}_{S\in\mathcal{S}}$ satisfy Properties 1 to 6. A decision maker is unaware of the event E if and only if for all events F with $S(F) = S(E)$, $f_F g \sim h_F g$ for all $f, g, h \in \mathcal{A}$.*

Proposition 2 may be restated using event exchangeability (de Finetti, 1937) albeit in an extreme form.

Definition 5 (Event Exchangeability) *A pair of disjoint events $E, E' \in \Sigma$ are exchangeable if for any $f, g, h \in \mathcal{A}$, $f_E h_{E'} g \sim h_E f_{E'} g$.*

Corollary 1 *Let $\{\succsim_S\}_{S\in\mathcal{S}}$ satisfy Properties 1 to 6. A decision maker is unaware of the event E if and only if any pair of disjoint events $F, F' \in \Sigma$ such that $S(F) = S(F') = S(E)$ are exchangeable.*

It is known that all null events are exchangeable and that in standard state-spaces exchangeability expresses a notion of equal likelihood (see Chew and Sagi, 2006). Unawareness structures allow for an extreme form of event exchangeability where *all* pairs of disjoint events with the same base-space may be exchangeable. The decision maker is "equally unaware" of all of them. Being unaware of one event means being unaware of any other event with the same base-space.

Consider now an outside observer who wishes to infer from choices of a decision maker whether she attaches subjective probability zero belief to the event E or whether she is unaware of the event E. The following proposition states the different behavioral implications of unawareness and subjective probability zero belief. With the structure in place, the proof is straight-forward.

Proposition 3 (Null versus Unawareness) *Let $\{\succsim_S\}_{S\in\mathcal{S}}$ satisfy Properties 1 to 6.*

(i) Unawareness: A decision maker is unaware of the event E if and only if $f_E g \sim h_E g$ and $f_{\neg E} g \sim h_{\neg E} g$ for all $f, g, h \in \mathcal{A}$.

(ii) Subjective Probability Zero Belief: A decision maker ascribes subjective probability zero to the event E if and only if $f_E g \sim h_E g$ and not $f_{\neg E} g \sim h_{\neg E} g$ for all $f, g, h \in \mathcal{A}$.

A decision maker is unaware of an event E if and only if she considers both E and the negation of E to be "null". This is different from assigning subjective probability zero to the event E which is characterized by considering E to be null but the negation of E to be nonnull.

5 Impersonal Expected Utility

In what sense could a probability zero approach "model" behavior under unawareness nevertheless?

Given a lattice of spaces $\mathcal{S}$, I follow Heifetz, Meier and Schipper (2009) in defining the *flattened state-space* associated with $\mathcal{S}$ simply by the union of all spaces, $\Omega = \bigcup_{S \in \mathcal{S}} S$. Note that the set of all subsets 2^{Ω} may contain elements that are not events in the unawareness structure (unless the lattice is trivially a singleton). That is, typically $\Sigma \subsetneqq 2^{\Omega}$.

A probability measure μ_S on S is extended to a probability measure φ_S on the flattened state-space Ω by

$$\varphi_S(E) \;:=\; \begin{cases} \mu_S(E \cap S) & \text{if } E \cap S \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Note that Ω is just a standard state-space. The extended probability measure does not have full support. It is extended by assigning probability zero to all subsets of Ω that are "not reasoned" about by the decision maker. Such subsets may not be events in the unawareness structure.

Consider a composite act of the form

$$f_{\{\omega\}}g(\omega') = \begin{cases} f(\omega') & \text{if } \omega = \omega' \\ g(\omega') & \text{otherwise.} \end{cases} \tag{1}$$

Although $\{\omega\}$ may not be an event in the unawareness structure, we still have $f_{\{\omega\}}g \in \mathcal{A}$ since for every $f, g \in \mathcal{A}$ we can define $h \in \mathcal{A}$ such that $h(\omega) = f(\omega)$ and $h(\omega') = g(\omega')$ for $\omega' \neq \omega$. $f_{\{\omega\}}g = h_{\{\omega\}^{\uparrow}}g$ and $h_{\{\omega\}^{\uparrow}}g \in \mathcal{A}$.

In the following remark is proved in the appendix. It characterizes "null" in the flattened state-space by S-null or unawareness.

Remark 7 *Properties 1 and 6 imply that $f_{\{\omega\}}g \sim_S h_{\{\omega\}}g$ for all $f, g, h \in \mathcal{A}$ if and only if $\omega \in \Omega$ is S-null or $\omega \notin S$.*

Definition 6 (IEU) *We say that* $\{\succsim_S\}_{S\in\mathcal{S}}$ *admits an* impersonal expected utility *(IEU) representation if there exists a collection of nonconstant von Neumann-Morgenstern utility functions* $\{u_S : X \longrightarrow \mathbb{R}\}_{S\in\mathcal{S}}$ *and a collection of probability measures* $\{\varphi_S \in \Delta(\Omega)\}_{S\in\mathcal{S}}$ *such that for all* $f, g \in \mathcal{A}$,

$$f \succ_S g \text{ if and only if } \int_\Omega u_S \circ f d\varphi_S > \int_\Omega u_S \circ g d\varphi_S,$$

and

$$\varphi_S(\{\omega\}) = 0 \text{ if and only if } \omega \text{ is } S\text{-null or } \omega \notin S.$$

Moreover, if there exists another collection of von Neumann-Morgenstern utility functions $\{v_S : X \longrightarrow \mathbb{R}\}_{S\in\mathcal{S}}$ *and a collection of probability measures* $\{\phi_S \in \Delta(\Omega)\}_{S\in\mathcal{S}}$*, then for any* $S \in \mathcal{S}$ *there are constants* $a_S > 0$ *and* b_S *such that* $v_S(x) = a_S u_S(x) + b_S$ *and* $\phi_S = \varphi_S$.

Compared to awareness-dependent subjective expected utility, we integrate over the union of spaces Ω and use the extended probability measure φ_S in impersonal expected utility. Moreover, for any state ω that is not "reasoned about" by the decision maker with awareness level S, the extended probability measure φ_S assigns probability zero as well.

Theorem 2 (Characterization) $\{\succsim_S\}_{S\in\mathcal{S}}$ *admits an impersonal expected utility representation if and only if it satisfies Properties 1 to 6.*

Corollary 2 $\{\succsim_S\}_{S\in\mathcal{S}}$ *admits an impersonal expected utility representation if and only if it admits an awareness-dependent subjective expected utility representation.*

Corollary 3 *Let* $\{\succsim_S\}_{S\in\mathcal{S}}$ *satisfy Properties 1 to 6. Denote by* $\{\mu_S \in \Delta(S)\}_{S\in\mathcal{S}}$ *the collection of subjective probability measures from the awareness-dependent subjective expected utility representation of* $\succsim_S$*, and by* $\{\varphi_S \in \Delta(\Omega)\}_{S\in\mathcal{S}}$ *the collection of probability measures from the impersonal representation of* $\succsim_S$*. Then for any* $S \in \mathcal{S}$, $\mu_S(E) = \varphi_S(E)$ *for all events* $E \in \Sigma$ *with* $S(E) \preceq S$.

6 Discussions

6.1 Which representation to select?

How to select among the two alternative representations of choice under unawareness? First, while both Theorem 1 and 2, provide characterizations of Properties 1 to 6, the characterization in Theorem 2 falls short of a representation in the following sense: The representation in Definition 6 does not distinguish between "two kinds" of probability zero. A decision maker assigns probability zero to a state if this state is null or if she is unaware of this state. Both, the notion of null-event and being unaware of an event are represented by probability zero in impersonal expected utility. Yet, we know already from Proposition 3 that unawareness and null have different behavioral implications. Thus, while both awareness-dependent expected utility and impersonal expected utility are behaviorally distinguishable, the representation of Definition 6 is impractical to capture the relevant behavioral distinction between the notions of null and unawareness. It "overburdens" the notion of probability zero by forcing it to represent two behaviorally and conceptually different states of mind: null and unaware. This may limit the use of impersonal expected utility in applications that seek to explicitly work out implications of unawareness.

Second, in order to claim that probability zero "models" behavior under unawareness in applications, we need to consider unawareness structures. However, using the unawareness structure in the first place makes the impersonal expected utility approach obsolete since behaviorally it is indistinguishable from awareness-dependent expected utility but latter but has the advantage of a transparent epistemic interpretation.

Third, probability measures in impersonal expected utility can not be interpreted as a "personal" or "subjective" probabilities of the decision maker. (Hence, the attribute "impersonal".) Statements like "I am assigning probability zero to the event E since I am unaware of it" are nonsensical since the very statement implies that I think about the event E. (Indeed, one of the epistemic properties of unawareness is that if a decision maker is aware that she is unaware of the event E then she is aware of the event E.[5]) Historically, it was precisely the goal of subjective expected utility theory to make sense of statements like "I find the event E more likely than the event F". For me the attraction of subjective expected utility theory is that choices provide a window into the decision maker's reasoning. This attraction is lost with impersonal expected utility but not with awareness-dependent expected utility. In latter representation, it makes sense to interpret the probability measures as "personal" or "subjective" beliefs of a decision maker given her awareness level. In contrast, the probability measures in impersonal expected utility can only be interpreted as an artificial construct ascribed to the decision maker by an outside observer. The issue here is more severe than the usual "as if" as-

[5]This is AU-introspection in Dekel, Lipman, and Rustichini (1998). It obtains in unawareness structures, see Heifetz, Meier, and Schipper (2009, Proposition 3).

sumption in decision theory. In subjective expected utility, the decision maker may not really reason with the subjective probabilities ascribed to her by her choices. But it is not impossible that she could use them for reasoning. Here, in impersonal expected utility, it is impossible that the decision maker uses herself such impersonal probabilities and at the same time be unaware of some events. The impossibility result by Dekel, Lipman, and Rustichini (1998) applies because the flattened state-space is a standard state-space.

Fourth, in a richer model with states of the world, in which states also describe beliefs of the decision maker like in an unawareness belief structure analogous to type spaces in Bayesian games, it can be shown that given a standard type space with zero probability, it is not always possible to find some unawareness belief structure with non-trivial unawareness (see Heifetz, Meier and Schipper, 2009, Section 2.13). That is, not every probability zero model actually "models" unawareness.

Finally, if we allow multiple players to interact in the richer model just mentioned, then the probability zero model needs additional restrictions on how probabilities agree among players. If player's unawareness of an event is "modeled" by assigning probability zero to this event, then she can not believe that others do not assign probability zero to this event etc. These restrictions may become quickly intractable but they fall naturally into place in an unawareness belief structure à la Heifetz, Meier and Schipper (2009).

6.2 When Facing Acts Influences Awareness

Property 6 implies that events of which the decision maker is unaware of do not affect her ranking of acts. This holds even for composite acts that condition on events that the decision maker is unaware of. More generally, it rules out that a decision maker becomes aware of an event merely by facing an act. While this is also the implicit assumption in standard decision theory (i.e., different acts do change the subset of "small worlds"), it may be unrealistic in some situations. Sometimes, when facing an act, a decision maker may become in very subtle ways a bit more careful with the "fine prints" of acts, and this care may lead her to become aware of events. E.g., a buyer facing a decision about whether or not to buy a certain insurance contract may become aware of events when reading all the fine prints of the contract. If ex ante an outside observer does not know how acts affect the awareness of a decision maker, can he still elicit whether or not a decision maker is unaware of an event? To answer this question, I considered also a modified framework. Acts may influence the awareness of a decision maker but the order in which the acts are presented to the decision maker does not.[6]

Denote by $\mathcal{A}_S$ the set of all acts that would make the decision maker at least aware of events with base-space S. We assume that if f and g are in $\mathcal{A}_S$ and E is an event with base-space $S(E) \preceq S$, then the composite act $f_E g$ is in $\mathcal{A}_S$. If $f \in \mathcal{A}_S$ and $g \in \mathcal{A}_{S'}$, and $\alpha \in [0,1]$, then $\alpha f + (1-\alpha)g$ is an act defined pointwise and by the join by $(\alpha f + (1-\alpha)g)(\omega) = \alpha f(\omega) + (1-\alpha)g(\omega) \in \mathcal{A}_{S \vee S'}$ for all $\omega \in \Omega$. Note that $\mathcal{A}_S$ is a mixture space for all $S \in \mathcal{S}$. Note also that $\mathcal{A}_S \subseteq \mathcal{A}_{S'}$ if $S \preceq S'$.

For each $S \in \mathcal{S}$, define a preference relation $\succsim_S$ on $\mathcal{A}_S$ and impose modified Properties 1' through 6' on $\{\succsim_S\}_{S \in \mathcal{S}}$, in which we replace $\mathcal{A}$ by $\mathcal{A}_S$ in each assumption.

When $\mathcal{A}$ is replaced by $\mathcal{A}_S$ in Definition 3, we can prove analogously to Theorem 1 a representation theorem by simply replacing $\mathcal{A}$ by $\mathcal{A}_S$ and Properties 1 to 6 by Properties 1' to 6' in the proof.

This modified theorem can then be used to investigate revealed unawareness analogous to Section 4. That is, we ask whether or not an outside observer can infer from choices alone that the decision maker is unaware of an event E. The outside observer does not know the preferences of the decision maker, nor does he know which preference relation is related to which awareness level (the mapping from state-spaces to binary relations over acts), nor does he know how acts would change the awareness of the decision maker (the mapping from state-spaces to subsets of acts). Clearly, Propositions 2 and 3 do not apply anymore to this setting. When an outside observer presents the decision maker with acts, he may change the decision maker's awareness. In a sense, the outside observer may destroy the unawareness of the decision maker with the experiment to measure it. However, it may still allow the outside observer to measure at least whether or not a decision maker was unaware of *some events ex ante* (i.e., before the experiment).

To see this, suppose that the outside observer sees the following sequence of choices: (1) $f \succeq g$, (2) $g \succeq h$ (or $h \succeq g$), and (3) $f \prec g$. Then the outside observer can conclude that the decision maker became aware of *some* event when facing h. Note that observed choices may not have properties of a preference relation anymore since it appears as if her choices are intransitive. Note further that the outside observer can not conclude of which event the decision maker became aware through h. Finally, note that the converse is not true. I.e., it is not true that if the decision maker is unaware of some event, then we can find such a choice experiment to reveal it. For instance, already with the first

[6] A decision maker's awareness level may increase to S after facing the act f and further to S' after facing the act g, but it can not be that her awareness level is $S'' \neq S'$ when facing f after g.

choice between f and g the decision maker may become aware of an event.

I conclude that if a decision maker's conceivable "small worlds" are affected by the acts the decision maker is facing, then the possibility of revealing unawareness is extremely limited. In such a case, the outside observer may need to use additional assumptions on how acts influence awareness in order to reveal a decision maker's unawareness.

6.3 Related Literature

Li (2008) analyzes in a different model unawareness versus zero probability. Her study is a bit more ambitious than mine as she considers a two-period model in which an initially unaware decision maker becomes aware in the second period. The decision maker chooses among bets defined on her first period "subjective" states. This requires her to specify how those "subjective bets" correspond to "objective" bets in the second period. In contrast, in my model acts are defined already on all states although the decision maker may have a limited understanding of them. Li (2008) considers various specifications, including one in which unawareness of an event may be though of "as if" the decision maker believes that the event does not obtain.

Ahn and Ergin (2010) study framing that may also be due to lack of awareness. They take more or less fine partitions of a state-space as the primitive. Since the set of all partitions forms a lattice, I believe that their analysis could be "translated" into unawareness structures. In their approach, acts are defined to be measurable with respect to some of the partitions. When a decision maker faces an act that is measurable with respect to some partition, then she evaluates the act on at least the events of that partition. Intuitively, they assume that a decision maker always reads the "fine prints" of an act presented. This is important for their aim of studying how decisions are affected by framing through acts. It is in contrast to my approach taken in Section 3 because - translated into their approach - I define acts on all partitions simultaneously. One interesting feature of their representation is a (not necessarily additive) set function from which the partition-dependent probability measure is defined. It allows them to relate beliefs across partitions. They discuss various interpretations of this set function. In particular, their approach is an extension and axiomatization of support theory in psychology.

Ahn and Ergin's (2010) notion of "completely unforeseen" differs from the notion of (propositional) unawareness in the epistemic literature. It is consistent with their model that an event is "completely unforeseen" while its negation is not. This is in contrast with the symmetry property of unawareness: if a decision maker can reason about the negation of an event, then she can reason about the event (and vice versa).

Grant and Quiggin (2008) study in a dynamic model under which conditions decisions taken by a decision maker within her "small worlds" are optimal also when being fully aware. While those conditions are quite stringent, the question is meaningful from a paternalistic point of view.

There is also a growing literature on "subjective state spaces" that are derived from preferences. The motivation of this literature can be viewed as a critique of my approach since analogous to Savage's state-space, I take the lattice of spaces as primitive. For instance, as Epstein, Marinacci, and Seo (2007) rightfully point out "even if we know how to model a 'coarse or incomplete state' and we redefined the Savage state-space accordingly" (as I do), "the resulting approach would still be unsatisfactory if, as in Savage, the state-space were adopted as a primitive and thus presumed observable by the modeler. Ideally, the agent's conceptualization of the future should be taken to be subjective - it should be derived from preferences, that is, from in principle observable behavior." In my approach, the choices of the decision maker effectively reveal the space among exogenously predefined state-spaces. I find it extremely intriguing to also derive the entire unawareness structure endogenously from (admittedly counterfactual) choices. This is beyond the scope of this paper and left for future research. In defense of my current approach, I like to point out that, first, the modest goal of my paper is to provide a simple tool for studying both unawareness and subjective expected utility. The literature on "subjective state-spaces" does not focus on unawareness as defined in the epistemic literature. Rather, it analyzes a decision maker facing "coarse" contingencies meaning that the decision maker lacks conception of some contingencies, knows that she lacks conception and may take this into account. A comprehensive model of unforeseen contingencies should have both, the absolute lack of conception of some contingencies as under unawareness as well as the suspicion of some other contingencies out there. Suspicion of unawareness may be conceptually questionable in models that epistemically preclude any unawareness.

Finally, Blume, Easley, and Halpern (2009) take a syntactic approach to subjective expected utility theory in which primitives in standard subjective expected utility theory such as the state-space, outcome space, and acts are replaced by syntactic descriptions. This requires a modified set of properties which are used to characterize subjective expected utility theory including the primitives. It is intriguing to extend their approach to unawareness structures. I believe some ideas from Heifetz, Meier, and Schipper (2008) can be used for that.

A Proofs

A.1 Proof of Remark 4

(i) If $S(E) \not\preceq S$, then $f_E g(\omega) = g(\omega)$ for all $\omega \in S$ for all $f, g \in \mathcal{A}$. Hence by Property 6, $f_E g \sim_S g$ for all $f, g \in \mathcal{A}$.

(ii) If $S(E) \not\preceq S$, then $f_E g(\omega) = h_E g(\omega)$ for all $\omega \in S$. Hence by Property 6, $f_E g \sim_S h_E g$. Since $S(E) = S(\neg E)$, we have by analogous arguments $f_{\neg E} g \sim_S h_{\neg E} g$. $\square$

A.2 Proof of Remark 5

If $S' \preceq S$, then $f_{S'^{\uparrow}} g(\omega) = f(\omega)$ and $h_{S'^{\uparrow}} g(\omega) = h(\omega)$ for all $\omega \in S'^{\uparrow} \cap S$. Since $S' \preceq S$, we have $S'^{\uparrow} \cap S = S$. Hence by Property 6 and transitivity (Property 1), $f_{S'^{\uparrow}} g \sim_S f \succsim_S h \sim_S h_{S'^{\uparrow}} g$ imply $f_{S'^{\uparrow}} g \succsim_S h_{S'^{\uparrow}} g$ and vice versa. $\square$

A.3 Proof of Remark 6

Assume that Properties 1, 4, and 6 hold, and suppose by contradiction that for some $S \in \mathcal{S}$ all states in S are S-null. Since S is finite, number states $1, ..., |S|$. Then for all $g, h \in \mathcal{A}$, $g \sim_S h_{\{\omega_1\}^{\uparrow}} g \sim_S h_{\{\omega_1, \omega_2\}^{\uparrow}} g \sim_S ... \sim_S h_{\{\omega_1, \omega_2, ..., \omega_{|S|-1}\}^{\uparrow}} g \sim_S h_{S^{\uparrow}} g \sim_S h$, where the last $\sim_S$ follows from Property 6. By transitivity (Property 1), we have $g \sim_S h$ for all $g, h \in \mathcal{A}$, a contradiction to Property 4. $\square$

A.4 Proof of Remark 7

"$\Leftarrow$": If $\omega \notin S$, then $f_{\{\omega\}} g(\omega') = g(\omega') = h_{\{\omega\}} g(\omega')$ for all $\omega' \in S$ and all $f, g, h \in \mathcal{A}$. Thus by Property 6, $f_{\{\omega\}} g \sim_S h_{\{\omega\}} g$ for all $f, g, h \in \mathcal{A}$.

State ω being S-null means $S(\{\omega\}^{\uparrow}) \preceq S$ and $f_{\{\omega\}^{\uparrow}} g \sim_S h_{\{\omega\}^{\uparrow}} g$ for all $f, g, h \in \mathcal{A}$. If $S(\{\omega\}^{\uparrow}) = S$, then $f_{\{\omega\}} g(\omega') = f_{\{\omega\}^{\uparrow}} g(\omega')$ and $h_{\{\omega\}} g(\omega') = h_{\{\omega\}^{\uparrow}} g(\omega')$ for all $\omega' \in S$ and all $f, g, h \in \mathcal{A}$. By Property 6, $f_{\{\omega\}} g \sim_S f_{\{\omega\}^{\uparrow}} g$ and $h_{\{\omega\}} g \sim_S h_{\{\omega\}^{\uparrow}} g$ for all $f, g, h \in \mathcal{A}$. Thus by Property 1, $f_{\{\omega\}} g = h_{\{\omega\}} g$ for all $f, g, h \in \mathcal{A}$. If $S(\{\omega\}^{\uparrow}) \prec S$, then $\omega \notin S$. Thus, in this case the result follows from above arguments.

"$\Rightarrow$": Suppose to the contrary that $f_{\{\omega\}}g \sim_S h_{\{\omega\}}g$ for all $f, g, h \in \mathcal{A}$ but ω is S-nonnull and $\omega \in S$. ω being S-nonnull and $\omega \in S$ means that $f_{\{\omega\}^{\uparrow}}g \succ_S h_{\{\omega\}^{\uparrow}}g$ for some $f, g, h \in \mathcal{A}$. Since $\omega \in S$, $f_{\{\omega\}^{\uparrow}}g(\omega') = f_{\{\omega\}}g(\omega')$ and $h_{\{\omega\}^{\uparrow}}g(\omega') = h_{\{\omega\}}g(\omega')$ for all $\omega' \in S$. By Property 6, $f_{\{\omega\}^{\uparrow}}g \sim_S f_{\{\omega\}}g$ and $h_{\{\omega\}^{\uparrow}}g \sim_S h_{\{\omega\}}g$. From Property 1 follows $f_{\{\omega\}}g \succ_S h_{\{\omega\}}g$, a contradiction. □

A.5 Proofs of Theorems 1 and 2

The proofs follows essentially Fishburn (1970, Chapter 13.1 and 13.2). We point out minor differences along the way. We present the proofs of both results side-by-side so that the interested reader can compare the differences. Moreover, this presentation helps to minimize redundancies.

First we show the following representation results in terms of state-dependent utilities or additively separable utilities.

Proposition 4 *$\{\succsim_S\}_{S\in\mathcal{S}}$ satisfies Properties 1 to 3 and 6 if and only if there exists a collection of functions $\{u_S : X \times S \longrightarrow \mathbb{R}\}_{S\in\mathcal{S}}$ such that for all $S \in \mathcal{S}$ and $f, g \in \mathcal{A}$,*

$f \succ_S g$ if and only if

$$\sum_{\omega\in S} \sum_{x\in supp(f(\omega))} u_S(x,\omega) f(\omega)(x) > \sum_{\omega\in S} \sum_{x\in supp(g(\omega))} u_S(x,\omega) g(\omega)(x). \tag{2}$$

Moreover, if $\{v_S : X \times S \longrightarrow \mathbb{R}\}_{S\in\mathcal{S}}$ is another collection of functions satisfying formula (2), then for each $S \in \mathcal{S}$ there exist constants $a_S \in \mathbb{R}_{++}$ and $b_S \in \mathbb{R}$ such $a_S u_S(\cdot,\omega) + b_S = v_S(\cdot,\omega)$ for each $\omega \in S$.

Proposition 5 *$\{\succsim_S\}_{S\in\mathcal{S}}$ satisfies Properties 1 to 3 and 6 if and only if there exists a collection of functions $\{w_S : X \times \Omega \longrightarrow \mathbb{R}\}_{S\in\mathcal{S}}$ such that for all $S \in \mathcal{S}$ and $f, g \in \mathcal{A}$,*

$f \succ_S g$ if and only if

$$\sum_{\omega\in \Omega} \sum_{x\in supp(f(\omega))} w_S(x,\omega) f(\omega)(x) > \sum_{\omega\in \Omega} \sum_{x\in supp(g(\omega))} w_S(x,\omega) g(\omega)(x). \tag{3}$$

Moreover, if $\{z_S : X \times \Omega \longrightarrow \mathbb{R}\}_{S\in\mathcal{S}}$ is another collection of functions satisfying formula (3), then for each $S \in \mathcal{S}$ there exist constants $a_S \in \mathbb{R}_{++}$ and $b_S \in \mathbb{R}$ such $a_S w_S(\cdot,\omega) + b_S = z_S(\cdot,\omega)$ for each $\omega \in \Omega$.

Proofs of Propositions. Under Properties 1 to 3 and 6, the existence of a collection of functions $\{U_S : \mathcal{A} \longrightarrow \mathbb{R}\}_{S\in\mathcal{S}}$ such that for $f, g \in \mathcal{A}$

$$f \succ_S g \text{ if and only if } U_S(f) > U_S(g) \tag{4}$$

and U_S being affine, i.e.,

$$U_S(\alpha f + (1-\alpha)g) = \alpha U_S(f) + (1-\alpha)U_S(g), \text{ for all } \alpha \in [0,1], \tag{5}$$

follows from applying the Mixture-Space Theorem (Herstein and Milnor, 1953, see also Fishburn, 1970, Section 8.4) for each $S \in \mathcal{S}$. Moreover, for each $S \in \mathcal{S}$, U_S is unique up to positive affine transformations.

We want to show that for $f \in \mathcal{A}$,

$$U_S(f) = \sum_{\omega \in S} \sum_{x \in supp(f(\omega))} u_S(x,\omega) f(\omega)(x) \tag{6}$$

for some function $u_S : X \times S \longrightarrow \mathbb{R}$ for every $S \in \mathcal{S}$.

We want to show that for $f \in \mathcal{A}$,

$$U_S(f) = \sum_{\omega \in \Omega} \sum_{x \in supp(f(\omega))} w_S(x,\omega) f(\omega)(x) \tag{7}$$

for some function $w_S : X \times \Omega \longrightarrow \mathbb{R}$ for every $S \in \mathcal{S}$.

The next step in the proof of Proposition 4 differs slightly from the Anscombe-Aumann approach.

We claim that Property 6 implies that

$$\frac{1}{|S|} f + \frac{|S|-1}{|S|} g \sim_S \sum_{\omega \in S} \frac{1}{|S|} f_{\{\omega\}\uparrow} g. \tag{8}$$

To see the claim, number states in S by $1, ..., |S|$, and observe that for any $\omega \in S$,

$$\begin{aligned} &\frac{1}{|S|} f(\omega) + \frac{|S|-1}{|S|} g(\omega) \\ = \ & \frac{1}{|S|} f_{\{\omega_1\}\uparrow} g(\omega) + \cdots + \frac{1}{|S|} f_{\{\omega_{|S|}\}\uparrow} g(\omega) \\ = \ & \sum_{\omega' \in S} \frac{1}{|S|} f_{\{\omega'\}\uparrow} g(\omega). \end{aligned}$$

Hence Property 6 implies the claim.

By equations (4) and (5), we have

$$\begin{aligned} &\frac{1}{|S|} U_S(f) + \frac{|S|-1}{|S|} U_S(g) \\ = \ & \frac{1}{|S|} \sum_{\omega \in S} U_S\left(f_{\{\omega\}\uparrow} g\right). \end{aligned} \tag{10}$$

Define $u_S : \Delta(X) \times S \longrightarrow \mathbb{R}$ by

$$u_S(p,\omega) := U_S\left(p_{\{\omega\}\uparrow} g\right) - \frac{|S|-1}{|S|} U_S(g). \tag{12}$$

For $f \in \mathcal{A}$,

$$u_S(f(\omega),\omega) = U_S\left(f_{\{\omega\}\uparrow} g\right) - \frac{|S|-1}{|S|} U_S(g). \tag{14}$$

Summing over $\omega \in S$ and dividing by $|S|$, we obtain

$$\begin{aligned} &\frac{1}{|S|} \sum_{\omega \in S} u_S(f(\omega),\omega) \\ = \ & \frac{1}{|S|} \sum_{\omega \in S} U_S\left(f_{\{\omega\}\uparrow} g\right) - \frac{|S|-1}{|S|} U_S(g). \end{aligned} \tag{16}$$

Comparing it with equation (10), we have

$$U_S(f) = \sum_{\omega \in S} u_S(f(\omega),\omega). \tag{18}$$

We claim

$$\frac{1}{|\Omega|} f + \frac{|\Omega|-1}{|\Omega|} g = \sum_{\omega \in \Omega} \frac{1}{|\Omega|} f_{\{\omega\}} g. \tag{9}$$

To see the claim, number states in Ω by $1, ..., |\Omega|$, and observe that for any $\omega \in S$,

$$\begin{aligned} &\frac{1}{|\Omega|} f(\omega) + \frac{|\Omega|-1}{|\Omega|} g(\omega) \\ = \ & \frac{1}{|\Omega|} f_{\{\omega_1\}} g(\omega) + \cdots + \frac{1}{|\Omega|} f_{\{\omega_{|\Omega|}\}} g(\omega) \\ = \ & \sum_{\omega' \in \Omega} \frac{1}{|\Omega|} f_{\{\omega'\}} g(\omega). \end{aligned}$$

By equations (4) and (5), we have

$$\begin{aligned} &\frac{1}{|\Omega|} U_S(f) + \frac{|\Omega|-1}{|\Omega|} U_S(g) \\ = \ & \frac{1}{|\Omega|} \sum_{\omega \in \Omega} U_S\left(f_{\{\omega\}} g\right). \end{aligned} \tag{11}$$

Define $w_S : \Delta(X) \times \Omega \longrightarrow \mathbb{R}$ by

$$w_S(p,\omega) := U_S\left(p_{\{\omega\}} g\right) - \frac{|\Omega|-1}{|\Omega|} U_S(g). \tag{13}$$

For $f \in \mathcal{A}$,

$$w_S(f(\omega),\omega) = U_S\left(f_{\{\omega\}} g\right) - \frac{|\Omega|-1}{|\Omega|} U_S(g). \tag{15}$$

Summing over $\omega \in \Omega$ and dividing by $|\Omega|$, we obtain

$$\begin{aligned} &\frac{1}{|\Omega|} \sum_{\omega \in \Omega} w_S(f(\omega),\omega) \\ = \ & \frac{1}{|\Omega|} \sum_{\omega \in \Omega} U_S\left(f_{\{\omega\}} g\right) - \frac{|\Omega|-1}{|\Omega|} U_S(g). \end{aligned} \tag{17}$$

Comparing it with equation (11), we have

$$U_S(f) = \sum_{\omega \in \Omega} w_S(f(\omega),\omega). \tag{19}$$

Combining equations (12) and (5) yields for $p, q \in \Delta(X)$

$$\begin{aligned} & u_S(\alpha p + (1-\alpha)q, \omega) \qquad (20) \\ = \quad & \alpha u_S(p,\omega) + (1-\alpha) u_S(q,\omega) \text{ for } \alpha \in [0,1] \end{aligned}$$

for $\omega \in S$.
For $x \in X$, let $u_S(x,\omega) = u_S(\delta_x, \omega)$, with δ_x being the Dirac measure with unit mass on x. Since the support of a simple probability measure is finite,

$$u_S(p,\omega) = \sum_{x \in supp(p)} u_S(x,\omega). \qquad (22)$$

Combining the representation in formula (4) with equation (18) yields inequality (2) for $f, g \in \mathcal{A}$. Repeat this construction for each $S \in \mathcal{S}$.

Uniqueness up to positive linear transformations follows from the uniqueness of U_S. If $v_S(\cdot,\omega)$ satisfies formula (2) in place of $u_S(\cdot,\omega)$, then

$$V_S(f) = \sum_{\omega \in S} \sum_{x \in supp(f(\omega))} v_S(x,\omega) f(\omega)(x),$$

$V_S = a_S U_S + b_S$, and $a_S > 0$. Holding $f(\omega')(x)$ fixed for all $\omega' \in S$, $\omega' \neq \omega$, it then follows that $v_S(\cdot,\omega) = a_S(\omega) u_S(\cdot,\omega) + b_S(\omega)$. This holds for each $\omega \in S$.
Note that $u_S(\cdot,\omega)$ is constant on X if and only if $\omega \in S$ is S-null. To see this, $\omega \in S$ being S-null means (with some slight abuse of notation) $x_{\{\omega\}\uparrow} g \sim_S g$ for all $x \in X$ and $g \in \mathcal{A}$, which is equivalent by formula (4) to $U_S(x_{\{\omega\}\uparrow} g) = U_S(g)$ for all $x \in X$ and $g \in \mathcal{A}$. $u_S(x,\omega) = U_S(g) - \frac{|S|-1}{|S|} U_S(g) = \frac{1}{|S|} U_S(g)$ which is independent of x and thus constant in x.

For the converse, we prove only the nonstandard Property 6. Suppose by contradiction that we have the representation in formula (2) but Property 6 is violated. Then there exist a space $S \in \mathcal{S}$ and acts $f, g \in \mathcal{A}$ with $f(\omega) = g(\omega)$ for all $\omega \in S$ but $f \succ_S g$. But this contradicts formula (2). $\square$
.

Combining equations (13) and (5) yields for $p, q \in \Delta(X)$

$$\begin{aligned} & w_S(\alpha p + (1-\alpha)q, \omega) \qquad (21) \\ = \quad & \alpha w_S(p,\omega) + (1-\alpha) w_S(q,\omega) \text{ for } \alpha \in [0,1] \end{aligned}$$

for $\omega \in \Omega$.
For $x \in X$, let $w_S(x,\omega) = w_S(\delta_x, \omega)$, with δ_x being the Dirac measure with unit mass on x. Since the support of a simple probability measure is finite,

$$w_S(p,\omega) = \sum_{x \in supp(p)} w_S(x,\omega). \qquad (23)$$

Combining the representation in formula (4) with equation (19) yields inequality (3) for $f, g \in \mathcal{A}$. Repeat this construction for each $S \in \mathcal{S}$.

Uniqueness up to positive linear transformations follows from the uniqueness of U_S. If $z_S(\cdot,\omega)$ satisfies formula (3) in place of $w_S(\cdot,\omega)$, then

$$Z_S(f) = \sum_{\omega \in \Omega} \sum_{x \in supp(f(\omega))} z_S(x,\omega) f(\omega)(x),$$

$Z_S = a_S U_S + b_S$, and $a_S > 0$. Holding $f(\omega')(x)$ fixed for all $\omega' \in \Omega$, $\omega' \neq \omega$, it then follows that $z_S(\cdot,\omega) = a_S(\omega) w_S(\cdot,\omega) + b_S(\omega)$. This holds for each $\omega \in \Omega$.
Note that $w_S(\cdot,\omega)$ is constant on X if and only if $\omega \in \Omega$ is S-null or $\omega \in \Omega \setminus S$. To see this, note that by Remark 7 (Property 1 and 6) $\omega \in \Omega$ being S-null or $\omega \notin S$ if and only if (with some slight abuse of notation) $x_{\{\omega\}} g \sim_S g$ for all $x \in X$ and $g \in \mathcal{A}$, which is equivalent by formula (4) to $U_S(x_{\{\omega\}\uparrow} g) = U_S(g)$ for all $x \in X$ and $g \in \mathcal{A}$. $u_S(x,\omega) = U_S(g) - \frac{|\Omega|-1}{|\Omega|} U_S(g) = \frac{1}{|\Omega|} U_S(g)$ which is independent of x and thus constant in x.

For the converse, we prove only the nonstandard Property 6. Suppose by contradiction that we have the representation in formula (3) but Property 6 is violated. Then there exist a space $S \in \mathcal{S}$ and acts $f, g \in \mathcal{A}$ with $f(\omega) = g(\omega)$ for all $\omega \in S$ but $f \succ_S g$.
Note that

$$\begin{aligned} & \sum_{\omega \in S} \sum_{x \in supp(f(\omega))} w_S(x,\omega) f(\omega)(x) \qquad (24) \\ & \quad = \sum_{\omega \in S} \sum_{x \in supp(g(\omega))} w_S(x,\omega) g(\omega)(x) \end{aligned}$$

follows from $f(\omega) = g(\omega)$ for all $\omega \in S$.

.

Note further that

$$\sum_{\omega\in\Omega\setminus S}\sum_{x\in supp(f(\omega))} w_S(x,\omega)f(\omega)(x) \qquad (25)$$
$$= \sum_{\omega\in\Omega\setminus S}\sum_{x\in supp(g(\omega))} w_S(x,\omega)g(\omega)(x)$$

since as we noted earlier $w_S(\cdot,\omega)$ is constant on X for all $\omega\in\Omega\setminus S$.

.

But this contradicts formula (3). □

We continue with the proof of Theorems 1 and 2 respectively. Fix a space $S\in\mathcal{S}$. By Remark 6, there exists a S-nonnull state $\omega^\circ\in S$. Let $p,q\in\Delta(X)$ and let $\omega\in S$ be any S-nonnull state.

For any $f \in \mathcal{A}$,

$$\sum_{x \in supp(p)} u_S(x,\omega)p(x) > \sum_{x \in supp(q)} u_S(x,\omega)q(x) \quad (26)$$

if and only if

$$U_S(p_{\{\omega\}^\uparrow} f) > U_S(q_{\{\omega\}^\uparrow} f) \quad (27)$$

if and only if by Proposition 4

$$p_{\{\omega\}^\uparrow} f \succ_S q_{\{\omega\}^\uparrow} f \quad (28)$$

if and only if by Property 5

$$p_{\{\omega^\circ\}^\uparrow} f \succ_S q_{\{\omega^\circ\}^\uparrow} f \quad (29)$$

if and only if by Proposition 4

$$U_S(p_{\{\omega^\circ\}^\uparrow} f) > U_S(q_{\{\omega^\circ\}^\uparrow} f) \quad (30)$$

if and only if

$$\sum_{x \in supp(p)} u_S(x,\omega^\circ)p(x) > \sum_{x \in supp(q)} u_S(x,\omega^\circ)q(x). \quad (31)$$

By the uniqueness of von Neumann-Morgenstern utilities, there exist constants $a_S(\omega) > 0$ and $b_S(\omega)$ such that

$$a_S(\omega)u_S(\cdot,\omega^\circ) + b_S(\omega) = u_S(\cdot,\omega). \quad (39)$$

For any $f \in \mathcal{A}$,

$$\sum_{x \in supp(p)} w_S(x,\omega)p(x) > \sum_{x \in supp(q)} w_S(x,\omega)q(x) \quad (32)$$

if and only if

$$U_S(p_{\{\omega\}^\uparrow} f) > U_S(q_{\{\omega\}^\uparrow} f). \quad (33)$$

To see this note that

$$\begin{aligned} U_S(p_{\{\omega\}^\uparrow} f) &= \sum_{x \in supp(p)} w_S(x,\omega)p(x) \\ &+ \sum_{\omega' \in \{\omega\}^\uparrow \setminus \{\omega\}} \sum_{x \in supp(p)} w_S(x,\omega')p(x) \\ &+ \sum_{\omega' \in \Omega \setminus \{\omega\}^\uparrow} \sum_{x \in supp(f(\omega'))} w_S(x,\omega')f(\omega')(x). \end{aligned}$$

It is sufficient to show

$$\begin{aligned} &\sum_{\omega' \in \{\omega\}^\uparrow \setminus \{\omega\}} \sum_{x \in supp(p)} w_S(x,\omega')p(x) \\ &= \sum_{\omega' \in \{\omega\}^\uparrow \setminus \{\omega\}} \sum_{x \in supp(q)} w_S(x,\omega')q(x). \end{aligned} \quad (34)$$

Since $\omega \in S$, $\omega' \in \{\omega\}^\uparrow \setminus \{\omega\}$ implies $\omega' \notin S$. By arguments in the proof of Proposition 5, $w_S(\cdot,\omega')$ is constant in X. This yields inequality (34). Inequality (33) holds if and only if by Proposition 5

$$p_{\{\omega\}^\uparrow} f \succ_S q_{\{\omega\}^\uparrow} f \quad (35)$$

if and only if by Property 5

$$p_{\{\omega^\circ\}^\uparrow} f \succ_S q_{\{\omega^\circ\}^\uparrow} f \quad (36)$$

if and only if by Proposition 5

$$U_S(p_{\{\omega^\circ\}^\uparrow} f) > U_S(q_{\{\omega^\circ\}^\uparrow} f) \quad (37)$$

if and only if (by analogous arguments as for inequality (33))

$$\sum_{x \in supp(p)} w_S(x,\omega^\circ)p(x) > \sum_{x \in supp(q)} w_S(x,\omega^\circ)q(x). \quad (38)$$

By the uniqueness of von Neumann-Morgenstern utilities, there exist constants $a_S(\omega) > 0$ and $b_S(\omega)$ such that

$$a_S(\omega)w_S(\cdot,\omega^\circ) + b_S(\omega) = w_S(\cdot,\omega). \quad (40)$$

For S-null states let $a_S(\omega) = 0$ since we observed in the proof of Proposition 4 that ω is S-null if and only if $u_S(\cdot, \omega)$ is constant on X.

Define $u_S(x) := u_S(x, \omega^\circ)$, i.e. $a_S(\omega^\circ) = 1$ and $b_S(\omega^\circ) = 0$. Then the representation in formula (2) becomes

$$f \succ_S g \text{ if and only if}$$

$$\sum_{\omega \in S} \sum_{x \in supp(f(\omega))} (a_S(\omega) u_S(x) + b_S(\omega)) f(\omega)(x) \quad (41)$$
$$> \sum_{\omega \in S} \sum_{x \in supp(g(\omega))} (a_S(\omega) u_S(x) + b_S(\omega)) g(\omega)(x)$$

which simplifies to

$$\sum_{\omega \in S} \left(b_S(\omega) + a_S(\omega) [\sum_{x \in supp(f(\omega))} u_S(x) f(\omega)(x)] \right)$$
$$> \sum_{\omega \in S} \left(b_S(\omega) + a_S(\omega) [\sum_{x \in supp(g(\omega))} u_S(x) g(\omega)(x)] \right).$$

We cancel $b_S(\omega)$, divide by $\sum_{\omega \in S} a_S(\omega)$, and define

$$\mu_S(\omega) := \frac{a_S(\omega)}{\sum_{\omega' \in S} a_S(\omega')} \quad (43)$$

to obtain

$$\sum_{\omega \in S} \left(\sum_{x \in supp(f(\omega))} u_S(x) f(\omega)(x) \right) \mu_S(\omega)$$
$$> \sum_{\omega \in S} \left(\sum_{x \in supp(g(\omega))} u_S(x) g(\omega)(x) \right) \mu_S(\omega).$$

If ω is S-null or $\omega \in \Omega \setminus S$, let $a_S(\omega) = 0$ since we observed in the proof of Proposition 5 that ω is S-null or $\omega \in \Omega \setminus S$ if and only if $w_S(\cdot, \omega)$ is constant on X.

Define $w_S(x) := w_S(x, \omega^\circ)$, i.e. $a_S(\omega^\circ) = 1$ and $b_S(\omega^\circ) = 0$. Then the representation in formula (3) becomes

$$f \succ_S g \text{ if and only if}$$

$$\sum_{\omega \in \Omega} \sum_{x \in supp(f(\omega))} (a_S(\omega) w_S(x) + b_S(\omega)) f(\omega)(x) \quad (42)$$
$$> \sum_{\omega \in \Omega} \sum_{x \in supp(g(\omega))} (a_S(\omega) w_S(x) + b_S(\omega)) g(\omega)(x)$$

which simplifies to

$$\sum_{\omega \in \Omega} \left(b_S(\omega) + a_S(\omega) [\sum_{x \in supp(f(\omega))} w_S(x) f(\omega)(x)] \right)$$
$$> \sum_{\omega \in \Omega} \left(b_S(\omega) + a_S(\omega) [\sum_{x \in supp(g(\omega))} w_S(x) g(\omega)(x)] \right).$$

We cancel $b_S(\omega)$, divide by $\sum_{\omega \in \Omega} a_S(\omega)$, and define

$$\varphi_S(\omega) := \frac{a_S(\omega)}{\sum_{\omega' \in \Omega} a_S(\omega')} \quad (44)$$

to obtain

$$\sum_{\omega \in \Omega} \left(\sum_{x \in supp(f(\omega))} w_S(x) f(\omega)(x) \right) \varphi_S(\omega)$$
$$> \sum_{\omega \in \Omega} \left(\sum_{x \in supp(g(\omega))} w_S(x) g(\omega)(x) \right) \varphi_S(\omega).$$

Repeating this construction for every $S \in \mathcal{S}$ yields representations of Theorems 1 and 2 respectively. □

A.6 Proof of Proposition 2

Suppose that Properties 1 to 6 hold. We need to show that $\mu_S \in \Delta(S)$, for $S \not\succeq S(E)$ if and only if for all events F such that $S(F) = S(E)$ we have $f_F \sim h_F g$ for all $f, g, h \in \mathcal{A}$.

"⇒": If $\mu_S \in \Delta(S)$ with $S \not\succeq S(E)$, then for all events F with $S(F) = S(E)$,

$$\sum_{\omega \in S} \left(\sum_{x \in supp(f_F g(\omega))} u_S(x) f_F g(\omega)(x) \right) \mu_S(\omega)$$

$$= \sum_{\omega \in S} \left(\sum_{x \in supp(h_F g(\omega))} u_S(x) h_F g(\omega)(x) \right) \mu_S(\omega). \quad (45)$$

for all $f, g, h \in \mathcal{A}$. By Theorem 1, we have $f_F g \sim_S h_F g$ for all events F such that $S(F) = S(E)$ and all $f, g, h \in \mathcal{A}$, and $S \not\preceq S(E)$.

"$\Leftarrow$": If for all events F with $S(F) = S(E)$ we have $f_F g \sim h_F g$ for all $f, g, h \in \mathcal{A}$, then $\sim = \sim_S$ with $S \not\preceq S(E)$ since otherwise it contradicts Remark 6. By Theorem 1, there exists an awareness-dependent expected utility for which equation (45) holds for all $f, g, h \in \mathcal{A}$ and $F \in \Sigma$ such that $S(F) = S(E)$. Thus $\mu_S \in \Delta(S)$ with $S \not\preceq S(E)$. □

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